Delay-Dependent Criterion for Asymptotic Stability of a Class of Neutral Equations

J. H. PARK
School of Electrical Engineering and Computer Science
Yeungnam University, 214-1 Dae-Dong
Kyongsan 712-749, Republic of Korea
jessie@yu.ac.kr

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Abstract—In this letter, a sufficient condition for all solutions of a class of neutral equations to approach zero at \( t \to \infty \) is presented. The condition, which is expressed in terms of linear matrix inequality, is delay-dependent and can be easily solved by various efficient convex optimization algorithm. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

The main purpose of this article is to investigate the asymptotic behaviors of solutions of the neutral delay equation

\[
\frac{d}{dt} [x(t) + px(t - \tau)] = -ax(t) + b \tanh x(t - \tau), \quad t \geq 0,
\]

where \( a, b, \tau, \) and \( \sigma \) are positive real numbers, \( \sigma \geq \tau \) and \( |p| < 1 \). With each solution \( x(t) \) of equation (1), we assume the initial condition:

\[
x(s) = \phi(s), \quad s \in [-\sigma, 0], \text{ where } \phi \in C([-\sigma, 0], \mathbb{R}).
\]

Delay differential equations of various types including equation (1) have been investigated by many authors for the study of the dynamic characteristics of neural networks of Hopfield type (see [1] and references cited therein).

Recently, the asymptotic stability of equation (1) has been discussed in [2,3], and the delay-independent sufficient conditions for the stability have been presented. In the work [2], only the case \( \sigma = \tau \) is considered. In general, abandonment of information on the delay causes conservativeness of the stability criteria especially when delays are small. Delay-dependent criteria are often less conservative than delay-independent criteria.
In this article, we shall establish a delay-dependent sufficient condition for all solutions of equation (1) to approach zero as \( t \to \infty \) using Lyapunov functional method and linear matrix inequality technique.

Through the article, * represents the elements below the main diagonal of a symmetric matrix. The notation \( X > Y \), where \( X \) and \( Y \) are matrices of same dimensions, means that the matrix \( X - Y \) is positive definite.

### 2. MAIN RESULT

Equation (1) can be written in the following form:

\[
\frac{dx(t)}{dt} + px(t - \tau) + b \int_{t-\tau}^t \tanh x(s) \, ds = -ax(t) + b \tanh x(t), \quad t \geq 0.
\]  

Define the operator \( D : C \to \mathcal{R} \) as

\[
D(x_t) = x(t) + px(t - \tau) + b \int_{t-\tau}^t \tanh x(s) \, ds.
\]

Now, we have the following theorem.

**Theorem 1.** For given \( \sigma > 0 \), every solution \( x(t) \) of equation (1) satisfies \( x(t) \to 0 \) as \( t \to \infty \), if the operator \( D \) is stable and there exist the positive scalars \( \varepsilon_1, \varepsilon_2, \varepsilon_3, \alpha, \) and \( \beta \) such that the linear matrix inequality holds

\[
\Sigma(\alpha, \beta, \varepsilon_1, \varepsilon_2, \varepsilon_3) = \begin{bmatrix}
-2a + \alpha + \sigma \beta + \varepsilon_1 b & \sqrt{b} & b & -ap & -\sigma ab \\
* & -\varepsilon_1 & 0 & 0 & 0 \\
* & * & -\varepsilon_2 & 0 & 0 \\
* & * & * & -\varepsilon_3 & 0 \\
* & * & * & * & \varepsilon_2 b^2 - \alpha \\
* & * & * & * & \varepsilon_3 \sigma b^3 - \beta \sigma
\end{bmatrix} < 0.
\]

**Proof.** Consider the Lyapunov functional defined by

\[
V(t) = D^2(x_t) + \alpha \int_{t-\tau}^t x^2(s) \, ds + \beta \int_{t-\tau}^t (\sigma - t + s) \tanh^2 x(s) \, ds,
\]

where \( \alpha \) and \( \beta \) are positive scalars to be chosen later.

The derivative of \( V(t) \) along the solution of equation (2) is given by

\[
\frac{dV}{dt} = 2 \left( x(t) + px(t - \tau) + b \int_{t-\tau}^t \tanh x(s) \, ds \right) (-ax(t) + b \tanh x(t))
\]

\[
+ \alpha x^2(t) - \alpha x^2(t - \tau) + \beta \sigma \tanh^2 x(t) - \beta \int_{t-\tau}^t \tanh^2 x(s) \, ds
\]

\[
= -2ax^2(t) + 2bx(t) \tanh x(t) - 2apx(t)(t - \tau) + 2pbx(t - \tau) \tanh x(t)
\]

\[
- 2abx(t) \int_{t-\tau}^t \tanh x(s) \, ds + 2b^2 \tanh x(t) \int_{t-\tau}^t \tanh x(s) \, ds + \alpha x^2(t)
\]

\[
- \alpha x^2(t - \tau) + \beta \sigma \tanh^2 x(t) - \beta \int_{t-\tau}^t \tanh^2 x(s) \, ds.
\]

Using the well-known inequality \( 2uv \leq \varepsilon u^2 + \varepsilon^{-1} v^2 \) for \( \varepsilon > 0 \), we have

\[
\frac{dV}{dt} \leq -2ax^2(t) + b \left( \varepsilon_1 x^2(t) + \varepsilon_1^{-1} \tanh^2 x(t) \right) - 2apx(t)(t - \tau)
\]

\[
+ b \left( \varepsilon_2 b^2 \sigma^2 (t - \tau) + \varepsilon_2^{-1} \tanh^2 x(t) \right) - 2abx(t) \int_{t-\tau}^t \tanh x(s) \, ds
\]

\[
+ b^2 \left[ \varepsilon_3^{-1} \tanh^2 x(t) + \varepsilon_3 \left( \int_{t-\tau}^t \tanh x(s) \, ds \right)^2 \right] + \alpha x^2(t)
\]

\[
- \alpha x^2(t - \tau) + \beta \sigma \tanh^2 x(t) - \beta \int_{t-\tau}^t \tanh^2 x(s) \, ds.
\]
Utilizing the relation \( \tanh^2 x(t) \leq x^2(t) \), we obtain

\[
\frac{dV}{dt} \leq (-2a + \varepsilon_1 b + \varepsilon_1^{-1} + \varepsilon_2^{-1} b + \varepsilon_3^{-1} b^2 + \alpha + \beta \sigma) x^2(t)
- 2apx(t)x(t - \tau) + \varepsilon_2 \beta^2 x^2(t - \tau) - 2a \beta x(t) \\
+ \varepsilon_3 b^2 \left( \int_{t-\sigma}^{t} \tanh x(s) ds \right)^2 - \alpha x^2(t - \tau) - \beta \int_{t-\sigma}^{t} \tanh^2 x(s) ds.
\]  

(7)

Here, for vector function \( y \), using the following inequality [4]

\[
\left[ \int_{t-\sigma}^{t} y(s) ds \right]^T \left[ \int_{t-\sigma}^{t} y(s) ds \right] \leq \sigma \int_{t-\sigma}^{t} y^T(s)y(s) ds,
\]

we obtain

\[
-\beta \int_{t-\sigma}^{t} \tanh^2 x(s) ds \leq -\beta \sigma \left[ \frac{1}{\sigma} \int_{t-\sigma}^{t} \tanh x(s) ds \right]^2.
\]

(8)

Using inequality (8), we have a new bound of \( \frac{dV}{dt} \) in the form of

\[
\frac{dV}{dt} \leq \chi^T(t) \begin{bmatrix}
-2a + \alpha + \beta \sigma + \varepsilon_1^{-1} b + \varepsilon_2^{-1} b + \varepsilon_3^{-1} b^2 & -ap & -\sigma ab \\
* & \varepsilon_2 \beta^2 - \alpha & 0 \\
* & * & \varepsilon_3 b^2 - \beta \sigma
\end{bmatrix} \chi(t)
\equiv \chi^T(t) \Omega \chi(t),
\]

(9)

where

\[
\chi(t) = \begin{bmatrix} x(t) \\ x(t - \tau) \\ \frac{1}{\sigma} \int_{t-\sigma}^{t} \tanh x(s) ds \end{bmatrix}.
\]

Therefore, if the matrix \( \Omega \) is negative definite, \( \frac{dV}{dt} \) is negative. By Schur complement [1], the fact that \( \Omega < 0 \) is equivalent to \( \Sigma(\cdot) < 0 \). Equation (3) implies that \( \frac{dV}{dt} \leq -\gamma \|x(t)\|^2 \) for sufficiently small \( \gamma > 0 \). Noting that the operator \( \mathcal{D} \) is stable, therefore, equation (1) is asymptotically stable according to Theorem 8.1 of [6, pp. 292–293].

REMARK 2. A simple condition for stability of the operator \( \mathcal{D} \) is \( |p| + \sigma b < 1 \).

REMARK 3. Problem (3) is to determine whether the problem is feasible or not. It is called the feasibility problem. The solutions of the problem can be found by solving eigenvalue problem with respect to \( \alpha, \beta, \varepsilon_1, \varepsilon_2, \) and \( \varepsilon_3 \), which is a convex optimization problem [5]. Various efficient convex optimization algorithms can be used to check whether the inequality (3) is feasible. In the letter, in order to solve the linear matrix inequality (3), we utilize MATLAB’s LMI Control Toolbox [7], which implements state-of-the-art interior-point algorithms, which is significantly faster than classical convex optimization algorithms [5]. Therefore, all solutions \((\alpha, \beta, \varepsilon_1, \varepsilon_2, \varepsilon_3)\) can be obtained at the same time.

REMARK 4. In the work [3], the delay-independent sufficient condition for stability of system (1) has been presented as

\[
1 + \frac{pa^2 + b^2}{2(1 - pb)} - a < 0 \quad \text{and} \quad pb < 1.
\]

(10)

To demonstrate the applications of main result, we give the following example.

EXAMPLE 5. Consider the following equation:

\[
\frac{d}{dt}[x(t) + 0.2x(t - 0.1)] = -0.6x(t) + 0.3 \tanh x(t - 0.3).
\]
By solving the linear matrix inequality (3), we found the positive solutions
\[ \alpha = 0.222, \quad \beta = 0.4236, \quad \varepsilon_1 = 1.0522, \quad \varepsilon_2 = 6.3146, \quad \varepsilon_3 = 5.89, \]
which implies that the equation is asymptotically stable. It is interesting to note that since
\[ 1 + \frac{pa^2 + b^2}{2(1 - pb)} - a = 0.4862 \leq 0, \]
the result of [3] cannot be satisfied.

REFERENCES