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Remarks on the space $\aleph_1$ in **ZF**

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ABSTRACT

We show:

(1) $\aleph_1$ with the order topology is effectively normal, i.e., there is a function associating to every pair (A, B) of disjoint closed subsets of $\aleph_1$ a pair (U, V) of disjoint open sets with $A \subseteq U$ and $B \subseteq V$.
(2) For every countable ordinal α the ordered space $\boldsymbol{\alpha}$ is metrizable. Hence, every closed subset of $\boldsymbol{\alpha}$ is a zero set and consequently the Čech–Stone extension of $\boldsymbol{\alpha}$ coincides with its Wallman extension.
(3) In the Feferman–Levy model where $\aleph_1$ is singular, the ordinal space $\aleph_1$ is base-Lindelöf but not Lindelöf.
(4) The Čech–Stone extension $\beta\aleph_1$ of $\aleph_1$ is compact iff its Wallman extension $\mathcal{W}(\aleph_1)$ is compact.
(5) The set L of all limit ordinals of $\aleph_1$ is not a zero set.

1. Notation and terminology

Let $\mathbf{X} = (X, T)$ be a topological space.

$\mathbf{X}$ is *compact* iff every open cover $\mathcal{U}$ of $\mathbf{X}$ has a finite subcover $\mathcal{V}$. Equivalently, $\mathbf{X}$ is compact iff for every family $\mathcal{G}$ of closed subsets of X having the *finite intersection property*, fip for abbreviation, $\bigcap \mathcal{G} \neq \emptyset$.

$\mathbf{X}$ is *countably compact* iff every countable open cover $\mathcal{U}$ of $\mathbf{X}$ has a finite subcover $\mathcal{V}$.

$\mathbf{X}$ is *Lindelöf* iff every open cover $\mathcal{U}$ of $\mathbf{X}$ has a countable subcover $\mathcal{V}$.

$\mathbf{X}$ is *effectively normal* if there exists a function F such that for every pair (A, B) of disjoint closed sets in $\mathbf{X}$, $F(A, B) = (C, D)$, where C, D are disjoint open sets in $\mathbf{X}$ including A and B respectively.

Let $(X, \leqslant)$ be a linearly ordered set. As usual, (∞, x), (x, ∞), (x, y), x, y, $x < y$ stand for the infinite open rays $\{z \in x\colon\ z < x\}$, $\{z \in x\colon\ z > x\}$ and the open interval $\{z \in x\colon\ x < z < y\}$ respectively.

The *canonical subbase* $\mathcal{S}$ for the order topology T on X consists of all open infinite rays, i.e. $\mathcal{S} = \{(\infty, x)\colon\ x \in X\} \cup \{(x, \infty)\colon\ x \in X\}$.

The *canonical base* $\mathcal{B}$ for the order topology T on X consists of all finite intersections of subbase elements, i.e. $\mathcal{B} = \mathcal{S} \cup \{(x, y)\colon\ x, y \in X,\ x < y\}$. Clearly, if X is well-orderable then $\mathcal{B}$ is well-orderable.

$\mathbf{X} = (X, T)$ is *subbase Lindelöf* provided that each cover by elements of $\mathcal{S}$ contains a countable one.

$\mathbf{X}$ is *base Lindelöf* provided that each cover by elements of $\mathcal{B}$ contains a countable one.

$\mathbf{X}$ *weakly Lindelöf* provided that each open cover has a countable open refinement (see E9 on p. 41 in [4]).

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Let $\aleph$ be an uncountable cardinal number endowed with the order topology. A set k is *unbounded* in $\aleph$ if and only if for any $\alpha < \aleph$, there is some $\beta \in k$ such that $\alpha < \beta$.

The *long ray* $\mathbf{L}$ is the set $\aleph_1 \times [0,1)$ endowed with the order topology produced from the lexicographical order on $\aleph_1 \times [0,1)$.

Let X be a non-empty set and let $\mathcal{E}$ be a collection of non-empty subsets of X. A $\mathcal{E}$-filter $\mathcal{F}$ is called *countably closed* if it satisfies the condition: If $\{F_i\colon i \in \omega\} \subset \mathcal{F}$ then $\bigcap\{F_i\colon i \in \omega\} \in \mathcal{F}$.

Let $\mathbf{X}$ be a T_1 space and $\mathcal{C}$ be a base for the closed subsets of $\mathbf{X}$. Then $\mathcal{C}$ is called a T_1 *base for* $\mathbf{X}$ if it satisfies:

(i) $\emptyset \in \mathcal{C}$.
(ii) If $B_1, B_2 \in \mathcal{C}$ then $B_1 \cap B_2 \in \mathcal{C}$ and $B_1 \cup B_2 \in \mathcal{C}$.
(iii) If $x \notin B \in \mathcal{C}$ then there is $B_x \in \mathcal{C}$ such that $x \in B_x$ and $B_x \cap B = \emptyset$.

If $(X, \leqslant)$ is a linearly ordered set then $\mathcal{C} = \{A \subset X\colon A = [x, \infty),\ x \in X \text{ or } A = (\infty, y],\ y \in X \text{ or } A \text{ is a finite union of closed intervals of } X\}$ is a T_1 base for the ordered space $\mathbf{X}$. We call $\mathcal{C}$ *the canonical* T_1 *base for* $\mathbf{X}$. We note that in case X is well-orderable, as is the case when X is an ordinal number, then $\mathcal{C}$ is well-orderable.

Let $\mathcal{C}$ be a T_1 base for $\mathbf{X}$. $\mathcal{W}(X,\mathcal{C})$ denotes the set $\{\mathcal{F} \subset \mathcal{C}\colon \mathcal{F} \text{ is a maximal } \mathcal{C}\text{-filter}\}$. The topology $T_{\mathcal{W}(X,\mathcal{C})}$ on $\mathcal{W}(X,\mathcal{C})$ having as a base for the closed sets the family

$$\mathcal{B} = \{A^*\colon A \in \mathcal{C}\}, \qquad A^* = \{\mathcal{F} \in \mathcal{W}(X,\mathcal{C})\colon A \in \mathcal{F}\}$$

is called the *Wallman topology on* $\mathcal{W}(X,\mathcal{C})$. $\mathcal{W}(\mathbf{X},\mathcal{C}) = (\mathcal{W}(X,\mathcal{C}), T_{\mathcal{W}(X,\mathcal{C})})$ is called the *Wallman extension of* $\mathbf{X}$ *corresponding to the base* $\mathcal{C}$. In case $\mathcal{C} = \mathcal{K}(\mathbf{X})$, the family of all closed subsets of $\mathbf{X}$, we shall denote $\mathcal{W}(X,\mathcal{C})$ simply as $\mathcal{W}(X)$.

A subset A of $\mathbf{X}$ is a *zero set* iff $A = f^{-1}(0)$ for some continuous real valued function f on $\mathbf{X}$. Clearly, if $\mathbf{X}$ is completely regular then the collection $\mathcal{Z}$ of all zero sets of $\mathbf{X}$ is a T_1 base for $\mathbf{X}$. We shall denote $\mathcal{W}(X,\mathcal{Z})$ and $\mathcal{W}(\mathbf{X},\mathcal{Z})$ by βX and $\beta\mathbf{X}$ respectively. $\beta\mathbf{X}$ is called the *Čech–Stone extension* of $\mathbf{X}$.

AC: Every family of non-empty sets has a choice function.
CAC: **AC** restricted to countable families.
CAC($\mathbb{R}$) (*Form 94 in [7]*) is **AC** restricted to countable families of subsets of the real line $\mathbb{R}$.
UF(ω) (*Form 70 in [7]*): *$\wp(\omega)$ has a free ultrafilter.*
BPI(ω) (*Form 225 in [7]*): *Every filter of $\wp(\omega)$ extends to an ultrafilter.*
R($\aleph_1$) (*Form 34 in [7]*): $\aleph_1$ is regular.

2. Introduction and some known results

As is well known (see [2]) in **ZFC** set theory:

(a) The Wallman extension of a T_1 space $\mathbf{X}$ is a T_1 compactification of $\mathbf{X}$.
(b) The Čech–Stone extension of a Tychonoff space $\mathbf{X}$ is the compact Hausdorff reflection of $\mathbf{X}$. The latter is usually denoted by $\beta\mathbf{X}$ and called the Čech–Stone compactification of $\mathbf{X}$.
(c) For a Tychonoff space $\mathbf{X}$ both extensions are isomorphic if and only if $\mathbf{X}$ is normal.

However in **ZF** set theory neither of these extensions needs to be compact.

Regarding Wallman compactifications of T_1 spaces the following results are known in **ZF**.

Proposition 1. *([9]) In* **ZF**, *the following statements are pairwise equivalent*:

(i) **AC**.
(ii) *For every T_1 space $\mathbf{X}$ and every T_1 base $\mathcal{C}$ for $\mathbf{X}$, $\mathcal{W}(\mathbf{X},\mathcal{C})$ is a compactification of $\mathbf{X}$.*
(iii) *For every non-compact T_1 space $\mathbf{X}$, for every T_1 base $\mathcal{C}$ for $\mathbf{X}$, there is a free maximal $\mathcal{C}$-filter.*
(iv) *For every T_1 space $\mathbf{X}$, for every T_1 base $\mathcal{C}$ for $\mathbf{X}$, every filterbase $\mathcal{H} \subset \mathcal{C}$ extends to a maximal $\mathcal{C}$-filter.*
(v) *"For every T_1 space X, $\mathcal{W}(\mathbf{X})$ is a compactification of $\mathbf{X}$" and* **CAC**.
(vi) *"For every non-compact T_1 space $\mathbf{X}$, every closed filterbase of $\mathbf{X}$ is contained in a maximal closed filter of $\mathbf{X}$" and* **CAC**.

In case $X = \omega$, the zero sets coincide with the closed sets, so that $\mathcal{W}(\omega) = \beta\omega$. In case $X = \omega_1$ however closed sets may fail to be zero sets. In particular, for regular $\aleph_1$ the set of all limit ordinals is closed in $\aleph_1$ but not a zero set since every continuous function $f: \aleph_1 \to \mathbb{R}$ is known to be finally constant. So the following questions arise:

Question 1. Is $\mathcal{W}(\omega_1) \simeq \beta\omega_1$?

Question 2. For $a \in \aleph_1$ is $\mathcal{W}(a) \simeq \beta a$?

Question 3. Does "$\mathcal{W}(\aleph_1)$ is compact" imply "$\beta\aleph_1$ is compact"? Does "$\beta\aleph_1$ is compact" imply "$\mathcal{W}(\aleph_1)$ is compact"?

Equivalently, in view of the next Proposition 2.

Proposition 2. *([9]) Let* $\mathbf{X}$ *be a* T_1 *topological space and* $\mathcal{C}$ *a* T_1 *base for* $\mathbf{X}$. *Then*:

(i) *Every filterbase of closed subsets of* $\mathbf{X}$ *extends to a maximal closed filter iff* $\mathcal{W}(\mathbf{X})$ *is compact. In particular,* $\mathbf{BPI}(\omega)$ *iff "*$\boldsymbol{\beta\omega}$ *is compact".*
(ii) *Every* $\mathcal{C}$*-filterbase of closed subsets of* $\mathbf{X}$ *extends to a maximal* $\mathcal{C}$*-filter iff* $\mathcal{W}(\mathbf{X},\mathcal{C})$ *is compact.*

Question 3 may be rephrased as:

Does the statement "every $\mathcal{Z}$-filterbase of $\aleph_1$ extends to a maximal $\mathcal{Z}$-filter" imply "every closed filterbase of $\aleph_1$ extends to a maximal closed filter"? Does the statement "every closed filterbase of $\aleph_1$ extends to a maximal closed filter" imply "every $\mathcal{Z}$-filterbase of $\aleph_1$ extends to a maximal $\mathcal{Z}$-filter"?

The research in this paper is motivated by these questions. Regarding Question 1 we show that the answer is yes (Theorem 18), even though $\mathcal{W}(\aleph_1)$ and $\beta\aleph_1$ fail to be equal (Theorem 14). Regarding Question 2 we show in Corollary 6 that the answer is in the affirmative. Regarding Question 3 we show in Theorem 18 that the answer is also in the affirmative.

Besides Questions 1–3, we prove in Theorem 8 that $\aleph_1$ with the ordered topology has a normality operator, i.e., a function F associating to every pair (A,B) of disjoint closed subsets of $\aleph_1$ a pair (U,V) of disjoint open sets with $A\subseteq U$ and $B\subseteq V$.

In Theorem 16 we extend the list of equivalents of the statement "$\aleph_1$ is regular" given in [7] by establishing its equivalence to each one of the following statements: "*The Wallman extension* $\mathcal{W}(\aleph_1)$ *of the ordered space* $\aleph_1$ *is homeomorphic to the one-point compactification of* $\aleph_1$" and "*the Čech–Stone extension* $\beta\aleph_1$ *of the ordered space* $\aleph_1$ *is homeomorphic to the one-point compactification of* $\aleph_1$".

Before we proceed with the main results we list a known one needed in the sequel.

Proposition 3. *([5])* $\mathbf{CAC}(\mathbb{R})$ *iff* $\mathbb{N}$ *is Lindelöf.*

3. Some properties of the long ray L and the ordered space $\aleph_1$ in ZF

The long ray $\mathbf{L}$ is a well-known counterexample in **ZFC**. For properties satisfied by $\mathbf{L}$ we refer the reader to [1] and [6]. In **ZF** we get, besides the curious results exhibited in [3], the following additional ones:

Theorem 4. *Assume that* $\aleph_1$ *is singular and that* $(a_n)_{n\in\mathbb{N}}$ *is a strictly increasing cofinal sequence in* $\aleph_1$. *Then the following hold with* $b_n=(a_n,0)$:

(i) *For each* $n\in\omega$ *the subspace* $[b_n,b_{n+1}]$ *of* $\mathbf{L}$ *is homeomorphic to the subspace* $[n,n+1]$ *of the space* $\mathbb{R}^+$ *of non-negative reals, but* $\mathbf{L}$ *is not homeomorphic to* $\mathbb{R}^+$.
(ii) *For each* $n\in\omega$ *the subspace* $[b_n,b_{n+1}]$ *of* $\mathbf{L}$ *is separable and metrizable, but* $\mathbf{L}$ *fails to be either separable or metrizable.*

Proof. (i) First we prove that the linearly ordered sets $X=[b_n,b_{n+1}]$ and $[0,1]$ are order isomorphic. Let Q be the set of rationals in $[0,1]$. Then Q is a countable, dense in itself ordered set with a first and a last element. Clearly, $P=[a_n,a_{n+1})\times(\mathbb{Q}\cap[0,1))\cup\{b_{n+1}\}$ is a countable dense in itself ordered subset of X. Using Cantor's back-and-forth argument, we construct an order isomorphism $f:Q\to P$. By the fact that the sets X and $[0,1]$ are ordered continuously, f can be extended uniquely to an order isomorphism $g:[0,1]\to X$. Since both spaces are equipped with their order topologies, they are homeomorphic. The conclusion now follows from the fact that $[\mathbf{0},\mathbf{1}]\simeq[\mathbf{n},\mathbf{n+1}]$.

(ii) By part (i) $[\mathbf{b}_n,\mathbf{b}_{n+1}]\simeq[\mathbf{n},\mathbf{n+1}]$ and $[\mathbf{n},\mathbf{n+1}]$ is both separable and metrizable.

$\mathbf{L}$ is not metrizable because it is known, see [3], that $\aleph_1$ is non-metrizable. The non-separability of $\mathbf{L}$ follows from the fact that separability would imply (as in the proof of (i)) that $\mathbf{L}$ is homeomorphic to $[\mathbf{0},\mathbf{1})$ and thus metrizable. □

Corollary 5. *For every* $b\in\aleph_1$, $[\mathbf{0},\mathbf{b}]$ *is homeomorphic to a closed subspace of* $[\mathbf{0},\mathbf{1}]$. *In particular, for every* $b\in\aleph_1$, $[\mathbf{0},\mathbf{b}]$ *is metrizable.*

Corollary 6. *For every ordinal* $\alpha\in\aleph_1$, $\beta\alpha=\mathcal{W}(\alpha)$.

Proof. This follows at once from Corollary 5 and the fact that in metric spaces closed sets are zero sets. (If d is a metric on α producing the order topology then, for every closed subset A of $\boldsymbol{\alpha}$, the mapping $h_A:\boldsymbol{\alpha}\to\mathbb{R}$ given by $h_A(x)=d(x,A)=\min\{d(x,y):\ y\in A\}$ is continuous such that $h^{-1}(0)=A$.) □

Clearly $\aleph_1$ is first countable and thus a Frechet space. Furthermore, it is easy to see that for every ordinal $b \leqslant \aleph_1$ the following statements:

$\mathbf{U}(b)$: *there is a family $(U_{n,x})_{n\in\mathbb{N},\ x\in b}$ such that, for each $x \in b$, $(U_{n,x})_{n\in\mathbb{N}}$ is a neighborhood base of x,*

and

$\mathbf{S}(b)$: *there is a family $(a_i)_{a\in L(b),\ i\in\omega}$, $L(b)$ is the set of all limit ordinals of b, of strictly increasing sequences satisfying $\lim_{i\to\infty} a_i = a$ for every $a \in L(b)$* are equivalent. Furthermore, if $b \in \aleph_1$ then $\mathbf{S}(b)$ is provable in **ZF**. Indeed, if d is the metric on $[0, b]$ which is guaranteed by Corollary 5, then for every limit ordinal a of b, $(a_n)_{n\in\mathbb{N}}$ where, $a_0 = 0$ and a_{n+1} is the first element of the open disc $D(a, 1/n)$ which is strictly greater than a_n, is a *fundamental sequence* for a, i.e., a strictly increasing cofinal sequence in a.

For every $A \subset L$, *the set of all limit ordinals of* $\aleph_1$, $\mathbf{S}(A)$ denotes the restriction of the statement $\mathbf{S}(\aleph_1)$ to A. The next proposition shows, as expected, that $\mathbf{S}(\aleph_1)$ is not provable in **ZF**.

Proposition 7. $\mathbf{S}(\aleph_1)$ *implies* $\mathbf{R}(\aleph_1)$.

Proof. This follows as in Lemmas 4.1 and 4.2 in [3]. □

We show next that for every pair (A, B) of disjoint closed subsets of $\aleph_1$ there is a choiceless construction of a pair (U, V) of disjoint open sets with $A \subseteq U$ and $B \subseteq V$.

Theorem 8. $\aleph_1$ *is effectively normal.*

Proof. Let H, K be two disjoint closed subsets of $\aleph_1$. We will define, without the use of the axiom of choice, two disjoint open sets U, V of $\aleph_1$ containing H and K respectively. Let $\mathcal{B}$ be the canonical base of $\aleph_1$. Since $\mathcal{B}$ is well-orderable, it follows that $[\mathcal{B}]^{<\omega}$ is well-orderable. We consider the following cases:

(i) $\aleph_1$ is regular. Clearly, $X = \aleph_1 + 1$ with the ordered topology is a compact space and $\mathcal{C} = \mathcal{B} \cup \{(x, \aleph_1]:\ x \in \aleph_1\}$ is a well-orderable base for **X**. We consider the following subcases:

(a) H, K are bounded. Clearly, $\mathcal{U}_H = \{B \in \mathcal{B}:\ B \cap H \neq \emptyset$ and $\overline{B} \cap K = \emptyset\}$ is an open cover of the compact subset H of $\aleph_1$. Let $\mathcal{U} \subset \mathcal{U}_H$ be the first element of $[\mathcal{B}]^{<\omega}$ which is covers H. Put $U = \bigcup \mathcal{U}$ and let $G = \overline{U}$. Clearly, $G \cap K = \emptyset$. Apply the previous step with K in place of H and G in place of K to obtain an open set V including K and being disjoint from G. Clearly, U, V are the required open sets separating H and K respectively.

(b) H is bounded and K is unbounded. Clearly, $G = cl_X(K) = \{\aleph_1\} \cup K$ and $H \cap G = \emptyset$. Apply step (a) to **X** to obtain disjoint open sets U, V including H and G respectively. Clearly, U and $V \setminus \{\aleph_1\}$ are the required open sets.

(c) H is unbounded and K is unbounded. Since $\aleph_1$ is regular, this case cannot occur.

(ii) $\aleph_1$ is singular. Fix $0 = a_0 < a_1 < \cdots$ a strictly increasing cofinal sequence of non-limit ordinals of $\aleph_1$. For our convenience, we assume that $0 \notin H \cup K$. For every $n \in \omega$, let $H_n = H \cap [a_n + 1, a_{n+1}]$, $K_n = K \cap [a_n + 1, a_{n+1}]$ and $\mathcal{B}_n = \{B \cap [a_n + 1, a_{n+1}]:\ B \in \mathcal{B}\}$. Since $[\mathcal{B}]^{<\omega}$ is well-orderable we may assume that each $[\mathcal{B}_n]^{<\omega}$, $n \in \omega$ inherits a well ordering from $[\mathcal{B}]^{<\omega}$. Without loss of generality we may assume that for every $n \in \omega$, $H_n \neq \emptyset$ and $K_n \neq \emptyset$. For every $n \in \omega$, let $\mathcal{B}_{K_n} = \{B \in \mathcal{B}_n:\ B \cap H_n \neq \emptyset$ and $B \cap \overline{K_n} \neq \emptyset\}$ and $\mathcal{B}_{H_n} = \{B \in \mathcal{B}_n:\ B \cap K_n \neq \emptyset$ and $B \cap \overline{H_n} \neq \emptyset\}$. As $\aleph_1$ is a T_3 space, it follows that $\mathcal{B}_{K_n}$ covers K_n and $\mathcal{B}_{H_n}$ covers H_n. Since K_n, H_n are closed subsets of the compact space $[a_n + 1, a_{n+1}]$ it follows that they are compact. Let $\mathcal{U}_n, \mathcal{V}_n$ be the first elements of $[\mathcal{B}_n]^{<\omega}$ which are finite subcoverings of $\mathcal{B}_{K_n}$ and $\mathcal{B}_{H_n}$ respectively. Clearly, $\mathcal{U} = \bigcup\{\mathcal{U}_n:\ n \in \omega\}$ and $\mathcal{V} = \bigcup\{\mathcal{V}_n:\ n \in \omega\}$, being countable unions of well ordered finite sets, have definable well orderings $\mathcal{U} = \{U_n:\ n \in \omega\}$ and $\mathcal{V} = \{V_n:\ n \in \omega\}$. Work now as in Theorem 16.8, p. 11 from [11] to define the disjoint open sets U and V. □

Corollary 9.

(i) *For every ordinal v, the ordered space* $\mathbf{v}$ *is effectively normal.*
(ii) *There exists a function F that associates with any pair (A, B) of disjoint closed sets of $\aleph_1$ a continuous map $F(A, B) : \aleph_1 \to$* $[\mathbf{0}, \mathbf{1}]$ *with $F(A, B)$ being* 0 *on A and* 1 *on B.*

Proof. (i) Follow the proof of Theorem 8.

(ii) By Theorem 8 $\aleph_1$ is effectively normal and, effective normality is enough to guarantee that the proof of Uryshon's Lemma, as is given in [11, p. 102], goes through for $\aleph_1$ without any use of choice. □

Theorem 10. *For an infinite ordinal x the following conditions are equivalent for the space $\mathbf{x} = (x, T)$, where T is the order topology on x.*

(i) $\mathbf{x}$ *is subbase Lindelöf.*
(ii) $\mathbf{x}$ *is base Lindelöf.*
(iii) $\mathbf{x}$ *is weakly Lindelöf.*
(iv) $\mathbf{x}$ *has a countable cofinal subset.*
(v) $\mathbf{x}$ *is compact or* $\mathbf{x}$ *has a countably infinite closed relatively discrete subset.*
(vi) $\mathbf{x}$ *is compact or* $\mathbf{x}$ *has an infinite closed relatively discrete subset.*

Proof. The implications (ii) → (iii) → (i) → (iv) → (v) ↔ (vi) are straightforward.

(vi) → (ii) If $\mathbf{x}$ is compact then there is nothing to show. Assume $\mathbf{x}$ is not compact and let A be an infinite closed relatively discrete subset of $\mathbf{x}$. Let a_0 be the first element of A and for $n \in \mathbb{N}$, a_{n+1} be the first element of A which is strictly greater that a_n. Put $a = \sup(\{a_i\colon i \in \omega\})$. $a = x$ as otherwise, by the fact that A is closed, $a \in A$ and A is not relatively discrete. Thus, $\{a_i\colon i \in \omega\}$ is a strictly increasing cofinal subset of x. Let $\mathcal{U}$ be a basic cover of x. Clearly, for every $i \in \omega$, $\mathcal{U}$ is a cover of $[0, a_i]$. Since $[0, a_i]$ is compact, it is covered by finitely many members of $\mathcal{U}$. As the set $[x]^{<\omega}$ of all finite subsets of x is well-orderable, we let $\mathcal{V}_i$ be the first finite subcover of $[0, a_i]$ of $\mathcal{U}$. Clearly, $\mathcal{V} = \bigcup\{\mathcal{V}_i\colon i \in \omega\}$, being a countable union of finite well ordered sets, is countable and the conclusion follows. □

Corollary 11. *Let x be an infinite ordinal. Then the ordered space* $\mathbf{x}$ *is countably compact iff x has no strictly increasing cofinal sequence.*

Proof. (→) This is straightforward.

(←) Assume on the contrary and let $\mathcal{G} = \{G_n\colon n \in \omega\}$ be a strictly descending family of closed subsets of x with empty intersection. Clearly, $\{g_n \in G_n \backslash G_{n+1}\colon n \in \omega\}$ is a closed relatively discrete subset of x. Working as in the proof of Theorem 10(vi) → (ii) we can construct a strictly increasing cofinal sequence $(a_n)_{n\in\omega}$ in x. Contradiction. □

It is known that there are models of **ZF**, see the Feferman–Levy model $\mathcal{M}9$ in [7], in which $\aleph_1$ is a singular cardinal. Clearly, in view of Corollary 11, in such models $\aleph_1$ fails to be countably compact. We show next that in these models, $\aleph_1$ with the ordered topology, is base Lindelöf.

Theorem 12.

(i) *Assume $\aleph_1$ to be singular. Then, every well ordered open cover $\mathcal{U}$ of $\aleph_1$ has a countable subcover. In particular, $\aleph_1$ is base Lindelöf.*
(ii) *It is consistent with* **ZF** *that $\aleph_1$ is base Lindelöf but not Lindelöf.*

Proof. (i) Assume on the contrary and let $\mathcal{U} = \{U_i\colon i \in k\}$ be a well ordered open cover of $\aleph_1$ without a countable subcover. Since k is well ordered, for every $v \in \aleph_1$ we pick the first member of $\mathcal{U}$ containing v. So, we may assume that $k = \aleph_1$ and finish off the proof of (i) as in the proof of Theorem 10(iv) → (ii).

The second assertion of (i) follows from the first part and the observation that the standard base is well-orderable.

(ii) In $\mathcal{M}9$, $\aleph_1$ is singular. Hence, by part (i), $\aleph_1$ is base Lindelöf. By Proposition 3 **CAC**($\mathbb{R}$) implies $\aleph_1$ is regular. Hence, **CAC**($\mathbb{R}$) fails in $\mathcal{M}9$. Fix $A = \{a_n\colon n \in \omega\}$ a strictly increasing cofinal set in $\aleph_1$. Clearly, A with the subspace topology is homeomorphic with the discrete space $\mathbb{N}$. Thus, if $\aleph_1$ is Lindelöf then A, being a closed subset of a Lindelöf space, is Lindelöf. Hence, $\mathbb{N}$ is Lindelöf and by Proposition 3 **CAC**($\mathbb{R}$) holds. Contradiction. Thus, $\aleph_1$ is not Lindelöf in $\mathcal{M}9$ as required. □

Remark 13.

(i) Let $\mathcal{C}$ be the canonical T_1 base for $\aleph_1$. Using the well orderability of $\mathcal{C}$, we can show via a straightforward transfinite induction, that every $\mathcal{C}$-filter extends to a $\mathcal{C}$-ultrafilter. Thus, by Proposition 2(ii) $\mathcal{W}(\aleph_1, \mathcal{C})$ is a compactification of $\aleph_1$ in **ZF**. We leave it for the reader to verify that $\mathcal{W}(\aleph_1, \mathcal{C})$ is actually a one-point compactification of $\aleph_1$.
(ii) In [3] it is claimed in Corollary 3.7 that $\aleph_1$ with the order topology is Lindelöf in the Feferman–Levy model $\mathcal{M}9$. However, in view of Theorem 12 this is not the case.

4. Zero sets of $\aleph_1$

We know that if $\aleph_1$ is a regular cardinal then the set L of all limit ordinals of $\aleph_1$ is not a zero set. However, if $\aleph_1$ is singular, then it is not known whether L is a zero set. We show next that L is always a closed non-zero subset of $\aleph_1$.

Theorem 14.

(i) *Every closed and bounded subset of* $\aleph_1$ *is a zero set. In particular, if* $\aleph_1$ *is singular, then every closed set can be expressed as a union of a countable discrete family of zero sets.*
(ii) *Every* C *in the canonical* T_1 *base* $\mathcal{C}$ *for* $\aleph_1$ *is a zero set of* $\aleph_1$.
(iii) *Let* L *be the set of all limit ordinals of* $\aleph_1$. *If* $A \subseteq L$ *is a zero set then there exists a family* (a_i), $a \in A$, $i \in \omega$ *of strictly increasing sequences satisfying* $\lim_{i\to\infty} a_i = a$ *for every limit ordinal* $a \in A$. *In particular,* L *is a closed but not a zero set of* $\aleph_1$.

Proof. (i) Fix A a closed bounded subset of $\aleph_1$. Clearly, by Corollary 5, A is a zero set of $[0, \sup(A)]$, hence of $\aleph_1$ also.

(ii) Fix $C \in \mathcal{C}$. We consider the following cases:

(1) C is bounded. By part (i) C is a zero set.
(2) $C = [a, \aleph_1)$. If $a = v + 1$ is a non-limit ordinal of $\aleph_1$ then $[a, \aleph_1) = (v, \aleph_1)$ is a clopen set and consequently the map $f : \aleph_1 \to \mathbb{R}$, $f([a, \aleph_1)) = \{0\}$, $f([a, \aleph_1)^c) = \{1\}$ is continuous.

If a is a limit ordinal of $\aleph_1$ then we fix a strictly increasing sequence $a_0 = 0 < a_1 < a_2 < \cdots$ with $\lim_{n\to\infty} a_n = a$. Define a function $f : \aleph_1 \to \mathbb{R}$, by requiring:

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, a_1], \\ 1/n & \text{if } x \in [a_n + 1, a_{n+1}], \\ 0 & \text{if } x \in [a, \aleph_1). \end{cases}$$

Clearly, f satisfies: $\forall x \in \aleph_1$, for every sequence $(x_n)_{n\in\omega}$ if $\lim_{n\to\infty} x_n = x$ then $\lim_{n\to\infty} f(x_n) = f(x)$. Hence, f is continuous.

(3) C is a finite union of disjoint closed intervals, say I_i, $i \leqslant n$. Let, by cases (1) and (2), for every $i \leqslant n$, f_i be a continuous real valued function with $f_i^{-1}(0) = I_i$. Clearly, $f : \aleph_1 \to \mathbb{R}$, $f(x) = \min\{f_i(x)\colon i \leqslant n\}$ is a continuous real valued function satisfying $f^{-1}(0) = C$ and C is a zero set as required.

(iii) Fix a zero set $A \subseteq L$ and let $f : \aleph_1 \to \mathbb{R}$ be a continuous function satisfying $f^{-1}(0) = A$. For every $a \in A$ we define a strictly increasing sequence, $(a_i)_{i\in\omega}$ satisfying $\lim_{i\to\infty} a_i = a$ as follows: Let $B = \{b_n\colon n \in \mathbb{N}\}$,

$$b_n = \sup(A_n), \quad A_n = f^{-1}\left(\left(\left(-\frac{1}{n}, -\frac{1}{n+1}\right] \cup \left[\frac{1}{n+1}, \frac{1}{n}\right)\right) \cap [0, a)\right).$$

Clearly, $b_n \leqslant a$. By the continuity of f, $f^{-1}(-\frac{1}{n+1}, \frac{1}{n+1})$ is an open set containing a. Therefore, there exists a basic neighborhood $(c, a]$ of a such that for all $x \in (c, a]$, $-\frac{1}{n+1} < f(x) < \frac{1}{n+1}$. Hence, c is an upper bound of A_n and consequently $b_n \leqslant c < a$.

$b = \sup\{b_n\colon n \in \mathbb{N}\} = a$. If $b < a$ then for every $x \in (b, a)$, $|f(x)| \geqslant 1$ but $f(a) = 0$ and f continuous at a means that there exists a $d \in a$ with $b \in d$ such that for all $x \in (d, a]$, $|f(x)| < 1/2$. Contradiction.

Let $a_0 = \min B$ and for $i = n + 1$, $a_i = \min\{b \in B\colon b > a_n\}$. Clearly, $(a_i)_{i\in\omega}$ is the required sequence. We observe that in the definition of $(a_i)_{i\in\omega}$ no choice form was involved, i.e., $(a_i)_{i\in\omega}$ is defined effectively.

The second assertion of (iii) follows at once from Proposition 7 in case $\aleph_1$ is singular and the remark preceding Question 1 in case $\aleph_1$ is regular. □

From the proof of the forthcoming Theorem 16(iv) → (v) it follows that if $A = \{a_i\colon i \in \omega\}$ is a strictly increasing closed subset of non-limit ordinals of $\aleph_1$ then (in **ZF**) A is a zero set. However, if A contains infinitely many limit points of $\aleph_1$ then it is not known whether A is, in **ZF**, a zero set. We show next that A is a zero set iff there exists a system of fundamental sequences for its limit ordinals.

Theorem 15. *Let* $A = \{a_i\colon i \in \omega\}$ *be a closed set of limit ordinals of* $\aleph_1$ *such that* $a_0 < a_1 < \cdots$. *Then, the following are equivalent*:

(i) *A is a zero set.*
(ii) *There exists a family* $(a_{i,n})_{i\in\omega,\, n\in\omega}$ *such the for every* $i \in \omega$, $(a_{i,n})_{n\in\omega}$ *is a strictly increasing cofinal sequence in* a_i *with* $a_{0,0} = 0$ *and* $a_{i,0} = a_{i-1}$ *for* $i \neq 0$.

Proof. It suffices to show (ii) → (i) as (i) → (ii) follows from Theorem 14(iii). Fix A as in the statement of the theorem. Without loss of generality we may assume that $0 = a_0$. Since A is closed and relatively discrete it follows, by Theorem 16(vii), that A is cofinal in $\aleph_1$. We define a function $f : \aleph_1 \to \mathbb{R}$ as follows

$$f(x) = \begin{cases} 0 & \text{if } x \in A, \\ \frac{1}{n+1} & \text{if } x \in \bigcup\{[a_{i,n} + 1, a_{i,n+1}]\colon n \in \omega,\ i \in \mathbb{N}. \end{cases}$$

We show that f is continuous. Fix $x \in \aleph_1$ and let $(x_n)_{n\in\omega}$ be a sequence of points of $\aleph_1$ converging to x. We consider the following cases:

(1) $x = a_i$, $i \in \mathbb{N}$. Fix a sequence $(x_m)_{m\in\omega} \subset [a_{i-1}, a_i)$ with $\lim_{m\to\infty} x_m = a_i$. Let $\varepsilon > 0$ and fix $n \in \omega$ with $\frac{1}{n+1} < \varepsilon$. Since, $\lim_{m\to\infty} x_m = a_i$, it follows that there exists $m_0 \in \mathbb{N}$ such that $\forall m \geqslant m_0$, $x_m \in (a_{i,n+1}, a_i]$ and consequently, by the definition of f, $\forall m \geqslant m_0$, $f(x_m) \leqslant \frac{1}{n+1} < \varepsilon$. Thus, $\lim_{m\to\infty} f(x_m) = f(x)$ as required.

(2) $x \notin \aleph_1$. In this case there exist $i \in \mathbb{N}, n \in \omega$ with $x \in [a_{i,n}, a_{i,n+1}]$. It is straightforward to verify that for every sequence $(x_n)_{n\in\omega}$ with $\lim_{n\to\infty} x_n = x$, $(f(x_n))_{n\in\omega}$ takes on eventually the value $f(x)$. Thus, $\lim_{n\to\infty} f(x_n) = f(x)$ and f is continuous. Hence, A is a zero set as required. □

5. Some characterizations of "$\aleph_1$ is regular"

In this section, we enlarge the list of the equivalents of the statement $\mathbf{R}(\aleph_1)$ ($= \aleph_1$ is regular) given in [7, p. 29].

Theorem 16. *The following are equivalent*:

(i) $\mathbf{R}(\aleph_1)$.
(ii) *The collection $\mathcal{K}$ of all closed unbounded sets of $\aleph_1$ is a countably closed, free, maximal closed filter.*
(iii) *$\mathcal{W}(\aleph_1)$ is a one-point compactification of $\aleph_1$.*
(iv) *$\beta\aleph_1$ is a one-point compactification of $\aleph_1$.*
(v) *Every infinite subset of $\aleph_1$ has a limit point.*
(vi) *Every sequence of $\aleph_1$ has a convergent subsequence.*
(vii) *$\aleph_1$ has no countable closed relatively discrete subsets.*
(viii) *$\aleph_1$ is countably compact.*
(ix) *$\aleph_1$ is not base Lindelöf.*
(x) *The collection $\mathcal{T}$ of all zero unbounded sets of $\aleph_1$ is a free, maximal zero filter.*

Proof. (i) → (ii) The fact that $\mathcal{K}$ is a countably closed free closed filter follows from Lemmas 7.3 and 7.4 from [8].

To see that $\mathcal{K}$ is maximal fix F a closed subset of $\aleph_1$ meeting non-trivially each member of $\mathcal{K}$. We show that $F \in \mathcal{K}$. As $[a, \aleph_1) \in \mathcal{K}$ for every $a \in \aleph_1$ it follows that $F \cap [a, \aleph_1) \neq \emptyset$ and consequently F is unbounded. Hence $F \in \mathcal{K}$ as required.

(ii) → (iii) It is easy to see that (ii) implies $\aleph_1$ is regular. (If $\aleph_1$ is a singular cardinal and $\{a_i\colon i \in \omega\}$ is a strictly increasing cofinal subset of $\aleph_1$, then $\{a_i\colon i \in \omega\}$ is a closed unbounded subset of $\aleph_1$. Hence, $A = \{a_{2i}\colon i \in \omega\}$ and $B = \{a_{2i+1}\colon i \in \omega\}$ are disjoint closed unbounded subsets of $\aleph_1$ and $\mathcal{K}$ is not a filter of $\aleph_1$.)

To complete the proof of (ii) → (iii), it suffices to show that for every T_2 compactification $\mathbf{Y}$ of $\aleph_1$, $Y \setminus \aleph_1$ is a singleton. This, given that $\aleph_1$ is regular, is straightforward and well known.

(iii) → (iv) Clearly, (iii) implies $\aleph_1$ is a regular cardinal. Indeed, if $\aleph_1$ is a singular cardinal and $\{a_i\colon i \in \omega\}$ is a strictly increasing cofinal subset of $\aleph_1$, then every subset of $\{a_i\colon i \in \omega\}$ is a closed subset of $\aleph_1$. Hence, by our hypothesis and Proposition 2, the filterbases of closed sets $\mathcal{A} = \{X \in \wp(\{a_{2i}\colon i \in \omega\})\colon |X^c| < \aleph_0\}$ and $\mathcal{B} = \{X \in \wp(\{a_{2i+1}\colon i \in \omega\})\colon |X^c| < \aleph_0\}$ are included in maximal closed filters of $\aleph_1$, say $\mathcal{F}$ and $\mathcal{Q}$ respectively. Clearly, $\mathcal{F}$ and $\mathcal{Q}$ are distinct and free. Hence, $\mathcal{W}(\aleph_1)$ is not a one-point compactification of $\aleph_1$. Contradiction.

In view of Theorem 14, $\mathcal{H} = \{[x, \aleph_1)\colon x \in \aleph_1\} \subset \mathcal{Z}$ is a closed filterbase of $\aleph_1$. Thus, by the regularity of $\aleph_1$, (i) → (ii) and our hypothesis, $\mathcal{K}$ is the only maximal closed filter of $\aleph_1$ extending $\mathcal{H}$. Clearly, $\mathcal{F} = \mathcal{K} \cap \mathcal{Z}$ is a free, maximal zero filter of $\aleph_1$. (If $K \in \mathcal{Z}$ meets non-trivially each member of $\mathcal{F}$ then $K \in \mathcal{K}$ and consequently $K \in \mathcal{F}$.) We show that $\mathcal{F}$ is the only free maximal zero filter of $\aleph_1$. Assume the contrary and let $\mathcal{Q}$, $\mathcal{Q} \neq \mathcal{F}$ be free maximal zero filter of $\aleph_1$. Fix $F \in \mathcal{F}$, $Q \in \mathcal{Q}$ such that $F \cap Q = \emptyset$. By our hypothesis and Proposition 2(i), there exists a free maximal closed filter $\mathcal{Q}'$ of $\aleph_1$ extending $\mathcal{Q}$. Since $F \in \mathcal{K}$ and $Q \in \mathcal{Q}'$, it follows that $\mathcal{K} \neq \mathcal{Q}'$ and consequently $\mathcal{W}(\aleph_1)$ is not a one-point compactification of $\aleph_1$. Contradiction.

(iv) → (v) Fix A an infinite subset of $\aleph_1$. Aiming for a contradiction, we assume that A has no limit point. Then, A is a closed relatively discrete subset of $\aleph_1$. Without loss of generality we may assume that $A = \{a_n\colon n \in \omega\}$ is countably infinite such that $a_0 < a_1 < a_2 < \cdots$. Clearly A is unbounded for otherwise $\sup(A)$ is a limit point of A. Thus, $\aleph_1$ is singular. By substituting, if necessary, each a_n with $a_n + 1$ we may also assume that the members of A are non-limit ordinals. Hence, every subset S of A is a clopen set and consequently a zero set of $\aleph_1$. In particular, $E = \{a_{2n}\colon n \in \omega\}$ and $O = \{a_{2n+1}\colon n \in \omega\}$ are zero sets. By our hypothesis and Proposition 2(ii), there exist zero maximal filters $\mathcal{F}$ and $\mathcal{H}$ extending $\{X \in \wp(E)\colon |E \setminus X| < \aleph_0\}$ and $\{X \in \wp(O)\colon |O \setminus X| < \aleph_0\}$ respectively. Clearly, $\mathcal{F}$, $\mathcal{H}$ are free and $\mathcal{F} \neq \mathcal{H}$. Thus, $\beta\aleph_1$ is not a one-point compactification of $\aleph_1$. Contradiction.

(v) → (vi) This is straightforward.

(vi) → (vii) Assume on the contrary and let A be a countable closed relatively discrete subset of $\aleph_1$. Define

$$a_0 = \min(A),$$
$$a_n = \min\bigl(A \setminus \{a_i\colon i \leqslant v\}\bigr) \quad \text{if } n = v + 1,\ v \in \omega.$$

As A is closed it follows that $a = \sup(\{a_n\colon n \in \omega\}) \in A$ or $a = \aleph_1$. Assume that $a \in A$. Since A is relatively discrete, it follows that there exists a neighborhood V_a of a meeting A in the singleton $\{a\}$. Contradiction. Hence, $a = \aleph_1$ and the sequence $(a_n)_{n\in\omega}$ has no convergent subsequence, contradicting our hypothesis. Thus, $\aleph_1$ has no countable closed relatively discrete subset as required.

(vii) → (viii) is straightforward and (viii) → (i) is a consequence of Corollary 11.

(i) ↔ (ix) This follows at once from Theorem 12.

(i) → (x) This follows from the observation that $\mathcal{T} \subset \mathcal{K}$ and the fact that finite intersections of zero sets are zero sets. (If $C_0, C_1, \ldots, C_n$ are zero sets and $f_i : \aleph_1 \to \mathbb{R}$, $i \leqslant n$ are continuous functions such that $f_i^{-1}(0) = C_i$ for every $i \leqslant n$, then the function $f : \aleph_1 \to \mathbb{R}$ given $f(x) = \max\{f_i(x)\colon i \leqslant n\}$ is continuous with $f^{-1}(0) = \bigcap\{C_i\colon i \leqslant n\}$.)

(x) → (i) If $\aleph_1$ is singular and $\{a_i\colon i \in \omega\}$ is a strictly increasing cofinal subset of $\aleph_1$ consisting in non-limit ordinals, then every subset of $\{a_i\colon i \in \omega\}$ is a clopen, hence zero, unbounded subset of $\aleph_1$. Since, $\{a_{2i}\colon i \in \omega\} \cap \{a_{2i+1}\colon i \in \omega\} = \emptyset$ it follows that $\mathcal{T}$ is not a filter. A contradiction terminating the proof of the theorem. □

6. Consequences of "$\mathcal{W}(\aleph_1)$ is compact"

Theorem 17.

(i) *"$\mathcal{W}(\aleph_1)$ is compact" implies either* $\mathbf{R}(\aleph_1)$ *or "$\boldsymbol{\beta\omega}$ is compact".*
(ii) *"$\mathcal{W}(\aleph_1)$ is compact" and "$\aleph_1$ is singular" imply "$\boldsymbol{\beta\omega}$ is compact".*
(iii) *"$\mathcal{W}(\aleph_1)$ is compact" and "$\boldsymbol{\beta\omega}$ is not compact" imply* $\mathbf{R}(\aleph_1)$.
(iv) *"$\mathcal{W}(\aleph_1)$ is compact" does not imply* $\mathbf{UF}(\omega)$. *In particular, "$\mathcal{W}(\aleph_1)$ is compact" does not imply "$\boldsymbol{\beta\omega}$ is compact".*
(v) *"$\aleph_1$ is singular" and "$\mathcal{W}(\aleph_1)$ is countably compact" together imply* $\mathbf{UF}(\omega)$.
(vi) $\mathbf{UF}(\omega)$ *implies "$\mathcal{W}(\aleph_1)$ is countably compact with respect to the base $\mathcal{B} = \{A^*\colon A \in \mathcal{K}(\aleph_1)\}$".*

Proof. (i) Assume $\aleph_1$ is singular. We show "$\boldsymbol{\beta\omega}$ is compact". To this end, it suffices to show that every filter on ω extends to an ultrafilter. Fix $A = \{a_i\colon i \in \omega\}$ a strictly increasing cofinal subset of $\aleph_1$. Identify ω with A and let $\mathcal{H}$ be a filter of A. Since A is clearly a closed relatively discrete subset of $\aleph_1$, it follows that $\mathcal{H}$ is a closed filter of $\aleph_1$. Hence, by the compactness of $\mathcal{W}(\aleph_1)$ and Proposition 2, $\mathcal{H}$ extends to a maximal closed filter $\mathcal{R}$ of $\aleph_1$. It is easy to verify that the trace $\mathcal{F}$ of $\mathcal{R}$ on A is the required ultrafilter of ω.

(ii), (iii) These are simply rewordings of (i).

(iv) This follows from the observation that "$\aleph_1$ is regular", hence "$\mathcal{W}(\aleph_1)$ is compact" also, hold in Solovay's model $\mathcal{M}5(\aleph)$ in [7], but $\mathbf{UF}(\omega)$ fails in $\mathcal{M}5(\aleph)$.

(v) Fix A as in the proof of (i) and identify A with ω. Let for every $n \in \omega$, $H_n = \{a_m\colon m \geqslant n\}$. Clearly, $\mathcal{H} = \{H_n^*\colon n \in \omega\}$ has the fip and consequently, by our hypothesis, $\bigcap\mathcal{H} \neq \emptyset$. Fix $\mathcal{G} \in \bigcap\mathcal{H}$ and let $\mathcal{F} = \{G \cap A\colon G \in \mathcal{G}\}$. Clearly, $\mathcal{F}$ is a free ultrafilter of ω.

(vi) Let $\mathcal{H}' = \{H_n^*\colon n \in \omega\} \subset \mathcal{B}$ have the fip. We show that $\bigcap\mathcal{H}' \neq \emptyset$. Put $\mathcal{H} = \{H_n\colon n \in \omega\}$. We show that $\bigcap\mathcal{H}' \neq \emptyset$. As $\mathcal{H}'$ has the fip it follows that $\mathcal{H}$ has the fip. Hence, we may assume that $\mathcal{H}$ is a strictly descending family of closed subsets of $\aleph_1$. If $\bigcap\mathcal{H} \neq \emptyset$ then the closed filter $\mathcal{F}_a$ generated by any $a \in \bigcap\mathcal{H}$ is a closed maximal filter included in $\bigcap\mathcal{H}'$. Thus, $\bigcap\mathcal{H}' \neq \emptyset$. Assume that $\bigcap\mathcal{H} = \emptyset$. Fix for every $n \in \omega, a_n \in H_n \backslash H_{n+1}$. Since $\bigcap\mathcal{H} = \emptyset$ it follows that $A = \{a_n\colon n \in \omega\}$ is a countable closed relatively discrete subset of $\aleph_1$. Identify A with ω and let, by our hypothesis, $\mathcal{G}$ be an ultrafilter of ω. Clearly, the closed filter $\mathcal{F}$ of $\aleph_1$ generated by $\mathcal{G}$ is in $\bigcap\mathcal{H}'$. Thus, $\bigcap\mathcal{H}' \neq \emptyset$ as required. □

We show next that the statement "$\mathcal{W}(\aleph_1)$ is compact" is equivalent to the proposition "$\beta\aleph_1$ is compact".

Theorem 18. *"$\mathcal{W}(\aleph_1)$ is compact" iff "$\beta\aleph_1$ is compact". In particular, "$\mathcal{W}(\aleph_1)$ is compact" implies $\mathcal{W}(\aleph_1) \simeq \beta\aleph_1$.*

Proof. If $\aleph_1$ is regular the conclusion follows from Theorem 16. So, assume that $\aleph_1$ is singular. Fix $0 = a_0 < a_1 < \cdots$ a strictly increasing cofinal sequence of non-limit ordinals of $\aleph_1$.

(→) Let $\mathcal{H}$ be a free zero filterbase. We show that $\mathcal{H}$ extends to a maximal zero filter of $\aleph_1$. Let, by our hypothesis, $\mathcal{G}$ be a maximal closed filter extending $\mathcal{H}$. We show that $\mathcal{F} = \mathcal{G} \cap \mathcal{Z}$ is the required maximal filter. If not, then there exists $Z \in \mathcal{Z} \backslash \mathcal{G}$ such that $\mathcal{F} \cup \{Z\}$ has the fip. By the maximality of $\mathcal{G}$, there exists $G \in \mathcal{G}$ such that $G \cap Z = \emptyset$. By Corollary 9, there exists a continuous function $f : \aleph_1 \to \mathbb{R}$ taking on the value 0 on G and the value 1 on Z. Hence, $F = f^{-1}(0) \in \mathcal{Z}$ with $G \subset F$ and $F \cap Z = \emptyset$. Since, $G \in \mathcal{G}$ and $\mathcal{G}$ is a filter, it follows that $F \in \mathcal{G}$ and consequently $F \in \mathcal{F}$ and $\mathcal{F} \cup \{Z\}$ does not have the fip. Contradiction.

(←) Let $\mathcal{H}$ be a free closed filterbase. We show that $\mathcal{H}$ extends to a maximal closed filter of $\aleph_1$. Let, by our hypothesis, $\mathcal{G}$ be a maximal zero filter extending the zero filterbase $\mathcal{H}' = \{Z \in \mathcal{Z}\colon H \subset Z$ for some $H \in \mathcal{H}\}$. Since, $\{Z^*\colon Z \in \mathcal{Z}\}$ is a base for the closed sets of $\beta\aleph_1$, it follows that $\{\mathcal{G}\} = \bigcap\{Z^*\colon Z \in \mathcal{Z}\}$ and consequently, $\{\mathcal{G}\} = \bigcap\{cl_{\beta\aleph_1}(B)\colon B \in \mathcal{K}(\aleph_1),\ \mathcal{G} \in cl_{\beta\aleph_1}(B)\}$. We show that $\mathcal{F} = \{B \in \mathcal{K}(\aleph_1)\colon \mathcal{G} \in cl_{\beta\aleph_1}(B)\}$ is the required maximal closed filter. Since, $cl_{\beta\aleph_1}(B \cap A) = cl_{\beta\aleph_1}(B) \cap cl_{\beta\aleph_1}(A)$, it follows that for every $B, A \in \mathcal{F}$, $(B \cap A) \in \mathcal{F}$. Hence, $\mathcal{F}$ is a closed filterbase.

Let $K \in \mathcal{K}(\aleph_1)$ satisfy $F \cap K \neq \emptyset$ for every $F \in \mathcal{F}$. We show that $K \in \mathcal{F}$. Assume, aiming for a contradiction, that $K \notin \mathcal{F}$. Then $\mathcal{G} \notin cl_{\beta\aleph_1}(K)$ and consequently, there exists a basic neighborhood $U^*_{\mathcal{G}}$ of $\mathcal{G}$ with $U^*_{\mathcal{G}} \cap cl_{\beta\aleph_1}(K) = \emptyset$. Hence, there is a closed set $B \subseteq U_{\mathcal{G}}$ with $B \in \mathcal{G}$. Therefore, $\mathcal{G} \in B^* \subseteq U^*_{\mathcal{G}}$, meaning that $B^* \in \mathcal{F}$ and $B^* \cap K = \emptyset$. Contradiction. Hence, $\mathcal{F}$ is a maximal closed filter.

$\mathcal{H} \subseteq \mathcal{F}$. Fix $H \in \mathcal{H}$. Clearly, $H = \bigcap\{Z \in \mathcal{Z}\colon\ H \subseteq Z\}$ and $\mathcal{G} \in \bigcap\{cl_{\beta\aleph_1}(Z)\colon\ Z \in \mathcal{Z},\ H \subseteq Z\}$. If $\mathcal{G} \notin cl_{\beta\aleph_1}(H)$ then there exists a clopen set U of $\aleph_1$ such that U^* is a neighborhood of $\mathcal{G}$ and $U^* \cap cl_{\beta\aleph_1}(H) = \emptyset$. Hence, $U \cap H = \emptyset$. Thus, $H \subset U^c \in \mathcal{Z}$ and $\emptyset = cl_{\beta\aleph_1}(U \cap U^c) = U^* \cap cl_{\beta\aleph_1}(U^c)$ meaning that $\mathcal{G} \in \bigcap\{cl_{\beta\aleph_1}(Z)\colon\ Z \in \mathcal{Z},\ H \subseteq Z\}$. Contradiction. Thus, $\mathcal{G} \in cl_{\beta\aleph_1}(H)$ and $H \in \mathcal{F}$ as required.

The second assertion follows as in the proof of E, p. 141 from [10]. □

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