A lower bound for the connectivity of directed Euler tour transformation graphs

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Abstract

Let $D$ be a directed Eulerian multigraph, $v$ be a vertex of $D$. We call the common value of $id(v)$ and $od(v)$ the degree of $v$, and simply denote it by $d_v$. Xia introduced the concept of the $T$-transformation for directed Euler tours and proved that any directed Euler tour $(T)$-transformation graph $E_T(D)$ is connected. Zhang and Guo proved that $E_T(D)$ is edge-Hamiltonian, i.e., any edge of $E_T(D)$ is contained in a Hamilton cycle of $E_T(D)$. In this paper, we obtain a lower bound

$$\frac{\sum_{v \in Q} (d_v - 1)(d_v - 2)}{2}$$

for the connectivity of $E_T(D)$, where $Q = \{v \in V(D) \mid d_v \geq 2\}$. Examples are given to show that this lower bound is in some sense best possible.

Keywords: Connectivity; Directed Euler tour; Transformation graph

1. Introduction

Let $D = (V, A)$ be a directed Eulerian multigraph. Then for any vertex $v$ of $D$, we have $id(v) = od(v)$. We simply denote the common value by $d_v$ and call it the degree of $v$. Let $E$ be a directed Euler tour of $D$. Then $E$ passes through each vertex $v$ exactly $d_v$ times. Thus we may write $E$ as

$$x_0'v_1x_1'v_2x_2'v_3 \cdots x_k'v_kx_k \cdots x_L.$$

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where \(x_0', x_1', \ldots, x_{d'-1}'\) are the arcs going into \(v\) and \(x_1, \ldots, x_{d}\) are the arcs going out of \(v\). A triple \((x_{i-1}', v, x_i')\) is called a transition of \(E\) through \(v\). A subsequence of \(E\) starting from \(v\) and ending at \(u\) (or \(v\)) which contains at least one arc is called a \(v - u\) (or \(v - v\)) segment of \(E\). Let \(S\) and \(S'\) be two arc-disjoint \(v - u\) segments of \(E\) such that \((S, S')\) is not a partition of \(E\). We call \(S\) and \(S'\) to be exchangeable. A directed Euler tour \(F\) is said to be obtained from \(E\) by a \(T\)-transformation at \(S\) and \(S'\), denoted by \(F = T(E)\), if \(F\) is obtained from \(E\) by exchanging \(S\) and \(S'\). The directed Euler tour graph of \(D\), denoted by \(E_u(D)\), is an undirected simple graph defined as follows: The vertices of \(E_u(D)\) are directed Euler tours of \(D\), and two directed Euler tours \(E\) and \(F\) are adjacent in \(E_u(D)\) if they can be obtained from each other by a \(T\)-transformation.

For more knowledge on Eulerian graphs, we refer the reader to Fleischner [1]. Xia [7] introduced the concept of the \(T\)-transformation for directed Euler tours and proved that any directed Euler tour \((T-)\)-transformation graph \(E_u(D)\) is connected. Zhang and Guo [9] proved that any edge of \(E_u(D)\) is contained in a Hamilton cycle of \(E_u(D)\). Now we will give a lower bound for the connectivity of \(E_u(D)\) and examples to show that this lower bound is in some sense best possible. First, we need the following preparations.

2. Preliminaries

Let \(Q = Q(D)\) be the set of vertices \(v\) of \(D\) such that \(d_v \geq 2\) and \(v\) is not a cut-vertex with degree 2. We assume that \(Q \neq \emptyset\), for otherwise, we get a trivial case that \(D\) has only one directed Euler tour. For \(v \in Q\), we denote by \(S_i\) the set of all directed Euler tours of \(D\) which contain the transition \((x_0', v, x_i)\). Then \(1 \leq i \leq d_v\) and \(S_1, S_2, \ldots\) form a partition of \(V(E_u(D))\). Obviously, \(S_j = \emptyset\) if and only if \(v\) is a cut-vertex and \(\{x_0', x_j\}\) is an arc-cut of \(D\). Thus, if \(d_v > 2\), there are at least \(d_v - 1\) non-empty \(S_i\), and if \(d_v = 2\), there are exactly two non-empty \(S_i\). Let \(H_i\) be the subgraph of \(E_u(D)\) induced by \(S_i\). Then \(H_i\) is isomorphic to \(E(D_i)\), where \(D_i\) is a directed Eulerian graph obtained from \(D\) in the following way: Replacing \(v\) by a pair of new vertices \(v'\) and \(v''\) and making the arc \(x_0'\) going into \(v'\) and \(x_i\) going out of \(v''\), whereas making the other arcs incident with \(v\) being incident with \(v''\) in the same manner as they are incident with \(v\).

In what follows, whenever \(S_i (i = 1, 2, \ldots)\) are mentioned, we mean the partition of \(V(E_u(D))\) through vertex \(v \in Q\) and with the starting arc \(x_0'\) incident with \(v\).

Lemma 1. Let \(S_i \) and \(S_j\) be non-empty with \(i \neq j\). Then for each \(E \in S_i\), there is at least one \(F \in S_j\) such that \(T(E) = F\), i.e., \(E\) is adjacent to \(F\) in \(E_u(D)\).

Proof. Since \(E \in S_i\), \(E\) contains the transition \((x_0', v, x_i)\). Since \(x_j\) is an arc going out of \(v\), \(E\) must contain a transition \((y', v, x_j)\). Hence \(E\) can be written as \(x_0'v x_1 \cdots y'v x_j \cdots\). We claim that there is a vertex \(u \neq v\) of \(D\) such that \(u\) appears in both the segment \(vx_i \cdots y'v\) and the segment \(vx_j \cdots x_0'v\) of \(E\), or \(v\) appears in the segment \(vx_j \cdots x_0'v\) of \(E\).
Otherwise, it is not difficult to see that \( \{x'_0, x_j\} \) would be an arc-cut and therefore \( S_j = \emptyset \), a contradiction. Therefore, we have that

\[
E: \frac{x'_0vx_1\cdots u\cdots yvx_j\cdots u\cdots}{s} \frac{s'}{s'}
\]

or,

\[
E: \frac{x'_0vx_1\cdots yvx_j\cdots v\cdots}{s} \frac{s'}{s'}
\]

In any case, we can use a \( T \)-transformation at \( S \) and \( S' \) (as indicated in the above) to transform \( E \) into a directed Euler tour belonging to \( S_j \). \( \square \)

In order to estimate the order of \( E_u(D) \), we introduce the following lemma.

Lemma 2. \(|V(E_u(D))| \geq \prod_{v \in Q} (d_v - 1)!\).

Proof. We use induction on \( \lambda(D) = \sum_{e \in Q} d_e \) to complete the proof.

Since \( Q \neq \emptyset \), \( \lambda(D) \geq 3 \). If \( \lambda(D) = 3 \), the conclusion holds clearly.

Suppose that the conclusion is true for any directed Eulerian graph \( D \) with \( \lambda(D) \leq m(\geq 3) \). If \( \lambda(D) = m + 1 \), there is a vertex \( v \) in \( Q \). Since \( S_i (i = 1, 2, \ldots) \) form a partition for \( V(E_u(D)) \), we have that

\[
|V(E_u(D))| = \sum_i |S_i| \geq (d_v - 1) \min \{|S_i|\} \quad (\text{at least } d_v - 1 \text{ non-empty } S_i).
\]

For each \( S_i \), since \( \lambda(D_i) \leq m \), from the induction hypothesis we know that

\[
|S_i| = |V(E_u(D_i))| \geq (d_v - 2)! \prod_{u \in Q(D)} (d_u - 1)!.
\]

Hence, we have

\[
|V(E_u(D))| \geq \prod_{v \in Q} (d_v - 1)!. \quad \square
\]

From Lemma 2, we see that, generally speaking, the order of a directed Euler tour transformation graph is considerably large. Thus, it is very difficult to give such a non-trivial concrete example.

Lemma 3. Let \( S_i \) and \( S_j \) be non-empty with \( i \neq j \). Then there are at least

\[
\sum_{u \in Q \cap r} (d_u - 1)(d_u - 2)/2 + (d_v - 2)(d_v - 3)/2 + 1
\]

independent edges (edges without any common end vertices) between \( S_i \) and \( S_j \).
**Proof.** We use induction on $\lambda(D) = \sum_{v \in Q} d_v$ to complete the proof.

If $\lambda(D) = 3$, the conclusion is obviously true.

Suppose that the conclusion is true for any directed Eulerian graph $D$ with $\lambda(D) \leq m (\geq 3)$. If $\lambda(D) = m + 1$, there is a vertex $v$ in $Q$. Consider the partition $S_1, S_2, \ldots, S_{d_v - 1}$ (or $S_{d_v}$) of $V(E_v(D))$. We claim that there is a vertex $u$ such that $u \in Q(D_i)$ for any $i = 1, 2, \ldots$. In fact, if $|Q| = 1$, then $d_v = \lambda(D) = m + 1 > 3$ and therefore $d_v > 3$ in any $D_i$. Thus $u = v$ is such a vertex. If $|Q| \geq 2$, there is a vertex $u \neq v$ such that $u \in Q$ and therefore $u \in Q(D_i)$ for any $i$. So, for each $i = 1, 2, \ldots, d_v - 1$ (or $d_v$) we can partition $V(E_u(D_i))$ through $u$ as $S_{i1}, S_{i2}, \ldots, S_{it}$ with $t = d_v - 1$ (or $d_v$).

We consider the following two cases.

**Case 1:** $|Q| \geq 2$. As stated in the above, $S_i$ and $S_j$ can be partitioned into $S_{i1}, S_{i2}, \ldots, S_{ik}, \ldots, S_{(d_v - 1)(d_v - 2) / 2 + 1}$ independent edges between $S_{ik}$ and $S_{(d_v - 1)(d_v - 2) / 2 + 1}$.

For a fixed $k$, consider how many independent edges there are between $S_{ik}$ and $S_{(d_v - 1)(d_v - 2) / 2 + 1}$.

**Case 2:** $|Q| = 1$. Let $Q = \{v\}$. We need prove that there are at least $(d_v - 2)(d_v - 3) / 2 + 1$ independent edges between $S_i$ and $S_j$. For $d_v = 4$, we can simply construct $E_v(D)$ to show the conclusion. So, we can assume that $d_v \geq 5$ in the following. As mentioned before, $S_i$ and $S_j$ can be partitioned into $S_{i1}, S_{i2}, \ldots, S_{(d_v - 2)$.
(or $S_{(d_v - 1)}$) and $S_{j1}, S_{j2}, \ldots, S_{jd_v - 2}$ (or $S_{j(d_v - 1)}$), respectively. For a fixed $k$, consider how many independent edges there are between $S_{ik}$ and $S_{jk}$. In a similar discussion as in Case 1, we know that there are at least $(d_v - 3)(d_v - 4)/2 + 1$ independent edges between $S_{ik}$ and $S_{jk}$. Fixing $i$ and $j$, running $k = 1, 2, \ldots, d_v - 2$ (or $d_v - 1$) and noticing that $S_i = \bigcup_k S_{ik}$, we obtain that there are at least

$$(d_v - 2)\left\{(d_v - 3)(d_v - 4)/2 + 1\right\} > (d_v - 2)(d_v - 3)/2 + 1$$

independent edges between $S_i$ and $S_j$. \(\square\)

3. Result

**Theorem 1.** Let $D$ be a directed Eulerian (multi-)graph. Then the connectivity of $E_u(D)$ is at least

$$\sum_{v \in Q} (d_v - 1)(d_v - 2)/2.$$

**Proof.** Let $v \in Q$. Then through $v$ we obtain a partition $S_1, S_2, \ldots, S_{d_v - 1}$ (or $S_{d_v}$) for $V(E_u(D))$. Again we use induction on $\lambda(D) = \sum_{v \in Q} d_v$ to complete the proof.

If $\lambda(D) = 3$, the conclusion is obviously true.

Suppose that the conclusion is true for any directed Eulerian graph $D$ with $\lambda(D) \leq m(\geq 3)$. If $\lambda(D) = m + 1$, since $\lambda(D) \leq m$, by the induction hypothesis, we know that each $H_i$ is

$$\sum_{u \in Q \setminus v} (d_u - 1)(d_u - 2)/2 + (d_v - 2)(d_v - 3)/2$$

closed. By Menger's theorem, we need show that for any pair of non-adjacent vertices $E$ and $F$ of $E_u(D)$, there are at least $\sum_{v \in Q} (d_v - 1)(d_v - 2)/2$ internally disjoint paths connecting $E$ and $F$ in $E_u(D)$. We consider the following two cases.

**Case 1:** There is an $i$ (1 $\leq i \leq d_v$) such that $E, F \in S_i$. Since $H_i$ is $\sum_{u \in Q \setminus v} (d_u - 1)(d_u - 2)/2 + (d_v - 2)(d_v - 3)/2$ connected, there are such many internally disjoint paths connecting $E$ and $F$ in $H_i$. By Lemma 1, for any $j \neq i$ with $S_j \neq \emptyset$, both $E$ and $F$ are adjacent to some vertices in $H_j$. Since there are at least $d_v - 2$ such $j$'s and $H_j$ is connected, there are in total at least

$$\left\{ \sum_{u \in Q \setminus v} (d_u - 1)(d_u - 2)/2 + (d_v - 2)(d_v - 3)/2 \right\} + (d_v - 2)$$

internally disjoint paths connecting $E$ and $F$ in $E_u(D)$. 


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Case 2: \( E \in S_i \) and \( F \in S_j \) with \( i \neq j \). We first prove that for any \( i \) and \( j \) with \( i \neq j \), the subgraph of \( E_u(D) \) induced by \( S_i \cup S_j \) is

\[
\sum_{u \in Q \setminus r} (d_u - 1)(d_u - 2)/2 + (d_v - 2)(d_v - 3)/2 + 1
\]

connected. In fact, if we delete a vertex-subset \( C \) from the subgraph induced by \( S_i \cup S_j \) with

\[
|C| < \sum_{u \in Q \setminus r} (d_u - 1)(d_u - 2)/2 + (d_v - 2)(d_v - 3)/2 + 1
\]

and the resultant subgraph is disconnected, then we shall deduce contradictions as follows.

First, there is at least one of \( S_i \) and \( S_j \), say \( S_i \) such that \( S_i \setminus C \) induces a connected subgraph of \( E_u(D) \). For otherwise, since the subgraphs of \( E_u(D) \) induced by \( S_i \) and \( S_j \) are \( H_i \) and \( H_j \), respectively, which are

\[
\sum_{u \in Q \setminus r} (d_u - 1)(d_u - 2)/2 + (d_v - 2)(d_v - 3)/2
\]

connected, we get that

\[
|C| \geq 2 \left\{ \sum_{u \in Q \setminus r} (d_u - 1)(d_u - 2)/2 + (d_v - 2)(d_v - 3)/2 \right\}
\]

\[
\geq \sum_{u \in Q \setminus r} (d_u - 1)(d_u - 2)/2 + (d_v - 2)(d_v - 3)/2 + 1,
\]

which contradicts the way that \( C \) was chosen.

Next, if \( (S_i \cup S_j) \setminus C \) induces a disconnected subgraph of \( E_u(D) \), then only the following two cases may happen.

(a) \( S_j \setminus C \) induces a disconnected subgraph of \( E_u(D) \). Let \( G_1, G_2, \ldots, G_t \) be all its components. Since \( S_j \) induces a subgraph of \( E_u(D) \) with connectivity at least

\[
\sum_{u \in Q \setminus r} (d_u - 1)(d_u - 2)/2 + (d_v - 2)(d_v - 3)/2,
\]

we have

\[
|S_j \cap C| \geq \sum_{u \in Q \setminus r} (d_u - 1)(d_u - 2)/2 + (d_v - 2)(d_v - 3)/2.
\]

Thus, \( |S_i \cap C| = 0 \). By Lemma 1, for each \( E \in G_k \) (\( k = 1, 2, \ldots, t \)) there is at least one vertex in \( S_i \) adjacent to \( E \). Hence, \( (S_i \cup S_j) \setminus C \) must induce a connected subgraph of \( E_u(D) \), a contradiction.

(b) \( S_j \setminus C \) induces a connected subgraph of \( E_u(D) \). On the one hand, we know that \( S_j \setminus C \) must also induce a connected subgraph of \( E_u(D) \); on the other hand, by Lemma 3, there are at least

\[
\sum_{u \in Q \setminus r} (d_u - 1)(d_u - 2)/2 + (d_v - 2)(d_v - 3)/2 + 1
\]
independent edges between \(S_i\) and \(S_j\). However,
\[
|C| < \sum_{u \in Q \setminus v} (d_u - 1)(d_u - 2)/2 + (d_v - 2)(d_v - 3)/2 + 1,
\]
which implies that \((S_i \cup S_j) \setminus C\) must induce a connected subgraph of \(E_u(D)\), again a contradiction.

Finally, we turn to finding \(\sum_{e \in Q} (d_e - 1)(d_e - 2)/2\) internally disjoint paths connecting \(E\) and \(F\) in \(E_u(D)\). Since \(E \in S_i\) and \(F \in S_j\) with \(i \neq j\), from the above we know that \(S_i \cup S_j\) induces a subgraph of \(E_u(D)\) with connectivity at least
\[
\sum_{u \in Q \setminus v} (d_u - 1)(d_u - 2)/2 + (d_v - 2)(d_v - 3)/2 + 1.
\]

Since \(E, F \in S_i \cup S_j\), there are at least
\[
\sum_{u \in Q \setminus v} (d_u - 1)(d_u - 2)/2 + (d_v - 2)(d_v - 3)/2 + 1
\]
internally disjoint paths connecting \(E\) and \(F\) in the subgraph induced by \(S_i \cup S_j\). On the other hand, for every \(k \neq i, j\) with \(S_k \neq \emptyset\), by Lemma 1 we know that both \(E\) and \(F\) have some neighbors in \(S_k\). Since \(H_k\), the subgraph induced by \(S_k\), is connected, there is a path connecting \(E\) and \(F\) and only passing through vertices in \(H_k\). Since there are at least \(d_v - 3\) such \(k\), there are in total at least
\[
\left\{ \sum_{u \in Q \setminus v} (d_u - 1)(d_u - 2)/2 + (d_v - 2)(d_v - 3)/2 + 1 \right\} + (d_v - 3)
\]
internally disjoint paths connecting \(E\) and \(F\) in \(E_u(D)\). \(\square\)

**Remark.** In [4] or [3], we obtained the exact connectivity for the Euler tour transformation graph \(E_u(G)\) of an undirected Eulerian (multi-)graph \(G\). From [6,2,8] we know that \(E_u(G)\) has many nice structural properties, for example, it is regular and it is the skeleton graph of a \((0,1)\)-polyhedron [5]. However, things are changed completely for \(E_u(D)\). It is not difficult to give examples to show that \(E_u(D)\) is neither regular nor the skeleton graph of any \((0,1)\)-polyhedron. Perhaps this is the reason why we have not yet found the exact connectivity for \(E_u(D)\).

4. **Concluding remark**

Let \(D = (V, A)\) be the directed Eulerian multigraph with \(V = \{1, 2, \ldots, n\}\) and \(A = \{(1, 2), (2, 3), \ldots, (n - 1, n), (n, 1), (1, 1), (1, 1), (2, 2), (2, 2), \ldots, (n, n), (n, n)\}\). It is not difficult to see that \(E_u(D) \cong K_2 \times K_2 \times \cdots \times K_2\) \((n\) copies of \(K_2\)). Hence, the
connectivity of the $E_u(D)$ is $n$. From the lower bound given in Theorem 1, we know that $E_u(D)$ is connected. In this sense, we say that the lower bound in Theorem 1 is best possible.

References