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Solutions of Nonlinear P.D.E.

G. ADOMIAN
General Analytics Corporation
155 Clyde Road, Athens, GA 30605, U.S.A.

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Abstract—Explicit solutions are calculated by the decomposition method for Burger's equation for comparison with other existing procedures such as similarity reduction, etc.

Explicit solutions of wide varieties of physically significant problems modelled by nonlinear partial differential equations are easily calculated by the decomposition method [1–7]. No similarity reductions are used to reduce a nonlinear partial differential equation to a system of simpler partial differential equations with a lower number of independent variables; the original nonlinear equation is directly solvable preserving the actual physics and involving much less calculation. No linearization, perturbation, or discretized methods which result in intensive computation are necessary. The reduction techniques become mathematical exercises with essentially undefined problems since an equation is considered without specification of initial/boundary conditions. The equations then represent no known physical situation.

A much-considered example is the Burger's equation [8,9]

$$u_t + uu_x - u_{xx} = 0.$$

This equation was only intended as an approach to the study of turbulence because it exhibited some essential characteristics of the more realistic (and difficult) equations. This equation involves nonlinearity, dissipation, and is relatively simple. Our interest arises only because comparison results are widely available. The decomposition method solves much more difficult systems. We consider it now to show the simplicity of a proper solution.

Let $L_t = \frac{\partial}{\partial t}$ and $L_{xx} = \frac{\partial^2}{\partial x^2}$. We have now

$$L_t u + uu_x = L_{xx} u.$$

As an initial-value problem, we write

$$L_t u = L_{xx} u - uu_x.$$

Operating with $L_t^{-1} = \int_0^t (\cdot)\, dt$,

$$L_t^{-1} L_t u = L_t^{-1} L_{xx} u - L_t^{-1} uu_x,$$
$$u - u(0) = L_t^{-1} L_{xx} u - L_t^{-1} uu_x.$$

Due to the death of this author, this work is published without the benefit of galley corrections. (Ed.)

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Decompose u into $\sum_{n=0}^{\infty} u_n$ with u_0 indentified as $u(0)$ with the nonlinearity uu_x written in terms of the Adomian Polynomials A_n, thus $uu_x = \sum_{n=0}^{\infty} A_n\{uu_x\}$,

$$u = u_0 + L_t^{-1} L_{xx} \sum_{n=0}^{\infty} u_n - L_t^{-1} \sum_{n=0}^{\infty} A_n.$$

Since $u = \sum_{n-0}^{\infty} u_n$, we can now write

$$\begin{aligned} u_1 &= L_t^{-1} L_{xx} u_0 - L_t^{-1} A_0, \\ u_2 &= L_t^{-1} L_{xx} u_1 - L_t^{-1} A_1, \\ &\vdots \\ u_{n+1} &= L_t^{-1} L_{xx} u_n - L_t^{-1} A_n, \end{aligned}$$

Thus, the components of u are calculable for $n \geq 0$. The A_n for this case are given by:

$$\begin{aligned} A_0 &= u_0 u_0', \\ A_1 &= u_1 u_0' + u_0 u_1', \\ A_2 &= u_2 u_0' + u_1 u_1' + u_0 u_2', \\ A_3 &= u_3 u_0' + u_2 u_1' + u_1 u_2' + u_0 u_3', \end{aligned}$$

etc., which can be written as

$$A_n = u_n u_0' + u_{n-1} u_1' + \cdots + u_1 u_{n-1}' + u_0 u_n'.$$

Now the u_n are determined. The n-term approximant

$$\varphi_n[u] = \sum_{m=0}^{n-1} u_m$$

converges rapidly to u, so it serves as an excellent approximation for small n. The problem is completely defined when the initial condition is specified. If we specify $u = x$ when $t = 0$, we have

$$\begin{aligned} u_0 &= u(t = 0) = x, \\ u_1 &= -L_t^{-1} A_0 = xt, \\ u_2 &= -L_t^{-1} A_1 = L_t^{-1}(-2xt) = \frac{xt^2}{2} \\ &\vdots \end{aligned}$$

Thus, $u = x(1 - t + (t^2/2) - \cdots)$ or $u = x/(1 + t)$.

We see that the efficiency of decomposition makes it the method of choice. It is to be noted that no linearization or perturbation was used and solutions can be said to be exact since any desired accuracy is obtainable by the increasing n, normally a very small number. Each additional term depends simply on the preceding term. Randomness in the initial term is handled without restrictive assumptions or closure approximants. The method has been generalized to systems of nonlinear partial differential equations, so a single ordinary or partial differential equation is an easily solved special case. It can be viewed as a dynamical systems theory which is quantitative rather than merely qualitative.

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