Linear Algebra and its Applications 437 (2012) 2366–2385

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

On disjoint range operators in a Hilbert space ☆

Chunyuan Deng [a], Yimin Wei [b,c], Qingxiang Xu [d,e,*], Chuanning Song [d]

[a] College of Mathematics Science, South China Normal University, Guangzhou 510631, PR China
[b] School of Mathematical Sciences, Fudan University, Shanghai 200433, PR China
[c] Key Laboratory of Mathematics for Nonlinear Sciences (Fudan University), Ministry of Education, Shanghai, PR China
[d] Department of Mathematics, Shanghai Normal University, Shanghai 200234, PR China
[e] School of Science, Shanghai Institute of Technology, Shanghai 201418, PR China

ARTICLE INFO

Article history:
Received 4 April 2012
Accepted 3 June 2012
Available online 28 June 2012

Submitted by P. Šemrl

AMS classification:
47A05
47A62
15A09

Keywords:
Positive operator
Moore–Penrose inverse
Projector

ABSTRACT

For a bounded linear operator M in a Hilbert space $\mathcal{H}$, various relations among the ranges $\mathcal{R}(M)$, $\mathcal{R}(M^*)$, $\mathcal{R}(M+M^*)$ and the null spaces $\mathcal{N}(M)$, $\mathcal{N}(M^*)$ are considered from the point of view of their relations to the known classes of operators, such as EP, co-EP, weak-EP, GP, DR, or SR. Particular attention is paid to the range projectors of the operators M, M^* and some further characteristics of these projectors are derived as well.

1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be separable, infinite dimensional, complex Hilbert spaces. We denote the set of all bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$ by $\mathcal{B}(\mathcal{H},\mathcal{K})$ and by $\mathcal{B}(\mathcal{H})$ when $\mathcal{H}=\mathcal{K}$. For $A\in\mathcal{B}(\mathcal{H},\mathcal{K})$, let A^*, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ be the adjoint, the range and the null space of A, respectively. An operator A is said to be positive if $(Ax,x)\geqslant 0$ for all $x\in\mathcal{H}$. If A is positive, the positive square root of A is denoted by $A^{\frac{1}{2}}$

☆ The first, third and fourth authors are supported by the National Natural Science Foundation of China under Grant 11171222. The second author is supported by the National Natural Science Foundation of China under Grant 10871051, Shanghai Municipal Education Committee, Shanghai Science & Technology Committee, and Doctoral Program of the Ministry of Education under Grant 20090071110003. The third and fourth authors are supported by the Innovation Program of Shanghai Municipal Education Commission under Grant 12ZZ129.

* Corresponding author at: Department of Mathematics, Shanghai Normal University, Shanghai 200234, PR China.
E-mail addresses: cydeng@scnu.edu.cn, cy-deng@263.net (C. Deng), ymwei@fudan.edu.cn, yimin.wei@gmail.com (Y. Wei), qxxu@shnu.edu.cn, qingxiang_xu@126.com (Q. Xu), songning@shnu.edu.cn, songning1962@163.com (C. Song).

0024-3795/$ - see front matter
http://dx.doi.org/10.1016/j.laa.2012.06.002

(see [16,32]). An operator $P \in \mathcal{B}(\mathcal{H})$ is said to be an orthogonal projector if $P^2 = P = P^*$. Clearly, any orthogonal projector is positive. The orthogonal projector onto a closed subspace $U \subset \mathcal{H}$ is denoted by P_U. The identity on U is denoted by I_U or I if there does not exist confusion. Let $\overline{K}$ denote the closure of $K \subset \mathcal{H}$. An operator $P \in \mathcal{B}(\mathcal{H})$ is said to be idempotent if $P^2 = P$. We use the usual notation $\overline{P} = I - P$. Let $P_{U,V}$ denote the idempotent with $\mathcal{R}(P_{U,V}) = U$ and $\mathcal{N}(P_{U,V}) = V$. The direct sum and the orthogonal direct sum are denoted by $U \oplus V$ and $U \oplus^{\perp} V$, respectively. A bounded linear idempotent $P_{U,V}$ induces a splitting of $\mathcal{H}$ into the direct sum of two subspaces: $\mathcal{R}(P_{U,V}) \oplus \mathcal{N}(P_{U,V}) = U \oplus V = \mathcal{H}$. This equation is equivalent to that the operator $P_U - P_V$ is invertible or equivalent to $\|P_U + P_V - I\| < 1$ [13,25]. It is clear that $\mathcal{R}(P_U) \oplus^{\perp} \mathcal{N}(P_U) = \mathcal{H}$.

An operator T is called generalized invertible, if there is an operator S such that (I) $TST = T$. The operator S is not unique in general. In order to guarantee its uniqueness, further conditions have to be imposed. The most likely convenient additional conditions are

$$\text{(II) } STS = S, \quad \text{(III) } (TS)^* = TS, \quad \text{(IV) } (ST)^* = ST, \quad \text{(V) } TS = ST.$$

Elements $S \in \mathcal{B}(\mathcal{H})$ satisfying (I, II, V) are called group inverses, denoted by $S = T^{\#}$. Similarly, (I, II, III, IV)-inverses are called Moore–Penrose inverses (for short MP-inverses), denoted by $S = T^{+}$. It is well known that T has the MP-inverse if and only if $\mathcal{R}(T)$ is closed, and the MP-inverse of T is unique and $T^{+} = T^*(TT^*)^{+}$ (see [8,15,38,40–42]). Moreover, (I, II, III, IV, V)-inverses are called EP elements (i.e., $T^{+} = T^{\#}$).

Recall that a linear operator M is said to be closed, if it satisfies the condition that $x_n \in \mathrm{dom}(M)$ converges to x and Mx_n converges to $y \in \mathcal{H}$, then $x \in \mathrm{dom}(M)$ and $y = Mx$. It is well-known that a densely defined, closed linear operator M is bounded if and only if $\mathrm{dom}(M) = \mathcal{H}$ [1]. In this paper, we only consider bounded linear operators in a Hilbert space $\mathcal{H}$. If $M \in \mathcal{B}(\mathcal{H})$ is a closed range operator, then $M^{+} \in \mathcal{B}(\mathcal{H})$ and the orthogonal projectors MM^{+} and $M^{+}M \in \mathcal{B}(\mathcal{H})$ have the relations:

$$\mathcal{R}(M) = \mathcal{R}(MM^{+}), \quad \mathcal{N}(M^{+}) = \mathcal{N}(MM^{+}), \quad \mathcal{H} = \mathcal{R}(M) \oplus^{\perp} \mathcal{N}(M^{+}),$$

and

$$\mathcal{R}(M^{+}) = \mathcal{R}(M^{+}M), \quad \mathcal{N}(M) = \mathcal{N}(M^{+}M), \quad \mathcal{H} = \mathcal{R}(M^{+}) \oplus^{\perp} \mathcal{N}(M).$$

The MP-inverse has been proved useful in systems theory, difference equations, differential equations and iterative procedures. It would be helpful if these results could be extended to infinite dimensional situations. Applications could then be made to denumerable systems theory, abstract Cauchy problems, infinite systems of linear differential equations, partial differential equations and other interesting topics (see, for example [12,35]).

The definition below introduces six types of operators which are important from the point of view of the present references (see, for example [5,6,10] for the matrix cases). Although these kinds of operators can be generalized to much more general settings, they have been studied specially in the space of complex matrices. In this paper, we study the EP, GP, RD, SR, co-EP and weak-EP on the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators in a Hilbert space $\mathcal{H}$. Since many of the usual techniques used in finite dimensional spaces (as pseudoinverses or singular value decompositions) are no longer available for general Hilbert spaces, we introduce new techniques, which allow us to show that almost all known properties which hold for matrices can be extended to operators acting in a Hilbert space $\mathcal{H}$, and to obtain simpler proofs. Indeed, several results of [2–7,9–11,19–22,28] are recovered in this paper, if we consider the finite dimensional spaces. On the other hand, we show several properties for general Hilbert spaces which are unknown even in the finite dimensional setting, particularly those results concerning the relationship between the projector and range relations. First we need to extend the notions to bounded linear operators in an infinite-dimensional Hilbert space.

Definition. The closed range operator $M \in \mathcal{B}(\mathcal{H})$ is called:

(1) GP whenever $\mathcal{R}(M) = \mathcal{R}(M^2)$ and $\mathcal{N}(M) = \mathcal{N}(M^2)$.
(2) EP whenever $\mathcal{R}(M) = \mathcal{R}(M^*)$.
(3) DR whenever $\mathcal{R}(M) \cap \mathcal{R}(M^*) = \{0\}$.
(4) SR whenever $\mathcal{R}(M) + \mathcal{R}(M^*) = \mathcal{H}$.

(5) co-EP whenever $\mathcal{R}(M) \oplus \mathcal{R}(M^*) = \mathcal{H}$.
(6) weak-EP whenever $P_{\mathcal{R}(M)}P_{\mathcal{R}(M^*)} = P_{\mathcal{R}(M^*)}P_{\mathcal{R}(M)}$.

The classes of EP and GP operators, the so-called range-Hermitian and group invertible operators, respectively, were extensively investigated in the literature (see [6, Lemma 2,8, Chapter 4.4,15, Chapter 4,28, Corollary 6]). The DR and SR operators, the so-called disjoint ranges and spanning ranges operators, respectively, were introduced by Baksalary and Trenkler [5, Definition 1]. The class of co-EP operators was investigated by Benítez and Rakočević [10]. It is obvious that, for a closed range operator $M \in \mathcal{B}(\mathcal{H})$ (see [5, Lemma 1] for the matrix cases),

(1) M is EP (resp. GP, DR, SR, co-EP and weak-EP)
$\Longleftrightarrow M^+$ is EP (resp. GP, DR, SR, co-EP and weak-EP)
$\Longleftrightarrow M^*$ is EP (resp. GP, DR, SR, co-EP and weak-EP).
(2) M is simultaneously EP and DR, if and only if $M = 0$.
(3) M is simultaneously EP and SR, if and only if M is invertible.
(4) M is simultaneously DR and SR, if and only if M is co-EP.

In [39], Šemrl discussed possible extensions of the concept of the minus partial order from matrices to bounded linear operators acting on an infinite-dimensional Hilbert space. Dolinar and Marovt in [20], using orthogonal projectors, introduced the equivalent definition of the star partial order on $\mathcal{B}(\mathcal{H})$. And some properties of the generalized concept of order relations on $\mathcal{B}(\mathcal{H})$, defined with the help of idempotent operators, are investigated in [20]. The aim of this paper is to present several representations of $M \in \mathcal{B}(\mathcal{H})$ in terms of operator matrix forms and several descriptions of range relations by using orthogonal projectors $P = MM^+$ and $Q = M^+M$, when $\mathcal{R}(M)$ is closed.

2. Some lemmas

In this section we shall recall some lemmas. If $T \in \mathcal{B}(\mathcal{H})$ and $G \in \mathcal{B}(\mathcal{K})$ both are invertible, and $Y, Z \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, then $T + YGZ^*$ is invertible if and only if $G^{-1} + Z^*T^{-1}Y$ is invertible (see [27,33]). In this case, we have the Sherman–Morrison–Woodbury formula (for short SMW-formula)

$$(T + YGZ^*)^{-1} = T^{-1} - T^{-1}Y(G^{-1} + Z^*T^{-1}Y)^{-1}Z^*T^{-1}.$$

The original SMW-formula is used to consider the inverse of 2×2 block matrices. In particular, the SMW formula implies the following result.

Lemma 2.1. *Let $A, B \in \mathcal{B}(\mathcal{H})$. $I - AB$ is invertible if and only if $I - BA$ is invertible. In this case,*

$$(I - AB)^{-1} = I + A(I - BA)^{-1}B. \tag{1}$$

Proof. Assume that $I - AB$ has the inverse $I - W$ and let $V = B(W - I)A$. Then $I - V$ is the inverse of $I - BA$ and (1) is the special case of the SMW formula by replacing T, Y, G, Z^* with $I, -A, I, B$, respectively. □

We also need the following well-known criteria about range. The following item (i) is from [24, Theorem 2.2].

Lemma 2.2 (see [23,29,24, Theorem 2.2])**.** *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

(i) *$\mathcal{R}(A) + \mathcal{R}(B) = \mathcal{R}((AA^* + BB^*)^{\frac{1}{2}})$ and $\mathcal{N}(AB) = \mathcal{N}(A^*AB)$.*
(ii) *$\mathcal{R}(A)$ is closed if and only if $\mathcal{R}(A) = \mathcal{R}(AA^*)$.*
(iii) *If $R(B)$ is closed, then $\mathcal{R}(AB) = \mathcal{R}(ABB^*)$ and $\mathcal{R}(B^+) = \mathcal{R}(B^*)$.*

(iv) *If* $A \geqslant 0$ *is a positive operator, then* $\overline{\mathcal{R}(A^{\frac{1}{2}})} = \overline{\mathcal{R}(A)}$, $\mathcal{R}(A) \subseteq \mathcal{R}(A^{\frac{1}{2}})$. $\mathcal{R}(A)$ *is closed if and only if* $\mathcal{R}(A) = \mathcal{R}(A^{\frac{1}{2}})$.

Let P and Q be two orthogonal projectors. Now, we consider the invertibility of $P - Q$. This problem is the theme of Buckholtz's papers [13,14], Koliha and Rakočević's paper [31] and a special case of [30, Theorem 3.1]. The next results were also proved in [15] in the setting of rings.

Lemma 2.3 (see [30, Corollary 3.2,14, Theorem 1])**.** *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be closed subspaces of a Hilbert space* $\mathcal{H}$ *and let* P *and* Q *be the orthogonal projectors with the ranges* $\mathcal{M}$ *and* $\mathcal{N}$*, respectively. The following statements are equivalent:*

(i) $P - Q$ *is invertible.*
(ii) $I - PQ$ *and* $P + Q - PQ$ *are invertible.*
(iii) $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$.
(iv) $P + Q$ *and* $I - PQ$ *are invertible.*

When we consider the operator matrix representation of $M \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$, we need the following lemmas.

Lemma 2.4 [34, Theorem 2.1]**.** *Let* $M = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ *be a bounded linear operator on* $\mathcal{H} \oplus \mathcal{K}$*. If* A_{22} *is invertible, then* M *is invertible if and only if the Schur complement* $S = A_{11} - A_{12}A_{22}^{-1}A_{21}$ *is invertible. In this case,*

$$M^{-1} = \begin{pmatrix} S^{-1} & -S^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}S^{-1} & A_{22}^{-1} + A_{22}^{-1}A_{21}S^{-1}A_{12}A_{22}^{-1} \end{pmatrix}.$$

As for the orthogonal projectors P and Q, we have the following 6×6 operator matrix representations.

Lemma 2.5 (see [22, Lemma 1] and [26])**.** *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be closed subspaces of a Hilbert space* $\mathcal{H}$ *and let* P *and* Q *be the orthogonal projectors with the ranges* $\mathcal{M}$ *and* $\mathcal{N}$*, respectively. Denote* $\mathcal{H}_1 = \mathcal{M} \cap \mathcal{N}$, $\mathcal{H}_2 = \mathcal{M} \cap \mathcal{N}^\perp$, $\mathcal{H}_3 = \mathcal{M}^\perp \cap \mathcal{N}$, $\mathcal{H}_4 = \mathcal{M}^\perp \cap \mathcal{N}^\perp$, $\mathcal{H}_5 = \mathcal{M} \ominus (\mathcal{H}_1 \oplus \mathcal{H}_2)$ *and* $\mathcal{H}_6 = \mathcal{H} \ominus (\oplus_{i=1}^5 \mathcal{H}_i)$*. Then* P *and* Q *can be represented as*

$$\begin{aligned} P &= I \oplus I \oplus 0 \oplus 0 \oplus \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \\ Q &= I \oplus 0 \oplus I \oplus 0 \oplus \begin{pmatrix} Q_0 & Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}}D \\ D^*Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}} & D^*(I-Q_0)D \end{pmatrix} \end{aligned} \tag{2}$$

with respect to the space orthogonal direct sum decomposition $\mathcal{H} = \oplus_{i=1}^6 \mathcal{H}_i$*, where* Q_0 *is a positive contraction on* $\mathcal{H}_5$ *such that neither* 0 *nor* 1 *belongs to the point spectrum of* Q_0*,* D *is a unitary operator from* $\mathcal{H}_6$ *onto* $\mathcal{H}_5$.

Lemma 2.6. *Let* $M \in \mathcal{B}(\mathcal{H})$*. Then*

(i) (See [16, page 38]) *According to the space decomposition* $\mathcal{H} = \overline{\mathcal{R}(M)} \oplus^\perp \mathcal{N}(M^*)$*,* M *has the* 2×2 *block operator matrix form*

$$M = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}, \quad \text{where } A \in \mathcal{B}(\overline{\mathcal{R}(M)}),\ B \in \mathcal{B}(\mathcal{N}(M^*), \overline{\mathcal{R}(M)}). \tag{3}$$

(ii) (See [17, Lemma 4]) *M is MP-invertible if and only if $\mathcal{R}(M) = \mathcal{R}(A) + \mathcal{R}(B)$ is closed. In this case, $\Delta = (AA^* + BB^*)^{-1}$ exists and*

$$M^+ = \begin{pmatrix} A^*(AA^* + BB^*)^{-1} & 0 \\ B^*(AA^* + BB^*)^{-1} & 0 \end{pmatrix} = \begin{pmatrix} A^*\Delta & 0 \\ B^*\Delta & 0 \end{pmatrix}. \tag{4}$$

(iii) *The orthogonal projectors $P = MM^+$ and $Q = M^+M$ have the forms*

$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \quad \textit{and} \quad Q = \begin{pmatrix} A^*\Delta A & A^*\Delta B \\ B^*\Delta A & B^*\Delta B \end{pmatrix}. \tag{5}$$

As we know, the ranges of GP, EP, idempotent and orthogonal projector are all closed. By Lemma 2.6, we have the following observations:

(1) M is GP if and only if A is invertible.
(2) M is EP if and only if A is invertible and $B = 0$.
(3) M is an idempotent if and only if $A = I$.
(4) M is an orthogonal projector if and only if $A = I$ and $B = 0$.

3. The operator matrix structures for DR, SR and co-EP operators

The type of square matrices M such that $MM^+ - M^+M$ is nonsingular was investigated by Benítez and Rakočević [10]. We first consider several characterizations of operator $MM^+ - M^+M$ in the case that $\mathcal{R}(M)$ is closed. In Lemma 2.5, if we set $\mathcal{M} = \mathcal{R}(M)$ and $\mathcal{N} = \mathcal{R}(M^*)$, then Lemma 2.5 and Lemma 2.6 imply that $A^*\Delta A$ as an operator on $\mathcal{H}_1 \oplus^\perp \mathcal{H}_2 \oplus^\perp \mathcal{H}_5$, $A^*\Delta B$ as an operator from $\mathcal{H}_3 \oplus^\perp \mathcal{H}_4 \oplus^\perp \mathcal{H}_6$ into $\mathcal{H}_1 \oplus^\perp \mathcal{H}_2 \oplus^\perp \mathcal{H}_5$ and $B^*\Delta B$ as an operator on $\mathcal{H}_3 \oplus^\perp \mathcal{H}_4 \oplus^\perp \mathcal{H}_6$ can be represented as diagonal operators:

$$A^*\Delta A = \begin{pmatrix} I & & \\ & 0 & \\ & & Q_0 \end{pmatrix}, \quad A^*\Delta B = \begin{pmatrix} 0 & & \\ & 0 & \\ & & Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}}D \end{pmatrix}, \quad B^*\Delta B = \begin{pmatrix} I & & \\ & 0 & \\ & & D^*(I-Q_0)D \end{pmatrix}, \tag{6}$$

respectively. Here, the omitted elements stand for zero operators of the appropriate sizes. We have the following equivalent relations:

Theorem 3.1. *Let $M \in \mathcal{B}(\mathcal{H})$ be such that $\mathcal{R}(M)$ is closed, M and M^+ be represented as in (3) and (4), respectively. The following statements are equivalent:*

(i) *$MM^+ - M^+M$ is EP.*
(ii) *$MM^+(I - M^+M)MM^+$ is EP.*
(iii) *$MM^+(I - M^+M)$ is MP-invertible.*
(iv) *$I - A^*\Delta A$ is EP.*
(v) *$B^*\Delta B$ is EP, where $\Delta = (AA^* + BB^*)^{-1}$.*

Proof. Let orthogonal projectors $P = MM^+$ and $Q = M^+M$. By (5) and (6), we have

$$(P - Q)^2 = \begin{pmatrix} I - A^*\Delta A & \\ & B^*\Delta B \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 & & \\ & I & \\ & & I-Q_0 \end{pmatrix} & \\ & \begin{pmatrix} I & & \\ & 0 & \\ & & D^*(I-Q_0)D \end{pmatrix} \end{pmatrix}. \tag{7}$$

Hence

$$P - Q \text{ is EP if and only if } (P - Q)^2 \text{ is EP if and only if } I - Q_0 \text{ is invertible,} \tag{8}$$

since D is unitary and Q_0 is a positive contraction on $\mathcal{H}_5$ and 1 is not the point spectrum of Q_0. Hence (i) $\Longleftrightarrow$ (iv) $\Longleftrightarrow$ (v). From

$$P(I-Q)P = \begin{pmatrix} I - A^*\Delta A & 0 \\ 0 & 0 \end{pmatrix},$$

we get (ii) $\Longleftrightarrow$ (iv). Note that $P(I-Q) = \begin{pmatrix} I - A^*\Delta A & -A^*\Delta B \\ 0 & 0 \end{pmatrix}$ and

$$P(I-Q)[P(I-Q)]^* = \begin{pmatrix} I - A^*\Delta A & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 & & \\ & I & \\ & & I-Q_0 \end{pmatrix} & \\ & 0 \end{pmatrix}.$$

Since $P(I-Q)$ is MP-invertible if and only if $P(I-Q)[P(I-Q)]^*$ is EP, if and only if $I-Q_0$ is invertible, we derive (iii) $\Longleftrightarrow$ (iv). $\square$

Note that $MM^+ - M^+M$ is EP if and only if $\mathcal{R}(MM^+ - M^+M)$ is closed if and only if $I - Q_0$ is invertible. In this case, by (7),

$$\mathcal{R}(MM^+ - M^+M) = \mathcal{R}(P-Q) = \mathcal{R}((P-Q)^2) = \mathcal{H}_2 \oplus^\perp \mathcal{H}_3 \oplus^\perp \mathcal{H}_5 \oplus^\perp \mathcal{H}_6.$$

Since $\mathcal{H} = \oplus_{i=1}^6 \mathcal{H}_i$, $\mathcal{H}_1 = \mathcal{R}(P) \cap \mathcal{R}(Q) = \mathcal{R}(M) \cap \mathcal{R}(M^*)$ and $\mathcal{H}_4 = \mathcal{R}(P)^\perp \cap \mathcal{R}(Q)^\perp = \mathcal{R}(M)^\perp \cap \mathcal{R}(M^*)^\perp$, we get

$$\mathcal{H} = \mathcal{R}(MM^+ - M^+M) \oplus^\perp [\mathcal{R}(M) \cap \mathcal{R}(M^*)] \oplus^\perp [\mathcal{R}(M)^\perp \cap \mathcal{R}(M^*)^\perp].$$

In particular, $MM^+ - M^+M$ is invertible if and only if M is co-EP, which can be found from the following results.

Theorem 3.2. *Let $M \in \mathcal{B}(\mathcal{H})$ be such that $\mathcal{R}(M)$ closed, M and M^+ be represented as in (3) and (4), respectively. The following statements are equivalent:*

(i) *$MM^+ - M^+M$ is invertible.*
(ii) *$MM^+ + M^+M$ is invertible and $\|MM^+ \cdot M^+M\| < 1$.*
(iii) *$\mathcal{H} = \mathcal{R}(M) \oplus \mathcal{R}(M^*)$ (resp. M is co-EP).*
(iv) *$I - A^*\Delta A$ and $B^*\Delta B$ are invertible, where $\Delta = (AA^* + BB^*)^{-1}$.*
(v) *According to the space decomposition $\mathcal{H} = \mathcal{R}(M) \oplus^\perp \mathcal{N}(M^*)$, M has the 2×2 block operator matrix form*

$$M = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}, \quad \text{where } B \text{ is invertible}. \tag{9}$$

(vi) *$MM^+ + M^+M$ is invertible and $\mathcal{R}(M) \cap \mathcal{R}(M^*) = \{0\}$.*
(vii) *$(M + M^*)(I - MM^+) \pm (I - MM^+)(M + M^*)$ is invertible.*
(viii) *$(M - M^*)(I - MM^+) \pm (I - MM^+)(M - M^*)$ is invertible.*

Proof. Let orthogonal projectors $P = MM^+$ and $Q = M^+M$. By Lemma 2.6, item (iii), we have

$$I - PQ = \begin{pmatrix} I - A^*\Delta A & -A^*\Delta B \\ 0 & I \end{pmatrix} \quad \text{and} \quad P + Q - PQ = \begin{pmatrix} I & 0 \\ B^*\Delta A & B^*\Delta B \end{pmatrix}.$$

Note that $\|PQ\| < 1 \Longleftrightarrow I - PQ$ is invertible (see also [9, Lemma 1.2]). By Lemma 2.3 we know (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv).

(iv) $\Longrightarrow$ (v) By (3) we know $B \in \mathcal{B}(\mathcal{N}(M^*), \mathcal{R}(M))$ and $BB^* \in \mathcal{B}(\mathcal{R}(M))$. So

$$\begin{aligned} I - A^*\Delta A \text{ is invertible } &\Longleftrightarrow I - \Delta AA^* \text{ is invertible} \quad [\text{by Lemma 2.1}] \\ &\Longleftrightarrow \Delta BB^* \text{ is invertible} \quad [\text{by } I = \Delta(AA^* + BB^*)] \\ &\Longleftrightarrow BB^* \text{ is invertible} \quad [\text{by Lemma 2.2.(ii)}] \\ &\Longleftrightarrow \mathcal{R}(B) = \mathcal{R}(M). \end{aligned} \tag{10}$$

Since $B^*\Delta B \in \mathcal{B}(\mathcal{N}(M^*))$ is invertible and $\mathcal{R}(B) = \mathcal{R}(M)$, we obtain

$$\mathcal{N}(M^*) = \mathcal{R}(B^*\Delta B) = \mathcal{R}(B^*).$$

Hence B, B^* are surjective and therefore B is invertible.

(v) $\Longrightarrow$ (i) By (3) and (4) we have

$$\begin{aligned} MM^+ - M^+M &= \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A^*\Delta & 0 \\ B^*\Delta & 0 \end{pmatrix} - \begin{pmatrix} A^*\Delta & 0 \\ B^*\Delta & 0 \end{pmatrix} \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} I - A^*\Delta A & -A^*\Delta B \\ -B^*\Delta A & -B^*\Delta B \end{pmatrix}. \end{aligned} \tag{11}$$

If B is invertible, then $(B^*\Delta B)^{-1} = B^{-1}\Delta^{-1}(B^*)^{-1}$ and the Schur complement

$$(I - A^*\Delta A) - (-A^*\Delta B)(-B^*\Delta B)^{-1}(-B^*\Delta A) = I.$$

Hence $MM^+ - M^+M$ is invertible by Lemma 2.4.

(vi) $\Longleftrightarrow$ (iii) It is clear (iii) $\Rightarrow$(vi). So we only need to show (vi)$\Rightarrow$(iii). Since $P + Q$ is invertible, $(P+Q)^{\frac{1}{2}}$ is invertible and, by Lemma 2.2,

$$\mathcal{R}(M) + \mathcal{R}(M^*) = \mathcal{R}(P) + \mathcal{R}(Q) = \mathcal{R}((P+Q)^{\frac{1}{2}}) = \mathcal{H}.$$

Hence $\mathcal{R}(M)$ and $\mathcal{R}(M^*)$ are complementary spaces and therefore (iii) holds.

(vii) $\Longleftrightarrow$ (v) By Lemma 2.6, we have

$$M + M^* = \begin{pmatrix} A + A^* & B \\ B^* & 0 \end{pmatrix}, \quad I - MM^+ = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$$

and

$$(M + M^*)(I - MM^+) \pm (I - MM^+)(M + M^*) = \begin{pmatrix} 0 & B \\ \pm B^* & 0 \end{pmatrix}.$$

Hence

$$(M + M^*)(I - MM^+) \pm (I - MM^+)(M + M^*)$$

is invertible if and only if B is invertible. The result follows immediately.

(viii) $\Longleftrightarrow$ (v) Similar to the proof of (vii) $\Longleftrightarrow$ (v). □

Theorem 3.3. *Let the assumptions of Theorem 3.2 hold and M be represented as in (9). Let also $\Delta = (AA^* + BB^*)^{-1}$.*

(i)

$$(I - A^*\Delta A)^{-1} = I + A^*(BB^*)^{-1}A \quad \textit{and} \quad (B^*\Delta B)^{-1} = I + B^{-1}AA^*(B^*)^{-1}.$$

(ii)

$$(M \pm M^*)^{-1} = \begin{pmatrix} 0 & \pm(B^*)^{-1} \\ B^{-1} & \mp B^{-1}(A \pm A^*)(B^*)^{-1} \end{pmatrix}.$$

(iii)

$$\begin{aligned}(MM^+ - M^+M)^{-1} &= (M + M^*)^{-1}(M^*M - MM^*)(M + M^*)^{-1} \\ &= \begin{pmatrix} I & -A^*(B^*)^{-1} \\ -B^{-1}A & -I \end{pmatrix}\end{aligned}$$

and

$$(MM^+ + M^+M)^{-1} = \begin{pmatrix} I & -A^*(B^*)^{-1} \\ -B^{-1}A & B^{-1}(2AA^* + BB^*)(B^*)^{-1} \end{pmatrix}.$$

Proof. (i) Let M be represented as in (9) and $\Delta = (AA^* + BB^*)^{-1}$. By (11), we get

$$(MM^+ - M^+M)^2 = \begin{pmatrix} I - A^*\Delta A & -A^*\Delta B \\ -B^*\Delta A & -B^*\Delta B \end{pmatrix}^2 = \begin{pmatrix} I - A^*\Delta A & 0 \\ 0 & B^*\Delta B \end{pmatrix}.$$

Next, we only prove $(I - A^*\Delta A)^{-1} = I + A^*(BB^*)^{-1}A$. The relation $(B^*\Delta B)^{-1} = I + B^{-1}AA^*(B^*)^{-1}$ can be proved in the same way. In fact,

$$\begin{aligned}(I - A^*\Delta A)[I + A^*(BB^*)^{-1}A] &= I - A^*\Delta A + A^*(BB^*)^{-1}A - A^*\Delta AA^*(BB^*)^{-1}A \\ &= I - A^*\Delta A + A^*\Delta(AA^* + BB^* - AA^*)(BB^*)^{-1}A \\ &= I\end{aligned}$$

and

$$\begin{aligned}[I + A^*(BB^*)^{-1}A](I - A^*\Delta A) &= I - A^*\Delta A + A^*(BB^*)^{-1}A - A^*(BB^*)^{-1}AA^*\Delta A \\ &= I - A^*\Delta A + A^*(BB^*)^{-1}(AA^* + BB^* - AA^*)\Delta A \\ &= I.\end{aligned}$$

(ii) It is obvious that

$$(M \pm M^*)^{-1} = \begin{pmatrix} A \pm A^* & B \\ \pm B^* & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & \pm(B^*)^{-1} \\ B^{-1} & \mp B^{-1}(A \pm A^*)(B^*)^{-1} \end{pmatrix}.$$

(iii) By item (i), we have

$$\begin{aligned}(MM^+ - M^+M)^{-1} &= (MM^+ - M^+M)(MM^+ - M^+M)^{-2} \\ &= \begin{pmatrix} I - A^*\Delta A & -A^*\Delta B \\ -B^*\Delta A & -B^*\Delta B \end{pmatrix}\begin{pmatrix} I + A^*(BB^*)^{-1}A & 0 \\ 0 & I + B^{-1}AA^*(B^*)^{-1} \end{pmatrix} \\ &= \begin{pmatrix} I & -A^*(B^*)^{-1} \\ -B^{-1}A & -I \end{pmatrix}.\end{aligned}$$

It follows from item (ii) that

$$\begin{aligned}\begin{pmatrix} I & -A^*(B^*)^{-1} \\ -B^{-1}A & -I \end{pmatrix} &= I - \begin{pmatrix} 0 & 0 \\ B^{-1}A & I \end{pmatrix} - \begin{pmatrix} 0 & A^*(B^*)^{-1} \\ 0 & I \end{pmatrix} \\ &= I - (M+M^*)^{-1}M - M^*(M+M^*)^{-1} \\ &= (M+M^*)^{-1}\big[(M+M^*)^2 - M(M+M^*) \\ &\quad - (M+M^*)M^*\big](M+M^*)^{-1} \\ &= (M+M^*)^{-1}(M^*M - MM^*)(M+M^*)^{-1}.\end{aligned}$$

Note that

$$MM^+ + M^+M = \begin{pmatrix} I + A^*\Delta A & A^*\Delta B \\ B^*\Delta A & B^*\Delta B \end{pmatrix}$$

and $B^*\Delta B$ is invertible. The representation for $(MM^+ + M^+M)^{-1}$ follows immediately by Lemma 2.4. □

The following is the immediate corollary of Theorem 3.2.

Corollary 3.4 (see [10, Theorem 2.9] for matrix case)**.** *Let $M \in \mathcal{B}(\mathcal{H})$ be such that $\mathcal{R}(M)$ is closed. The following statements are equivalent:*

(i) $\mathcal{H} = \mathcal{R}(M) \oplus^{\perp} \mathcal{R}(M^*)$ (or, $\mathcal{R}(M^*) = \mathcal{N}(M^*)$).

(ii) *According to the space decomposition $\mathcal{H} = \mathcal{R}(M) \oplus^{\perp} \mathcal{N}(M^*)$, M has the 2×2 block operator matrix form*

$$M = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}, \quad \text{where } B \text{ is invertible}. \tag{12}$$

Let P and Q be orthogonal projectors. If $P, Q \in \mathbb{C}^{n\times n}$, where $\mathbb{C}^{n\times n}$ is the set of $n \times n$ complex matrices, an important tool in constructing orthogonal projectors onto given column spaces is provided in the literature. Baksalary and Trenkler (see (2.16) and (2.17) in [3]) showed that

$$P_{\mathcal{R}(P)\cap\mathcal{R}(Q)} = I_n - (I_n - PQ)(I_n - PQ)^+,$$

and

$$P_{\mathcal{N}(P)+\mathcal{N}(Q)} = (I_n - PQ)(I_n - PQ)^+.$$

Several alternative formulae for $P_{\mathcal{R}(P)\cap\mathcal{R}(Q)}$ are given in [36, Theorem 4], with

$$P_{\mathcal{R}(P)\cap\mathcal{R}(Q)} = 2P(P+Q)^+Q$$

proved by Groß [37, Corollary 3] and

$$P_{\mathcal{R}(P)\cap\mathcal{R}(Q)} = P - P(P\overline{Q})^+, \quad P_{\mathcal{R}(P)+\mathcal{R}(Q)} = P + \overline{P}(\overline{P}Q)^+$$

provided by Baksalary and Trenkler in [4, Lemma 7]. If $P, Q \in \mathcal{B}(\mathcal{H})$ and $I - PQ$ (or $P - Q$) is MP-invertible, then the relations

$$P_{\mathcal{R}(P)\cap\mathcal{R}(Q)} = P - P(P\overline{Q})^+, \quad P_{\mathcal{R}(P)+\mathcal{R}(Q)} = P + \overline{P}(\overline{P}Q)^+$$

still hold (see, for example [18, Theorem 2.6]). In the following theorem, we consider some range relations under the case that $\mathcal{R}(MM^+ - M^+M)$ is closed.

Theorem 3.5. *Let $M \in \mathcal{B}(\mathcal{H})$ be represented as in (3) so that $\mathcal{R}(M)$ and $\mathcal{R}(MM^+ - M^+M)$ are closed.*

(i) M *is DR* $\Longleftrightarrow \mathcal{R}(B) = \mathcal{R}(M)$ (i.e., B is surjective).
(ii) M *is SR* $\Longleftrightarrow \mathcal{R}(B^*) = \mathcal{N}(M^*)$ (i.e., B^* is surjective).
(iii) M *is co-EP* $\Longleftrightarrow$ *B is invertible.*

Proof. Since $\mathcal{R}(M)$ is closed, M^+ exists. Let M, M^+, $P = MM^+$ and $Q = M^+M$ be represented as in (3)–(6), respectively.

(i) Since $\mathcal{R}(P-Q)$ is closed, $P-Q$ is MP-invertible. Since $P-Q$ is selfadjoint, we know that $P-Q$ is EP. Hence, by (8), $P-Q$ is MP-invertible if and only if $I-Q_0$ is invertible. By (5) and (6), we have

$$P\overline{Q} = \begin{pmatrix} I - A^*\Delta A & -A^*\Delta B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 & & \\ & I & \\ & & I-Q_0 \end{pmatrix} & \begin{pmatrix} 0 & & \\ & 0 & \\ & & -Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}}D \end{pmatrix} \\ 0 & 0 \end{pmatrix}$$

and $P\overline{Q}(P\overline{Q})^* = \begin{pmatrix} \begin{pmatrix} 0 & & \\ & I & \\ & & I-Q_0 \end{pmatrix} & \\ & 0 \end{pmatrix}$. Similarly,

$$\overline{P}Q = \begin{pmatrix} 0 & 0 \\ B^*\Delta A & B^*\Delta B \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \begin{pmatrix} 0 & & \\ & 0 & \\ & & D^*Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}} \end{pmatrix} & \begin{pmatrix} 0 & & \\ & I & \\ & & D^*(I-Q_0)D \end{pmatrix} \end{pmatrix}$$

and $\overline{P}Q(\overline{P}Q)^* = \begin{pmatrix} 0 & \\ & \begin{pmatrix} 0 & & \\ & I & \\ & & D^*(I-Q_0)D \end{pmatrix} \end{pmatrix}$. By Lemma 2.2, item (ii), $\mathcal{R}(P\overline{Q})$ (or $\mathcal{R}(\overline{P}Q)$) is closed if and only if $\mathcal{R}(I-Q_0)$ is closed. Since 1 is not the point spectrum of the positive contraction operator Q_0 and D is a unitary operator by Lemma 2.5, $\mathcal{R}(I-Q_0)$ is closed if and only if $I-Q_0$ is invertible. Now, we deduce that $P-Q$ is MP-invertible if and only if $P\overline{Q}$ is MP-invertible, if and only if $\overline{P}Q$ is MP-invertible, if and only if $I-Q_0$ is invertible (see also [38, Proposition 7]). Moreover, $(P\overline{Q}(P\overline{Q})^*)^+ = \begin{pmatrix} \begin{pmatrix} 0 & & \\ & I & \\ & & (I-Q_0)^{-1} \end{pmatrix} & \\ & 0 \end{pmatrix}$,

$$(P\overline{Q})^+ = (P\overline{Q})^*(P\overline{Q}(P\overline{Q})^*)^+ = \begin{pmatrix} \begin{pmatrix} 0 & & \\ & I & \\ & & I \end{pmatrix} & 0 \\ \begin{pmatrix} 0 & & \\ & 0 & \\ & & -D^*Q_0^{\frac{1}{2}}(I-Q_0)^{-\frac{1}{2}} \end{pmatrix} & 0 \end{pmatrix}$$

and

$$\begin{aligned} P - P(P\overline{Q})^+ &= \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 & & \\ & I & \\ & & I \end{pmatrix} & 0 \\ \begin{pmatrix} 0 & & \\ & 0 & \\ & & -D^*Q_0^{\frac{1}{2}}(I-Q_0)^{-\frac{1}{2}} \end{pmatrix} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \begin{pmatrix} I & & \\ & 0 & \\ & & 0 \end{pmatrix} & \\ & 0 \end{pmatrix} = P_{\mathcal{R}(P)\cap\mathcal{R}(Q)}. \end{aligned}$$

Observing that, for arbitrary MP-invertible operator T, it has $T^+ = T^*(TT^*)^+$. So

$$\begin{aligned} \begin{pmatrix} I - A^*\Delta A & -A^*\Delta B \\ 0 & 0 \end{pmatrix}^+ &= \begin{pmatrix} I - A^*\Delta A & 0 \\ -B^*\Delta A & 0 \end{pmatrix} \left[\begin{pmatrix} I - A^*\Delta A & -A^*\Delta B \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I - A^*\Delta A & 0 \\ -B^*\Delta A & 0 \end{pmatrix} \right]^+ \\ &= \begin{pmatrix} I - A^*\Delta A & 0 \\ -B^*\Delta A & 0 \end{pmatrix} \begin{pmatrix} (I - A^*\Delta A)^+ & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} (I - A^*\Delta A)(I - A^*\Delta A)^+ & 0 \\ -B^*\Delta A(I - A^*\Delta A)^+ & 0 \end{pmatrix}. \end{aligned}$$

Hence, from $\mathcal{R}(P) = \mathcal{R}(M)$ and $\mathcal{R}(Q) = \mathcal{R}(M^*)$ we have

$$\begin{aligned} P_{\mathcal{R}(M)\cap\mathcal{R}(M^*)} &= P_{\mathcal{R}(P)\cap\mathcal{R}(Q)} = P - P(P\overline{Q})^+ \\ &= \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \left[\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I - A^*\Delta A & -A^*\Delta B \\ -B^*\Delta A & I - B^*\Delta B \end{pmatrix} \right]^+ \\ &= \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I - A^*\Delta A & -A^*\Delta B \\ 0 & 0 \end{pmatrix}^+ \\ &= \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (I - A^*\Delta A)(I - A^*\Delta A)^+ & 0 \\ -B^*\Delta A(I - A^*\Delta A)^+ & 0 \end{pmatrix} \\ &= \begin{pmatrix} I - (I - A^*\Delta A)(I - A^*\Delta A)^+ & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Hence,

$$\begin{aligned} M \text{ is DR} &\iff \mathcal{R}(M) \cap \mathcal{R}(M^*) = \{0\} \\ &\iff \mathcal{R}(P) \cap \mathcal{R}(Q) = \{0\} \\ &\iff I - A^*\Delta A \text{ is invertible} \\ &\iff \mathcal{R}(B) = \mathcal{R}(M) \quad [\text{by } (10)]. \end{aligned} \tag{13}$$

(ii) Note that $\Delta = \Delta^* = (AA^* + BB^*)^{-1}$ is invertible and M is SR if and only if $\mathcal{R}(M) + \mathcal{R}(M^*) = \mathcal{H}$. Observe that

$$\begin{aligned} \begin{pmatrix} 0 & 0 \\ B^*\Delta A & B^*\Delta B \end{pmatrix}^+ &= \begin{pmatrix} 0 & A^*\Delta B \\ 0 & B^*\Delta B \end{pmatrix} \left[\begin{pmatrix} 0 & 0 \\ B^*\Delta A & B^*\Delta B \end{pmatrix} \begin{pmatrix} 0 & A^*\Delta B \\ 0 & B^*\Delta B \end{pmatrix} \right]^+ \\ &= \begin{pmatrix} 0 & A^*\Delta B \\ 0 & B^*\Delta B \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & (B^*\Delta B)^+ \end{pmatrix} \\ &= \begin{pmatrix} 0 & (A^*\Delta B)(B^*\Delta B)^+ \\ 0 & (B^*\Delta B)(B^*\Delta B)^+ \end{pmatrix}. \end{aligned}$$

From

$$\begin{aligned} P_{\mathcal{R}(M)+\mathcal{R}(M^*)} &= P_{\mathcal{R}(P)+\mathcal{R}(Q)} = P + \overline{P}(\overline{P}Q)^+ \\ &= \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \left[\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A^*\Delta A & A^*\Delta B \\ B^*\Delta A & B^*\Delta B \end{pmatrix} \right]^+ \\ &= \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & 0 \\ B^*\Delta A & B^*\Delta B \end{pmatrix}^+ \\ &= \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & (A^*\Delta B)(B^*\Delta B)^+ \\ 0 & (B^*\Delta B)(B^*\Delta B)^+ \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ 0 & (B^*\Delta B)(B^*\Delta B)^+ \end{pmatrix} \end{aligned}$$

we know that

$$\mathcal{R}(M)+\mathcal{R}(M^*)=\mathcal{H} \text{ if and only if } B^*\Delta B=(\Delta^{\frac{1}{2}}B)^*(\Delta^{\frac{1}{2}}B) \text{ is invertible,} \tag{14}$$

which is equivalent to that

$$\mathcal{R}(B^*)=\mathcal{R}((\Delta^{\frac{1}{2}}B)^*)=\mathcal{R}[(\Delta^{\frac{1}{2}}B)^*(\Delta^{\frac{1}{2}}B)]=\mathcal{N}(M^*)=\mathcal{R}(M)^{\perp}.$$

(iii) See Theorem 3.2. □

As for the weak-EP, we have the following equivalent conditions.

Theorem 3.6. *Let $M\in\mathcal{B}(\mathcal{H})$ be represented as in (3) such that $\mathcal{R}(M)$ is closed. If we set the orthogonal projectors $P=MM^+$, $Q=M^+M$ and the closed subspaces $\mathcal{M}=\mathcal{R}(P)$, $\mathcal{N}=\mathcal{R}(Q)$, then the following statements are equivalent:*

(i) *M is weak-EP.*

(ii) *$A^*\Delta B=0$, where $\Delta=(AA^*+BB^*)^{-1}$.*

(iii) *$\mathcal{H}=\oplus_{i=1}^4\mathcal{H}_i$, where $\mathcal{H}_i$ is defined by Lemma 2.5.*

Proof. By (5), it is clear that (i) $\Longleftrightarrow$ (ii). Let P, Q be represented as in (2). Note that Q_0 is a positive contraction on $\mathcal{H}_5$ such that neither 0 nor 1 belongs to the point spectrum of Q_0, D is a unitary operator from $\mathcal{H}_6$ onto $\mathcal{H}_5$. By (6), $A^*\Delta B=0 \Longleftrightarrow \mathcal{H}_5\oplus\mathcal{H}_6=\{0\}$. Hence (ii) $\Longleftrightarrow$ (iii). □

Theorem 3.6 implies that, if M is weak-EP with

$$\mathcal{H}_2=\mathcal{M}\cap\mathcal{N}^{\perp}=\mathcal{R}(M)\cap\mathcal{N}(M)=\{0\}$$

and

$$\mathcal{H}_3=\mathcal{M}^{\perp}\cap\mathcal{N}=\mathcal{N}(M^*)\cap\mathcal{R}(M^*)=\{0\},$$

then M is EP. Moreover, by Theorem 3.5, it is easy to get the following orthogonal direct sum decomposition of Hilbert space $\mathcal{H}$.

Corollary 3.7. *Let the assumptions of Theorem 3.5 hold and let $P=MM^+$.*

(1) *M is DR $\Longleftrightarrow \mathcal{R}(M\overline{P})\oplus^{\perp}\mathcal{N}(M^*)=\mathcal{H}$.*
(2) *M is SR $\Longleftrightarrow \mathcal{R}(M)\oplus^{\perp}\mathcal{R}(\overline{P}M^*)=\mathcal{H}$.*
(3) *M is co-EP $\Longleftrightarrow \mathcal{R}(M\overline{P})\oplus^{\perp}\mathcal{R}(\overline{P}M^*)=\mathcal{H}$.*

Corollary 3.8. *Let the assumptions of Theorem 3.5 hold.*

(i) *M is co-EP if and only if M is DR and SR.*
(ii) *M is co-EP and GP if and only if M has the 2×2 block operator matrix form*

$$M=\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}, \quad \text{where } A, B \text{ are invertible}$$

according to the space decomposition $\mathcal{H}=\mathcal{R}(M)\oplus^{\perp}\mathcal{N}(M^*)$.

Furthermore, if $M\in\mathcal{B}(\mathcal{H})$ is DR or SR, $\mathcal{R}(M+M^*)$ has the following range relation.

Theorem 3.9. *Let $M\in\mathcal{B}(\mathcal{H})$ be represented as in (3) such that $\mathcal{R}(M)$ and $\mathcal{R}(MM^+-M^+M)$ are closed. Let $P=MM^+$.*

(i) *If M is DR, then*

$$\mathcal{R}(M+M^*)=\mathcal{R}(M)\oplus^{\perp}\mathcal{R}(\overline{P}M^*)$$

and

$$(M+M^*)^+ = \begin{pmatrix} 0 & (B^*)^+ \\ B^+ & -B^+(A+A^*)(B^+)^* \end{pmatrix}.$$

Furthermore,

$$M(M+M^*)^+M = M, \quad M(M+M^*)^+M^* = 0,$$

$M(M+M^*)^+$ *is an idempotent and* $S^{-1}(M+M^*)^+M^*$ *is an orthogonal projector with*

$$\mathcal{R}(M(M+M^*)^+) = \mathcal{R}(S^{-1}(M+M^*)^+M^*) = \mathcal{R}(M),$$

where $S = \begin{pmatrix} I & 0 \\ -B^+A & I \end{pmatrix}$.

(ii) *If M is simultaneously DR and SR, then*

$$\mathcal{R}(M+M^*) = \mathcal{R}(M) + \mathcal{R}(M^*),$$

$M \pm M^*$ *is invertible and* $(M \pm M^*)^{-1}$ *has the representation as in Theorem 3.3* (iii).

Proof. (i) By Theorem 3.5, M is DR if and only if B is surjective, if and only if B^+ exists such that $BB^+ = I_{\mathcal{R}(M)}$. Let $S = \begin{pmatrix} I & 0 \\ -B^+A & I \end{pmatrix}$. Then

$$S^*M = M, \quad MS(MS)^+ = P_{\mathcal{R}(M)}.$$

Since

$$S^*(M+M^*)S = \begin{pmatrix} I & -A^*(B^+)^* \\ 0 & I \end{pmatrix} \begin{pmatrix} A+A^* & B \\ B^* & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ -B^+A & I \end{pmatrix} = \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}.$$

From

$$[S^*(M+M^*)S]^+ = S^{-1}(M+M^*)^+(S^*)^{-1}$$

and

$$\begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}^+ = \begin{pmatrix} 0 & (B^*)^+ \\ B^+ & 0 \end{pmatrix},$$

we get

$$(M+M^*)^+ = S\begin{pmatrix} 0 & (B^*)^+ \\ B^+ & 0 \end{pmatrix}S^* = \begin{pmatrix} 0 & (B^*)^+ \\ B^+ & -B^+(A+A^*)(B^+)^* \end{pmatrix}$$

and

$$(M+M^*)(M+M^*)^+ = \begin{pmatrix} I & 0 \\ 0 & B^*(B^*)^+ \end{pmatrix}.$$

Hence

$$\mathcal{R}(M+M^*) = \mathcal{R}(M) \oplus^{\perp} \mathcal{R}(B^*) = \mathcal{R}(M) \oplus^{\perp} \mathcal{R}(\overline{P}M^*).$$

Moreover

$$M(M+M^*)^+ = \begin{pmatrix} I & -A^*(B^+)^* \\ 0 & 0 \end{pmatrix}$$

is an idempotent on $\mathcal{R}(M)$,

$$S^{-1}(M+M^*)^+M^* = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

is an orthogonal projector on $\mathcal{R}(M)$,

$$M(M+M^*)^+M = MS\begin{pmatrix} 0 & (B^*)^+ \\ B^+ & 0 \end{pmatrix}S^*M = M$$

and

$$M(M+M^*)^+M^* = MS\begin{pmatrix} 0 & (B^*)^+ \\ B^+ & 0 \end{pmatrix}S^*M^* = 0.$$

(ii) Note that M is simultaneously DR and SR $\Longleftrightarrow$ M is co-EP $\Longleftrightarrow$ B is invertible. The results follow immediately. $\square$

4. An orthogonal projector approach for range relations

Let $M \in \mathcal{B}(\mathcal{H})$ be such that $\mathcal{R}(M)$ is closed. Let orthogonal projectors $P = MM^+$ and $Q = M^+M$. If we set closed subspaces

$$\mathcal{M} = \mathcal{R}(P) = \mathcal{R}(M), \quad \mathcal{N} = \mathcal{R}(Q) = \mathcal{R}(M^*)$$

in Lemma 2.5, then the orthogonal projectors P and Q have the representations as in (2) and

$$\begin{aligned}
P-Q &= 0\oplus I\oplus -I\oplus 0\oplus \begin{pmatrix} I-Q_0 & -Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}}D \\ -D^*Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}} & -D^*(I-Q_0)D \end{pmatrix},\\
P+Q &= 2I\oplus I\oplus I\oplus 0\oplus \begin{pmatrix} I+Q_0 & Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}}D \\ D^*Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}} & D^*(I-Q_0)D \end{pmatrix},\\
(P-Q)^2 &= 0\oplus I\oplus I\oplus 0\oplus \begin{pmatrix} I-Q_0 & 0 \\ 0 & D^*(I-Q_0)D \end{pmatrix},\\
PQ &= I\oplus 0\oplus 0\oplus 0\oplus \begin{pmatrix} Q_0 & Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}}D \\ 0 & 0 \end{pmatrix},\\
QP &= I\oplus 0\oplus 0\oplus 0\oplus \begin{pmatrix} Q_0 & 0 \\ D^*Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}} & 0 \end{pmatrix},\\
PQ(PQ)^* &= I\oplus 0\oplus 0\oplus 0\oplus \begin{pmatrix} Q_0 & 0 \\ 0 & 0 \end{pmatrix},\\
\overline{P}Q &= 0\oplus 0\oplus I\oplus 0\oplus \begin{pmatrix} 0 & 0 \\ D^*Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}} & D^*(I-Q_0)D \end{pmatrix},\\
P\overline{Q} &= 0\oplus I\oplus 0\oplus 0\oplus \begin{pmatrix} I-Q_0 & -Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}}D \\ 0 & 0 \end{pmatrix}
\end{aligned} \tag{15}$$

with respect to the space orthogonal direct sum decomposition $\mathcal{H} = \oplus_{i=1}^{6}\mathcal{H}_i$, where $\mathcal{R}(Q_0)$ and $\mathcal{R}(I-Q_0)$ are dense in $\mathcal{H}_5$, since Q_0 is a positive operator and 0, 1 are not the point spectrums of Q_0. As for the MP-inverse of the related projectors, we have

$$\begin{aligned}
PQ \text{ is MP-invertible} &\Longleftrightarrow \mathcal{R}(PQ) \text{ is closed}\\
&\Longleftrightarrow \mathcal{R}(PQP) \text{ is closed} \quad [\text{by Lemma (2.2)}]\\
&\Longleftrightarrow \mathcal{R}(Q_0) \text{ is closed} \quad [\text{by (2)}]\\
&\Longleftrightarrow Q_0 \text{ is invertible} \quad [\text{by Lemma (2.5)}].
\end{aligned} \tag{16}$$

Similarly,

$$P - Q \text{ is MP-invertible} \Longleftrightarrow I - Q_0 \text{ is invertible.} \tag{17}$$

Hence, if $\mathcal{R}(PQ)$ is closed, by the sixth equation in (15), $(PQ(PQ)^*)^+ = I \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} Q_0^{-1} & 0 \\ 0 & 0 \end{pmatrix}$. So, by the fourth equation in (15),

$$\begin{aligned}
(PQ)^+ &= (PQ)^*[(PQ)(PQ)^*]^+ \\
&= \left[I \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} Q_0 & 0 \\ D^*Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}} & 0 \end{pmatrix}\right]\left[I \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} Q_0^{-1} & 0 \\ 0 & 0 \end{pmatrix}\right] \\
&= I \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} I & 0 \\ D^*Q_0^{-\frac{1}{2}}(I-Q_0)^{\frac{1}{2}} & 0 \end{pmatrix}.
\end{aligned}$$

Similarly, if $\mathcal{R}(PQ)$ or $\mathcal{R}(P-Q)$ is closed, we have

$$\begin{aligned}
(QP)^+ &= (QP)^*[(QP)(QP)^*]^+ = I \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} I & Q_0^{-\frac{1}{2}}(I-Q_0)^{\frac{1}{2}}D \\ 0 & 0 \end{pmatrix}, \\
(\bar{P}Q)^+ &= (\bar{P}Q)^*[(\bar{P}Q)(\bar{P}Q)^*]^+ = 0 \oplus 0 \oplus I \oplus 0 \oplus \begin{pmatrix} 0 & Q_0^{\frac{1}{2}}(I-Q_0)^{-\frac{1}{2}}D \\ 0 & I \end{pmatrix}, \\
(P-Q)^+ &= (P-Q)^*[(P-Q)(P-Q)^*]^+ \\
&= 0 \oplus I \oplus -I \oplus 0 \oplus \begin{pmatrix} I & -Q_0^{\frac{1}{2}}(I-Q_0)^{-\frac{1}{2}}D \\ -D^*Q_0^{\frac{1}{2}}(I-Q_0)^{-\frac{1}{2}} & -I \end{pmatrix}.
\end{aligned} \tag{18}$$

As for the range relations, we observe that

$$\begin{aligned}
&\mathcal{R}(P) = \mathcal{R}(M) = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_5, \\
&\mathcal{R}(Q) = \mathcal{R}(M^*) = \mathcal{H}_1 \oplus \mathcal{H}_3 \oplus \mathcal{R}(Q\,|_{\mathcal{H}_5 \oplus \mathcal{H}_6}), \\
&\mathcal{R}(P-Q) = \mathcal{R}((P-Q)^+) = \mathcal{H} \ominus (\mathcal{H}_1 \oplus \mathcal{H}_4) \text{ if } P-Q \text{ is MP-invertible}, \\
&\mathcal{R}((PQ)^+) = \mathcal{R}(QP) = \mathcal{R}(Q) \ominus \mathcal{H}_3 \text{ if } PQ \text{ is MP-invertible}, \\
&\mathcal{R}((QP)^+) = \mathcal{R}(PQ) = \mathcal{R}(P) \ominus \mathcal{H}_2 \text{ if } PQ \text{ is MP-invertible}.
\end{aligned} \tag{19}$$

In general, if we set $E = \begin{pmatrix} Q_0 & 0 \\ D^*Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}} & 0 \end{pmatrix}$, then $E^*E = \begin{pmatrix} Q_0 & 0 \\ 0 & 0 \end{pmatrix}$ and

$$\begin{aligned}
EE^* &= \begin{pmatrix} Q_0^2 & Q_0^{\frac{3}{2}}(I-Q_0)^{\frac{1}{2}}D \\ D^*Q_0^{\frac{3}{2}}(I-Q_0)^{\frac{1}{2}} & D^*Q_0(I-Q_0)D \end{pmatrix} \\
&= \begin{pmatrix} Q_0 & Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}}D \\ D^*Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}} & D^*(I-Q_0)D \end{pmatrix}\begin{pmatrix} Q_0 & 0 \\ 0 & D^*Q_0D \end{pmatrix}.
\end{aligned}$$

We obtain

$$\overline{\mathcal{R}(PQ)} = \mathcal{H}_1 \oplus \overline{\mathcal{R}(E^*)} = \mathcal{H}_1 \oplus \overline{\mathcal{R}(Q_0)} = \mathcal{H}_1 \oplus \mathcal{H}_5\,, \quad (Q_0 \text{ is injective and positive})$$

and

$$\begin{aligned}
\overline{\mathcal{R}(QP)} &= \mathcal{H}_1 \oplus \overline{\mathcal{R}(E)} = \mathcal{H}_1 \oplus \mathcal{R}\left(\begin{pmatrix} Q_0 & Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}}D \\ D^*Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}} & D^*(I-Q_0)D \end{pmatrix}\right) \\
&= \mathcal{H}_1 \oplus \mathcal{R}(Q\,|_{\mathcal{H}_5 \oplus \mathcal{H}_6}).
\end{aligned}$$

Hence

$$\overline{\mathcal{R}(PQ)} \cap \overline{\mathcal{R}(QP)} = \mathcal{R}(P) \cap \mathcal{R}(Q). \tag{20}$$

Moreover, we have the following results.

Theorem 4.1. *Let $M \in \mathcal{B}(\mathcal{H})$ be such that $\mathcal{R}(M)$ is closed. If we set the orthogonal projectors $P = MM^+$, $Q = M^+M$ and the closed subspaces $\mathcal{M} = \mathcal{R}(P)$, $\mathcal{N} = \mathcal{R}(Q)$, then*
(i) $P = M(M^*M)^+M^*, \quad Q = M^*(MM^*)^+M.$
(ii) $\mathcal{M}_0 = \mathcal{R}(M) \cap [\mathcal{R}(M) \cap \mathcal{N}(M)]^\perp = \mathcal{H}_1 \oplus \mathcal{H}_5$ *and*

$$\mathcal{N}_0 = \mathcal{N}(M) \oplus [\mathcal{R}(M) + \mathcal{N}(M)]^\perp = \oplus_{i=2}^4 \mathcal{H}_i \oplus \mathcal{R}(T_0).$$

Furthermore

$$\mathcal{M}_0 \cap \mathcal{N}_0 = \{0\}, \quad \overline{\mathcal{M}_0 + \mathcal{N}_0} = \mathcal{H},$$

where $\mathcal{H}_i$, $i = 1, 2, \cdots, 5$ are defined as in Lemma 2.5 and

$$T_0 = \begin{pmatrix} I - Q_0 & -Q_0^{\frac{1}{2}}(I - Q_0)^{\frac{1}{2}}D \\ -D^*Q_0^{\frac{1}{2}}(I - Q_0)^{\frac{1}{2}} & D^*Q_0D \end{pmatrix}.$$

(iii) *If $\mathcal{R}(PQ)$ is closed, then $P_{\mathcal{M}_0,\mathcal{N}_0} = (QP)^+$.*

Proof. Let $T = M(M^*M)^+M^*$. It is clear that $T = T^* = T^2$, i.e., T is an orthogonal projector. By Lemma 2.2,

$$\begin{aligned} \mathcal{R}(T) &= M\mathcal{R}((M^*M)^+M^*) = M\mathcal{R}(M^*MM^*) = M\mathcal{R}(M^*M) \\ &= \mathcal{R}(MM^*M) = \mathcal{R}(MM^*) = \mathcal{R}(M). \end{aligned}$$

These give the assertion that $P = M(M^*M)^+M^*$. Analogously, we have $Q = M^*(MM^*)^+M$.

(ii) By Lemma 2.5, we have

$$P_{\mathcal{N}(M)} = \overline{Q} = 0 \oplus I \oplus 0 \oplus I \oplus \begin{pmatrix} I - Q_0 & -Q_0^{\frac{1}{2}}(I - Q_0)^{\frac{1}{2}}D \\ -D^*Q_0^{\frac{1}{2}}(I - Q_0)^{\frac{1}{2}} & D^*Q_0D \end{pmatrix}$$

and

$$P_{\mathcal{R}(M)} + P_{\mathcal{N}(M)} = I \oplus 2I \oplus 0 \oplus I \oplus \begin{pmatrix} 2I - Q_0 & -Q_0^{\frac{1}{2}}(I - Q_0)^{\frac{1}{2}}D \\ -D^*Q_0^{\frac{1}{2}}(I - Q_0)^{\frac{1}{2}} & D^*Q_0D \end{pmatrix}.$$

Hence, $\mathcal{R}(M) \cap \mathcal{N}(M) = \mathcal{H}_2$ and

$$\mathcal{M}_0 = \mathcal{R}(M) \cap [\mathcal{R}(M) \cap \mathcal{N}(M)]^\perp = \mathcal{H}_1 \oplus \mathcal{H}_5.$$

Since $P_{\mathcal{R}(M)} + P_{\mathcal{N}(M)} \geq 0$, its diagonal element $\begin{pmatrix} 2I - Q_0 & -Q_0^{\frac{1}{2}}(I - Q_0)^{\frac{1}{2}}D \\ -D^*Q_0^{\frac{1}{2}}(I - Q_0)^{\frac{1}{2}} & D^*Q_0D \end{pmatrix}$ is positive. If there exists a vector $x = (x_1, x_2)$ such that

$$\begin{pmatrix} 2I - Q_0 & -Q_0^{\frac{1}{2}}(I - Q_0)^{\frac{1}{2}}D \\ -D^*Q_0^{\frac{1}{2}}(I - Q_0)^{\frac{1}{2}} & D^*Q_0D \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0,$$

that is,

$$\begin{cases} (2I-Q_0)x_1-Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}}Dx_2=0\,, \\ -D^*Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}}x_1+D^*Q_0Dx_2=0\,. \end{cases} \tag{21}$$

By Lemma 2.5, Q_0 is a positive contraction, Q_0 and $I-Q_0$ are injective and D is unitary. By second equation in (21), we deduce that $(I-Q_0)^{\frac{1}{2}}x_1-Q_0^{\frac{1}{2}}Dx_2=0$ and, hence, $(I-Q_0)x_1-Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}}Dx_2=0$. By first equation in (21), we obtain $x_1=0$ and $x_2=0$. So $\begin{pmatrix} 2I-Q_0 & -Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}}D \\ -D^*Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}} & D^*Q_0D \end{pmatrix}$ is an injective positive operator.

$$\overline{\mathcal{R}(P_{\mathcal{R}(M)}+P_{\mathcal{N}(M)})}=\mathcal{H}\ominus\mathcal{H}_3.$$

Hence

$$\mathcal{N}_0=\mathcal{N}(M)\oplus[\mathcal{R}(M)+\mathcal{N}(M)]^{\perp}=\oplus_{i=2}^{4}\mathcal{H}_i\oplus\mathcal{R}(T_0),$$

where

$$T_0=\begin{pmatrix} I-Q_0 & -Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}}D \\ -D^*Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}} & D^*Q_0D \end{pmatrix}.$$

To show $\mathcal{M}_0\cap\mathcal{N}_0=\{0\}$ and $\overline{\mathcal{M}_0+\mathcal{N}_0}=\mathcal{H}$, it is sufficient to establish that

$$\mathcal{H}_5\cap\mathcal{R}(T_0)=\{0\}\quad\text{and}\quad\overline{\mathcal{H}_5+\mathcal{R}(T_0)}=\mathcal{H}_5\oplus\mathcal{H}_6.$$

Since $\begin{pmatrix} 2I-Q_0 & -Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}}D \\ -D^*Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}} & D^*Q_0D \end{pmatrix}$ is injective positive operator,

$$\overline{\mathcal{H}_5+\mathcal{R}(T_0)}=\mathcal{R}\left(\begin{pmatrix} 2I-Q_0 & -Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}}D \\ -D^*Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}} & D^*Q_0D \end{pmatrix}\right)=\mathcal{H}_5\oplus\mathcal{H}_6$$

by Lemma 2.2. For every $(z,0)\in\mathcal{H}_5\cap\mathcal{R}(T_0)$, there exists $(x_1,x_2)\in\mathcal{H}_5\oplus\mathcal{H}_6$ such that

$$\begin{pmatrix} I-Q_0 & -Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}}D \\ -D^*Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}} & D^*Q_0D \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} z \\ 0 \end{pmatrix},$$

that is,

$$\begin{cases} (I-Q_0)x_1-Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}}Dx_2=z\,, \\ -D^*Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}}x_1+D^*Q_0Dx_2=0\,. \end{cases} \tag{22}$$

Since Q_0 and $I-Q_0$ are injective, we obtain $z=0$ and $\mathcal{H}_5\cap\mathcal{R}(T_0)=\{0\}$.

(iii) By (16), if $\mathcal{R}(PQ)$ is closed, then Q_0 is invertible. Lemma 2.4 implies that

$$\begin{pmatrix} 2I-Q_0 & -Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}}D \\ -D^*Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}} & D^*Q_0D \end{pmatrix}$$

is invertible. Hence $\mathcal{M}_0+\mathcal{N}_0=\mathcal{H}$ (i.e., $\mathcal{M}_0$ and $\mathcal{N}_0$ are complementary spaces). By first equation in (18) we know that $(QP)^+$ is an idempotent with

$$\mathcal{R}((QP)^+)=\mathcal{M}_0\quad\text{and}\quad\mathcal{N}((QP)^+)=\mathcal{R}(I-(QP)^+)=\mathcal{N}_0.$$

Hence (iii) holds. □

Next a characterization of the range inclusions will be presented.

Theorem 4.2. *Let $M \in \mathcal{B}(\mathcal{H})$ be such that $\mathcal{R}(M)$ is closed. Then, the following statements hold.*
(i) $\mathcal{R}(M^*) \subset \mathcal{R}(M) \iff M = MMM^+$.
(ii) $\mathcal{N}(M) \subset \mathcal{N}(M^*) \iff M = M^+MM$.

Proof. Note that $\mathcal{R}(M) = \mathcal{R}(MM^+) = \mathcal{N}(I - MM^+)$ and $\mathcal{N}(M) = \mathcal{N}(M^+M) = \mathcal{R}(I - M^+M)$. Hence

$$\mathcal{R}(M^*) \subset \mathcal{R}(M) \iff (I - MM^+)M^* = 0 \iff M(I - MM^+) = 0$$

and

$$\mathcal{N}(M) \subset \mathcal{N}(M^*) \iff M^*(I - M^+M) = 0 \iff (I - M^+M)M = 0. \quad \square$$

In [6, Theorem 2], Baksalary and Trenkler proved that, if $M \in \mathbb{C}^{n\times n}$, then $P + Q$ is nonsingular if and only if M is SR; $P - Q$ is nonsingular if and only if M is DR and SR. Applying range projectors P, Q and the result in Lemma 2.4, we get the following range relations (see also [5, Theorem 1] for finite matrix case).

Theorem 4.3. *Let $M \in \mathcal{B}(\mathcal{H})$ be such that $\mathcal{R}(M)$ is closed. Let also $P = MM^+$ and $Q = M^+M$ be such that $\mathcal{R}(PQ)$ and $\mathcal{R}(P - Q)$ are closed.*

(i) *M is DR*
$$\begin{aligned}
&\iff \mathcal{R}(P) \cap \mathcal{R}(Q) = \{0\} \iff \mathcal{R}(PQ) \cap \mathcal{R}(QP) = \{0\}\\
&\iff \mathcal{R}(P - Q) = \mathcal{R}(P + Q) = \mathcal{R}(P) + \mathcal{R}(Q) \quad (\textit{by Lemma } 2.2)\\
&\iff \mathcal{R}(PQ - QP) = \mathcal{R}(PQ + QP) = \mathcal{R}(PQ) + \mathcal{R}(QP)\\
&\iff \mathcal{R}(P - Q) = \mathcal{R}(P + Q - PQ)\\
&\iff \mathcal{R}(P\overline{Q}) \oplus^{\perp} \mathcal{R}(\overline{P}Q) = \mathcal{R}(P + Q)\\
&\iff \mathcal{R}(I - PQ) = \mathcal{H}.
\end{aligned}$$

(ii) *M is SR*
$$\begin{aligned}
&\iff \mathcal{R}(P) + \mathcal{R}(Q) = \mathcal{H} \iff \mathcal{N}(P) \cap \mathcal{N}(Q) = \{0\}\\
&\iff \mathcal{R}(P - \overline{Q}) = \mathcal{R}(PQ + QP)\\
&\iff \mathcal{R}(P + \overline{P}\,\overline{Q}) = \mathcal{R}(P + QPQ)\\
&\iff \mathcal{R}(P + \overline{P}Q) = \mathcal{H}\\
&\iff \mathcal{R}(Q + P\overline{Q}) = \mathcal{H}.
\end{aligned}$$

(iii) *M co-EP $\iff$ any one of the conditions in* (i) *and*
any one of the conditions in (ii) *hold simultaneously*.

Proof. From (16) and (17) we know $\mathcal{R}(PQ)$ is closed $\iff Q_0$ is invertible, $\mathcal{R}(P - Q)$ is closed $\iff I - Q_0$ is invertible.

(i) By (13) and (20) we get

$$M \text{ is DR} \iff \mathcal{R}(P) \cap \mathcal{R}(Q) = \{0\} \iff \mathcal{R}(PQ) \cap \mathcal{R}(QP) = \{0\}.$$

Analogously, it is trivial to show that the remaining items are all equivalent to $\mathcal{R}(P) \cap \mathcal{R}(Q) = \{0\}$.

(ii) From (6) and (14), we get

$$M \text{ is SR} \iff \mathcal{R}(P) + \mathcal{R}(Q) = \mathcal{H} \iff \mathcal{N}(P) \cap \mathcal{N}(Q) = \{0\}.$$

Similarly, we can show that the remaining items are all equivalent to that $\mathcal{N}(P) \cap \mathcal{N}(Q) = \{0\}$.

(iii) See Corollary 3.8.(i). □

Acknowledgments

A part of this paper was written while the first author was visiting School of Mathematical Sciences and Key Laboratory of Mathematics for Nonlinear Sciences, Fudan University during the summer of 2011, he would like to thank the faculty and administration of those units for their warm hospitality.

The authors would like to thank Professor Peter Šemrl and the referee for their very detailed reports.

References

[1] T. Ando, Unbounded or bounded idempotent operators in Hilbert space, Linear Algebra Appl. (2011) , http://dx.doi.org/10.1016/j.laa.2011.06.047.
[2] J.K. Baksalary, G.P.H. Styan, Generalized inverses of partitioned matrices in Banachiewicz–Schur form, Linear Algebra Appl. 354 (2002) 41–47.
[3] O.M. Baksalary, G. Trenkler, Revisitation of the product of two orthogonal projectors, Linear Algebra Appl. 430 (2009) 2813–2833.
[4] O.M. Baksalary, G. Trenkler, Eigenvalues of functions of orthogonal projectors, Linear Algebra Appl. 431 (2009) 2172–2186.
[5] O.M. Baksalary, G. Trenkler, On disjoint range matrices, Linear Algebra Appl. 435 (2011) 1222–1240.
[6] O.M. Baksalary, G. Trenkler, On the projectors FF^+ and F^+F, Appl. Math. Comput. 217 (2011) 10213–10223.
[7] O.M. Baksalary, G.P.H. Styan, G. Trenkler, On a matrix decomposition of Hartwig and Spindelböck, Linear Algebra Appl. 430 (2009) 2798–2812.
[8] A. Ben-Israel, T.N.E. Greville, Generalized Inverses: Theory and Applications, second ed., Springer-Verlag, New York, 2003.
[9] J. Benítez, V. Rakočević, On the spectrum of linear combinations of two projectors in C*-algebras, Linear and Multilinear Algebra 58 (2010) 673–679.
[10] J. Benítez, V. Rakočević, Matrices A such that $AA^+ - A^+A$ are nonsingular, Appl. Math. Comput. 217 (2010) 3493–3503.
[11] A. Böttcher, I. Spitkovsky, A gentle guide to the basics of two projections theory, Linear Algebra Appl. 432 (2010) 1412–1459.
[12] R.H. Bouldin, Generalized inverses and factorizations, recent applications of generalized inverses, Pitman Ser. Res. Notes Math. 66 (1982) 233–249.
[13] D. Buckholtz, Inverting the difference of Hilbert space projections, Amer. Math. Monthly 104 (1997) 60–61.
[14] D. Buckholtz, Hilbert space idempotents and involutions, Proc. Amer. Math. Soc. 128 (2000) 1415–1418.
[15] S.L. Campbell, C.D. Meyer, Generalized Inverses of Linear Transformations, Pitman Publishing Limited, 1979, Reprinting, SIAM, 2010.
[16] J. Conway, A Course in Functional Analysis, Springer-Verlag, New York, 1990.
[17] C. Deng, H. Du, Representations of the Moore–Penrose inverse for a class of 2-by-2 block operator valued partial matrices, Linear and Multilinear Algebra 58 (2010) 15–26.
[18] C. Deng, Y. Wei, Further results on the Moore–Penrose invertibility of projection operators and its applications, Linear and Multilinear Algebra 60 (2012) 109–129.
[19] D.S. Djordjević, Characterizations of normal, hyponormal and EP operators, J. Math. Anal. Appl. 329 (2007) 1181–1190.
[20] G. Dolinar, J. Marovt, Star partial order on $\mathcal{B}(\mathcal{H})$, Linear Algebra Appl. 434 (2011) 319–326.
[21] H. Du, C. Deng, Moore–Penrose inverses of products and differences of orthogonal projections, Acta Anal. Funct. Appl. 8 (2006) 104–109.
[22] H. Du, C. Deng, A new characterization of gaps between two subspaces, Proc. Amer. Math. Soc. 133 (2005) 3065–3070.
[23] R.G. Douglas, On majorization, factorization, and range inclusion of operators on Hilbert spaces, Proc. Amer. Math. Soc. 17 (1966) 413–416.
[24] P.A. Fillmore, I.P. Williams, On operator ranges, Adv. Math. 7 (1971) 244–281.
[25] A. Galántai, Projectors and projection methods, in: Advances in Mathematics (Dordrecht), vol. 6, Kluwer Academic Publishers, Boston, MA, 2004.
[26] P. Halmos, Trans. Amer. Math. Soc. 144 (1969) 381–389.
[27] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1985.
[28] R.E. Hartwig, K. Spindelbock, Matrices for which A and A^* commute, Linear and Multilinear Algebra 14 (1984) 241–256.
[29] Mohammad Khadivi, Range inclusion and operator equations, J. Math. Anal. Appl. 197 (1996) 630–633.
[30] J.J. Koliha, V. Rakočević, Fredholm properties of the difference of orthogonal projections in a Hilbert space, Integral Equations Operator Theory 52 (2005) 125–134.
[31] J.J. Koliha, V. Rakočević, Invertibility of the difference of idempotents, Linear and Multilinear Algebra 51 (2003) 97–110.
[32] G.J. Murphy, C*-Algebras and Operator Theory, Academic Press, San Diego, 1990.
[33] K.S. Riedel, A Sherman–Morrison–Woodbury identity for rank augmenting matrices with application to centering, SIAM J. Matrix Anal. Appl. 13 (1992) 659–662.
[34] Tzon-Tzer Lu, Sheng-Hua Shiou, Inverses of 2×2 block matrices, Comput. Math. Appl. 43 (2002) 119–129.
[35] Phailaung Phohomsiri, Byungrin Han, An alternative proof for the recursive formulae for computing the Moore–Penrose M-inverse of a matrix, Appl. Math. Comput. 174 (2006) 81–97.
[36] R. Piziak, P.L. Odell, R. Hahn, Constructing projections on sums and intersections, Comput. Math. Appl. 37 (1999) 67–74.
[37] J. Groß, On the product of orthogonal projectors, Linear Algebra Appl. 289 (1999) 141–150.
[38] Y. Li, The Moore–Penrose inverses of products and differences of projections in a C*-algebra, Linear Algebra Appl. 428 (2008) 1169–1177.

[39] P. Šemrl, Automorphisms of $B(H)$ with respect to minus partial order, J. Math. Anal. Appl. 369 (2010) 205–213.
[40] Yongge Tian, The Moore–Penrose inverses of $m \times n$ block matrices and their applications, Linear Algebra Appl. 283 (1998) 35–60.
[41] Yimin Wei, Jiu Ding, Representations for Moore–Penrose inverses in Hilbert spaces, Appl. Math. Lett. 14 (2001) 599–604.
[42] Q. Xu, Y. Wei, Y. Gu, Sharp norm-estimations for Moore–Penrose inverses of stable perturbations of Hilbert C^*-module operators, SIAM J. Numer. Anal. 47 (2010) 4735–4758.