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The Solution to a Generalized Toda Lattice and Representation Theory

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CONTENTS

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0. Introduction

0.1. One of the main results of this paper concerns the existence of certain classical mechanical systems, generalizing the (finite, nonperiodic) Toda lattice (see, e.g., [8, 10, 19, 24, 25]) which one can explicitly integrate for all values of time. These systems are related to Dynkin diagrams. One knows these diagrams classify semi-simple Lie groups. The integration of the system associated to a Dynkin diagram is expressible in terms of the finite-dimensional representations of the corresponding group. In fact, more than that is true. The integration of the system and the theory of the finite-dimensional representation theory of semi-simple Lie groups are in a sense equivalent. Indeed, the integration of the system completely determines and is determined by the weight structure of the fundamental representations of the corresponding group.

In more detail, recall the Hamilton–Jacobi theory as it would apply to a system of particles moving on a line with respect to some potential. We envision then n particles, say with masses m_i, $i = 1,..., n$, and $\mathbb{R}^{2n}$ with linear canonical coordinate functions p_i, q_j, $i, j = 1,..., n$ (i.e., phase space), so that any $x \in \mathbb{R}^{2n}$ is a (classical) state of the system and $p_i(x)$, $q_i(x)$ are, respectively, the momentum and position of the ith particle in the state x. Assume the potential U is smooth (i.e., $U \in C^\infty(\mathbb{R}^{2n})$), and of course, depends only on the q's (i.e., $(\partial/\partial p_i)U = 0$, $i = 1, 2,..., n$). The Hamiltonian $H \in C^\infty(\mathbb{R}^{2n})$ of the system is then given by

$$H = \sum_{i=1}^{n} \frac{p_i^2}{2m_i} + U. \tag{0.1.1}$$

Now if $x \in \mathbb{R}^{2n}$ and $x(t) \in \mathbb{R}^{2n}$ is the state of the system at time t such that $x(0) = x$ then $x(t)$ is determined by Hamilton's equations. That is,

$$-\frac{\partial H}{\partial q_j}(x(t)) = \frac{dp_j(x(t))}{dt}, \qquad p_j(x(t)) = m_j \frac{dq_j(x(t))}{dt}. \tag{0.1.2}$$

The fundamental problem of the Hamilton–Jacobi theory is to integrate (0.1.2), that is, to determine the function $t \to x(t)$. It is clear from the second equation in (0.1.2) that it is enough to determine the position functions $t \to q_j(x(t))$, $j = 1,..., n$. For a general potential function U of course one cannot do this. It is one of the main results of this paper to show that for certain very special U having to do with Dynkin diagrams one can explicitly integrate (0.1.2) using representation theory.

Let $\mathcal{Q} \subseteq C^\infty(\mathbb{R}^{2n})$ be the n-dimensional vector space spanned by q_j, $j = 1,..., n$. Let $l \leqslant n$ and let $\psi_i \in \mathcal{Q}$, $i = 1,..., l$, be l linear independent functions. Thus we may write

$$\psi_i = \sum_{j=1}^{n} a_{ij} q_j, \qquad i = 1,..., l, \tag{0.1.3}$$

where (a_{ij}) is a constant $l \times n$ matrix. The potentials we shall consider are those of the form

$$U = r_1 e^{\psi_1} + \cdots + r_l e^{\psi_l}, \tag{0.1.4}$$

where the r_i are some positive constants. For example, if $l = n - 1$, $m_j = r_i = 1$ and $\psi_i = q_i - q_{i+1}$, $i = 1,..., l$, then the system is the Toda lattice. The interaction between the particles is that of exponential forces between nearest neighbors. (Throughout this paper "Toda lattice" will mean "finite nonperiodic Toda lattice".)

What is crucial for us in our result here is the geometry of the ψ's, that is, the lengths of the vectors $\psi_i \in \mathscr{Q}$ and the angles between them. We hasten to add that there is in fact a natural geometry in $\mathscr{Q}$, defined by the Hamiltonian H. Indeed let $[\varphi, \psi] \in C^\infty(\mathbb{R}^{2n})$ be the Poisson bracket for any pair of functions $\varphi, \psi \in C^\infty(\mathbb{R}^{2n})$. It is easy to see that if $\varphi, \psi \in \mathscr{Q}$ then $[\varphi, [\psi, H]]$ is a scalar times the identity function 1. A positive definite symmetric bilinear form B_H is then defined in $\mathscr{Q}$ by the relation

$$B_H(\varphi, \psi)1 = [\varphi, [\psi, H]]. \tag{0.1.5}$$

One has in fact

$$B_H(q_i, q_j) = \delta_{ij}/m_i . \tag{0.1.6}$$

We will say that the ψ's define a Dynkin diagram if there exists a real split semi-simple Lie algebra $\mathfrak{g}$ of rank l with an invariant bilinear form Q, and with a split Cartan subalgebra $\mathfrak{h}$ with simple roots $\alpha_1 ,..., \alpha_l$ such that

$$B_H(\psi_i , \psi_j) = Q(\alpha_i , \alpha_j) \tag{0.1.7}$$

for $i, j = 1,..., l$. In such a case, as one knows, the possible angles between the ψ's are 90, 120, 135, or 150°. The Toda lattice satisfies this condition. Here $\mathfrak{g}$ is the Lie algebra of $Sl(n, \mathbb{R})$. The angle between $q_{i-1} - q_i$ and $q_i - q_{i+1}$ is 120°. We will give some further examples. (See Section 7.4 here and [3; 7, Section 30] for others. See also [3] for suggested physical interpretations.)

$$\text{(1)} \quad H = \sum_{i=1}^{n} \frac{p_i^2}{2} + e^{q_1-q_2} + \cdots + e^{q_{n-1}-q_n} + e^{q_n}.$$

Example (1) is like the Toda lattice except that the last particle also interacts with a fixed mass. The diagram here is that of B_n ; i.e., $\mathfrak{g}$ is isomorphic to the Lie algebra of $SO(n, n + 1)$.

$$\text{(2)} \quad H = \sum_{i=1}^{4} \frac{p_i^2}{2} + e^{q_1-q_2} + e^{q_2-q_3} + e^{q_3} + e^{(q_4-q_1-q_2-q_3)/2}.$$

This is a four-body problem where the first three particles are as in (1) but where, also, the center of mass of these three particles interacts exponentially with a fourth particle. The Dynkin diagram here is that of the exceptional Lie algebra F_4.

Of course many different systems may correspond to the same diagram. For example, the following two four-body problems both correspond to D_4.

$$(3)\quad \sum_{i=1}^{4} \frac{p_i^2}{2} + e^{q_1} + e^{q_2} + e^{q_3} + e^{(q_4-q_1-q_2-q_3)/2}.$$

$$(4)\quad \sum_{i=1}^{4} \frac{p_i^2}{2} + e^{q_1-q_2} + e^{q_2-q_3} + e^{q_3-q_4} + e^{q_3+q_4}.$$

Example (4) is like the usual Toda lattice with four particles except that the center of mass of the third and fourth particles interacts exponentially with a fixed mass. In Section 7.8 we will apply the result Theorem 7.5 to work out $q_j(x(t))$ for the three-body problem whose potential is $e^{q_1-q_2} + e^{q_2-q_3} + e^{2q_3}$. The Lie algebra in question is that of the symplectic group $\mathrm{Sp}(6, \mathbb{R})$.

Before describing the main result we would like to remark that although the condition on the ψ's is quite rigid there is in fact some flexibility. Namely, by adjusting the masses m_i one has, given a potential of form (0.1.4), an n-parameter degree of freedom (see (0.1.6)), which one may use to possibly satisfy the condition.

Assume now that the ψ's define a Dynkin diagram and that $\mathfrak{g}$, Q, $\mathfrak{h}$, and the α's are as above. We will occasionally then refer to the mechanical system whose potential is (0.1.4) as a generalized Toda lattice. Let π_i, $i = 1,\ldots, l$, be the l-fundamental finite-dimensional irreducible representation of $\mathfrak{g}$ (or $\mathfrak{g}_{\mathbb{C}}$, its complexification). Paraphrasing Theorem 7.5 one has the following solution to Hamilton's equation for the generalized Toda lattice.

THEOREM A. *If $m(i)$ is the number of distinct weights of π_i there exist constants c_k and d_k, $k = 1,\ldots, m(i)$, which depend upon the initial state $x \in \mathbb{R}^{2n}$ of the system, whose potential is* (0.1.4), *and on the weights of π_i (see Theorem* 7.5 *for the exact dependence) such that if*

$$\Phi_i(t) = c_1 e^{-td_1} + c_2 e^{-td_2} + \cdots + c_{m(i)} e^{-td_{m(i)}} \tag{0.1.8}$$

then $\Phi_i(t) > 0$ and except for a linear term (which vanishes if $l = n$ and all $r_k = 1$) one has

$$q_j(x(t)) = -\frac{2}{m_j} \sum_{i=1}^{l} \frac{a_{ij}}{Q(\alpha_i, \alpha_i)} \log \Phi_i(t) \tag{0.1.9}$$

for $j = 1,\ldots, n$. The a_{ij} are given by (0.1.3).

Remark 0.1. A solution to the generalized Toda lattice has appeared in [28]. This solution (see Corollary 1 in [28]) is stated in terms of the diagonal component $h(\exp ty)$ of the exponential of a symmetric Jacobi element y. The reduction of the problem to $h(\exp ty)$ is also an early side step in the proof of Theorem 7.5. It is implicit in Theorem 6.8.2 here. The reduction of the problem to $h(\exp ty)$ has been known and widely discussed by the author for a number of years. The main problem lies not with the reduction to $h(\exp ty)$ —this is relatively easy—but with the determination of $h(\exp ty)$. This is where representation theory enters. Also entering here is an analysis of the variety Z (a coadjoint orbit) of normalized Jacobi elements. See Sections 2–6.[1]

Fixing i the ordering of the weight may be made so that c_1, d_1 corresponds to the highest weight of π_i and $c_{m(i)}$, $d_{m(i)}$ correspond to the lowest weight. One then has

$$d_1 > d_k > d_{m(i)} \tag{0.1.10}$$

for all $1 > k > m(i)$. Write $d_{ji} = d_j$ and $c_{ji} = c_j$. Now if b_i, $i = 1,\dots, l$, are l numbers and

$$s_j = \frac{2}{m_j} \sum_{i=1}^{l} \frac{a_{ij}}{Q(\alpha_i, \alpha_i)}\, b_i$$

then (1) put $v_j^+ = s_j$ in case $b_i = d_{m(i)i}$, (2) put $v_j^- = s_j$ in case $b_i = d_{1i}$, (3) put $u_j^+ = -s_j$ in case $b_i = \log c_{m(i)i}$, and (4) put $u_j^- = -s_j$ in case $b_i = \log c_{1i}$. (One knows that $c_{m(i)i}$ and c_{1i} are positive.)

We can now express the scattering of the system as $t \to \pm\infty$ in terms of the highest and lowest weights of the fundamental representations. Ignoring a trivial linear term as in Theorem A, the following paraphrases Theorem 7.6.

THEOREM B. *For any $j = 1,\dots, n$ the curve $(t, q_j(x(t)))$ in the t, q_j plane is asymptotic to the line $v_j^- t + u_j^-$ as $t \to -\infty$ and to the line $v_j^+ t + u_j^+$ as $t \to +\infty$. That is, $v_j^\pm$ are the limiting velocities of the ith particle as $t \to \pm\infty$ and $u_j^\pm$ are the limiting phases.*

Moser's result (see 4.3 in [19]) on the scattering of the Toda lattice may be readily recovered from Theorem B. See Section 7.7.

We also remark that we can reverse our considerations here. That is, if solution (0.1.9) of Hamilton's equations is given one solves for $\Phi_i(t)$ and, over all initial conditions, one obtains weights of the fundamental representation π_i. In particular one obtains the highest weight, which of course determines π_i.

0.2. The mechanical system in Theorem A is completely integrable in a classical sense. This means the Poisson commutativity of a suitable family of functions. In Section 1 we establish the Poisson commutativity in a much more general setting than that used to establish Theorem A. In particular

[1] *Added in proof.* A solution of the Toda lattice using representation theory has appeared in [29]. However, the solution in [29] is expressed in terms of integrals. See (18) in [29]. In effect, the results in Section 5 of our paper determine these integrals.

one has an "isospectral flow" for all suitably translated coadjoint orbits of a parabolic subalgebra.

The technique for dealing with the usual Toda lattice involves the set of Jacobi matrices. In particular it involves a decomposition of this set into isospectral classes. See, e.g., [19, 25]. In Section 2, first of all, the notion of Jacobi matrix is generalized and one introduces, using the simple negative and positive root vectors, the notion of Jacobi element in a real split or complex semi-simple Lie algebra. One is readily reduced to the study of the $2l$-dimensional variety Z of normalized Jacobi elements. Section 2 is devoted to an analysis of Z, especially in the complex case. Moser used the theory of continued fractions and a method going back to Stieljes to obtain a rationality result for the case of diagonalizable Jacobi matrices. Among other structure theorems this result is generalized in Section 2. The isomorphisms β_y and $\beta_{(w)}$ establish, in explicit terms, the rationality of the isospectral leaves $Z(\gamma)$. Since we can take the field to be $\mathbb{C}$ in Section 2 the elements in $Z(\gamma)$ can even be (principal) nilpotent. The results are, we believe, new even in the matrix case, and lead, for example, to a simple iterative procedure for diagonalizing a diagonalizable Jacobi matrix once one knows its eigenvalues. The results in Section 2 make notable use of the results in Section 1 of [17].

The situation in the real case is considerably more subtle. Using an interplay between the polar and Bruhat–Gelfand decompositions we obtain in Section 3 a completeness result for $Z(\gamma)$ (not true over $\mathbb{C}$) which later guarantees the integration of certain Hamiltonian vector fields for all values of t. Using an isomorphism $\beta_{(w)}\colon G_0{}^w \to Z(\gamma)$ one obtains (later to be seen) the action angle coordinates in terms of the characters of a split Cartan subgroup. In fact $\beta_{(w)}$ leads to a parametrization $Z \simeq H \times \mathfrak{h}_+$ where $H \simeq \mathbb{R}^l$ is the split Cartan subgroup and $\mathfrak{h}_+$ is an open Weyl chamber. The isolation of $G_0{}^w$ (which is not the identity component) among all the connected components of the centralizer $\bar{G}^w$ of w appears to be rather remarkable. It brings into focus a special automorphism, denoted by m in Section 3.5, of $\mathfrak{g}$. This automorphism also plays a special role in the Whittaker theory (see, e.g., the element h_o in [9, p. 106]), to which this paper will be related elsewhere.

In Section 4 it is shown that $\beta_{(w)}\colon G_0{}^w \to Z(\gamma)$ is an isomorphism of complete, flat, affinely connected manifolds. This sets a correspondence between the cosets of one-parameter subgroups in $G_0{}^w$ and the trajectories of the (Hamiltonian) vector fields ξ_I in $Z(\gamma)$. It reduces our integration problem to the determination of $h(g \exp(-t)w)^\lambda$, as will be seen in Section 6.

In Section 5 we determine $h(g \exp(-t)w)^\lambda$ using the representation theory of $\mathfrak{g}$. A particular role is played by the explicit formulas for $\bar{n}_f(w)^{-1}$ and $\bar{n}_{-f}(w)$. It is also seen in Section 5 that the scattering theory is determined by the special element $d(w) \in H$. The relationship between the isospectral leaves $Z(\gamma)$ and the elements $d(w)$ is rather mysterious (to us) and (we believe) remarkable.

Section 6 is devoted to the underlying symplectic theory and coadjoint orbits. It is shown that Z itself is just one coadjoint orbit (suitably translated) of the Borel subgroup $\bar{B}$. The basic formula $\varphi_i(\exp t\xi \cdot y) = \log h(g \exp(-t)w)^{-\alpha_i}$ is established here. See (6.8.4).

Section 7 starts off with the Hamilton–Jacobi theory. It can be initially read without reference to the other sections. However, the results of the preceding sections are later brought together yielding Theorem 7.5, which gives the formula for $q_i(x(t))$. What is surprising here is the special role of the fundamental representations among all the others. The Cartan matrix initially enters the formula for $q_i(x(t))$ but since we are using the fundamental representations it is canceled out, leaving the coefficients a_{ij} in (0.1.9) the same as the coefficients a_{ij} in (0.1.3).

0.3. There is by now an extensive literature on both the periodic and nonperiodic Toda lattice. There is an even larger literature on completely integrable Hamiltonian systems. As examples we cite references [1, 3, 8, 10, 19, 20, 24, 25, 27, 28, 29]. This, of course, is only a small part of a complete list. There are also connections with semi-simple Lie groups. See, e.g., [3, 20]. We would like to remark that there is a considerable distance between establishing the Poisson commutativity of certain functions—and actually integrating and finding the solution. In fact, there is a generalization of the periodic Toda lattice in that one replaces the Dynkin diagram by the extended Dynkin diagram. Independently this has been observed in [3]. We have been able to establish complete integrability in the sense above for this case as well, but, as yet, we have not been able to solve for $q_i(x(t))$, which presumably, as in [25], requires Abelian integrals.

We have been principally influenced by Refs. [19, 25]. In fact the starting point for us was our recognition that the symplectic form written by van Moerbeke [25, formula (42), p. 76] on the space of symmetric Jacobi matrices (and also cited by Moser in a lecture) was in fact the symplectic structure of a coadjoint orbit of the Borel subgroup $\bar{B}$.

Translation by the element f (see (1.5.4)) plays an important role here for establishing complete integrability. On the other hand our proof of what is referred to as the Kostant–Symes splitting theorem in [27] (another complete integrability statement) involved other methods. We wish to thank S. Sternberg for pointing out that the proof of Theorem 1.4 here may be readily adapted so that the splitting theorem in [27] is a special case ($f = 0$).

The coadjoint orbit Z of $\bar{B}$ is not only the setting for classical mechanics as considered in this paper but it also, recalling geometric quantization, as applied in [2], defines a unitary representation π_Z of $\bar{B}$. Elsewhere it will be seen that the generalized Toda lattice is completely integrable and solvable not only in the classical sense, as established here, but also in the quantum sense. That is, one can write down the simultaneous spectral resolution of a commuting family of operators which includes the Schrödinger operator associated to the Hamiltonian (0.1.1), where U is given by (0.1.4) and the ψ's define a Dynkin diagram. This uses π_Z and the Whittaker theory as developed in [17].

1. Poisson Commutativity of Translated Invariants

1.1. Assume A is a commutative (associative) algebra over a field F, where F has characteristic zero. In fact in our applications here F will be either the field of real, $\mathbb{R}$, or complex, $\mathbb{C}$, numbers. We will say that A has a Poisson structure if there is a bilinear map $A \times A \to A$, $(a, b) \to [a, b]$ with respect to which (1) A is a Lie algebra and is such that (2)

$$[a, bc] = [a, b]c + b[a, c] \tag{1.1.1}$$

for any $a, b, c \in A$.

If V is a finite-dimensional vector space over F then $S(V) = \bigoplus_{k=0}^{\infty} S_k(V)$ will denote the symmetric algebra over V with its usual grading. In particular, $V = S_1(V)$ and $F = S_0(V)$. We may regard $S(V)$ as the algebra of polynomial functions on the dual space V' to V. If $v \in V$ and $v' \in V'$ the pairing of v and v' is denoted by $\langle v', v\rangle$ so that one has $v(v') = \langle v', v\rangle$.

Now assume that $\mathfrak{a}$ is a finite-dimensional Lie algebra over F. For any $g \in \mathfrak{a}'$ let $\partial(g)$ be the derivation of degree -1 of $S(\mathfrak{a})$ such that $\partial(g)x = \langle g, x\rangle$ for any $x \in \mathfrak{a}$. Now if $u \in S(\mathfrak{a})$ then as a polynomial function on $\mathfrak{a}'$ the differential du defines a polynomial map $\mathfrak{a}' \to \mathfrak{a}$, $g \mapsto (du)(g)$. Explicitly one has

$$(du)(g) = \sum_{i=1}^{n} ((\partial(g_i)u)(g))\, x_i\,, \tag{1.1.2}$$

where $x_i \in \mathfrak{a}$, $i = 1,..., n$, is a basis of $\mathfrak{a}$ and $g_i \in \mathfrak{a}'$ is the dual basis.

Proposition 1.1. *If $u, v \in S(\mathfrak{a})$ and $g \in \mathfrak{a}'$ then*

$$[u, v](g) = \langle g, [(du)(g), (dv)(g)]\rangle \tag{1.1.3}$$

defines a Poisson structure on $S(\mathfrak{a})$ which extends the given Lie algebra structure on $\mathfrak{a}$.

Proof. If $u \in \mathfrak{a}$ then clearly $(du)(g) = u$ so that (1.1.3) agrees with the bracket structure on $\mathfrak{a}$. From the differentiation properties of the exterior derivative it is clear that (1.1.1) is satisfied. One only has to see that the Jacobi identity is satisfied for any $u, v, w \in S(\mathfrak{a})$. If $adu(w) = [u, w]$ this amounts to showing that

$$ad[u, v] = [adu, adv]. \tag{1.1.4}$$

However, both sides of (1.1.4) are derivations of the associative structure of $S(\mathfrak{a})$ by (1.1.1) and hence one has equality (1.1.4) when applied to $w = w_1 w_2$ if it already holds for w_1 and w_2. The proposition then follows easily since $\mathfrak{a}$ generates $S(\mathfrak{a})$ and (1.1.4) holds for $u, v \in \mathfrak{a}$ when applied to $w \in \mathfrak{a}$. Q.E.D.

Henceforth $S(\mathfrak{a})$, for any finite-dimensional Lie algebra $\mathfrak{a}$ will always have the Poisson structure given by Proposition 1.1.

Remark 1.2. Since $\mathfrak{a}$ generates $S(\mathfrak{a})$ note that the Poisson structure on $S(\mathfrak{a})$ is the unique one which reduces on $\mathfrak{a}$ to the given Lie algebra structure on $\mathfrak{a}$.

1.2. Now assume that $\mathfrak{g}$ is a finite-dimensional Lie algebra over F and $Q(x, y) \in F$ is a nonsingular invariant symmetric bilinear form on $\mathfrak{g}$. Invariance means that $Q([x, z], y) = Q(x, [z, y])$ for $x, y, z \in \mathfrak{g}$. If $x \in \mathfrak{g}$ and $g_x \in \mathfrak{g}'$ is defined so that $\langle g_x, y\rangle = Q(x, y)$ then $x \to g_x$ defines an isomorphism $\mathfrak{g} \to \mathfrak{g}'$. We regard $S(\mathfrak{g})$ as the algebra of polynomial functions on $\mathfrak{g}$ itself where if $u \in S(\mathfrak{g})$, $x \in \mathfrak{g}$ then $u(x) = u(g_x)$. In particular if $u \in \mathfrak{g}$ then

$$u(x) = Q(u, x). \tag{1.2.1}$$

For any $x \in \mathfrak{g}$ let $i(x)$ be the derivation of degree -1 on $S(\mathfrak{g})$ so that $i(x)y = Q(x, y)$ when $y \in \mathfrak{g}$. Let $u \in S(\mathfrak{g})$. To avoid confusion with du we will let δu be the differential of u when regarded as a function on $\mathfrak{g}$. Thus δu is the polynomial map $\mathfrak{g} \to \mathfrak{g}$ given by

$$(\delta u)(x) = \sum_j ((i(x_j)u)(x))\, y_j \tag{1.2.2}$$

for $x \in \mathfrak{g}$, where x_i, y_j are two bases of $\mathfrak{g}$ such that $Q(x_i, y_j) = \delta_{ij}$.

By Proposition 1.1 one immediately has

LEMMA 1.2.1. *The Poisson structure in $S(\mathfrak{g})$ is given by*

$$[u, v](x) = Q(x, [(\delta u)(x), (\delta v)(x)]) \tag{1.2.3}$$

for any $u, v \in S(\mathfrak{g})$, $x \in \mathfrak{g}$.

Now assume that F is either $\mathbb{R}$ or $\mathbb{C}$ and $\mathfrak{g}$ is a semi-simple Lie algebra over F. Let Q be a fixed invariant symmetric bilinear form on $\mathfrak{g}$ which on each simple component is a positive multiple of the Killing form. Also let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$. Thus $\mathfrak{k}$ is the Lie algebra of a maximal compact subgroup of the adjoint group of $\mathfrak{g}$ and $\mathfrak{p}$ is the orthogonal complement to $\mathfrak{k}$ with respect to Re Q. Now let θ be the corresponding Cartan involution. Thus $\theta = 1$ on $\mathfrak{k}$ and $\theta = -1$ on $\mathfrak{p}$. For any $x \in \mathfrak{g}$ let $x^* = -\theta x$. Since Q is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$ one defines an inner product Q_* on $\mathfrak{g}$ (i.e., $\mathfrak{g}$ is a Hilbert space over F with respect to Q_*) by putting

$$Q_*(x, y) = Q(x, y^*). \tag{1.2.4}$$

One notes that $(\lambda x)^* = \bar{\lambda} x^*$ for $\lambda \in F$ and $x \in \mathfrak{g}$, where the bar denotes conjugation. The relation $Q_*(x, y) = \overline{Q_*(y, x)}$ follows immediately from the readily verified relation $Q(x^*, y^*) = \overline{Q(x, y)}$. Also, the positive definiteness of Q_* is immediate from the fact that $\operatorname{Re} Q(x, y) = 0$ for $x \in \mathfrak{k}$ and $y \in \mathfrak{p}$.

Now if $\mathfrak{a} \subseteq \mathfrak{g}$ is any subspace let $\mathfrak{a}^* = \{x^* \mid x \in \mathfrak{a}\}$. Clearly $\mathfrak{a}^*$ is a subspace of the same dimension as $\mathfrak{a}$. There are two other subspaces, $\mathfrak{a}^\perp$, and $\mathfrak{a}^o$, which are also associated with $\mathfrak{a}$. They both have the same dimension as $\mathfrak{g}/\mathfrak{a}$. The subspace $\mathfrak{a}^\perp$ is defined as the orthogonal complement to $\mathfrak{a}$ with respect to Q_* and $\mathfrak{a}^o$ is the orthogonal subspace to $\mathfrak{a}$ with respect to Q. Thus $\mathfrak{a} \to \mathfrak{a}^*$, $\mathfrak{a}^\perp$, and $\mathfrak{a}^o$ define three involutory operations on the set of all subspaces of $\mathfrak{g}$. The following lemma clarifies the relations between them. It asserts that together with the identity operation they define an action of the Klein 4-group on the set of all subspaces.

LEMMA 1.2.2. *The three operations $\mathfrak{a} \to \mathfrak{a}^*$, $\mathfrak{a}^\perp$, and $\mathfrak{a}^o$ commute with one another and the composite of any two distinct operations is the third.*

The proof is straightforward and is left to the reader. A particular case of Lemma 1.2.2 is the relation

$$(\mathfrak{a}^*)^\perp = \mathfrak{a}^o. \tag{1.2.5}$$

Now if $\mathfrak{a} \subseteq \mathfrak{g}$ is a subspace we regard $S(\mathfrak{a})$ as a subalgebra of $S(\mathfrak{g})$. The fact that Q_* is an inner product implies that $\mathfrak{a}$ and $\mathfrak{a}^*$ are non-singularly paired by Q.

LEMMA 1.2.3. *Let $u \in S(\mathfrak{a})$. Then for any $x \in \mathfrak{g}$, $y \in \mathfrak{a}^o$ one has*

$$u(x + y) = u(x). \tag{1.2.6}$$

Further, the restriction map $u \to u \mid \mathfrak{a}^*$ defines an isomorphism of $S(\mathfrak{a})$ onto the algebra of all polynomial functions on $\mathfrak{a}^*$.

Proof. Clearly $v(y) = 0$ for all $v \in \mathfrak{a}$. Since u is generated by elements in $\mathfrak{a}$ one has (1.2.6). On the other hand since $\mathfrak{a}$ and $\mathfrak{a}^*$ are non-singularly paired by Q the map $v \to v \mid \mathfrak{a}^*$ defines an isomorphism of $S(\mathfrak{a})$ onto the algebra of all polynomial functions on $\mathfrak{a}^*$. Q.E.D.

Now let $P_\mathfrak{a}: \mathfrak{g} \to \mathfrak{a}$ be the orthogonal projection of $\mathfrak{g}$ on $\mathfrak{a}$ with respect to the inner product Q_*. If $u \in S(\mathfrak{g})$ we recall (see (1.2.2)) that δu is a polynomial map $\mathfrak{g} \to \mathfrak{g}$. We now define $\delta_{\mathfrak{a}} u$ to be the map $\mathfrak{g} \to \mathfrak{a}$ given by

$$(\delta_{\mathfrak{a}} u)(x) = P_{\mathfrak{a}}((\delta u)(x)). \tag{1.2.7}$$

Obviously if $\mathfrak{a} = \mathfrak{a}_1 + \mathfrak{a}_2$ is an orthogonal direct sum with respect to Q_* then

$$\delta_{\mathfrak{a}} u = \delta_{\mathfrak{a}_1} u + \delta_{\mathfrak{a}_2} u. \tag{1.2.8}$$

In particular one has

$$\delta u = \delta_{\mathfrak{a}} u + \delta_{\mathfrak{a}^\perp} u. \tag{1.2.9}$$

Explicitly one notes

LEMMA 1.2.4. *Let x_i, $i = 1,\ldots, d$, be any basis of $\mathfrak{a}$ and let $y_j \in \mathfrak{a}^*$ be the basis of $\mathfrak{a}^*$ such that $Q(x_i, y_j) = \delta_{ij}$. Then $\delta_{\mathfrak{a}} u$ is the polynomial map $\mathfrak{g} \to \mathfrak{a}$ given by*

$$(\delta_{\mathfrak{a}} u)(x) = \sum_{j=1}^{d} (i(y_j)u)(x)\, x_j \tag{1.2.10}$$

for any $x \in \mathfrak{g}$.

Proof. Let x_i, $i = d+1,\ldots, n = \dim \mathfrak{g}$, be any basis of $\mathfrak{a}^\perp$ and let y_i, $i = d+1,\ldots, n$, be the basis of $(\mathfrak{a}^\perp)^*$ such that $Q(x_i, y_j) = \delta_{ij}$ for $i, j \geqslant d+1$. Put $w_2 = \sum_{j=d+1}^{n} (i(y_j)u)(x)\, x_j$ and let w_1 be given by the right side of (1.2.10). Now since $\mathfrak{a}^* = (\mathfrak{a}^\perp)^o$ and $(\mathfrak{a}^\perp)^* = \mathfrak{a}^o$ by Lemma 1.2.2 it follows that $Q(x_i, y_j) = \delta_{ij}$ for $i, j = 1,\ldots, n$. Thus $w = w_1 + w_2 = (\delta u)(x)$ by (1.2.9). But then $w_1 = (\delta_{\mathfrak{a}} u)(x)$. This proves the lemma. Q.E.D.

LEMMA 1.2.5. *Let $\mathfrak{a} \subseteq \mathfrak{g}$ be any subspace and let $u \in S(\mathfrak{a})$. Then*

$$\delta u = \delta_{\mathfrak{a}} u \tag{1.2.11}$$

so that δu is a polynomial map $\mathfrak{g} \to \mathfrak{a}$.

Proof. By (1.2.9) we have only to show that $\delta_{\mathfrak{a}^\perp} u = 0$. But now applying (1.2.10) where $\mathfrak{a}^\perp$ replaces $\mathfrak{a}$ one notes that $i(y_j)u = 0$ since $y_j \in (\mathfrak{a}^\perp)^* = \mathfrak{a}^o$, using Lemma 1.2.2. Thus $\delta_{\mathfrak{a}^\perp} u = 0$. Q.E.D.

Now if $\mathfrak{a}$ is a Lie subalgebra of $\mathfrak{g}$ then $S(\mathfrak{a})$ inherits a Poisson structure from the Lie algebra structure of $\mathfrak{a}$. This has been defined using the dual space $\mathfrak{a}'$ to $\mathfrak{a}$. See Proposition 1.1. We now note that $\mathfrak{a}'$ may be replaced by $\mathfrak{a}^* \subseteq \mathfrak{g}$.

PROPOSITION 1.2. *Let $\mathfrak{a}$ be a Lie subalgebra of $\mathfrak{g}$. Then $S(\mathfrak{a})$ is a Lie subalgebra of $S(\mathfrak{g})$ with respect to the Poisson structure on $S(\mathfrak{g})$. Furthermore the Lie algebra structure thus induced on $S(\mathfrak{a})$ is the same as the Poisson structure on $S(\mathfrak{a})$ which $S(\mathfrak{a})$ would normally inherit from the Lie algebra structure in $\mathfrak{a}$. Thus if $u, v \in S(\mathfrak{a})$ then $[u, v]$ is unambiguous and is in $S(\mathfrak{a})$. Moreover it is given by*

$$[u, v](x) = Q(x, [(\delta u)(x), (\delta v)(x)]) \tag{1.2.12}$$

for $x \in \mathfrak{g}$ but is in fact determined by the restriction $[u, v] \mid \mathfrak{a}^*$. In particular, $[u, v] = 0$ if and only if the right side of (1.2.12) vanishes for all $x \in \mathfrak{a}^*$.

Proof. The first statement follows from (1.1.1). The second and hence the third follows from Remark 1.2. Relation (1.2.12) is just (1.2.3). Since $[u, v] \in S(\mathfrak{a})$ it is determined using Lemma 1.2.3, by the restriction $[u, v] \mid \mathfrak{a}^*$. Q.E.D.

1.3. Now let $G = \operatorname{Aut} \mathfrak{g}$ be the adjoint group of $\mathfrak{g}$. The action of $a \in G$ on $x \in \mathfrak{g}$ is denoted simply by $ax \in \mathfrak{g}$. The algebra $S(\mathfrak{g})$ then becomes a G-module where if $a \in G$, $u \in S(\mathfrak{g})$, and $x \in \mathfrak{g}$ one has $au(ax) = u(x)$. From the invariance of Q it follows that $a(i(x)u) = i(ax)\, au$. It then follows easily from (1.2.2) that

$$a((\delta u)(x)) = (\delta(au))(ax) \tag{1.3.1}$$

for any $u \in S(\mathfrak{g})$, $a \in G$, and $x \in \mathfrak{g}$.

Now let $l = \operatorname{rank} \mathfrak{g}$ and let $S(\mathfrak{g})^G$ be the algebra of G-invariants in $S(\mathfrak{g})$. By Chevalley's theorem one knows there exist homogeneous elements $I_j \in S(\mathfrak{g})^G$, $j = 1, 2, \ldots, l$, referred to as the fundamental invariants, which are algebraically independent and which generate $S(\mathfrak{g})^G$. That is, by abuse of notation,

$$S(\mathfrak{g})^G = F[I_1, \ldots, I_l]. \tag{1.3.2}$$

Now for any $x \in \mathfrak{g}$ let $\mathfrak{g}^x$ denote the centralizer of x in $\mathfrak{g}$ and let $\operatorname{cent} \mathfrak{g}^x$ be the center of $\mathfrak{g}^x$. Also let G^x be the centralizer (or rather stabilizer) of x in G.

PROPOSITION 1.3. *For any $x \in \mathfrak{g}$ and $I \in S(\mathfrak{g})^G$ one has $(\delta I)(x) \in \operatorname{cent} \mathfrak{g}^x$. In particular one has*

$$[x, (\delta I)(x)] = 0. \tag{1.3.3}$$

Proof. For any $a \in G$ one has $aI = I$. Thus $a((\delta I)(x)) = (\delta I)(ax)$ by (1.3.1). But then if $a \in G^x$ one has $a((\delta I)(x)) = (\delta I)(x)$. That is, if $\mathfrak{g}^{(G^x)}$ is the set of fixed elements for the action of G^x in $\mathfrak{g}$ one has $(\delta I)(x) \in \mathfrak{g}^{(G^x)}$. But $\mathfrak{g}^{(G^x)} \subseteq \mathfrak{g}^{(\mathfrak{g}^x)}$, where $\mathfrak{g}^{(\mathfrak{g}^x)}$ is the centralizer of $\mathfrak{g}^x$ in $\mathfrak{g}$. However, since $x \in \mathfrak{g}^x$ one has $\mathfrak{g}^{(\mathfrak{g}^x)} =$ cent $\mathfrak{g}^x$. This proves the proposition. Q.E.D.

1.4. Now let $\mathfrak{a}$ be a Lie subalgebra of $\mathfrak{g}$. We shall regard $\mathfrak{a}$ as fixed. This will enable us to suppress $\mathfrak{a}$ in notation which in fact depends upon $\mathfrak{a}$.

Now let $f \in \mathfrak{g}$ and $u \in S(\mathfrak{g})$. The map $x \mapsto u(f+x)$ is clearly a polynomial function on $\mathfrak{g}$. In particular its restriction to $\mathfrak{a}^*$ is a polynomial function. By Lemma 1.2.3 therefore there exists a unique element $u^f \in S(\mathfrak{a})$ such that

$$u^f(x) = u(f+x) \tag{1.4.1}$$

for all $x \in \mathfrak{a}^*$. The correspondence $u \to u^f$ therefore defines a homomorphism

$$S(\mathfrak{g}) \to S(\mathfrak{a}). \tag{1.4.2}$$

We are particularly interested in the case where $u \in S(\mathfrak{g})^G$ and especially when $\mathfrak{a}$ satisfies the following condition.

We say that $\mathfrak{a}$ is a Lie summand in case $\mathfrak{a}^o$ is also a Lie subalgebra of $\mathfrak{g}$. The terminology stems from the fact that $\mathfrak{a}$ is a Lie summand if and only if $\mathfrak{a}^\perp$ is a Lie subalgebra of $\mathfrak{g}$. This is clear since, by Lemma 1.2.2, $\mathfrak{a}^\perp = (\mathfrak{a}^o)^*$ and one has $[x, y]^* = [y^*, x^*]$.

Example. Any parabolic subalgebra of $\mathfrak{g}$ is a Lie summand. This is clear since if $\mathfrak{a}$ is parabolic then $\mathfrak{a}^o$ is just the nilradical of $\mathfrak{a}$.

It will be seen later that the following theorem leads to the complete integrability of the Toda lattice and to extensive generalizations of it.

Theorem 1.4. *Let $\mathfrak{g}$ be a semi-simple Lie algebra over $\mathbb{R}$ or $\mathbb{C}$. Let Q be a bilinear form on $\mathfrak{g}$ which on each simple component is a fixed positive multiple of the Killing form and let Q_* be the inner product in $\mathfrak{g}$ defined by Q and a Cartan decomposition of $\mathfrak{g}$. See* (1.2.4).

Let $\mathfrak{a} \subseteq \mathfrak{g}$ be a Lie summand. That is, $\mathfrak{a}$ and its orthocomplement $\mathfrak{a}^\perp$ with respect to Q_* are Lie subalgebras of $\mathfrak{g}$. Let $f \in \mathfrak{g}$ be any element such that

$$Q(f, [\mathfrak{a}, \mathfrak{a}] + [\mathfrak{a}^\perp, \mathfrak{a}^\perp]) = 0. \tag{1.4.3}$$

Then, using the notation of (1.4.1) *the elements I^f, J^f in $S(\mathfrak{a})$ Poisson commute for any pair of invariants $I, J \in S(\mathfrak{g})^G$.*

Proof. Using Proposition 1.2 it is enough to show that

$$Q(x, [(\delta I^f)(x), (\delta J^f)(x)]) = 0 \tag{1.4.4}$$

for any $x \in a^*$. But now we assert that

$$(\delta I^f)(x) = (\delta_a I)(f + x) \tag{1.4.5}$$

for $x \in a^*$. Indeed for any y, $z \in g$ and $u \in S(g)$ one has $(i(y)u)(z) = (d/dt)\, u(z + ty)\,|_{t=0}$. It follows therefore that if $x, y \in a^*$ one has $(i(y)\, I^f)(x) = (i(y)I)(f + x)$. Thus recalling (1.2.10) one has $(\delta_a I^f)(x) = (\delta_a I)(f + x)$ for $x \in a^*$. But $\delta_a I^f = \delta I^f$ by (1.2.11). This proves (1.4.5).

But now $[(\delta_a I)(f + x), (\delta_a J)(f + x)] \in [a, a]$. However, f is Q-orthogonal to $[a, a]$. That is,

$$Q(f, [(\delta_a I)(f + x), (\delta_a J)(f + x)]) = 0. \tag{1.4.6}$$

But then adding this relation to the left side of (1.4.4) and recalling (1.4.5) it is enough to show that

$$Q(f + x, [(\delta_a I)(f + x), (\delta_a J)(f + x)]) = 0. \tag{1.4.7}$$

But now by (1.3.3) when $f + x$ replaces x one has

$$[f + x, (\delta I)(f + x)] = 0. \tag{1.4.8}$$

Thus by the invariance of Q one does in fact have

$$Q(f + x, [(\delta I)(f + x), (\delta_a J)(f + x)]) = 0. \tag{1.4.9}$$

But $(\delta I)(f + x) = (\delta_a I)(f + x) + (\delta_{a^\perp} I)(f + x)$ by (1.2.9). Substituting in (1.4.9) it then suffices to prove

$$Q(f + x, [(\delta_{a^\perp} I)(f + x), (\delta_a J)(f + x)]) = 0. \tag{1.4.10}$$

But (1.4.8) is valid if J replaces I. Thus by the invariance of Q one does have (1.4.10) if δJ replaces $\delta_a J$. Thus it suffices to show

$$Q(f + x, [(\delta_{a^\perp} I)(f + x), (\delta_{a^\perp} J)(f + x)]) = 0.$$

But by (1.4.3) one also has (1.4.6) when $a^\perp$ replaces a. Thus it suffices only to show that

$$Q(x, [(\delta_{a^\perp} I)(f + x), (\delta_{a^\perp} J)(f + x)]) = 0. \tag{1.4.11}$$

But since $a^\perp$ is a Lie algebra the commutator in (1.4.11) is an element in $a^\perp$. However, $x \in a^*$ and $a^* = (a^\perp)^o$ by Lemma 1.2.2. This proves (1.4.11).

Q.E.D.

1.5. We consider an example now which illustrates Theorem 1.4 and which plays a major role in the remainder of the paper. No special assumptions need be made if $F = \mathbb{C}$. However, if $F = \mathbb{R}$ then for this example we assume that $\mathfrak{g}$ is split. We may then find, as one knows, using, say, Weyl's normal form (see, e.g., Section 5 in [13]), a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ (which is split if $F = \mathbb{R}$ and root vectors $e_\varphi \in \mathfrak{g}$, $\varphi \in \Delta$, where $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ is the corresponding set of roots such that

$$e_\varphi^* = e_{-\varphi} \tag{1.5.1}$$

and

$$Q(e_\varphi, e_{-\varphi}) = 1. \tag{1.5.2}$$

Let $\Delta_+ \subseteq \Delta$ be a fixed chosen system of positive roots and let $\{\alpha_1, \ldots, \alpha_l\} = \Pi \subseteq \Delta_+$ be the set of simple positive roots.

Remark 1.5.1. One notes that the root vectors $e_{-\alpha_i}$, $i = 1, \ldots, l$, may initially be chosen arbitrarily. Then e_{α_i} are determined by (1.5.2). This then, using (1.5.1), determines the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$.

Now

$$\bar{\mathfrak{b}} = \mathfrak{h} + \sum_{\varphi \in \Delta_+} Fe_{-\varphi} \qquad \text{and} \qquad \mathfrak{b} = \mathfrak{h} + \sum_{\varphi \in \Delta_+} Fe_\varphi \tag{1.5.3}$$

are "opposing" Borel subalgebras and $\bar{\mathfrak{n}} = [\bar{\mathfrak{b}}, \bar{\mathfrak{b}}]$ and $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ are the corresponding nilradicals.

THEOREM 1.5. *Let $\mathfrak{a} = \bar{\mathfrak{b}}$ and let*

$$f = \sum_{i=1}^{l} e_{-\alpha_i}. \tag{1.5.4}$$

Then $\mathfrak{a}$ and f satisfy the conditions of Theorem 1.4. *That is, $\bar{\mathfrak{b}}$ is a Lie summand and f satisfies* (1.4.3). *In particular the elements I^f and J^f in $S(\bar{\mathfrak{b}})$ Poisson commute for any $I, J \in S(\mathfrak{g})^G$.*

Proof. If $\varphi \in \Delta$ then as one knows $Q(x, e_\varphi) = Q(e_\psi, e_\varphi) = 0$ for any $x \in \mathfrak{h}$, and $\psi \in \Delta$, where $\psi \neq -\varphi$. Since

$$\mathfrak{n} = \sum_{\varphi \in \Delta_+} Fe_\varphi \qquad \text{and} \qquad \bar{\mathfrak{n}} = \sum_{\varphi \in \Delta_+} Fe_{-\varphi} \tag{1.5.5}$$

one then has $(\bar{\mathfrak{b}})^o = \bar{\mathfrak{n}}$. Hence $\bar{\mathfrak{b}}$ is a Lie summand. But $[\mathfrak{a}, \mathfrak{a}] = [\bar{\mathfrak{b}}, \bar{\mathfrak{b}}] = \bar{\mathfrak{n}} = (\bar{\mathfrak{b}})^o$. Since $f \in \bar{\mathfrak{b}}$ this implies $Q(f, [\mathfrak{a}, \mathfrak{a}]) = 0$. But by Lemma 1.2.2 one also has $\mathfrak{a}^\perp = (\mathfrak{a}^o)^* = (\bar{\mathfrak{n}})^* = \mathfrak{n}$. Thus $[\mathfrak{a}^\perp, \mathfrak{a}^\perp]$ is spanned by all e_φ, where $\varphi \in \Delta_+$ is not simple. Thus $Q(f, [\mathfrak{a}^\perp, \mathfrak{a}^\perp]) = 0$. Hence f satisfies (1.4.3). The remaining statement follows from Theorem 1.4. Q.E.D.

Remark 1.5.2. If $N \subseteq G$ is the subgroup corresponding to $\mathfrak{n}$ then in Section 1.3 of [17] we defined an action of N on $S(\bar{\ell})$ so that if $S(\bar{\ell})^N$ is the algebra of invariants then when $\mathfrak{a} = \bar{\ell}$

$$S(\bar{\ell})^N = \{I^f \mid I \in S(\mathfrak{g})^G\}. \tag{1.5.6}$$

Theorem 1.5 then says that the elements of the algebra $S(\bar{\ell})^N$ Poisson commute with one another. This algebra is very explicit in the case where $\mathfrak{g}$ is the Lie algebra of $Sl(n, F)$. In that case choices can be made so that $f + \ell = f + \mathfrak{a}^*$ is the set of all traceless matrices y of the form

$$y = \begin{pmatrix} a_{11} & a_{12} & \cdots & & a_{1n} \\ 1 & & & & \vdots \\ \vdots & & & & \vdots \\ \vdots & & & & a_{n-1n} \\ 0 & \cdots & & 1 & a_{nn} \end{pmatrix}. \tag{1.5.7}$$

By abuse of notation regard a_{ij} as a linear functional on ℓ and thereby identify it with an element in $\bar{\ell}$. The nontrivial coefficients $u_k \in S(\bar{\ell})$, $k = 1,..., l = n - 1$, of the characteristic polynomial of y, as polynomials in the a_{ij}, are of the form $u_k = I_k{}^f$, where the I_k are the fundamental invariants. The algebra $S(\bar{\ell})^N$ is just the algebra generated by the u_k. Theorem 1.5 asserts that they Poisson commute with one another. It is this example which applies directly to the Toda lattice.

1.6. Let $\mathfrak{a}$ and f be as in Theorem 1.5 so that $\mathfrak{a}$ is a Lie summand in $\mathfrak{g}$ and $f \in \mathfrak{g}$ satisfies (1.4.3). Now consider the translation $f + \mathfrak{a}^*$ of $\mathfrak{a}^*$ by f. For convenience put $f + \mathfrak{a}^* = (\mathfrak{a}^*)_f$

$$\tau_f\colon \mathfrak{a}^* \to (\mathfrak{a}^*)_f, \qquad x \to f + x. \tag{1.6.1}$$

We will understand that $(\mathfrak{a}^*)_f$ has the structure of an affine variety and that τ_f is an isomorphism of affine varieties.

LEMMA 1.6.1. *For any $I \in S(\mathfrak{g})^G$ and $y \in (\mathfrak{a}^*)_f$ one has $[y, (\delta_{\mathfrak{a}} I)(y)] \in \mathfrak{a}^*$.*

Proof. Put $w = (\delta_{\mathfrak{a}^\perp} I)(y)$ so that $w + (\delta_{\mathfrak{a}} I)(y) = (\delta I)(y)$ by (1.2.9). But $(\delta I)(y)$ commutes with y by (1.3.3). Thus

$$[y, (\delta_{\mathfrak{a}} I)(y)] = [w, y]. \tag{1.6.2}$$

But now write $y = f + x$, where $x \in \mathfrak{a}^*$. We assert that

$$[w, f] \in \mathfrak{a}^*. \tag{1.6.3}$$

Indeed since $\mathfrak{a}^* = (\mathfrak{a}^\perp)^o$ it suffices to show that $[w, f]$ is Q-orthogonal to $\mathfrak{a}^\perp$. But $Q(\mathfrak{a}^\perp, [w, f]) = Q([\mathfrak{a}^\perp, w], f) = 0$ by (1.4.3) since $w \in \mathfrak{a}^\perp$. This proves (1.6.3). Thus to prove the lemma it suffices by (1.6.2) to show that $[w, x] \in \mathfrak{a}^*$. But to prove this it certainly suffices to show that

$$[\mathfrak{a}^\perp, \mathfrak{a}^*] \subseteq \mathfrak{a}^*. \tag{1.6.4}$$

But since $\mathfrak{a}^* = (\mathfrak{a}^\perp)^o$ it suffices to show that $[\mathfrak{a}^\perp, (\mathfrak{a}^\perp)^o]$ is Q-orthogonal to $\mathfrak{a}^\perp$. However, $Q(\mathfrak{a}^\perp, [\mathfrak{a}^\perp, (\mathfrak{a}^\perp)^o]) \subseteq Q([\mathfrak{a}^\perp, \mathfrak{a}^\perp], (\mathfrak{a}^\perp)^o) = 0$ since $\mathfrak{a}^\perp$ is a Lie algebra. This proves (1.6.4). Q.E.D.

Now $\mathfrak{g}$, whether or not $F = \mathbb{R}$ or $\mathbb{C}$, has the structure of a vector space over $\mathbb{R}$. Thus if $y \in \mathfrak{g}$ the tangent space to $\mathfrak{g}$ at y is naturally identified with $\mathfrak{g}$. In this identification if $y \in (\mathfrak{a}^*)_f$ then the tangent space to $(\mathfrak{a}^*)_f$ at y is clearly $\mathfrak{a}^*$. Let $I \in S(\mathfrak{g})^G$. The map given by $y \to [y, (\delta_{\mathfrak{a}} I)(y)]$ is clearly a polynomial map on $\mathfrak{g}$ and hence by Lemma 1.6.1 there exists a smooth vector field η_I on $(\mathfrak{a}^*)_f$ such that

$$(\eta_I)_y = [y, (\delta_{\mathfrak{a}} I)(y)] \tag{1.6.5}$$

for any $y \in (\mathfrak{a}^*)_f$.

Now let $\tau_f^{-1}\colon (\mathfrak{a}^*)_f \to \mathfrak{a}^*$ be the isomorphism which is inverse to τ_f. See (1.6.1). Thus τ_f^{-1} "carries" vector fields on $(\mathfrak{a}^*)_f$ to vector fields on $\mathfrak{a}^*$. In particular, using the same notation for this map of vector fields, $\eta_I \to \tau_f^{-1}\eta_I$ the vector field $\tau_f^{-1}\eta_I$ can be applied to an element $u \in S(\mathfrak{g})$ since by restriction u defines a function on $\mathfrak{a}^*$. We express this action in terms of Poisson brackets.

LEMMA 1.6.2. *For any $I \in S(\mathfrak{g})^G$, $u \in S(\mathfrak{a})$, $x \in \mathfrak{a}^*$ one has*

$$((\tau_f^{-1}\eta_I)u)(x) = [I^f, u](x). \tag{1.6.6}$$

Proof. Let $y = f + x = \tau_f x$. We express the left side of (1.6.6) in terms of the pairing of vectors and covectors. Let c be the left side of (1.6.6). Recalling the definition (see (1.2.2)) of δu one has

$$\begin{aligned} c &= Q([y, (\delta_{\mathfrak{a}} I)(y)], (\delta u)(x)) \\ &= Q(y, [(\delta_{\mathfrak{a}} I)(y), (\delta u)(x)]). \end{aligned}$$

But $(\delta_{\mathfrak{a}} I)(f + x) = (\delta I^f)(x)$ by (1.4.5). Thus

$$c = Q(y, [(\delta I^f)(x), (\delta u)(x)]).$$

But $y = f + x$. Hence $c = [I^f, u](x) + Q(f, [(\delta I^f)(x), (\delta u)(x)])$, recalling the formula (1.2.3). It suffices therefore to show that

$$Q(f, [(\delta I^f)(x), (\delta u)(x)] = 0. \tag{1.6.7}$$

But since I^f, $u \in S(\mathscr{a})$ the commutator in (1.6.7) is in $[\mathscr{a}, \mathscr{a}]$ by Lemma 1.2.5. Thus (1.6.7) follows from (1.4.3). Q.E.D.

Theorems 1.6.1 and 1.6.2 below are corollaries of Theorem 1.5. They are familiar consequences in a symplectic context. However, the latter does not quite apply here.

THEOREM 1.6.1. *Let $\mathscr{g}$ be a semi-simple Lie algebra over $\mathbb{R}$ or $\mathbb{C}$. Let $\mathscr{a} \subseteq \mathscr{g}$ and $f \in \mathscr{g}$ satisfy the conditions of Theorem* 1.5 *and put $(\mathscr{a}^*)_f = f + \mathscr{a}^*$. For any invariant $I \in S(\mathscr{g})^G$ let η_I be the vector field on $(\mathscr{a}^*)_f$ defined by* (1.6.5). *Then*

$$\eta_I J = 0 \tag{1.6.8}$$

on $(\mathscr{a}^*)_f$ for any $I, J \in S(\mathscr{g})^G$.

Proof. Apply Lemma 1.6.2, where $u = J^f$. But then by Theorem 1.5 one has $((\tau_f^{-1}\eta_I)u)(x) = 0$ for any $x \in \mathscr{a}^*$. However, if $y = f + x$ then $((\tau_f^{-1}\eta_I)u)(x) = (\eta_I(u \circ \tau_f^{-1}))(y)$. But clearly $u \circ \tau_f^{-1} = J \mid (\mathscr{a}^*)_f$. This proves the theorem. Q.E.D.

Now let $P_{\mathscr{a}^*}: \mathscr{g} \to \mathscr{a}^*$ be the Q_*-orthogonal projection of $\mathscr{g}$ onto $\mathscr{a}^*$. For any $y \in (\mathscr{a}^*)_f$ let

$$\mathscr{a}^*(y) = \{P_{\mathscr{a}^*}[x, y] \mid x \in \mathscr{a}\}. \tag{1.6.9}$$

Remark 1.6.1. If $A \subseteq G$ is the subgroup corresponding to $\mathscr{a}$ then one may define an action of A on $(\mathscr{a}^*)_f$ by translating to $(\mathscr{a}^*)_f$, using τ_f, the coadjoint action of A on $\mathscr{a}^*$. The subspace $\mathscr{a}^*(y)$ of $\mathscr{a}^*$ then turns out to be nothing more than the tangent space at y to the A orbit containing y. By Lemma 1.6.1 one has $P_{\mathscr{a}^*}[y, (\delta_{\mathscr{a}} I)(y)] = [y, (\delta_{\mathscr{a}} I)(y)]$. Since $(\delta_{\mathscr{a}} I)(y) \in \mathscr{a}$ it follows that

$$(\eta_I)_y \in \mathscr{a}^*(y) \qquad \text{for any} \quad I \in S(\mathscr{g})^G. \tag{1.6.10}$$

THEOREM 1.6.2. *Let $\mathscr{g}$ be a semi-simple Lie algebra over $\mathbb{C}$ or $\mathbb{R}$. Let $\mathscr{a} \subseteq \mathscr{g}$ and $f \in \mathscr{g}$ satisfy the conditions of Theorem* 1.5 *and put $(\mathscr{a}^*)_f = f + \mathscr{a}^*$. For any invariant $I \in S(\mathscr{g})^G$ let η_I be the vector field on $(\mathscr{a}^*)_f$ defined by* (1.6.5). *Then*

$$[\eta_I, \eta_J] = 0 \tag{1.6.11}$$

for any $I, J \in S(\mathfrak{g})^G$. *Furthermore if* $y \in (\mathfrak{a}^*)_f$ *then*

$$(\eta_I)_y = 0 \Leftrightarrow Q(\mathfrak{a}^*(y), (\delta I)(y)) = 0, \tag{1.6.12}$$

where $\mathfrak{a}^*(y)$ *is defined by* (1.6.9) *and* δI *is defined by* (1.2.2).

Proof. We may iterate the formula (1.6.6). Let $I, J \in S(\mathfrak{g})^G$ and $u \in S(\mathfrak{a})$. That is, if $x \in \mathfrak{a}^*$, then

$$((\tau_f^{-1}\eta_I)(\tau_f^{-1}\eta_J)u)(x) = [I^f, [J^f, u]](x). \tag{1.6.13}$$

But then $((\tau_f^{-1}[\eta_I, \eta_J])u)(x) = [[I^f, J^f]u](x)$. However, $[I^f, J^f] = 0$ by Theorem 1.5. But since the set of functions $u \mid \mathfrak{a}^*$, $u \in S(\mathfrak{a})$ contains a coordinate system (e.g., let $u \in \mathfrak{a}$) it follows that $\tau_f^{-1}[\eta_I, \eta_J] = 0$ and hence one has (1.6.11).

Now if $x \in \mathfrak{a}$ and $y \in (\mathfrak{a}^*)_f$ then

$$\begin{aligned} Q(x, (\eta_I)_y) &= Q(x, [y, (\delta_{\mathfrak{a}} I)(y)]) \\ &= Q([x, y], (\delta_{\mathfrak{a}} I)(y)). \end{aligned} \tag{1.6.14}$$

But now since $(\mathfrak{a}^*)^\perp = \mathfrak{a}^o$ we may replace $[x, y]$ in (1.6.14) by $P_{\mathfrak{a}^*}[x, y]$. But then having done that we may replace $(\delta_{\mathfrak{a}} I)(y)$ by $(\delta I)(y)$ since $(\mathfrak{a}^\perp)^o = \mathfrak{a}^*$. This uses (1.2.9). Thus

$$Q(x, (\eta_I)_y) = Q(P_{\mathfrak{a}^*}[x, y], (\delta I)(y)). \tag{1.6.15}$$

But since $(\eta_I)_y \in \mathfrak{a}^*$ by Lemma 1.6.1 and $\mathfrak{a}$ and $\mathfrak{a}^*$ are non-singularly paired by Q it follows that $(\eta_I)_y \neq 0$ if and only if there exists $x \in \mathfrak{a}$ such that (1.6.15) is not zero. This implies (1.6.12). Q.E.D.

Remark 1.6.2. Recalling Remark 1.6.1 one may interpret (1.6.12) as follows: Let O denote the A-orbit of y with respect to the f-translated coadjoint action of A on $(\mathfrak{a}^*)_f$ described in Remark 1.6.1. Then $(\eta_I)_y \neq 0$ if and only if the differential $d(I \mid O)_y \neq 0$, where of course $I \mid O$ is the restriction of I to O. In fact if $F = \mathbb{R}$ then O has the structure of a symplectic manifold and $\eta_I \mid O$ (see Section 6.4 for the case where $O = Z$) is just the Hamiltonian vector field corresponding to the function $I \mid O$.

2. The Variety Z of Normalized Jacobi Elements

2.1. Henceforth we restrict our attention to the case of the example in Section 1.5. We recall that there is no restriction on $\mathfrak{g}$ if $F = \mathbb{C}$ but $\mathfrak{g}$ is split if $F = \mathbb{R}$. Furthermore $\mathfrak{h}$ (which is split if $F = \mathbb{R}$), Δ, Δ_+, $\Pi = \{\alpha_1, \ldots, \alpha_l\}$, e_φ, for $\varphi \in \Delta$, satisfying (1.5.1) and (1.5.2) are as in Section 1.5.

Now let $H \subseteq G$ be the subgroup corresponding to $\mathfrak{h}$. Let $\mathfrak{h}' = \mathrm{Hom}_F(\mathfrak{h}, F)$ and where $\mathbb{Z}$ is the set of integers let

$$\mathfrak{h}'(H) = \{\nu \in \mathfrak{h}' \mid \langle \nu, x\rangle \in 2\pi i\mathbb{Z} \text{ for } x \in \mathfrak{h}, \text{ where } \exp x = 1\}. \tag{2.1.1}$$

Remark 2.1. One knows that $\mathfrak{h}'(H)$ is the lattice $\sum_{i=1}^{l} \mathbb{Z}\alpha_i$ if $F = \mathbb{C}$ and $\mathfrak{h}'(H) = \mathfrak{h}'$ if $F = \mathbb{R}$.

Let $F^* = \exp F$ so that F^* is the multiplicative group of nonzero complex numbers if $F = \mathbb{C}$ and F^* is the multiplicative group of positive numbers if $F = \mathbb{R}$. Then any $\nu \in \mathfrak{h}'(H)$ defines a homomorphism $H \to F^*$, $h \to h^\nu$, where $h^\nu = \exp\langle \nu, x\rangle$ for $x \in \mathfrak{h}$ such that $h = \exp x$. One of course has $\Delta \subseteq \mathfrak{h}'(H)$ and if $h \in H$, $\varphi \in \Delta$ then

$$he_\varphi = h^\varphi e_\varphi . \tag{2.1.2}$$

Now let $x_0 \in \mathfrak{h}$ be that unique element such that $\langle \alpha_i, x_0\rangle = 1$ for $i = 1,..., l$. The eigenvalues of $\operatorname{ad} x_0$ are in $\mathbb{Z}$. One puts $\mathfrak{d}_j = \{x \in \mathfrak{g} \mid [x_0, x] = jx\}$ for $j \in \mathbb{Z}$. We shall refer to the $\mathfrak{d}_j$ as the diagonals of $\mathfrak{g}$. Of course $\mathfrak{d}_0 = \mathfrak{h}$ and one has the direct sum decomposition

$$\mathfrak{g} = \bigoplus \mathfrak{d}_j , \tag{2.1.3}$$

where the sum is over $\mathbb{Z}$.

DEFINITION 2.1. If $0 \neq x \in \mathfrak{g}$ and $x_j \in \mathfrak{d}_j$ denotes the component of x in $\mathfrak{d}_j$ relative to (2.1.3) then the minimal j such that $x_j \neq 0$ will be called the minimal diagonal degree of x. The element x_j is called the minimal diagonal component of x. Similarly we will speak of the maximal diagonal degree and component of x.

One of course has, for $i, j \in \mathbb{Z}$,

$$[\mathfrak{d}_i , \mathfrak{d}_j] \subseteq \mathfrak{d}_{i+j} . \tag{2.1.4}$$

Now let $\mathfrak{b}$, $\bar{\mathfrak{b}}$, $\mathfrak{n}$, and $\bar{\mathfrak{n}}$ be the Borel subalgebras and their commutators defined as in Section 1.5 so that

$$\mathfrak{b} = \sum_{j\geqslant 0} \mathfrak{d}_j , \qquad \bar{\mathfrak{b}} = \sum_{j\leqslant 0} \mathfrak{d}_j \tag{2.1.5}$$

and

$$\mathfrak{n} = \sum_{j\geqslant 1} \mathfrak{d}_j , \qquad \bar{\mathfrak{n}} = \sum_{j\leqslant 1} \mathfrak{d}_j . \tag{2.1.6}$$

Henceforth unless otherwise stated as in the example of Section 1.5 the subalgebra $\mathfrak{a}$ of Sections 1.2–1.5 will be just $\bar{\mathfrak{b}}$. Thus all notation and results referring to $\mathfrak{a}$ now apply solely to $\bar{\mathfrak{b}}$. Furthermore the element f is fixed and is given by (1.5.4). Thus in particular, $u^f \in S(\bar{\mathfrak{b}})$ for any $u \in S(\mathfrak{g})$. Furthermore $\mathfrak{a}^* = \mathfrak{b}$ by (1.5.1) and $(\mathfrak{a}^*)_f = f + \mathfrak{b}$ which of course we now write as $\mathfrak{b}_f$. For the case of $Sl(n, F)$ we note that choices can be made so that the typical element $y \in \mathfrak{b}_f$ is of form (1.5.7).

Now if $x \in \mathfrak{g}$ then for the centralizer $\mathfrak{g}^x$ of x one knows that $\dim \mathfrak{g}^x \geqslant l$ and the set $R = \{x \in \mathfrak{g} \mid \dim \mathfrak{g}^x = l\}$ is an open dense subset of $\mathfrak{g}$. The elements of R are called regular.

LEMMA 2.1.1. *One has*

$$\mathfrak{b}_f \subseteq R. \tag{2.1.7}$$

Proof. It clearly suffices, by complexification, to assume that $F = \mathbb{C}$. But then Lemma 2.1.1 is Lemma 10 in [14, p. 370]. Q.E.D.

If $y \in \mathfrak{b}_f$ then it is not in general easy to describe the centralizer $\mathfrak{g}^y$ of y. However, one has

LEMMA 2.1.2. *For any $y \in \mathfrak{b}_f$ one has*

$$\mathfrak{g}^y \cap \mathfrak{b} = 0. \tag{2.1.8}$$

Proof. It clearly suffices to assume that $F = \mathbb{C}$. But then Lemma 2.1.2 is (1.2.4) in [17, p. 109]. Q.E.D.

Now

$$\mathfrak{g} = \bar{\mathfrak{b}} \oplus \mathfrak{n} \tag{2.1.9}$$

is a Q_*-orthogonal direct sum since $(\bar{\mathfrak{b}})^\perp = ((\bar{\mathfrak{b}})^o)^* = (\bar{\mathfrak{n}})^* = \mathfrak{n}$. Thus the decomposition (1.2.9) becomes $\delta u = \delta_{\bar{\mathfrak{b}}} u + \delta_{\mathfrak{n}} u$ for any $u \in S(\mathfrak{g})$.

We recall that I_j, $j = 1,\ldots, l$, are the generators of $S(\mathfrak{g})^G$.

PROPOSITION 2.1. *Let $y \in \mathfrak{b}_f$ be arbitrary and put*

$$(\delta_{\bar{\mathfrak{b}}}(S(\mathfrak{g})^G))(y) = \{(\delta_{\bar{\mathfrak{b}}} I)(y) \mid I \in S(\mathfrak{g})^G\}.$$

Then the elements $(\delta_{\bar{\mathfrak{b}}} I_j)(y)$, $j = 1,\ldots, l$, are linearly independent and are a basis of $(\delta_{\bar{\mathfrak{b}}}(S(\mathfrak{g})^G))(y)$. Furthermore $[\mathfrak{n}, y] \subseteq \mathfrak{b}$ and one has

$$Q((\delta_{\bar{\mathfrak{b}}}(S(\mathfrak{g}))^G))(y), [\mathfrak{n}, y]) = 0. \tag{2.1.10}$$

In fact $[\mathfrak{n}, y]$ *has codimension* l *in* $\mathfrak{b}$ *and is the* Q*-orthocomplement of* $(\delta_{\mathfrak{z}}(S(\mathfrak{g})^G))(y)$ *in* $\mathfrak{b}$.

Proof. Assume that there is a nontrivial relation $\sum c_i(\delta_{\mathfrak{z}} I_i)(y) = 0$. This implies that if $w = \sum c_i(\delta I_i)(y)$ then $w \in \mathfrak{n}$ by (2.1.9). But $w \in \mathfrak{g}^y$ by (1.3.3). However, $\mathfrak{g}^y \cap \mathfrak{n} = 0$ by (2.1.8). Thus $w = 0$. But $y \in R$ by (2.1.7). This contradicts Theorem 9 in [14, p. 382], which asserts that $(\delta I_i)(y)$, $i = 1,..., l$, are linearly independent for any $y \in R$. Thus the $(\delta_{\mathfrak{z}} I_j)(y)$ are linearly independent. From the differentiation properties of the map $I \to (\delta_{\mathfrak{z}} I)(y)$ it is clear that they span $(\delta_{\mathfrak{z}}(S(\mathfrak{g})^G))(y)$. This proves the first statement of the proposition.

Now the minimal diagonal degree (see Definition 2.1) of y is -1. Thus $[\mathfrak{n}, y] \subseteq \mathfrak{b}$ by (2.1.4), (2.1.5), and (2.1.6). Thus if $I \in S(\mathfrak{g})$ one has $Q((\delta_{\mathfrak{n}} I)(y),$ $[\mathfrak{n}, y]) = 0$ since $\mathfrak{b}^o = \mathfrak{n}$. However, $Q((\delta I)(y), [\mathfrak{n}, y]) = 0$ by the invariance of Q and the fact that $[(\delta I)(y), y] = 0$. See (1.3.3). Thus, subtracting, $Q((\delta_{\mathfrak{b}} I)(y), [\mathfrak{n}, y]) = 0$. This proves (2.1.10). But now (2.1.8) implies that $\dim[\mathfrak{n}, y] = \dim \mathfrak{n}$. However, $\dim \mathfrak{b} = l + \dim \mathfrak{n}$. Also, $\mathfrak{b}$ and $\bar{\mathfrak{b}}$ are nonsingularly paired by Q and we have shown that $\dim(\delta_{\mathfrak{z}}(S(\mathfrak{g}))^G)(y) = l$. This proves the last statement of the proposition. Q.E.D.

2.2. Now it is clear that the diagonal $\mathfrak{d}_1$ is the l-dimensional subspace given by

$$\mathfrak{d}_1 = \sum_{i=1}^{l} F e_{\alpha_i}. \tag{2.2.1}$$

Let

$$\tilde{\mathfrak{d}}_1 = \left\{ \sum_{i=1}^{l} a_i e_{\alpha_i} \mid \text{all } a_i \neq 0 \text{ if } F = \mathbb{C};\ \text{all } a_i > 0 \text{ if } F = \mathbb{R} \right\}. \tag{2.2.2}$$

Remark 2.2.1. If $F = \mathbb{C}$ note that $\tilde{\mathfrak{d}}_1$ is the set of all principal nilpotent elements in $\mathfrak{d}_1$ by Theorem 5.3 in [12].

Now put

$$Z_o = \mathfrak{h} + \tilde{\mathfrak{d}}_1 \tag{2.2.3}$$

so that Z_o is a $2l$-dimensional submanifold of $\mathfrak{b}$. Our primary concern is with the $2l$-dimensional submanifold Z of $\mathfrak{b}_f$ obtained by translating Z_o by f. That is,

$$Z = f + Z_o \subseteq \mathfrak{b}_f. \tag{2.2.4}$$

Let $\bar{B} \subseteq G$ be the subgroup corresponding to $\bar{\mathfrak{b}}$.

Remark 2.2.2. Note that $\bar{B}$ in the notation of Remark 1.6.2 plays the role of A. We shall not be concerned with this at present but it is easy to prove (see Section 6.4) that Z, in the sense of Remark 1.6.2, is an orbit of $\bar{B}$ in $\mathfrak{b}_f$.

For the case of $Sl(n, F)$ note that choices can be made so that Z is the set of all traceless Jacobi matrices of the form

$$y = \begin{pmatrix} b_1 & a_1 & & \cdots & 0 \\ 1 & b_2 & a_2 & & \vdots \\ \vdots & 1 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & a_{n-1} \\ 0 & \cdots & & 1 & b_n \end{pmatrix}, \tag{2.2.5}$$

where $a_i, b_i \in F$ and $a_i \neq 0$ if $F = \mathbb{C}$, $a_i > 0$ if $F = \mathbb{R}$.

The notion of Jacobi matrix generalizes to $\mathfrak{g}$. Let

$$\operatorname{Jac} \mathfrak{g} = \Big\{ x + \sum_{i=1}^{l} (a_{-i} e_{-\alpha_i} + a_i e_{\alpha_i}) \mid x \in \mathfrak{h},\ a_{-i} a_i \neq 0 \text{ if } F = \mathbb{C};\ a_{-i}, a_i > 0 \text{ if } F = \mathbb{R} \Big\}. \tag{2.2.6}$$

Clearly one has

$$Z \subseteq \operatorname{Jac} \mathfrak{g}. \tag{2.2.7}$$

Of course the definition of $\operatorname{Jac} \mathfrak{g}$ depends upon the choice of positive and negative simple root vectors. Also, it appears at first in considering Z that we would be restricting ourselves in any investigation of $\operatorname{Jac} \mathfrak{g}$. This, however, is not the case. We are in fact just factorizing out the trivial action of H on $\operatorname{Jac} \mathfrak{g}$. That is, one has

PROPOSITION 2.2. *The map*

$$H \times Z \to \operatorname{Jac} \mathfrak{g}, \qquad (a, x) \to ax \tag{2.2.8}$$

is bijective.

Proof. It is clear that $\operatorname{Jac} \mathfrak{g}$ is stable under the action of H. On the other hand since G is the adjoint group the map

$$H \to (F^*)^l, \qquad h \to (h^{\alpha_1}, \ldots, h^{\alpha_l}) \tag{2.2.9}$$

is bijective. It follows that h leaves no element of $\operatorname{Jac} \mathfrak{g}$ fixed in case $h \neq 1$. It also follows (replacing α_i by $-\alpha_i$) that every H-orbit in $\operatorname{Jac} \mathfrak{g}$ contains a unique element in Z. This proves the proposition. Q.E.D.

We refer to Z as the manifold of normalized Jacobi elements.

Now if $y \in Z$ the tangent space to Z at y is clearly given by

$$T_y(Z) = \mathscr{h} + \mathscr{d}_1 . \tag{2.2.10}$$

On the other hand for any $y \in \mathscr{b} = \mathscr{a}^*$ we have defined a subspace $\mathscr{a}^*(y)$ now written as $\mathscr{b}(y)$, by (1.6.9).

LEMMA 2.2.1. *For any $y \in Z$ one also has*

$$\mathscr{b}(y) = \mathscr{h} + \mathscr{d}_1 . \tag{2.2.11}$$

Proof. The projection $P_{\mathscr{a}^*}$ of Section 1.6 is just $P_{\mathscr{b}}$ and the kernel of $P_{\mathscr{b}}$ is $\bar{\mathscr{n}}$ since $\mathscr{b}^\perp = (\mathscr{b}^o)^* = \mathscr{n}^* = \bar{\mathscr{n}}$. By definition, $\mathscr{b}(y) = P_{\mathscr{b}}[\bar{\mathscr{b}}, y]$. But since the maximal diagonal degree of y is 1 (see Definition 2.1) one has $\mathscr{b}(y) \subseteq \mathscr{h} + \mathscr{d}_1$ by (2.1.4) and (2.1.5). On the other hand since $[e_{-\alpha_i}, e_{\alpha_j}] = 0$ if $i \neq j$ and the elements $[e_{-\alpha_i}, e_{\alpha_i}]$ are a basis of $\mathscr{h}$ it follows that $P_{\mathscr{b}}[\mathscr{d}_{-1}, y] = \mathscr{d}_0 = \mathscr{h}$. But also since $[\mathscr{d}_0, \mathscr{d}_0] = 0$ one clearly has $P_{\mathscr{b}}[\mathscr{d}_0, y] = \mathscr{d}_1$. Thus $\mathscr{h} + \mathscr{d}_1 \subseteq \mathscr{b}(y)$. This proves the lemma. Q.E.D.

Remark 2.2.3. Recalling Remarks 1.6.1 and 2.2.2 note that Lemma 2.2.1 is anticipated by the fact that Z is an orbit of $\bar{B}$ in $\mathscr{b}$.

Now for any $y \in Z$ note that

$$[y, [\bar{\mathscr{n}}, \bar{\mathscr{n}}]] \subseteq \bar{\mathscr{n}}. \tag{2.2.12}$$

This is clear since the maximal diagonal degree of y is 1 and the maximal diagonal degree of any nonzero element in $[\bar{\mathscr{n}}, \bar{\mathscr{n}}]$ is -2. Now for $y \in Z$ let

$$\mathscr{n}_{(y)} = \{x \in \mathscr{n} \mid Q(x, [y, [\bar{\mathscr{n}}, \bar{\mathscr{n}}]]) = 0\}. \tag{2.2.13}$$

LEMMA 2.2.2. *For any $y \in Z$ one has* $\dim \mathscr{n}_{(y)} = l$. *Furthermore* $\dim(\mathscr{h} + \mathscr{d}_1) \cap [\mathscr{n}, y] = l$ *and one has*

$$[\mathscr{n}_{(y)}, y] = (\mathscr{h} + \mathscr{d}_1) \cap [\mathscr{n}, y]. \tag{2.2.14}$$

Proof. One first has that

$$\mathscr{g}^y \cap \bar{\mathscr{b}} = 0. \tag{2.2.15}$$

Indeed if $e \in \tilde{\mathscr{d}}_1$ is the maximal diagonal component of y then $y \in e + \bar{\mathscr{b}}$. On the other hand no normalization was made with regard to the simple root vector so that we can interchange the roles of positive and negative roots where

$\bar{b}$ replaces b and e replaces f. But then (2.2.15) follows from (2.1.8). But now one knows

$$[\bar{n}, \bar{n}] = \sum_{j \leqslant -2} d_j \tag{2.2.16}$$

so that $[\bar{n}, \bar{n}]$ has codimension l in $\bar{n}$. However, $\operatorname{ad} y$ has no kernel in $[\bar{n}, \bar{n}] \subseteq \bar{b}$ by (2.2.15) and hence $[y, [\bar{n}, \bar{n}]]$ also has codimensional l in $\bar{n}$, recalling (2.2.12). This proves $\dim n_{(y)} = l$. But then by (2.1.8) one also has

$$\dim[n_{(y)}, y] = l. \tag{2.2.17}$$

Again by (2.1.8) any element in $[n, y]$ is uniquely of the form $[x, y]$ for $x \in n$. On the other hand $h + d_1$ by (2.2.16) is clearly the Q-orthocomplement of $[\bar{n}, \bar{n}]$ in b. Thus $[x, y]$ is in the right side of (2.2.14) if and only if $Q([x, y], [\bar{n}, \bar{n}]) = 0$. Therefore by the invariance of Q the element $[x, y]$ is in the right side of (2.2.14) if and only if $x \in n_{(y)}$, using the definition (2.2.13) of $n_{(y)}$. This proves (2.2.14) and the lemma follows from (2.2.17). Q.E.D.

Now recall (see (1.6.5)) that η_I for any invariant $I \in S(g)^G$ is the vector field on b_f such that

$$(\eta_I)_y = [y, (\delta_{\bar{b}} I)(y)]. \tag{2.2.18}$$

By (2.2.10), (2.2.11), and (1.6.10), η_I is tangent to the submanifold Z. Let

$$\xi_I = \eta_I \mid Z \tag{2.2.19}$$

and let z be the space of vector fields on Z defined by

$$z = \{\xi_I \mid I \in S(g)^G\}. \tag{2.2.20}$$

Also for any $y \in Z$, let

$$z_y = \{(\xi_I)_y \mid I \in S(g)^G\}. \tag{2.2.21}$$

THEOREM 2.2. *Let g be any complex or real split semi-simple Lie algebra. Let $l = \operatorname{rank} g$ and let Z be the $2l$-dimensional manifold of normalized Jacobi elements in g. See (2.2.4). For any invariant $I \in S(g)^G$ let ξ_I be the vector field on Z defined by putting*

$$(\xi_I)_y = [y, (\delta_{\bar{b}} I)(y)] \tag{2.2.22}$$

for any $y \in Z$. The element $(\delta_{\bar{b}} I)(y)$ is given by (1.2.10).

Now let $\mathfrak{z}$ be the space spanned by ξ_I for all $I \in S(\mathfrak{g})^G$. Then $\mathfrak{z}$ is a commutative Lie algebra of vector fields. Furthermore for any $y \in Z$ the subspace $\mathfrak{z}_y \subseteq T_y(Z)$ defined by (2.2.21) *is l-dimensional and the elements $(\xi_{I_j})_y$, $j = 1,\ldots, l$, are a basis of $\mathfrak{z}_y$, where the $I_j \in S(\mathfrak{g})^G$ are the fundamental invariants. Moreover if $\mathfrak{n}_{(y)} \subseteq \mathfrak{n}$ is defined by* (2.2.13) *then the elements $(\delta_{\mathfrak{n}} I_j)(y)$, $j = 1,\ldots, l$, are a basis of $\mathfrak{n}_{(y)}$ and*

$$\mathfrak{z}_y = [\mathfrak{n}_{(y)}, y]. \tag{2.2.23}$$

Proof. It follows immediately from Theorem 1.6.2 that $\mathfrak{z}$ is a commutative Lie algebra of vector fields. Now let $y \in Z$. By Proposition 2.1, $(\delta_{\bar{\mathfrak{b}}} I)(y)$ is in the span of $(\delta_{\bar{\mathfrak{b}}} I_j)(y)$, $j = 1,\ldots, l$, for any $I \in S(\mathfrak{g})^G$. Thus the elements $(\delta_{I_j})_y$ span $\mathfrak{z}_y$. But also by Proposition 2.1 the elements $(\delta_{\bar{\mathfrak{b}}} I_j)(y)$ are linearly independent. Furthermore $\mathfrak{g}^y \cap \bar{\mathfrak{b}} = 0$ by (2.2.15). Thus the elements $[y, (\delta_{\bar{\mathfrak{b}}} I_j)(y)] = (\xi_{I_j})_y$ are also linearly independent and hence are a basis of $\mathfrak{z}_y$. But $[y, (\delta I)(y)] = 0$ by (1.3.3). Thus for any $I \in S(\mathfrak{g})^G$

$$\begin{aligned}(\xi_I)_y &= [y, (\delta_{\bar{\mathfrak{b}}} I)(y)] \\ &= [(\delta_{\mathfrak{n}} I)(y), y].\end{aligned} \tag{2.2.24}$$

But $(\xi_I)_y \in T_y(Z) = \mathfrak{h} + \mathfrak{a}_1$. Thus by (2.2.24), $(\xi_I)_y \in (\mathfrak{h} + \mathfrak{a}_1) \cap [\mathfrak{n}, y]$. Thus $[(\delta_{\mathfrak{n}} I)(y), y] \in [\mathfrak{n}_{(y)}, y]$ by (2.2.14). But since $\mathfrak{g}^y \cap \mathfrak{n} = 0$ by (2.1.8) this implies $(\delta_{\mathfrak{n}} I)(y) \in \mathfrak{n}_{(y)}$. However, by (2.2.24) the elements $(\delta_{\mathfrak{n}} I_j)(y)$ must be linearly independent. But then they are a basis of $\mathfrak{n}_{(y)}$ by Lemma 2.2.2. Furthermore (2.2.24) then immediately implies (2.2.23). Q.E.D.

2.3. Now the fundamental invariants I_j define a differentiable map

$$\mathscr{I}: \mathfrak{g} \to F^l, \tag{2.3.1}$$

where $\mathscr{I}(x) = (I_1(x),\ldots, I_l(x))$.

For each γ in the image, $\mathscr{I}(Z)$, of Z under $\mathscr{I}$ one defines a nonempty closed subset $Z(\gamma)$ of Z by putting

$$Z(\gamma) = \mathscr{I}^{-1}(\gamma) \cap Z. \tag{2.3.2}$$

One of course then has the disjoint union

$$Z = \bigcup_{\gamma \in \mathscr{I}(Z)} Z(\gamma). \tag{2.3.3}$$

Remark 2.3. For the case of $Sl(n, F)$ note that the equivalence relation in Z defined by regarding (2.3.3) as a union of equivalence classes is the relation

$x \sim y$ when x and y are normalized Jacobi matrices with the same characteristic polynomial.

Consider the functions $I_i \mid Z$ on Z defined by restricting the fundamental invariants to Z. The following guarantees that the $Z(y)$ are submanifolds, although at this point it is not clear whether or not they are connected.

PROPOSITION 2.3.1. *For any $y \in Z$ the differentials $(d(I_i \mid Z))_y$, $i = 1,\ldots, l$, are linearly independent.*

Proof. Now as in (2.2.10) the tangent space to Z at y is identified with $\mathfrak{h} + \mathfrak{a}_1$. With respect to this identification if $w \in \mathfrak{h} + \mathfrak{a}_1$ then by (1.1.2) and definition (1.2.2) one has

$$\langle w, (d(I \mid Z))_y \rangle = Q(w, (\delta I)(y)). \tag{2.3.4}$$

Thus $(d(I \mid Z))_y = 0$ if and only if $(\delta I)(y)$ is Q-orthogonal to $\mathfrak{h} + \mathfrak{a}_1$. But $\mathfrak{h} + \mathfrak{a}_1 = \mathfrak{b}(y)$ by (2.2.11). Thus $(d(I \mid Z))_y = 0$ if and only if $(\xi_I)_y = 0$ by (1.6.12). However, the vectors $(\xi_{I_i})_y$ are linearly independent by Theorem 2.2. Thus the differentials $(d(I_i \mid Z))_y$ are then also linearly independent. Q.E.D.

Let $\bar{N}, N \subseteq G$ be the subgroups respectively corresponding to $\bar{\mathfrak{n}}$ and $\mathfrak{n}$. We return briefly to $\mathfrak{b}_f = f + \mathfrak{b}$.

PROPOSITION 2.3.2. *Let $y_1, y_2 \in \mathfrak{b}_f$. Then there exists $n \in N$ such that $ny_1 = y_2$ if and only if $\mathscr{I}(y_1) = \mathscr{I}(y_2)$. Furthermore in such a case n is unique.*

Proof. If such an element $n \in N$ exists then by invariance one has $\mathscr{I}(y_1) = \mathscr{I}(y_2)$. Conversely assume $\mathscr{I}(y_1) = \mathscr{I}(y_2)$. We must show there exists a unique $n \in N$ such that $ny_1 = y_2$. Assume first that $F = \mathbb{C}$.

Let $x_o \in \mathfrak{h}$ be as in Section 2.1. It is clear from (2.1.4) that $[f, \mathfrak{n}] \subseteq \mathfrak{b}$ and $[f, \mathfrak{n}]$ is stable under ad x_o. Let $\mathfrak{s} \subseteq \mathfrak{b}$ be any ad x_o stable subspace such that

$$\mathfrak{b} = [f, \mathfrak{n}] + \mathfrak{s}$$

is a direct sum. Now by Theorem 1.2 in [17] there exist elements $n_i \in N$, $x_i \in f + \mathfrak{s}$ such that $n_i x_i = y_i$, $i = 1, 2$. Clearly $\mathscr{I}(x_i) = \mathscr{I}(y_i)$. However, since $\mathscr{I}(y_1) = \mathscr{I}(y_2)$ one then has $\mathscr{I}(x_1) = \mathscr{I}(x_2)$. But then $x_1 = x_2$ by Theorem 7 in [14, p. 381]. (See also Remark 19′ in [14, p. 375]. This proves there exists $n \in N$ such that $ny_1 = y_2$. The uniqueness of n follows from Theorem 1.2 in [17].

Now assume $F = \mathbb{R}$. Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g}$ and let $G_{\mathbb{C}}$ be the adjoint group of $\mathfrak{g}_{\mathbb{C}}$. Let $\mathfrak{n}_{\mathbb{C}} = \mathfrak{n} + i\mathfrak{n}$ and let $N_{\mathbb{C}} \subseteq G_{\mathbb{C}}$ be the subgroup corresponding to $\mathfrak{n}_{\mathbb{C}}$. By the result above there exists a unique $n \in N_{\mathbb{C}}$ such that $ny_1 = y_2$. Now regarding $N \subseteq N_{\mathbb{C}}$ we have only to show that $n \in N$. Write $n = \exp z$, where $z \in n_{\mathbb{C}}$, and write $z = u + iv$, where $u, v \in n$. It is enough to show that $v = 0$. Assume not and let j and v_j be respectively the minimal diagonal degree of v and the minimal diagonal component of v. Then $j \geqslant 1$ since $v \in n$ and hence

$$[v_j, f] \in d_{j-1} \subseteq b. \tag{2.3.5}$$

But now

$$\left(\sum_m \frac{(\mathrm{ad}(u + iv)^m)}{m!}\right)(f + x_1) = y_2, \tag{2.3.6}$$

where we have written $y_1 = f + x_1$ so that $x_1 \in b$.

Now the minimal diagonal degree of any nonzero element in $[v, b]$ is at least j and any further bracketing with u or v would only increase the degree. Thus upon writing $g_{\mathbb{C}} = \sum d_k + i \sum d_k$ as a real direct sum it follows that the component of the left side of (2.3.6) in id_{j-1} is just $i[v_j, f]$. But $[v_j, f] \neq 0$ by (2.1.8). This contradicts the fact that $y_2 \in g$. Thus $v = 0$. Q.E.D.

2.4. Here and in Sections 2.5 and 2.6 we assume that $F = \mathbb{C}$. Now the Weyl group $W = W(g, h)$ operates on h and on its dual h'. For each $\sigma \in W$ let $s(\sigma) \in G$ be the unique element in the normalizer of h, corresponding to σ, such that

$$s(\sigma)\, e_{\alpha_i} = e_{\sigma\alpha_i}, \qquad i = 1,\ldots, l. \tag{2.4.1}$$

The Bruhat–Gelfand decomposition of G asserts that

$$G = \bigcup_{\sigma \in W} Ns(\sigma)\, HN \tag{2.4.2}$$

is a disjoint union. Now let $\kappa \in W$ be the unique element such that $\kappa\Delta_+ = -\Delta_+$. Then $s(\kappa)\, Ns(\kappa)^{-1} = \bar{N}$. Multiplying the components of (2.4.2) on the left by $s(\kappa)$ one also has the disjoint union

$$G = \bigcup_{\tau \in W} \bar{N}s(\tau)\, HN. \tag{2.4.3}$$

Now if τ is the identity then $\bar{N}s(\tau)\, HN$ is just $\bar{N}HN$ Now put

$$G_* = \bar{N}HN. \tag{2.4.4}$$

One knows that G_* is a Zariski open subset of G and the map

$$\bar{N} \times H \times N \to G_*, \tag{2.4.5}$$

where $(\bar{n}, h, n) \to \bar{n}hn$ is an isomorphism of algebraic varieties. Thus for any $d \in G_*$ there exist unique elements $\bar{n}_d \in \bar{N}$, $h_d \in H$, and $n_d \in N$ such that

$$d = \bar{n}_d h_d n_d. \tag{2.4.6}$$

We will retain this notation throughout. One notes in particular that by inverting (2.4.5) the map

$$G_* \to N, \qquad d \mapsto n_d \tag{2.4.7}$$

is a morphism of algebraic varieties.

Now for any $y \in G$ we recall that G^y is the centralizer of y in G. Thus G^y is a Zariski closed algebraic subgroup of G. In case y is regular, i.e., $y \in R$, this group is of a particularly simple form.

PROPOSITION 2.4. *If $y \in R$ then G^y is an Abelian connected algebraic group of complex dimension l. That is, as algebraic groups one has the isomorphism*

$$G^y \cong (\mathbb{C}^*)^p \times \mathbb{C}^q \tag{2.4.8}$$

for some p, q where $p + q = l$.

Proof. The first statement is contained in Proposition 14 in [14, p. 362]. The second statement follows then from the well-known classification of connected Abelian complex algebraic groups. Q.E.D.

Now let $y \in Z$. Then $y \in R$ so that Proposition 2.4 applies to y. Let

$$G_*^y = G^y \cap G_* \tag{2.4.9}$$

so that G_*^y is a Zariski open subset of G^y. Since $1 \in G_*^y$ it follows that G_*^y is not empty and hence is dense in G^y. Furthermore since G^y is connected and nonsingular it follows that G_*^y is a connected, nonsingular algebraic variety of dimension l. We recall that a variety of dim l is called rational if its function field is isomorphic to $\mathbb{C}(X_1, \ldots, X_l)$, where the X_i are indeterminates. Since G^y is an algebraic group it is rational. (This is of course obvious from (2.4.8).) Since G_*^y is Zariski dense in G^y it has the same function field so that it too is rational.

Now for any $d \in G_*^y$ one has the decomposition (2.4.6). That is, $d = n_d h_d \bar{n}_d$ and by restriction (2.4.7) the map

$$G_*^y \to N, \qquad d \to n_d \tag{2.4.10}$$

is a morphism of algebraic varieties.

Now if γ is the image $\mathscr{I}(Z)$ of the map $\mathscr{I}$ it is clear that the "isospectral" set $Z(\gamma)$ is a closed subvariety of the variety Z of normalized Jacobi elements. However, it is not clear just what variety it is and in fact whether or not it is connected. These questions are settled by

THEOREM 2.4. *Let $\mathfrak{g}$ be a complex semi-simple Lie algebra and let Z be the* *set of normalized Jacobi elements defined by* (2.2.4). *Let* $y \in Z$, $\gamma = \mathscr{J}(y)$, *and* $Z(\gamma) \subseteq Z$ *be defined by* (2.3.2) (*so that* $Z(\gamma)$ *is the set of all elements in* Z *which are conjugate to* y).

Now let G *be the adjoint group of* $\mathfrak{g}$. *Let* G^y *be the centralizer of* y *in* G *and let* G_*^y *be the Zariski open subset of* G^y *defined by* (2.4.9). *Then for any* $d \in G_*^y$ *one has* $n_d y \in Z(\gamma)$. (*See* (2.4.6).) *Furthermore if* $\beta_y(d) = n_d y$ *then*

$$\beta_y\colon G_*^y \to Z(\gamma) \tag{2.4.11}$$

is an isomorphism of algebraic varieties. In particular $Z(\gamma)$ *is an irreducible, nonsingular, rational algebraic variety of dimension* l *where* $l = \operatorname{rank}\mathfrak{g}$ (*so that* $Z(\gamma)$ *is a connected complex manifold of dimension* l.

Proof. Now since $y \in Z \subseteq \mathscr{E}_f$ and $\mathscr{E}_f$ is stable under the action of N it is clear that $\beta_y(d) \in \mathscr{E}_f$. Let $w = \beta_y(d)$ and let j and w_j be respectively the maximal diagonal degree and component of w. See Definition 2.1. To prove that $w \in Z$ it suffices then to show that $j = 1$ and $w_j \in \tilde{\mathscr{A}}_1$. Now clearly $n_d = h^{-1}(\bar{n})^{-1}d$, where we have written $\bar{n} = \bar{n}_d$, $h = h_d$. Thus $w = h^{-1}(\bar{n})^{-1}\,dy = h^{-1}(\bar{n})^{-1}y$ since $dy = y$. That is,

$$\beta_y(d) = h_d^{-1}(\bar{n}_d)^{-1}\,y. \tag{2.4.12}$$

But if y_1 is the maximal diagonal component of y then $y_1 \in \tilde{\mathscr{A}}_1$. However, clearly the maximal diagonal component is unchanged by the action of any element in $\bar{N}$. Thus y_1 is also the maximal diagonal component of $(\bar{n})^{-1}y$. But now obviously $h^{-1}y_1$ is the maximal diagonal component of $h^{-1}(\bar{n})^{-1}y = w$. That is, there exists $x \in \mathscr{h}$ so that

$$\beta_y(d) = f + x + h_d^{-1}y_1\,. \tag{2.4.13}$$

Since $\tilde{\mathscr{A}}_1$ is stable under H this proves that $\beta_y(d) \in Z$. However, then certainly $\beta_y(d) = n_d y \in Z(\gamma)$ by the invariance of $\mathscr{J}$. This defines the map (2.4.11). Assume now that $\beta_y(d_1) = \beta_y(d_2)$, where $d_1, d_2 \in G_*^y$. That is, $n_1 y = n_2 y$, where $n_i = n_{d_i}$, $i = 1, 2$. But then by the uniqueness in Proposition 2.3.2 one has $n_1 = n_2$. But then $h_1^{-1}y_1 = h_2^{-1}y_1$ by (2.4.13), where $h_i = h_{d_i}$. Since H operates in a simple transitive way on $\tilde{\mathscr{A}}_1$ it follows that $h_1 = h_2$. Finally if $\bar{n}_i = \bar{n}_{d_i}$ one has $(\bar{n}_1)^{-1}y = (\bar{n}_2)^{-1}y$ by (2.4.12). But then by Proposition 2.3.2 with the roles of $\mathscr{E}$ and $\bar{\mathscr{E}}$, N and $\bar{N}$, f and y_1 reversed, one has $\bar{n}_1 = \bar{n}_2$ by uniqueness since $y \in y_1 + \bar{\mathscr{E}}$. This proves $d_1 = d_2$ so that β_y is injective.

Now let $w \in Z(\gamma)$. Since $w, y \in \mathscr{E}_f$ this implies there exists $n \in N$ by Proposition 2.3.2 such that

$$ny = w. \tag{2.4.14}$$

Now let y_1 and w_1 be respectively the maximal diagonal components of y and w.

Then $y_1, w_1 \in \tilde{d}_1$ and hence by the transitivity of H on $\tilde{d}_1$ there exists $h \in H$ such that $hw_1 = y_1$. Thus $hw, y \in y_1 + \ell$ and $\mathscr{I}(hw) = \mathscr{I}(y) = \gamma$ by invariance of $\mathscr{I}$ under conjugation. But then by Proposition 2.3.2 with the roles reversed as above (in the proof of injectivity) there exists $\bar{n} \in N$ such that

$$\bar{n}hw = y. \tag{2.4.15}$$

Substituting for w using (2.4.14) one has $\bar{n}hny = y$. Thus $\bar{n}hn = d \in G^y$ and $n = n_d$. Thus $w = \beta_y(d)$ by (2.4.14). This proves the surjectivity of β_y and hence β_y is bijective. On the other hand β_y is clearly a morphism of algebraic varieties since (2.4.10), as noted, is such a morphism. Thus $Z(\gamma)$ is an irreducible variety. But, by Proposition 2.3.1, $Z(\gamma)$ is certainly nonsingular, using, e.g., Zariski's criterion. Thus $Z(\gamma)$ is a nonsingular irreducible variety. In particular it is normal. But then since β_y is bijective it is an isomorphism of algebraic varieties by the Zariski main theorem. Q.E.D.

Remark 2.4.1. We thank D. Mumford and D. Kazhdan for pointing out that our isomorphism (2.4.11) implies the rationality of $Z(\gamma)$. For the special case of $Sl(n)$ and where y has distinct eigenvalues this result is due to J. Moser. The rationality in that case was established by Moser, using continued fractions.

Remark 2.4.2. Since $Z(\gamma)$ is a nonsingular variety it also has the structure of a smooth manifold. Moreover as such it is a submanifold of Z. Furthermore since β_y is an isomorphism of algebraic varieties one knows then that β_y is also a diffeomorphism of smooth manifolds. See, e.g., Chapter VII, Section 1, in [22].

2.5. As in Section 2.4 we assume that $F = \mathbb{C}$. Now one can easily determine the image $\mathscr{I}(Z)$ of Z under the map $\mathscr{I}$. See (2.3.1). Let $e \in d_1$ and put $g = f + e$. Let

$$\mathscr{I}_g : \mathfrak{h} \to \mathbb{C}^l \tag{2.5.1}$$

be the map defined by putting $\mathscr{I}_g(x) = \mathscr{I}(g + x)$ for $x \in \mathfrak{h}$. Recall that W is the Weyl group $W(\mathfrak{g}, \mathfrak{h})$. For any finite set S let $| S |$ denote its cardinality.

PROPOSITION 2.5.1. *Let $\gamma \in \mathbb{C}^l$ be arbitrary. Then $\mathscr{I}_g^{-1}(\gamma)$ is finite and in fact*

$$1 \leqslant | \mathscr{I}_g^{-1}(\gamma) | \leqslant | W |. \tag{2.5.2}$$

In particular the map $\mathscr{I}_g$ is surjective.

Proof. Let $J_i \in S(\mathfrak{h})$ be defined by putting $J_i = I_i \mid \mathfrak{h}$. (We are of course identifying $S(\mathfrak{h})$ with its dual using $Q \mid \mathfrak{h}$.) If $\deg I_i = m_i + 1$ then of course J_i is also homogeneous of degree $m_i + 1$. On the other hand by Chevalley the J_i are algebraically independent and the algebra $S(\mathscr{h})^W$ of Weyl group invariants is exactly the polynomial algebra, $\mathbb{C}[J_1,..., J_l]$, generated by the J_i. Furthermore it is also due to Chevalley that $S(\mathscr{h})$ is a free $S(\mathscr{h})^W$ with $|W|$ generators. (See, e.g., Theorem 4.15.28 in [26, p. 386].) That is, there exists a graded subspace $D \subseteq S(\mathscr{h})$ of dimension $|W|$ such that $S(\mathscr{h}) = S(\mathscr{h})^W \otimes D$. But then since D is graded it is easy to see that

$$S(\mathscr{h}) = A_g \otimes D, \tag{2.5.3}$$

where if $J_i^g \in S(\mathscr{h})$ is defined by $J_i^g(x) = J_i(g + x)$ for $x \in \mathscr{h}$ then $A = \mathbb{C}[J_1^g,..., J_l^g]$. One needs only the obvious fact that $J_i - J_i^g$ is a polynomial of degree at most m_i.

Thus $S(\mathscr{h})$ is a free A_g module with $|W|$ generators. In particular $S(\mathscr{h})$ is integral over A_g. On the other hand A_g is just the pullback of the affine algebra on $\mathbb{C}^l$ with respect to the map $\mathscr{I}_g$. Thus $\mathscr{I}_g$ is a finite map. See, e.g., [22, p. 48]. But then $\mathscr{I}_g$ is surjective (see, e.g., Theorem 4 in [22, p. 48]) so that $1 \leqslant |\mathscr{I}_g^{-1}(\gamma)|$ for any $\gamma \in \mathbb{C}^l$. On the other hand if (A) denotes the quotient field of an integral domain A then (2.5.3) easily implies $(S(\mathscr{h})) = (A_g) \otimes D$ so that $|W|$ is the degree of the map $\mathscr{I}_g$. See, e.g., [22, p. 116]. But then one knows $|\mathscr{I}_g^{-1}(\gamma)| \leqslant W$ for any $\gamma \in \mathbb{C}^l$. See, e.g., Theorem 6 in [22, p. 116]. Q.E.D.

The set of unramified points is of course nonempty and Zariski open. That is,

PROPOSITION 2.5.2. *Let the notation be as in Proposition* 2.5.1. *Then the set*

$$\mathbb{C}_g^{\,l} = \{\gamma \in \mathbb{C}^l \mid |\mathscr{I}_g^{-1}(\gamma)| = |W|\} \tag{2.5.4}$$

is a nonempty open subset of $\mathbb{C}^l$.

Proof. Since $|W|$ is the degree of the map $\mathscr{I}_g$ as noted in the proof of Proposition 2.5.1 the result is just Theorem 7 in [22, p. 117]. Q.E.D.

Summarizing one has

THEOREM 2.5. *Let* $\mathfrak{g}$ *be a complex semi-simple Lie algebra. Let* $l = \operatorname{rank} \mathfrak{g}$ *and let* Z *be the* $2l$*-dimensional variety of normalized Jacobi elements in* $\mathfrak{g}$ *defined by* (2.2.4). *For any* $\gamma \in \mathbb{C}^l$ *let* $Z(\gamma) \subseteq Z$ *be the subvariety defined by* (2.3.2) *in terms of the fundamental invariants* I_j, $j = 1,..., l$. *Then* $Z(\gamma)$ *is an* l*-dimensional nonsingular, irreducible closed rational subvariety of* Z *and hence*

$$Z = \bigcup_{\gamma \in \mathbb{C}^l} Z(\gamma) \tag{2.5.5}$$

is a foliation of Z *by such subvarieties.*

Proof. Recall the notation and statement of Proposition 2.5.1. Since the map $\mathscr{I}_g$ is surjective it of course follows that $\mathscr{I}$ is surjective. That is,

$$\mathscr{I}(Z) = \mathbb{C}^l. \tag{2.5.6}$$

The result then follows from Theorem 2.4 and (2.3.3). Q.E.D.

Henceforth we will speak of the sets $Z(\gamma)$ as the isospectral leaves of Z.

Remark 2.5. Although the elements of Z are regular one notes that by (2.5.6) the elements of Z are not in general semi-simple. In fact since two regular elements $x, y \in \mathfrak{g}$ are G-conjugate if and only if $\mathscr{I}(x) = \mathscr{I}(y)$ (see (3.8.4) in [14]) it follows from (2.5.6) that every regular element is G-conjugate to a normalized Jacobi element. In particular taking $\gamma = 0$ in $\mathbb{C}^l$ the isospectral leaf $Z(0)$ is the set of all nilpotent elements in Z. Since $Z \subseteq R$ any element in $Z(0)$ is in fact a principal nilpotent element.

Here again as in Sections 2.4 and 2.5 we assume $F = \mathbb{C}$.

2.6. Now if $\gamma \in \mathbb{C}^l$ then the isospectral leaf $Z(\gamma)$ of Z has been described by Theorem 2.4 in terms of a Zariski open subset of the centralizer G^y of an element $y \in Z$. However, the centralizer G^y is itself fairly complicated. We wish now to describe $Z(\gamma)$ in terms of the centralizer G^w of an element w which is more readily accessible.

Now let $w_0 \in \mathfrak{h}$ and put $w = f + w_0$. Thus $w \in \mathfrak{b}_f$ and hence $w \in R$ by (2.1.7). Thus G^w by Proposition 2.4 is a connected Abelian subgroup of complex dimension l. Now since $G_* = \bar{N}HN$ is Zariski open in G the same is true for its translate $s(\kappa)\, G_*$. Thus if we put

$$G_{(*)} = s(\kappa)\, G_* \tag{2.6.1}$$

then $G_{(*)}$ is a Zariski open subset of G and for any $g \in G_{(*)}$ there exist unique elements $\bar{n}(g) \in \bar{N}$, $h(g) \in H$, and $n(g) \in N$ such that

$$g = s(\kappa)\, \bar{n}(g)\, h(g)\, n(g). \tag{2.6.2}$$

This notation as with (2.4.6) will be retained throughout. By inverting (2.4.5) it is clear that the map

$$G_{(*)} \to N, \qquad g \to n(g) \tag{2.6.3}$$

is a morphism of algebraic varieties

$$G^w_{(*)} = G^w \cap G_{(*)} \tag{2.6.4}$$

so that $G^w_{(*)}$ is Zariski open in G^w.

Remark 2.6.1. Unlike the case of G_*^y for $y \in Z$ it is not immediately obvious (at least to us) whether $G_{(*)}^w$ is empty or not. It is in fact nonempty as stated in Theorem 2.6 below.

Now for any $g \in G_{(*)}^w$ one has the decomposition (2.6.2). That is, $s(\kappa)^{-1}g = \bar{n}(g)\, h(g)\, n(g)$. Furthermore by restricting (2.6.3) the map

$$G_{(*)}^w \to N, \qquad g \to n(g) \tag{2.6.5}$$

is a morphism of algebraic varieties.

LEMMA 2.6.1. *Let $\gamma = \mathscr{I}(w)$ and let $g \in G_{(*)}^w$. Then $n(g)w \in Z(\gamma)$.*

Proof. Let $y = n(g)w$. Since $\mathscr{E}_f$ is stable under the action of N it is clear that $y \in \mathscr{E}_f$. Let j and y_j be respectively the maximal diagonal degree and component of y. We have to show that $j = 1$ and $y_1 \in \tilde{\mathscr{A}}_1$. But the maximal diagonal component of $\bar{n}(g)\, h(g) y$ is clearly just $h(g)\, y_j$. However, $\bar{n}(g)\, h(g) y = \bar{n}(g)\, h(g)\, n(g) w = s(\kappa)^{-1}\, gw$. But $gw = w$ and $s(\kappa)^{-1} w = \kappa^{-1} w_o + e$, where, by (2.4.1) and (1.5.4), $e = \sum_{i=1}^{l} e_{\alpha_i}$. Thus $h(g)\, y_j = e$. Hence $j = 1$ and $y_1 = h(g)^{-1} e$. That is,

$$y_1 = \sum_{i=1}^{l} h(g)^{-\alpha_i}\, e_{\alpha_i}. \tag{2.6.6}$$

Note also that one has

$$\bar{n}(g)\, h(g) y = \kappa^{-1} w_o + e. \tag{2.6.7}$$

But now (2.6.6) proves $y_1 \in \tilde{\mathscr{A}}_1$ and hence $y \in Z$. Also, since y and w are G-conjugate one has $\mathscr{I}(y) = \mathscr{I}(w) = \gamma$ so that $y \in Z(\gamma)$. Q.E.D.

Now let the notation be as in Lemma 2.6.1. Put $y = n(g)w$ so that $y \in Z(\gamma)$. Since $n(g)$ carries w to y one clearly has

$$n(g)\, G^w n(g)^{-1} = G^y. \tag{2.6.8}$$

But now for any $a \in G^w$ one has $g^{-1}a \in G^w$ and hence

$$\psi_g \colon G^w \to G^y \tag{2.6.9}$$

is an isomorphism of varieties (not of groups), where

$$\psi_g(a) = n(g)\, g^{-1} a n(g)^{-1}. \tag{2.6.10}$$

LEMMA 2.6.2. *Recalling that* $G^v_* = G^v \cap \bar{N}HN$ *and* $G^w_{(*)} = G^w \cap s(\kappa)\,\bar{N}HN$ *one has*

$$\psi_g(G^w_{(*)}) = G^v_* . \tag{2.6.11}$$

Proof. Let $a \in G^w$ and put $c = g^{-1}a$ so that $s(\kappa)^{-1}a = s(\kappa)^{-1}gc = \bar{n}hnc$, where we have put $\bar{n} = \bar{n}(g)$, $h = h(g)$, and $n = n(g)$. But $nc = ncn^{-1}n = \psi_g(a)n$. Thus if we put $d = \psi_g(a)$ then $d \in G^v$ by (2.6.9) and

$$s(\kappa)^{-1}a = \bar{n}hdn. \tag{2.6.12}$$

But now by the decomposition (2.4.3) there exists a unique $\sigma \in W$ such that $d \in \bar{N}s(\sigma)\,HN$. Since $s(\sigma)$ normalizes H this implies $s(\kappa)^{-1}a \in \bar{N}s(\sigma)\,HN$. Thus $d = \psi_g(a) \in G^v_*$ if and only if $s(\kappa)^{-1}a \in \bar{N}HN$ or if and only if $a \in G^w_{(*)}$. This proves the lemma. Q.E.D.

Remark 2.6.2. Let the notation be as in the proof above. If $a \in G^w_{(*)}$ so that $d \in \bar{N}HN$, note that (2.6.12) implies $n(a) = n_d n$ using the notation of (2.4.6). That is, for any g, $a \in G^w_{(*)}$ one has

$$n(a) = n_d n(g) \tag{2.6.13}$$

where $d = \psi_g(a)$.

The element $w = f + w_o$ in our considerations here can be chosen so that $\mathscr{I}(w) = \gamma$ where γ is any given element in $\mathbb{C}^l$. That is, one has

PROPOSITION 2.6. *Given any* $\gamma \in \mathbb{C}^l$ *there exists a* $w_o \in \mathfrak{h}$, *unique up to conjugacy by* W *such that* $\mathscr{I}(w_o) = \gamma$. *Furthermore if* $w = f + w_o$ *then one also has*

$$\mathscr{I}(w) = \mathscr{I}(w_o) \tag{2.6.14}$$

so that $\mathscr{I}(w) = \gamma$.

Proof. The first statement is a well-known result of Chevalley. It follows, for example, from the surjectivity part of the proof of Proposition 10 in [14, p. 355] together with Lemma 9.2 in [12]. The second statement is part of Lemma 11 in [14, p. 371]. Q.E.D.

THEOREM 2.6. *Let* $\mathfrak{g}$ *be a complex semi-simple Lie algebra and let* Z *be the set of normalized Jacobi elements in* $\mathfrak{g}$. *This set is defined by* (2.2.4). *Let* $\gamma \in \mathbb{C}^l$ *and let* $Z(\gamma) \subseteq Z$ *be the isospectral leaf in* Z *defined by* γ. *See* (2.3.2). *Let* w_o *be any element in the Cartan subalgebra* $\mathfrak{h}$ *such that* $\mathscr{I}(w_o) = \gamma$. *Such an element* w_o *exists. Let* $w = f + w_o$, *where we recall* f *is defined by* (1.5.4). *Let* $G^w_{(*)}$ *be the Zariski open subset of the centralizer of* w *defined by* (2.6.4). *Then* $G^w_{(*)}$ *is not empty so that it is a complex, connected l-dimensional, nonsingular rational variety.*

Furthermore $Z(\gamma)$ is also such a variety. Moreover if $a \in G^w_{(*)}$ and $n(a) \in N$ is defined by (2.6.2) then $n(a)w \in Z(\gamma)$ and if $\beta_{(w)}(a) = n(a)w$ then

$$\beta_{(w)}\colon G^w_{(*)} \to Z(\gamma) \tag{2.6.15}$$

is an isomorphism of algebraic varieties.

Proof. By Theorem 2.5, $Z(\gamma)$ is not empty and by Proposition 2.6 there exists an element $w_o \in \mathscr{h}$ such that $\mathscr{I}(w_o) = \gamma$.

Now let $y \in Z(\gamma)$. Then $y, w \in \mathscr{b}_f$ and $\mathscr{I}(y) = \mathscr{I}(w) = \gamma$ by (2.6.14). But then by Proposition 2.3.2 there exists $n \in N$ such that

$$nw = y. \tag{2.6.16}$$

But now if $y_1 \in \tilde{\mathscr{d}}_1$ is the maximal diagonal component of y there exists $h \in H$ such that

$$e = hy_1, \tag{2.6.17}$$

where

$$e = \sum_{i=1}^{l} e_{\alpha_i}. \tag{2.6.18}$$

Thus $hy \in e + \bar{\mathscr{b}}$. On the other hand $s(\kappa)^{-1}w = \kappa^{-1}w_o + e$ since it is clear from (2.4.1) that

$$s(\kappa)e = f. \tag{2.6.19}$$

Hence hy and $s(\kappa)^{-1}w$ are elements in $e + \bar{\mathscr{b}}$ and $\mathscr{I}(hy) = \mathscr{I}(s(\kappa)^{-1}w) = \gamma$. Thus, as argued in the proof of Theorem 2.4, Proposition 2.3.2 applies with the roles of $\mathscr{b}$ and $\bar{\mathscr{b}}$, N and $\bar{N}$, and now f and e reversed. That is, there exists $\bar{n} \in \bar{N}$ such that $\bar{n}hy = s(\kappa)^{-1}w$ or $s(\kappa)\,\bar{n}hy = w$. But since $nw = y$ this implies that $g \in G^w$, where $g = s(\kappa)\,\bar{n}hn$. Hence $g \in G^w_{(*)}$ so that $G^w_{(*)}$ is not empty. Moreover note that $n = n(g)$. But now by Lemma 2.6.1 the map $\beta_{(w)}$ exists and is a morphism of algebraic varieties since (2.6.5) is such a morphism. Furthermore by (2.6.16) one has

$$\beta_{(w)}(g) = n(g)y = w. \tag{2.6.20}$$

Now recall (see (2.4.11)) the map $\beta_y\colon G^y_* \to Z(\gamma)$, where if $d \in G^y_*$ then $\beta_y(d) = n_d y$, and the map $G^w_{(*)} \to G^y_*$, defined by restricting ψ_g to $G^w_{(*)}$. See (2.6.11), where we recall $\psi_g(a) = n(g)\,g^{-1}an(g)^{-1}$. We assert the diagram

$$\begin{tikzcd} G^w_{(*)} \arrow[r, "\beta_{(w)}"] \arrow[d, "\psi_g"'] & Z(\gamma) \\ G^y_* \arrow[ur, "\beta_y"'] & \end{tikzcd} \tag{2.6.21}$$

is commutative. Indeed if $a \in G^w_{(*)}$ and $d = \psi_g(a)$ then $\beta_y(\psi_g(a)) = \beta_y(d) = n_d y$. However, $y = n(g)w$ by (2.6.16). But $n_d n(g) = n(a)$ by (2.6.13). Thus $\beta_y(\psi_g(a)) = n(a)w = \beta_{(w)}(a)$. This proves that diagram (2.6.21) is commutative. But β_y is an isomorphism by Theorem 2.4. Also since ψ_g is an isomorphism the vertical map in (2.6.11) is an isomorphism by (2.6.11). Thus $\beta_{(w)}$ is an isomorphism of algebraic varieties. By Proposition 2.4, $G^w_{(*)}$ and hence $Z(\gamma)$ is isomorphic to a nonempty Zariski open subset of $(\mathbb{C}^*)^p \times \mathbb{C}^q$ for some p, q where $p + q = l$ and hence has the properties described in the statement of the theorem. Q.E.D.

3. The Parametrization $Z \cong H \times \mathfrak{h}_+$ in the Real Case

3.1. Throughout Section 3 we assume that $F = \mathbb{R}$. We will see that the situation is more complicated but perhaps more interesting. In order to make use of the results of Sections 2.4–2.6 we put $\mathfrak{g}_\mathbb{C} = \mathfrak{g} + i\mathfrak{g}$ and let $G_\mathbb{C}$ be the complex adjoint group. Of course we may assume that $G \subseteq G_\mathbb{C}$. Also let $\mathfrak{n}_\mathbb{C}$, $\mathfrak{h}_\mathbb{C}$, and $\bar{\mathfrak{n}}_\mathbb{C}$ be the complexifications of $\mathfrak{n}$, $\mathfrak{h}$, and $\bar{\mathfrak{n}}$ and let $N_\mathbb{C}$, $H_\mathbb{C}$, and $\bar{N}_\mathbb{C}$ be the corresponding subgroups of $G_\mathbb{C}$.

Now let $K \subseteq G$ be the subgroup corresponding to $\mathfrak{k}$ where we recall $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the given Cartan decomposition of $\mathfrak{g}$. See Section 1.2. Thus K is a maximal compact subgroup. Put $P = \exp \mathfrak{p}$ so that as one knows, any $g \in G$ can be uniquely written

$$g = kp, \tag{3.1.1}$$

where $k \in K$, $p \in P$. One refers to (3.1.1) as the polar decomposition of g.

One also knows that the map

$$\mathfrak{p} \to P, \qquad x \mapsto \exp x \tag{3.1.2}$$

is a diffeomorphism. This clearly implies the familiar fact that the map

$$P \to P, \qquad p \to p^2 \tag{3.1.3}$$

is also a diffeomorphism. For any $p \in P$ its unique square root in P will be denoted by $p^{1/2}$.

Now the real form $\mathfrak{g}$ of $\mathfrak{g}_\mathbb{C}$ defines a conjugation operation on $\mathfrak{g}_\mathbb{C}$ and $G_\mathbb{C}$. That is, if $z \in \mathfrak{g}_\mathbb{C}$ and we write $z = x + iy$ for $x, y \in \mathfrak{g}$ then $z^c \in \mathfrak{g}_\mathbb{C}$ is defined by putting

$$z^c = x - iy. \tag{3.1.4}$$

Since $G_{\mathbb{C}}$ is the adjoint group the operation induces an automorphism $g \to g^c$ of $G_{\mathbb{C}}$, where of course for any $x \in \mathfrak{g}_{\mathbb{C}}$

$$(\exp x)^c = \exp x^c. \tag{3.1.5}$$

Now the Cartan involution θ (see Section 1.2) extends by linearity to an automorphism of $\mathfrak{g}_{\mathbb{C}}$ and, since $G_{\mathbb{C}}$ is the adjoint group, it also defines an automorphism of $G_{\mathbb{C}}$ which we continue to denote by θ. Now recall that the $*$-operation on $\mathfrak{g}$ was defined by putting $x^* = \theta(-x)$. We now extend the operation to $\mathfrak{g}_{\mathbb{C}}$ as a conjugate linear map by putting $x^* = \theta(-x^c)$ for any $x \in \mathfrak{g}_{\mathbb{C}}$. Thus if

$$\mathfrak{g}_u = \mathfrak{k} + i\mathfrak{p} \tag{3.1.6}$$

then $\mathfrak{g}_u$ is a compact form of $\mathfrak{g}_{\mathbb{C}}$ and

$$x^* = -x \qquad \text{if and only if} \quad x \in \mathfrak{g}_u. \tag{3.1.7}$$

One also defines a $*$-operation on $G_{\mathbb{C}}$ by putting $g^* = \theta((g^c)^{-1})$ for any $g \in G_{\mathbb{C}}$. Clearly if $a, b \in G_{\mathbb{C}}$, $x, y \in \mathfrak{g}_{\mathbb{C}}$ one has

$$(ab)^* = b^*a^* \qquad \text{and} \qquad [x, y]^* = [y^*, x^*] \tag{3.1.8}$$

and also

$$(\exp x)^* = \exp x^*. \tag{3.1.9}$$

Now let $G_u \subseteq G_{\mathbb{C}}$ be the (maximal compact) subgroup of $G_{\mathbb{C}}$ corresponding to $\mathfrak{g}_u$ (see (3.1.6)). Also let $P_u = \exp i\mathfrak{g}_u$ so that as one knows P_u is a closed submanifold of $G_{\mathbb{C}}$ and the map

$$G_u \times P_u \to G_{\mathbb{C}}, \qquad (w, q) \mapsto wq \tag{3.1.10}$$

is a diffeomorphism. In fact the decomposition of $G_{\mathbb{C}}$ defined by (3.1.10) is just a polar decomposition. One has $K \subseteq G_u$, $P \subseteq P_u$ and restricting (3.1.10) one has a diffeomorphism

$$K \times P \to G, \tag{3.1.11}$$

which of course defines our given polar decomposition of G.

Now it is clear that G is stable under the $*$-operation and in fact if $g \in G$ is given by (3.1.1) then by (3.1.8) one sees that

$$g^* = pk^{-1}. \tag{3.1.12}$$

This also uses (3.1.7) and (3.1.9), which imply $a = a^*$ if $a \in P_u$ and $a^* = a^{-1}$ if $a \in G_u$.

Now note that

$$\mathfrak{h} \subseteq \mathfrak{p}. \tag{3.1.13}$$

Indeed for any $\varphi \in \Delta$ let

$$h_\varphi = [e_\varphi, e_{-\varphi}] \in \mathfrak{h}. \tag{3.1.14}$$

But then since $e_{-\varphi} = e_\varphi^*$ (see (1.5.1)) one has $h_\varphi^* = h_\varphi$ by (3.1.8) so that $h_\varphi \in \mathfrak{p}$. On the other hand the root normals h_φ span $\mathfrak{h}$ so that one has (3.1.13). Since $\dim \mathfrak{h} = l$ the Cartan subalgebra $\mathfrak{h}$ is of course a Cartan subspace of $\mathfrak{p}$ and by exponentiating (3.1.13) one has

$$H \subseteq P. \tag{3.1.15}$$

Now let T be the subgroup of $G_{\mathbb{C}}$ corresponding to $i\mathfrak{h}$. Then by (3.1.6) and (3.1.13) one has $T \subseteq G_u$ (in fact T is clearly a maximal torus in G_u) and clearly the restriction of (3.1.10) to $T \times H$ defines a diffeomorphism

$$T \times H \rightarrow H_{\mathbb{C}}. \tag{3.1.16}$$

One notes then that if $a \in H_{\mathbb{C}}$ and we write $a = th$, where $t \in T$ and $h \in H$ then

$$a^* = ht^{-1}. \tag{3.1.17}$$

Now let $\tilde{M}$ be the set of all elements of order 2 or 1 in $H_{\mathbb{C}}$. Obviously

$$\tilde{M} \subseteq T \tag{3.1.18}$$

by (3.1.16) and since T is a torus of rank l it is clear that $\tilde{M}$ is a finite Abelian group and in fact its cardinality $|\tilde{M}|$ is given by

$$|\tilde{M}| = 2^l. \tag{3.1.19}$$

LEMMA 3.1.1. *Let $a \in H_{\mathbb{C}}$. Then $a = a^*$ if and only if $a \in \tilde{M}H$.*

Proof. This is immediate from (3.1.16) and (3.1.17) since $H_{\mathbb{C}}$ is commutative. Q.E.D.

Let $M = \tilde{M} \cap G$.

LEMMA 3.1.2. *Let $g \in G$ and assume $g = g^*$. Then $g \in K$ if and only if $g^2 = 1$. Furthermore one has*

$$M = K \cap T. \tag{3.1.20}$$

Proof. If $g = kp$ is the polar decomposition of g then by (3.1.12)

$$kp = pk^{-1}. \tag{3.1.21}$$

Thus if $g \in K$ one has $g = k = k^{-1}$ so that $g^2 = 1$. On the other hand by (3.1.21) one has $g^2 = p^2 \in P$. Thus if $g^2 = 1$ then $p^2 = 1$ which implies $p = 1$. See (3.1.3). Hence $g \in K$. This proves the first statement. Now $M \subseteq K \cap T$ by the first statement and Lemma 3.1.1. On the other hand if $g \in K \subseteq G$ then $g^c = g$ recalling (3.1.4) and (3.1.5). But if $g \in T = \exp i\mathfrak{h}$ then also $g^c = g^{-1}$. Thus $K \cap T \subseteq \tilde{M}$. However, $K \cap T \subseteq G$ also, so that one has (3.1.20). Q.E.D.

For any $x \in \mathfrak{g}$ let $G_e{}^x$ be the identity component of the centralizer G^x of x in G.

PROPOSITION 3.1. *Assume $x \in \mathfrak{p}$ is regular. Then*

$$g = g^* \qquad \text{for any} \quad g \in G^x. \tag{3.1.22}$$

On the other hand one has

$$G^x \cap P = G_e{}^x. \tag{3.1.23}$$

Furthermore as Lie groups one has the isomorphism

$$G_e{}^x \cong \mathbb{R}^l. \tag{3.1.24}$$

Proof. From the conjugation theory of Cartan subspaces of $\mathfrak{p}$ one knows that x is K-conjugate to an element in $\mathfrak{h}$. But since the $*$-operation commutes with the conjugacy action of K and P is stable under this action it then suffices to assume, as we shall, that $x \in \mathfrak{h}$. But now clearly $G^x \subseteq G_{\mathbb{C}}{}^x$, where $G_{\mathbb{C}}{}^x$ is the centralizer of x in $G_{\mathbb{C}}$. On the other hand $G_{\mathbb{C}}{}^x = H_{\mathbb{C}}$ since $G_{\mathbb{C}}{}^x$ is connected by Proposition 2.4 and $\mathfrak{g}_{\mathbb{C}}{}^x = \mathfrak{h}_{\mathbb{C}}$ by regularity. Thus

$$G^x = H_{\mathbb{C}} \cap G. \tag{3.1.25}$$

But now comparing the diffeomorphisms (3.1.11) and (3.1.16) one has $G^x = (K \cap T)H$ by (3.1.25). But $K \cap T = M$ by (3.1.20) so that $G^x = MH$. But then in fact restricting (3.1.16) one has a diffeomorphism

$$M \times H \to G^x. \tag{3.1.26}$$

But then by the uniqueness of the polar decomposition this implies

$$G^x \cap P = H. \tag{3.1.27}$$

However, one also clearly has $G_e^x = H$ by (3.1.26). This proves (3.1.23). Furthermore one also has (3.1.22) by Lemma 3.1.1 and (3.1.26).

Now the restriction of the bijection (3.1.2) to $\mathfrak{h}$ is clearly an isomorphism $\mathfrak{h} \to H$ of Abelian Lie groups. This proves (3.1.24). Q.E.D.

3.2. Now put

$$G_* = \bar{N}HN. \tag{3.2.1}$$

PROPOSITION 3.2. *Assume $x \in \mathfrak{p}$ is regular. Then*

$$G^x \cap G_* = G_e^x, \tag{3.2.2}$$

where we recall that G_e^x is the identity component of G^x.

Proof. Assume that g is in the left side of (3.2.2). Write $g = \bar{n}hn$ for $\bar{n} \in \bar{N}$, $h \in H$ and $n \in N$. Then $g^* = n^*h\bar{n}^*$. But now by (1.5.1) and (3.1.9) one has

$$N^* = \bar{N} \qquad \text{and} \qquad (\bar{N})^* = N. \tag{3.2.3}$$

Thus $n^* \in \bar{N}$ and $(\bar{n})^* \in N$. However, $g = g^*$ by (3.1.22). But then by the isomorphism (2.4.5) applied to $N_{\mathbf{C}}$, $H_{\mathbf{C}}$, and $\bar{N}_{\mathbf{C}}$ one has $n^* = \bar{n}$. Hence $g = n^*hn$. Thus if we put $h^{1/2}n = a \in G$ one has $g = a^*a$. However, for any $b \in G$ one has

$$b^*b \in P. \tag{3.2.4}$$

Indeed if $b = kp$ is the polar decomposition of b then $b^*b = p^2 \in P$, proving (3.2.4). Thus $g \in P$ and hence $g \in G^x \cap P$. But then $g \in G_e^x$ by (3.1.23).

Now the Iwasawa decomposition asserts that the map

$$K \times H \times N \to G, \qquad (k, a, n) \to kan \tag{3.2.5}$$

is a diffeomorphism. Thus if $g \in G$ we can uniquely write $g = kan$, where $k \in K$, $a \in H$, and $n \in N$, and we refer to this as the Iwasawa decomposition of g.

Now recalling that $G_* = \bar{N}HN$ we assert that

$$P \subseteq G_*. \tag{3.2.6}$$

Indeed let $p \in P$ and let $p^{1/2} = kan$ be the Iwasawa decomposition of the square root $p^{1/2}$ of p. Then $(p^{1/2})^* = n^*ak^{-1}$. Since $(p^{1/2})^* = p^{1/2}$ this implies

$$p = (p^{1/2})^*(p^{1/2}) = n^*a^2n. \tag{3.2.7}$$

But $n^* \in \bar{N}$ and $a^2 = h \in H$. This proves (3.2.6).

Now to prove the proposition we have only to show that the right side of (3.2.2) is contained in the left side. Let $g \in G_e^x$. Then $g \in P$ by (3.1.23). Thus $g \in G_*$ by (3.2.6). Hence g is contained in the left side of (3.2.2). Q.E.D.

Now recalling (2.3.1) let $\gamma \in \mathscr{I}(Z)$. We wish to obtain results about the closed subset $Z(\gamma)$ in the present case (i.e., where $F = \mathbb{R}$) which are analogous to those given in Theorem 2.6. If $y \in Z$ then even though y is not necessarily in $\not{p}$ it is H-conjugate to an element in $\not{p}$ so, as we now observe, the statement of Proposition 3.2 holds for y.

Lemma 3.2. *Let $y \in Z$. Then there exists a unique $c \in H$ such that*

$$cy \in \not{p}. \tag{3.2.8}$$

Also, one has

$$G^y \cap G_* = G_e^y. \tag{3.2.9}$$

Proof. Let $x \in \not{h}$ and let $a_i \in \mathbb{R}^*$ (i.e., $a_i > 0$), $i = 1,\ldots, l$, be such that $y = f + x + \sum_i a_i e_{\alpha_i}$. Now for any $c \in H$ one has

$$cy = x + \sum_i (c^{-\alpha_i} e_{-\alpha_i} + c^{\alpha_i} a_i e_{\alpha_i}). \tag{3.2.10}$$

But now recalling the isomorphism (2.2.8) it follows that there is a unique $c \in H$ such that $c^{-\alpha_i} = c^{\alpha_i} a_i$, namely, that element c such that $c^{\alpha_i} = a_i^{-1/2}$. However, $x^* = x$ by (3.1.13) and $e_{\alpha_i}^* = e_{-\alpha_i}$ by (1.5.1). Thus only for this value of c does one have $(cy)^* = cy$ or equivalently $cy \in \not{p}$. This proves the first statement of the lemma.

Now fix $c \in H$ so that one has (3.2.8). But $y \in R$ by (2.1.7). Thus cy is a regular element of $\not{p}$ and hence by (3.2.2) one has $G^{cy} \cap G_* = G_e^{cy}$. On the other hand one certainly has $c^{-1}G^{cy}c = G^y$ and a similar relation for the identity components. Since $\bar{N}$, H, and N are stable under conjugation by c it follows that G_* is stable under conjugation by c. This proves (3.2.9). Q.E.D.

Now we may apply the results of Sections 2.4–2.6 to the complexification $G_{\mathbb{C}}$ of G. Thus recalling (2.4.5) the map $\bar{N}_{\mathbb{C}} \times H_{\mathbb{C}} \times N_{\mathbb{C}} \to \bar{N}_{\mathbb{C}} H_{\mathbb{C}} N_{\mathbb{C}}$ defined by multiplication is an algebraic isomorphism of nonsingular affine varieties.

In particular then it is a diffeomorphism. Thus, by dimension, G_* is an open connected subset of G and, by restriction the map

$$\bar{N} \times H \times N \to G_*\,, \qquad (\bar{n}, h, n) \to \bar{n}hn \tag{3.2.11}$$

is a diffeomorphism. Thus using the notation of (2.4.6) for the complexification $G_{\mathbb{C}}$ one has the decomposition

$$d = \bar{n}_d h_d n_d \tag{3.2.12}$$

for any $d \in G_*$, where now $\bar{n}_d \in \bar{N}$, $h_d \in H$, and $n_d \in N$. Furthermore upon restriction of (2.4.7) the map

$$G_* \to N, \qquad d \to n_d \tag{3.2.13}$$

is smooth.

THEOREM 3.2. *Let $\mathfrak{g}$ be a real split semi-simple Lie algebra. Let l* = rank $\mathfrak{g}$ *and let G be the adjoint group of $\mathfrak{g}$. Let $Z \subseteq \mathfrak{g}$ be the 2l dimensional manifold of normalized Jacobi elements in $\mathfrak{g}$. The manifold Z is defined by* (2.2.3) *and* (2.2.4). *Now let $\mathscr{I}: \mathfrak{g} \to \mathbb{R}^l$ be the map (see* (2.3.1)) *defined by the fundamental G-invariant polynomials on $\mathfrak{g}$ and for any $\gamma \in \mathscr{I}(Z)$ let $Z(\gamma) = \mathscr{I}^{-1}(\gamma) \subseteq Z$. Then the isospectral set $Z(\gamma)$ is a closed connected submanifold of dimension l in Z. In fact one has a diffeomorphism*

$$Z(\gamma) \cong \mathbb{R}^l. \tag{3.2.14}$$

Furthermore for any $y \in Z(\gamma)$ the identity component G_e^y of the centralizer of y in G is also isomorphic (as Lie groups) to $\mathbb{R}^l$. Moreover using the notation of (3.2.1) *one has $G_e^y \subseteq G_*$ and if for any $d \in G_e^y$ the element $n_d \in N$ is defined by* (3.2.12) *then $n_d y \in Z(\gamma)$ and the map*

$$\beta_y\colon G_e^y \to Z(\gamma), \qquad d \mapsto n_d y \tag{3.2.15}$$

is a diffeomorphism.

Proof. Let $(\mathscr{d}_1)_{\mathbb{C}}$ be the complexification of the diagonal $\mathscr{d}_1$ so that $(\mathscr{d}_1)_{\mathbb{C}}$ is stable under the $H_{\mathbb{C}}$ and let $(\tilde{\mathscr{d}}_1)_{\mathbb{C}}$ be defined by (2.2.2), where $F = \mathbb{C}$. Put $Z_{\mathbb{C}} = f + \mathscr{h}_{\mathbb{C}} + (\tilde{\mathscr{d}}_1)_{\mathbb{C}}$ so that we can apply Theorem 2.4, where $Z_{\mathbb{C}}$ replaces Z. But then regarding $S(\mathfrak{g}_{\mathbb{C}})^{G_{\mathbb{C}}}$ as the complexification of $S(\mathfrak{g})^G$ one has $y \in Z_{\mathbb{C}}(\gamma)$. Furthermore if we let $(\beta_{\mathbb{C}})_y$ now denote the β_y of Theorem 2.4 then by Theorem 2.4 one has an isomorphism

$$(\beta_{\mathbb{C}})_y\colon (G_{\mathbb{C}}^y)_* \to Z_{\mathbb{C}}(\gamma). \tag{3.2.16}$$

But now by (3.2.9) one has $G_e{}^y \subseteq (G_{\mathbb{C}}{}^y)_*$ and clearly for $d \in G_e{}^y$ the definition of h_d and n_d whether given by (3.2.12) or (2.4.6) is the same. But now $Z(\gamma) \subseteq Z_{\mathbb{C}}(\gamma)$ and if $d \in G_e{}^y$ we assert that $(\beta_{\mathbb{C}})_y(d) \in Z(\gamma)$. Indeed $(\beta_{\mathbb{C}})_y(d) = n_d y$ so that $(\beta_{\mathbb{C}})_y(d) \in \mathfrak{g}$. But then recalling (2.4.13) one must have $x \in \mathfrak{h}$. Furthermore again using the notation of (2.4.13) if we write $y_1 = \sum a_i e_{\alpha_i}$ one has $a_i > 0$ since $y \in Z$. But then

$$h_d^{-1} y_1 = \sum (h_d^{-\alpha_i} a_i)\, e_{\alpha_i}. \tag{3.2.17}$$

Since $h_d^{-\alpha_i} > 0$ this implies by (2.4.13) that $(\beta_{\mathbb{C}})_y(d) \in Z(\gamma)$ and hence one has a map (3.2.15) and the map (3.2.15) is just the restriction of (2.4.11) to $G_e{}^y$. We next assert that (3.2.15) is surjective. Let $w \in Z(\gamma)$. Recalling the argument of the surjectivity of β_y in the proof of Theorem 2.4 we first observe that the element $n \in N_{\mathbb{C}}$ of (2.4.14) is in fact in N. This is clear from Proposition 2.3.2 since y and w are in $\mathfrak{b}_f$ and not just $f + \mathfrak{b}_{\mathbb{C}}$. Similarly the unique element $h \in H_{\mathbb{C}}$ such that $hw_1 = y_1$ is in fact in H since $w_1, y_1 \in \tilde{\mathfrak{a}}_1$. Finally the element $\bar{n} \in \bar{N}_{\mathbb{C}}$ is in $\bar{N}$ again by Proposition 2.3.2 and (2.4.15). Thus $d = \bar{n}hn$ is in G and we recall $(\beta_{\mathbb{C}})_y(d) = w$. But clearly $d \in G^y \cap G_*$. Thus $d \in G_e{}^y$ by (3.2.9) and hence (3.2.15) is surjective.

Now $Z(\gamma)$, by Proposition 2.3.1, is a (not necessarily connected) submanifold of dimension l in Z. But in fact (2.4.11) is an isomorphism of nonsingular varieties and hence as noted in Remark 2.4.2 is a diffeomorphism of manifolds. Since (3.2.15) is surjective and is the restriction of (2.4.11) to $G_e{}^y$ it follows that (3.2.15) is a diffeomorphism. Now, using the notation of Lemma 3.2, $G_e{}^y$ is isomorphic to G_e^{cy}. But G_e^{cy} is isomorphic to $\mathbb{R}^l$ as Lie groups by (3.1.24). Thus $G_e{}^y$ and hence $Z(\gamma)$ are isomorphic to $\mathbb{R}^l$ as manifolds. This proves the theorem. Q.E.D.

Remark 3.2. One aspect of Theorem 3.2 which we think ought to be emphasized is the connectivity of $Z(\gamma)$. If Z was defined by (2.2.3) and (2.2.4) except that, say, $a_i < 0$ instead of $a_i > 0$, then it would not be necessarily true that $Z(\gamma)$ is connected. An example of this disconnectivity is easily constructed for the case where $\mathfrak{g}$ is the Lie algebra of $Sl(2, \mathbb{R})$.

3.3. Now let $\mathfrak{h}_+$ be the open Weyl chamber in $\mathfrak{h}$ defined by putting

$$\mathfrak{h}_+ = \{x \in \mathfrak{h} \mid \langle \varphi, x \rangle > 0 \text{ for all } \varphi \in \Delta_+\}. \tag{3.3.1}$$

Also let $\bar{\mathfrak{h}}_+$ be the closure of $\mathfrak{h}_+$ in $\mathfrak{h}$ so that $\bar{\mathfrak{h}}_+$ is a closed Weyl chamber. As one knows and easily sees, $\mathfrak{h} \subseteq \mathfrak{g}^x$ for any $x \in \bar{\mathfrak{h}}_+$ and equality holds if and only if $x \in \mathfrak{h}_+$. That is,

$$\mathfrak{h}_+ = \bar{\mathfrak{h}}_+ \cap R. \tag{3.3.2}$$

Now recalling the map (2.3.1) one has

PROPOSITION 3.3.1. *$\mathscr{I}(\mathfrak{h}_+)$ is a connected open subset of $\mathbb{R}^l$ and the map*

$$\mathfrak{h}_+ \to \mathscr{I}(\mathfrak{h}_+), \qquad x \to \mathscr{I}(x) \tag{3.3.3}$$

is a diffeomorphism.

Proof. The bilinear form Q is positive definite on $\mathfrak{h}$ and hence we can find a basis x_i of $\mathfrak{g}$ such that x_i for $i \leqslant l$ is an orthonormal basis of $\mathfrak{h}$ and $Q(x_i, x_j) = 0$ for $i \leqslant l < j$. But now if $x \in \mathfrak{h}_+$ then $\mathfrak{g}^x = \mathfrak{h}$. But then for any $I \in S(\mathfrak{g})^G$ one has $(\delta I)(x) \in \mathfrak{h}$ by (1.3.1). But then using the basis x_i of $\mathfrak{g}$ and its dual basis the summands in (1.2.2) must vanish for $i > l$. That is, one has $(\delta I)(x) = \sum_{j=1}^{l} (i(x_j)I)(x)\, x_j$. On the other hand as already noted in the proof of Proposition 2.1, since $x \in R$, the elements $(\delta I_k)(x)$, $k = 1,..., l$, are linearly independent (see Theorem 9 in [14, p. 382]). Thus the $l \times l$ matrix $M_{jk} = (i(x_j)\, I_k)(x)$ is nonsingular. But this matrix is just the Jacobian of the map (3.3.3) at x with respect to the coordinate system on $\mathfrak{h}_+$ defined by the x_j, $1 \leqslant j \leqslant l$. Thus $\mathscr{I}(\mathfrak{h}_+)$ is open and (3.3.3) is a local diffeomorphism. It is connected since $\mathfrak{h}_+$ is clearly connected. It suffices then only to show that (3.3.3) is injective. However, if $x, y \in \mathfrak{h}_+$ and $\mathscr{I}(x) = \mathscr{I}(y)$ then by Chevalley's theorem x and y are W-conjugate. But, as one knows, $\bar{\mathfrak{h}}_+$ is a fundamental domain for the action of W on $\mathfrak{h}$. Thus $x = y$. Q.E.D.

An element $x \in \mathfrak{g}$ is called semi-simple if $\operatorname{ad} x$ is diagonalizable over $\mathbb{C}$. It is called real semi-simple if $\operatorname{ad} x$ is diagonalizable over $\mathbb{R}$. The following is well known.

LEMMA 3.3.1. *Let $x \in \mathfrak{g}$. Then x is real semi-simple if and only if x is G-conjugate to an element in $\mathfrak{h}$.*

Proof. See, e.g., Proposition 2.4 in [16] and Theorem 2(2) in [23, p. 383]. Q.E.D.

Now let R_+ denote the set of all regular real semi-simple elements in $\mathfrak{g}$. It is clear that R_+ is stable under the action of G.

LEMMA 3.3.2. *Let $x \in \mathfrak{g}$. The following conditions are all equivalent.*

(1) $x \in R_+$.

(2) *x is G-conjugate to an element in $\mathfrak{h}_+$.*

(3) $\mathscr{I}(x) \in \mathscr{I}(\mathfrak{h}_+)$.

Proof. Using the fact that $\bar{\mathfrak{h}}_+$ is a fundamental domain for the action of the Weyl group W on $\mathfrak{h}$ the equivalence of (1) and (2) follows from Lemma 3.3.1 and (3.3.2). Obviously (2) implies (3). It suffices to show that (3) implies (1).

Assume $\mathscr{I}(x) \in \mathscr{I}(\mathscr{h}_+)$. Let $z \in \mathscr{h}_+$ be such that $\mathscr{I}(x) = \mathscr{I}(z)$. Then since $z \in R$ one has $x \in \overline{G_{\mathbb{C}} z}$ by Theorem 3 in [14, p. 365] (see in particular (3.8.7) in [14]), where the bar denotes closure. However, since z is semi-simple $G_{\mathbb{C}} z$ is closed by Lemma 5 in [14, p. 353]. Thus $x \in G_{\mathbb{C}} z$. But then $\operatorname{ad} x$ is semi-simple with real eigenvalues. But then x is real semi-simple. Furthermore $x \in R$ since $z \in R$. Thus $x \in R_+$. Q.E.D.

PROPOSITION 3.3.2. *One has*

$$Z \subseteq R_+ . \tag{3.3.4}$$

Furthermore

$$\mathscr{I}(Z) \subseteq \mathscr{I}(\mathscr{h}_+). \tag{3.3.5}$$

Proof. The second statement follows from (3.3.4) and Lemma 3.3.2. Thus it suffices to prove (3.3.4). Let $y \in Z$ then $y \in R$ by (2.1.7). But now y is conjugate to an element in $\mathscr{p}$ by Lemma 3.2. However, the elements of $\mathscr{p}$ are real semi-simple. In fact any element in $\mathscr{p}$ is K-conjugate to an element in $\mathscr{h}$ by the conjugacy theorem of Cartan subspaces. Thus y is real semi-simple by Lemma 3.1. Hence $y \in R_+$. Q.E.D.

Remark 3.3. In the complex case we found (see (2.5.6)) that $\mathscr{I}(Z) = \mathbb{C}^l$. From the nonconjugacy of Cartan subalgebras in $\mathscr{g}$ in the present case it is clear from Proposition 3.3 that we cannot now expect an analogous result.

3.4. Recall that (see (3.1.1)) $z \to z^c$ is the conjugate linear automorphism of $\mathscr{g}_{\mathbb{C}}$ given by putting $z^c = x - iy$, where we have written $z = x + iy$ for $x, y \in \mathscr{g}$. We recall also that this automorphism induces an automorphism

$$G_{\mathbb{C}} \to G_{\mathbb{C}}, \qquad a \to a^c \tag{3.4.1}$$

of $G_{\mathbb{C}}$. One of course has

$$a^c z^c = (az)^c \tag{3.4.2}$$

for $a \in G_{\mathbb{C}}$, $z \in \mathscr{g}_{\mathbb{C}}$. One also has

$$(\exp z)^c = \exp(z^c). \tag{3.4.3}$$

Now let

$$\tilde{G} = \{a \in G_{\mathbb{C}} \mid a^c = a\}. \tag{3.4.4}$$

One easily has

PROPOSITION 3.4. *$\tilde{G}$ is the set of all elements $g \in G_{\mathbb{C}}$ which stabilize $\mathfrak{g}$. In particular one has*

$$G \subseteq \tilde{G}. \tag{3.4.5}$$

Proof. Clearly an element in $G_{\mathbb{C}}$ is determined (since it is $\mathbb{C}$-linear) by its restriction to $\mathfrak{g}$. The first statement and hence the second of the proposition then follows from (3.4.2) since, in the notation of (3.4.2), $z^c = z$ if and only if $z \in \mathfrak{g}$. Q.E.D.

Recall (see (3.1.18)) that $\tilde{M}$ is the set of all $a \in H_{\mathbb{C}}$ such that $a^2 = 1$.

LEMMA 3.4.1. *One has*

$$\tilde{G} \cap H_{\mathbb{C}} = \tilde{M}H. \tag{3.4.6}$$

Proof. Now recalling (3.1.16) and the notation of (3.1.16) any element $a \in H_{\mathbb{C}}$ can be uniquely written $a = sh$ when $s \in T$ and $h \in H$. But then one clearly has $a^c = s^{-1}h$. Thus $a = a^c$ if and only if $s = s^{-1}$, that is, if and only if $s \in \tilde{M}$. Q.E.D.

For any $x \in \mathfrak{g}$ let $\tilde{G}^x$ denote the centralizer of x in $\tilde{G}$.

LEMMA 3.4.2. *For any $x \in \mathfrak{h}_+$ one has*

$$\tilde{G}^x = \tilde{M}H. \tag{3.4.7}$$

Proof. One has $\tilde{G}^x = \tilde{G} \cap G_{\mathbb{C}}^x$. But since x is a regular element of $\mathfrak{g}_{\mathbb{C}}$ and $x \in \mathfrak{h}_+$ then

$$G_{\mathbb{C}}^x = H_{\mathbb{C}} \tag{3.4.8}$$

by Proposition 2.4. The result then follows from (3.4.6). Q.E.D.

Remark 3.4. We note then by (3.1.16) and (3.4.7) that $\tilde{G}^x$ has 2^l connected components for any $x \in \mathfrak{h}_+$. Recalling Lemma 3.3.2 the same statement is then true for any $x \in R_+$. Furthermore since $H = G_e^x = \tilde{G}_e^x$ for $x \in \mathfrak{h}_+$ one has

$$G_e^x = \tilde{G}_e^x \qquad \text{for any} \quad x \in R_+. \tag{3.4.9}$$

LEMMA 3.4.3. *One has*

$$\tilde{G} \cap \bar{N}_{\mathbb{C}} H_{\mathbb{C}} N_{\mathbb{C}} = \bar{N}\tilde{M}HN \tag{3.4.10}$$

and any element g in this set can be uniquely written

$$g = \bar{n}mhn, \tag{3.4.11}$$

where $\bar{n} \in \bar{N}$, $m \in \tilde{M}$, $h \in H$, *and* $n \in N$.

Proof. Clearly the right side of (3.4.10) is contained in the left side. Let g be in the left side so that we can uniquely write $g = \bar{n}an$, where $\bar{n} \in \bar{N}_{\mathbb{C}}$, $a \in H_{\mathbb{C}}$, and $n \in N_{\mathbb{C}}$. But $g^c = (\bar{n})^c a^c n^c$. However, $\bar{N}_{\mathbb{C}}$, $H_{\mathbb{C}}$, and $N_{\mathbb{C}}$ are clearly stable under (3.4.1) by (3.4.3). Thus $\bar{n}^c = \bar{n}$, $a^c = a$, and $n^c = n$ by uniqueness. But the map exp: $\mathfrak{n}_{\mathbb{C}} \to N_{\mathbb{C}}$ is a bijection. Thus $n \in N$ by (3.4.3). Similarly $\bar{n} \in \bar{N}$. One also has $a \in \tilde{M}H$ by (3.4.6). Thus g is in the right side of (3.4.10). The uniqueness of (3.4.11) follows from the injectivity of (2.4.5) and (3.1.16). Q.E.D.

Now for any $\sigma \in W$ let $s(\sigma) \in G_{\mathbb{C}}$ be the unique element in the normalizer of $\mathfrak{h}$ in $G_{\mathbb{C}}$ such that

$$s(\sigma)\, e_{\alpha_i} = e_{\sigma\alpha_i}, \qquad i = 1,\dots, l. \tag{3.4.12}$$

LEMMA 3.4.4. *One has* $s(\sigma) \in \tilde{G}$ *for any* $\sigma \in W$.

Proof. By Proposition 3.4 it suffices to show that $s(\sigma)$ stabilizes $\mathfrak{g}$. Since $\mathfrak{g}$ is generated by e_{α_i}, $e_{-\alpha_i}$, $i = 1,\dots, l$, it suffices to show that $s(\sigma)\, e_{-\alpha_i} \in \mathfrak{g}$ for all i. Thus if we let $\lambda_i \in \mathbb{C}$ be defined by $s(\sigma)\, e_{-\alpha_i} = \lambda_i e_{-\sigma\alpha_i}$ it suffices to show $\lambda_i \in \mathbb{R}$. But now $Q(e_\varphi, e_{-\varphi})$ is a nonvanishing real number for any $\varphi \in \Delta$. But if we extend Q to $\mathfrak{g}_{\mathbb{C}}$ by $\mathbb{C}$-linearity one has $Q(s(\sigma)\, e_{-\alpha_i}, e_{\sigma\alpha_i}) = \lambda_i Q(e_{\sigma\alpha_i}, e_{-\sigma\alpha_i})$ so that it suffices to show $Q(s(\sigma)\, e_{-\alpha_i}, e_{\sigma\alpha_i}) \in \mathbb{R}$. But by the invariance of Q one has $Q(s(\sigma)\, e_{-\alpha_i}, e_{\sigma\alpha_i}) = Q(e_{-\alpha_i}, s(\sigma)^{-1} e_{\sigma\alpha_i}) = Q(e_{-\alpha_i}, e_{\alpha_i}) \in \mathbb{R}$. Q.E.D.

Now it is clear from Proposition 3.4 that G is the identity component of $\tilde{G}$. On the other hand $G_* = \bar{N}HN$ is an open connected subset of $\tilde{G}$. But $s(\kappa) \in \tilde{G}$ by Lemma 3.4.4. Thus if we put

$$G_{(*)} = s(\kappa)\, G_* \tag{3.4.13}$$

then $G_{(*)}$ is an open connected subset of $\tilde{G}$. Furthermore using the notation of (2.6.2) for the complexification $G_{\mathbb{C}}$ of G one has the decomposition

$$s(\kappa)^{-1} g = \bar{n}(g)\, h(g)\, n(g) \tag{3.4.14}$$

for any $g \in G_{(*)}$, where now $\bar{n}(g) \in \bar{N}$, $h(g) \in H$, and $n(g) \in N$. Furthermore by restriction the map

$$G_{(*)} \to N, \qquad g \to n(g) \tag{3.4.15}$$

is smooth.

3.5. The following lemma is no doubt known. Because of the overriding importance for us of the element $\bar{n}_f(w)$ in Lemma 3.5.2 we prove Lemma 3.5.1 for completeness.

LEMMA 3.5.1. *For any $\bar{n} \in \bar{N}$ and $x \in \mathfrak{h}_+$ one has $\bar{n}x \in x + \bar{\mathfrak{n}}$ so that one has a map*

$$\bar{N} \times \mathfrak{h}_+ \to \mathfrak{h}_+ + \bar{\mathfrak{n}}, \qquad (\bar{n}, x) \to \bar{n}x. \tag{3.5.1}$$

Furthermore the map (3.5.1) *is a diffeomorphism.*

Proof. Let $x \in \mathfrak{h}_+$ and $\bar{n} \in \bar{N}$. Since x and $\bar{n}x$ have the same maximal diagonal component (see Definition 2.1) one has $\bar{n}x \in x + \bar{\mathfrak{n}}$. This proves the first statement of the proposition and establishes the map (3.5.1). But now by (3.4.8) one has $G^x \cap N = (1)$ so that (3.5.1) is injective. Now let $y \in \mathfrak{h}_+ + \bar{\mathfrak{n}}$ so that $y = z + v$, where $z \in \mathfrak{h}_+$ and $v \in \bar{\mathfrak{n}}$. But $\mathscr{I}(y) = \mathscr{I}(z)$ by Proposition 17 in [14, p. 369]. Thus $\mathscr{I}(y) \in \mathscr{I}(\mathfrak{h}_+)$ and hence by Proposition 3.3.1 and Lemma 3.3.2 there exists $g \in G$ such that $gz = y$. But now applying the decomposition (2.4.3) to $G_{\mathbb{C}}$ there exist $\sigma \in W$, $n \in N_{\mathbb{C}}$, $h \in H_{\mathbb{C}}$, $\bar{n} \in \bar{N}_{\mathbb{C}}$ such that $g = \bar{n}s(\sigma)\,hn$. Put $w = (\bar{n})^{-1}y$ so that if w_o is the component of w in $\mathfrak{d}_0$ $(=\mathfrak{h})$ relative to (2.1.3) then $w_o = z$. However, $w = s(\sigma)\,hnz$ since $gz = y$. But the component of hnz in $\mathfrak{d}_0$ relative to (2.1.3) is also z and hence the component of $w = s(\sigma)\,hnz$ in $\mathfrak{d}_0$ relative to (2.1.3) is σz. Thus $\sigma z = z$. However, z is regular and hence as one knows this implies σ is the identity. Thus $s(\sigma) = 1$ so that $g = \bar{n}hn$. But then $\bar{n} \in \bar{N}$, $h \in \tilde{M}H$, and $n \in N$ by (3.4.10). On the other hand the equation becomes $w = hnz$. But the maximal diagonal degree of w, and hence of nz, is zero. Thus $nz = z$. This implies $n = 1$ since $G^z \cap N = (1)$ by (3.4.8). But then $y = gz = \bar{n}hz = \bar{n}z$. Thus y is in the image of the map (3.5.1) so that the map is bijective.

Obviously the map (3.4.16) is smooth. If $x \in \mathfrak{h}_+$ then the tangent space to $\bar{N} \times \mathfrak{h}_+$ at $(1, x)$ may be identified with $(\bar{\mathfrak{n}}, \mathfrak{h})$ and the tangent space to $\mathfrak{h}_+ + \bar{\mathfrak{n}}$ at x may be identified with $\mathfrak{h} + \bar{\mathfrak{n}}$. The differential of the map at $(1, x)$ carries $(\bar{\mathfrak{n}}, \mathfrak{h})$ to $\mathfrak{h} + [\bar{\mathfrak{n}}, x]$. But $[\bar{\mathfrak{n}}, x] = \bar{\mathfrak{n}}$ since x is regular. Thus (3.5.1) is a local diffeomorphism at $(1, x)$. Translation by $\bar{N}$ then implies that there is a local diffeomorphism at all points. Since the map (3.5.1) is bijective it then follows that it is a diffeomorphism. Q.E.D.

LEMMA 3.5.2. *Let $w_o \in \mathfrak{h}_+$ and (using the notation of* (1.5.4)*) put $w = f + w_o$. Then there exists a unique element $\bar{n}_f(w) \in \bar{N}$ such that*

$$\bar{n}_f(w)\, w_o = w. \tag{3.5.2}$$

Furthermore the map

$$\mathscr{k}_+ \to \bar{N}, \qquad w_0 \mapsto \bar{n}_f(w) \tag{3.5.3}$$

is smooth.

Proof. The first statement follows from Lemma 3.5.1. Consider the map

$$\mathscr{k}_+ \to \mathscr{k}_+ + \bar{n}, \qquad w_0 \to w = f + w. \tag{3.5.4}$$

Obviously (3.5.4) is smooth. On the other hand (3.5.3) is just the composite of (3.5.4) with the inverse to (3.5.1) and then with the projection onto $\bar{N}$. This proves (3.5.3) is smooth. Q.E.D.

Remark 3.5.1. The map (3.5.3) will play an important role in Section 5. We give an explicit formula for $\bar{n}_f(w)^{-1}$ in Section 5.8.

Now by (2.2.9) (for the case where $F = \mathbb{C}$) there exists a unique element $m \in H_{\mathbb{C}}$ such that

$$me_{\alpha_i} = -e_{\alpha_i}, \qquad i = 1,..., l. \tag{3.5.5}$$

Clearly $m \in \tilde{M}$. Now let $\tilde{M} = \{m_j\}$, $j = 0,..., 2^l - 1$, be some ordering of the set $\tilde{M}$, where $m_0 = m$ and $m_1 = 1$, the identity element of G. Also put

$$H_j = m_j H, \qquad j = 0,..., 2^l - 1. \tag{3.5.6}$$

LEMMA 3.5.3. *Let $w_0 \in \mathscr{k}_+$ and as usual put $w = f + w_0$. Then if $\bar{n}_f(w) \in \bar{N}$ is defined by* (3.5.2) *one has $\bar{n}_f(w)\, H_j(\bar{n}_f(w))^{-1} \subseteq \tilde{G}^w$. In fact if we put*

$$G_j^w = \bar{n}_f(w)\, H_j(\bar{n}_f(w))^{-1} \tag{3.5.7}$$

then the G_j^w for $j = 0, 1,..., 2^l - 1$ are the cosets of the identity component of $\tilde{G}^w$ so that

$$\tilde{G}^w = \bigcup_{j=0} G_j^w \tag{3.5.8}$$

is a disjoint union.

Proof. Since $w_0 \in \mathscr{k}_+$ one has $\tilde{G}^{w_0} = \tilde{M}H$ by (3.4.7). Thus the H_j are the cosets of the identity component of $\tilde{G}^{w_0}$. But since $\bar{n}_f(w)\, w_0 = w$ it follows immediately that the G^w, defined by (3.5.7), are the cosets of the identity component of $\tilde{G}^w$. Q.E.D.

Remark 3.5.2. One notes that G_1^w in the notation of (3.5.8) is in fact the identity component of $\tilde{G}^w$.

Our interest is not in G_1^w but in G_0^w. The following result is somewhat surprising (at least to us).

PROPOSITION 3.5. *If* $w_o \in \mathfrak{h}_+$ *and* $w = f + w_o$ *then in the notation of* (3.4.13) *and* (3.5.8) *one has*

$$\tilde{G}^w \cap G_{(*)} = G_0{}^w. \tag{3.5.9}$$

Proof. If $w_o \in \mathfrak{h}_+$ and $w = f + w_o$ we may apply the results of Section 2.6 to $G_{\mathbb{C}}{}^w$. That is, if $(G_{\mathbb{C}}{}^w)_{(*)} = G_{\mathbb{C}}{}^w \cap (G_{\mathbb{C}})_{(*)}$, where $(G_{\mathbb{C}})_* = s(\kappa)\,\bar{N}_{\mathbb{C}} H_{\mathbb{C}} N_{\mathbb{C}}$, then, by Theorem 2.6, $(G_{\mathbb{C}}{}^w)_{(*)}$ is a nonempty Zariski open subset of $G_{\mathbb{C}}{}^w$.

Now for $0 \leqslant j \leqslant 2^l - 1$ let

$$(\mathfrak{h}_+)_j = \{w_o \in \mathfrak{h}_+ \mid G_j{}^w \cap G_{(*)} \text{ is not empty}\}. \tag{3.5.10}$$

If $w_o \in (\mathfrak{h}_+)_j$ we assert that

$$\tilde{G}^w \cap G_{(*)} = G_j{}^w. \tag{3.5.11}$$

In order to prove (3.5.11) we first show that the right side is contained in the left side. Let $a \in G_j{}^w$. We must show $a \in G_{(*)}$ since of course $a \in \tilde{G}^w$ by (3.5.8). Now by assumption there exists $g \in G_j{}^w \cap G_{(*)}$. But $G_{(*)} \subseteq (G_{\mathbb{C}})_{(*)}$. On the other hand if we let $Z_{\mathbb{C}}(\gamma)$, where $\gamma = \mathscr{I}(w_o)$, be defined as in the proof of Theorem 3.2 and let $(\beta_{\mathbb{C}})_{(w)}$ be the map $\beta_{(w)}$ of Theorem 2.6 then by Theorem 2.6 one has an algebraic isomorphism

$$(\beta_{\mathbb{C}})_{(w)}\colon (G_{\mathbb{C}})_{(*)} \to Z_{\mathbb{C}}(\gamma), \tag{3.5.12}$$

where $(\beta_{\mathbb{C}})_{(w)}(c) = n(c)w$ for any $c \in (G_{\mathbb{C}})_{(*)}$ and $n(c) \in N_{\mathbb{C}}$ is defined by (2.6.2). But now since $g \in G_{(*)}$ one has $n(g) \in N$ and hence if $y = (\beta_{\mathbb{C}})_{(w)}(g) = n(g)w$ then $y \in \mathfrak{g}$. Furthermore the maximal diagonal component y_1 of y is given by (2.6.6). But $h(g) \in H$ so that $h(g)^{-\alpha_i} > 0$. Thus

$$\mathscr{I}(w_o) = \gamma \in \mathscr{I}(Z) \qquad \text{and} \qquad y = (\beta_{\mathbb{C}})_{(w)}(g) \in Z(\gamma). \tag{3.5.13}$$

Now since g and a are in the same connected component of $\tilde{G}^y$ one has $g^{-1}a \in \tilde{G}_e{}^w$ and hence if $d = n(g)(g^{-1}a)\,n(g)^{-1}$ then $d \in \tilde{G}_e{}^y = G_e{}^y$ (recalling (3.3.4) and (3.4.9)). But $G_e{}^y \subseteq G_*$ by (3.2.9). Thus $d \in (G_{\mathbb{C}}{}^y)_*$. But then by (2.6.11) there exists $a' \in (G_{\mathbb{C}}{}^w)_{(*)}$ such that $\psi_g(a') = n(g)\,g^{-1}a'n(g)^{-1} = d$. But then from the injectivity of (2.6.9) one has $a = a'$ so that $a \in (G_{\mathbb{C}}{}^w)_{(*)}$. Thus in particular $a \in s(\kappa)\,\bar{N}_{\mathbb{C}} H_{\mathbb{C}} N_{\mathbb{C}}$. But then, by Lemma 3.4.4, $s(\kappa)^{-1}a \in \tilde{G} \cap \bar{N}_{\mathbb{C}} H_{\mathbb{C}} N_{\mathbb{C}}$. Thus $s(\kappa)^{-1}a \in \bar{N}\tilde{M}HN$ by (3.4.10) so that we can write, using (2.6.2),

$$h(a) = m'h \tag{3.5.14}$$

for $m' \in \tilde{M}$, $h \in H$.

But if $z = (\beta_{\mathbb{C}})_{(w)}(a)$ then by the commutative diagram (2.6.21) one has $z = (\beta_{\mathbb{C}})_y(d) = \beta_y(d)$. But then $z \in Z(\gamma)$ by (3.2.15) since $d \in G_e{}^y$. Thus if z_1 is the maximal diagonal component of z then $z_1 = \sum r_i e_{\alpha_i}$, where $r_i > 0$. But by (2.6.6) one has $r_i = h(a)^{-\alpha_i}$. However, if m' in (3.5.14) is not the identity in $\tilde{M}$ there clearly exists $1 \leqslant j \leqslant l$ such that $m' e_{\alpha_j} = -e_{\alpha_j}$. This contradicts the positivity of r_j. Hence $m' = 1$ so that $a \in s(\kappa)\, \bar{N}HN = G_{(*)}$. Thus we have proved the right side of (3.5.11) is contained in the left side. We have also shown that if $w_o \in (\mathscr{h}_+)_j$ then

$$(\beta_{\mathbb{C}})_{(w)}(G_j{}^w) \subseteq Z(\gamma) \qquad \text{for} \quad \gamma = \mathscr{I}(w). \tag{3.5.15}$$

Now assume that b is in the left side of (3.5.11). Thus $b \in G_k{}^w$ for some k. Let g be as above. We have to show that $k = j$. But now $b \in \tilde{G}^w \cap G_{(*)} \subseteq (G_{\mathbb{C}}{}^w)_{(*)}$ and $(\beta_{\mathbb{C}})_{(w)}(b) = n(b)w \in Z_{\mathbb{C}}(\gamma)$, where $\gamma = \mathscr{I}(w)$. But $n(b) \in N$ so that if $z = (\beta_{\mathbb{C}})_{(w)}(b)$ then $z \in \mathscr{g}$. On the other hand if z_1 is the maximal diagonal component of z then $z_1 = \sum h(b)^{-\alpha_i} e_{\alpha_i}$ by (2.6.6). But $h(b) \in H$ so that $h(b)^{-\alpha_i} > 0$. Thus $z \in Z(\gamma)$ and hence

$$(\beta_{\mathbb{C}})_{(w)}(b) \in Z(\gamma). \tag{3.5.16}$$

Now let $a \in (G_{\mathbb{C}}{}^w)_{(*)}$ be any element such that $(\beta_{\mathbb{C}})_{(w)}(a) \in Z(\gamma)$. Put $v = (\beta_{\mathbb{C}})_{(w)}(a)$. Then by the commutative diagram (2.6.21) one has $v = (\beta_{\mathbb{C}})_y(d)$ for $d = \psi_g(a)$ using the notation of (3.5.13), where of course $d \in (G_{\mathbb{C}}{}^y)_*$ and $y = n(g)w$. But by Theorem 3.2 (see (3.2.15)) there exists $d' \in G_e{}^y$ such that $v = \beta_y(d')$. But since $(\beta_{\mathbb{C}})_y$ is an isomorphism one has $d' = d$ or $d = \psi_g(a) = n(g)\, g^{-1} a n(g)^{-1} \in G_e{}^y$. Thus $g^{-1}a \in \tilde{G}_e{}^w$. Hence $a \in G_j{}^w$. Thus we have proved

$$(\beta_{\mathbb{C}})^{-1}_{(w)}(Z(\gamma)) \subseteq G_j{}^w \qquad \text{in case} \quad w_o \in (\mathscr{h}_+)_j\,. \tag{3.5.17}$$

Thus $b \in G_j{}^w$ by (3.5.16) so that $j = k$. This proves the assertion.

Now for any $w_o \in \mathscr{h}_+$ and $0 \leqslant j \leqslant 2^l - 1$ let

$$m_j(w) = \bar{n}_f(w)\, m_j \bar{n}_f(w)^{-1}. \tag{3.5.18}$$

Since $m_j \in H_j$ one has

$$m_j(w) \in G_j{}^w \tag{3.5.19}$$

by (3.5.7). Now let σ_j be the map

$$\sigma_j\colon \mathscr{h}_+ \to \tilde{G}, \qquad w_o \to m_j(w). \tag{3.5.20}$$

It follows from the smoothness of (3.5.3) that σ_j is smooth. We assert

$$(\mathscr{k}_+)_j = \sigma_j^{-1}(G_{(*)}). \tag{3.5.21}$$

Indeed if $w_o \in (\mathscr{k}_+)_j$ then $G_j^w \subseteq G_{(*)}$ by (3.5.11). Thus $\sigma_j(w_o) \in G_{(*)}$. Conversely if $\sigma_j(w_o) \in G_{(*)}$ then $G_j^w \cap G_{(*)}$ is not empty. Hence $w_o \in (\mathscr{k}_+)_j$ by definition. This proves (3.5.21). We next assert that

$$\bigcup_{j=0} \mathscr{I}((\mathscr{k}_+)_j) = \mathscr{I}(Z). \tag{3.5.22}$$

We first observe that the left side of (3.5.22) is contained in the right side of (3.5.13). Now let $\gamma \in \mathscr{I}(Z)$. Then $\gamma = \mathscr{I}(w_o)$ for some $w \in \mathscr{k}_+$ by (3.3.5). Now let $y \in Z(\gamma)$. By Theorem 2.6 there exists $g \in (G_{\mathbb{C}}{}^w)_{(*)}$ such that $(\beta_{\mathbb{C}})_{(w)}(g) = n(g)w = y$, where, of course, $n(g) \in N_{\mathbb{C}}$. But $w, y \in \mathscr{b}_f$ so that by Proposition 2.3.2 one has $n(g) \in N$. But now using the notation of (2.6.6) one has $h(g)^{-\alpha_i} > 0$ since $y_1 \in \tilde{\mathscr{d}}_1$. Thus $h(g) \in H$. Furthermore recalling (2.6.7) and its notation one must have $h(g)y$, $\kappa^{-1}w_o + e \in e + \mathscr{b}$ and by (2.6.7) these elements are conjugate under $\bar{n}(g)$. But then $\bar{n}(g) \in \bar{N}$ by Proposition 2.3.2 with a reversal of positive and negative roots and a reversal of e and f. Thus $s(\kappa)^{-1}g = \bar{n}(g)\, h(g)\, n(g) \in G$. Consequently $g \in \tilde{G}^w \cap G_{(*)}$. But then $g \in G_j^w \cap G_{(*)}$ for some j by (3.5.8). But then $w_o \in (\mathscr{k}_+)_j$ by definition and hence $\gamma = \mathscr{I}(w_o) \in \mathscr{I}((\mathscr{k}_+)_j)$. This proves equality (3.5.22).

We next assert that $(\mathscr{k}_+)_j$ for any j is closed in $\mathscr{k}_+$. Indeed let $w_o \in (\mathscr{k}_+)_j \cap \mathscr{k}_+$. We first show that

$$(G_{\mathbb{C}}{}^w)_{(*)} \cap G_j^w \qquad \text{is not empty.} \tag{3.5.23}$$

Indeed $(G_{\mathbb{C}}{}^w)_{(*)}$ is not an empty Zariski open subset of $G_{\mathbb{C}}{}^w$ by Theorem 2.6. On the other hand $\mathscr{g}_{\mathbb{C}}{}^w$ is the complexification of $\mathscr{g}^w$ so that for any $g \in G_j^w$ the tangent space to G_j^w at g is a real form of the tangent space to $G_{\mathbb{C}}{}^w$ at g. Thus any regular function on $G_{\mathbb{C}}{}^w$ which vanishes on G_j^w must vanish on $G_{\mathbb{C}}{}^w$. Thus G_j^w is Zariski dense in $G_{\mathbb{C}}{}^w$. This proves (3.5.23).

Let $g \in (G_{\mathbb{C}}{}^w)_{(*)} \cap G_j^w$. But then $s(\kappa)^{-1}g \in \tilde{G} \cap \bar{N}_{\mathbb{C}}H_{\mathbb{C}}N_{\mathbb{C}}$ and hence by Lemma 3.4.3 one has $\bar{n}(g) \in \bar{N}$, $n(g) \in N$ and we can write $h(g) = m'h$, where $m' \in \tilde{M}$ and $h \in H$. Now if $m' = 1$ then $g \in G_{(*)}$ and hence $G_j^w \cap G_{(*)}$ is not empty. But then $w_o \in (\mathscr{k}_+)_j$. Thus in order to prove that $(\mathscr{k}_+)_j$ is closed in $\mathscr{k}_+$ it suffices to prove that $m' = 1$. Assume $m' \neq 1$. Then there exists $1 \leqslant k \leqslant l$ such that $m'e_{\alpha_k} = -e_{\alpha_k}$. But then if $y = n(g)w$ and we use the notation of (2.6.6) one has $y_1 = \sum r_i e_{\alpha_i}$, where $r_k < 0$. Now let $w_o^{(m)} \in (\mathscr{k}_+)_j$, $m = 1, 2, \ldots$, be a sequence such that $w_o^{(m)}$ converges to w_o. Put $w^{(m)} = f + w_o^{(m)}$. But by (3.5.7) there exists $g_o \in H_j$ so that $g = \bar{n}_f(w)\, g_o \bar{n}_f(w)^{-1}$. Put $g^{(m)} = \bar{n}_f(w^{(m)})\, g_o \bar{n}_f(w^{(m)})^{-1}$ so that $g^{(m)}$ converges to g. On the other hand $g^{(m)} \in G_j^{w^{(m)}}$ by (3.5.7). But since $w_o^{(m)} \in (\mathscr{k}_+)_j$ it follows that $g^{(m)} \in \tilde{G}^{w^{(m)}} \cap G_{(*)} \subseteq (G_{\mathbb{C}}^{w^{(m)}})_{(*)}$

by (3.5.11). Furthermore if $y^{(m)} = (\beta_{\mathbb{C}})_{(w^{(m)})}(g^{(m)}) = n(g^{(m)})\, w^{(m)}$ then $y^{(m)} \in Z$ by Theorem 2.6. But clearly $y^{(m)}$ converges to y since $g^{(m)}$ and $w^{(m)}$ respectively converge to g and w. Thus if $y_1^{(m)} = \sum_i r_i^{(m)} e_{\alpha_i}$ is the maximal diagonal component of $y^{(m)}$ then $r_k^{(m)}$ converges to $r_k < 0$. But $r_k^{(m)} > 0$ since $y^{(m)} \in Z$. This is a contradiction and hence $(\mathfrak{h}_+)_j$ is closed in $\mathfrak{h}_+$.

But now by (3.5.7), and recalling Lemma 3.5.2, $(\mathfrak{h}_+)_j$ is open in $\mathfrak{h}_+$ since $G_{(*)}$ is open in $\tilde{G}$. But clearly $\mathfrak{h}_+$ is connected. Hence for any $1 \leqslant j \leqslant 2^l - 1$ either $(\mathfrak{h}_+)_j$ is empty or $(\mathfrak{h}_+)_j = \mathfrak{h}_+$. Furthermore clearly by (3.5.11) the sets $(\mathfrak{h}_+)_j$, over all j, are mutually disjoint. But by (3.5.22), since Z is not empty, it follows that there exists a unique $0 \leqslant k \leqslant 2^l - 1$ such that

$$(\mathfrak{h}_+)_k = \mathfrak{h}_+ \qquad \text{and} \qquad (\mathfrak{h}_+)_j \text{ is empty if } j \neq k. \tag{3.5.24}$$

It follows then from (3.5.11) that

$$\tilde{G}^w \cap G_{(*)} = G_k{}^w \qquad \text{for any} \quad w_o \in \mathfrak{h}_+. \tag{3.5.25}$$

One also notes that (3.5.22) and (3.5.24) imply that

$$\mathscr{I}(\mathfrak{h}_+) = \mathscr{I}(Z). \tag{3.5.26}$$

It remains only to prove that $k = 0$.

Now if h_φ is given by (3.1.14) then for any $x \in \mathfrak{h}_{\mathbb{C}}$ one has $Q(x, h_\varphi) = \langle \varphi, x\rangle\, Q(e_\varphi, e_{-\varphi})$ by the invariance of Q. However, $Q(e_\varphi, e_{-\varphi}) = 1$ by (1.5.2). Thus if we consider the isomorphism

$$\mathfrak{h}_{\mathbb{C}} \to \mathfrak{h}'_{\mathbb{C}} \tag{3.5.27}$$

defined by Q then

$$h_\varphi \to \varphi \tag{3.5.28}$$

with respect to (3.5.27). Now one knows that the root normals h_{α_i} are a basis of $\mathfrak{h}$. Let $x_o \in \mathfrak{h}$ be as in Section 2.1 so that $\langle \alpha_i, x_o\rangle = 1$. Let $s_i \in \mathbb{R}$ be such that $x_o = \sum s_i h_{\alpha_i}$. We assert that

$$s_i > 0. \tag{3.5.29}$$

Indeed one identifies $\mathfrak{h}_{\mathbb{C}}$ with $\mathfrak{h}'_{\mathbb{C}}$ by (3.5.27) and if we write $x_o = \sum t_i \alpha_i$ then as one knows (see, e.g., Lemma 8.3 in [6, p. 166]), $t_i > 0$ since $Q(x_o, \alpha_i) > 0$. However, one has $s_i = t_i$ by (3.5.28). But this implies (3.5.29). (See also the argument in Lemma 15 in [18, p. 790.])

Now put

$$e_o = \sum s_i e_{\alpha_i}. \tag{3.5.30}$$

Since, as one knows, $[e_{\alpha_i}, e_{-\alpha_j}] = 0$ if $i \neq j$ it follows that $[e_o, f] = x_o$ so that if α is the $\mathbb{R}$-span of x_o, e_o, and f then α is a Lie subalgebra whose complexification $\alpha_{\mathbb{C}}$ is a principal TDS of $\mathfrak{g}_{\mathbb{C}}$ in the notation of [12]. See Section 5 in [12]. Furthermore we note that $e_o \in \tilde{\mathscr{A}}_1$ by (3.5.30) and hence there exists $a \in H$ such that

$$ae = e_o, \tag{3.5.31}$$

where e is given by (2.6.18).

Now let A and $A_{\mathbb{C}}$ be the subgroups of $G_{\mathbb{C}}$ corresponding to α and $\alpha_{\mathbb{C}}$. Since all the eigenvalues of $\operatorname{ad} x_o$ are integers it follows that $A_{\mathbb{C}}$ is isomorphic to the adjoint group of $\alpha_{\mathbb{C}}$. In particular all of our previous results and notation with respect to $\mathfrak{g}$, $\mathfrak{g}_{\mathbb{C}}$, G, and $G_{\mathbb{C}}$ may be applied to α, $\alpha_{\mathbb{C}}$, A, and $A_{\mathbb{C}}$. To indicate that such an application is being made we will use $L(A)$ for any previously defined notation L (e.g., $L = \bar{B}$, N, etc.).

Now recall that $m \in \tilde{M}$ is defined by the relations $me_{\alpha_j} = -e_{\alpha_j}$ for all j. But then clearly $m = \exp i\pi x_o$. Thus $m \in A_{\mathbb{C}}$. Now $x_o \in \mathfrak{h}_+$ and hence using our previous notation, where $w_o \in \mathfrak{h}_+$, we now fix w_o so that $w_o = x_o$. Note then that w_o and $w = f + w_o$ are both in α_o. If we now put $\bar{n}(A) = \mathbb{R}f$ and recall that if $\bar{n}_f(w) \in \bar{N}$ is the unique element given by (3.5.2) (so that $\bar{n}_f(w)\, w_o = w$) one clearly has

$$1 \neq \bar{n}_f(w) \in \bar{N}(A). \tag{3.5.32}$$

But now since obviously $mf = -f$ it follows that $\bar{n}_f(w)m = m(\bar{n}_f(w))^{-1}$. Thus if we put

$$g = \bar{n}_f(w)\, m\bar{n}_f(w)^{-1} \tag{3.5.33}$$

one has $g = (\bar{n}_f(w))^2 m$ and hence $g \in \bar{N}(A)\, \tilde{M}(A)$ where $\mathfrak{h}(A) = \mathbb{R}x_o$ so that

$$\{1, m\} = \tilde{M}(A). \tag{3.5.34}$$

Now $g \in \tilde{G}(A)$ and hence there exist $c, c', c'' \in \mathbb{R}$ such that $ge_o = c''e_o + c'x_o + cf$. We assert that

$$c > 0. \tag{3.5.35}$$

In fact for any $\bar{n} \in \bar{N}(A)$ it is clear that we may find $d(\bar{n})$, $d' \in \mathbb{R}$ such that $\bar{n}e_o = e_o + d'x_o + d(\bar{n})f$. We first prove that

$$d(\bar{n}) \leqslant 0 \qquad \text{and} \qquad d(\bar{n}) < 0 \qquad \text{if and only if} \quad \bar{n} \neq 1. \tag{3.5.36}$$

Indeed write $\bar{n} = \exp rf$, where $r \in \mathbb{R}$. However, $[f, [f, e_o]] = [x_o, f] = -f$.

Thus $d(\bar{n}) = -r^2/2$, which proves (3.5.36). But now $g = (\bar{n}_f(w))^2 m$, where $\bar{n}_f(w)^2 \neq 1$ by (3.5.32). However, $me_o = -e_o$. Thus (3.5.36) implies (3.5.35).

Now let $\mathfrak{n}(A) = \mathbb{R}e_o$. Furthermore from the bracket relations of f, x_o, and e_o it is clear that we can introduce a compact form of $\mathfrak{a}_{\mathbb{C}}$ and hence a $*$-operation, written $x \to x^{(*)}$ (to avoid confusion with $*$-operation in $\mathfrak{g}_{\mathbb{C}}$), so that $e_o^{(*)} = f$. But then we can fix $e(A) = e_o$ and $f(A) = f$. Put $s = s(\kappa)(A)$ so that $sx_o = -x_o$, $se_o = f$. If $a \in H$ is defined by (3.5.31) we assert that

$$sa = s(\kappa). \tag{3.5.37}$$

Indeed if $h = s(\kappa)^{-1} sa$ then clearly $hx_o = x_o$ since one of course has $\kappa x_o = -x_o$. But then $h \in G_{\mathbb{C}}^{x_o} = H_{\mathbb{C}}$ (recalling (3.4.8)). However, $he = e$ so that $h^{\alpha_i} = 1$ for i. Thus $h = 1$, proving (3.5.37).

But now the Gelfand–Bruhat decomposition of $A_{\mathbb{C}}$ becomes the disjoint union

$$A_{\mathbb{C}} = N_{\mathbb{C}}(A)\, sB_{\mathbb{C}}(A) \cup B_{\mathbb{C}}(A). \tag{3.5.38}$$

However, $B_{\mathbb{C}}(A)$ normalizes $\mathbb{C}e$ so that by (3.5.38) one must have $g \in N_{\mathbb{C}}(A)\, sB_{\mathbb{C}}(A)$. Thus there exist $n', n \in N_{\mathbb{C}}(A)$ and $h' \in H_{\mathbb{C}}(A)$ so that $g = n'sh'n$. Thus if $\bar{n}_1 = s^{-1}n's$ one has $\bar{n}_1 \in \bar{N}_{\mathbb{C}}$ and $s^{-1}g = \bar{n}_1 h'n$. But now $s^{-1}g \in \tilde{G}(A)$ so that by (3.4.11) one has $\bar{n}_1 \in \bar{N}(A)$ $n \in N(A)$ and $h' = m'h_1$, where $h_1 \in H(A)$ and m' is either m or 1 by (3.5.34). Put $b = s^{-1}g$ and write $be_o = re_o + r'x_o + r''f$, where $r, r', r'' \in \mathbb{R}$. Since $s^{-1}f = e_o$ it then follows from (3.5.35) that $r > 0$. However, $b = \bar{n}_1 h'n$. Hence $be_o = \bar{n}_1 h'e_o = (h')^{\alpha}\, \bar{n}_1 e_o$, where $\alpha = \alpha_1(A)$. But then $r = (h')^{\alpha}$ since $\bar{n}_1 e \in e + \bar{\mathfrak{b}}(A)$. Thus $(h')^{\alpha} > 0$ so that $m' = 1$. But then $s^{-1}g = \bar{n}_1 h_1 n$. However, $s^{-1} = as(\kappa)^{-1}$ by (3.5.37). Thus if we put $\bar{n} = a^{-1}\bar{n}_1 a$ and $h = a^{-1}h_1$ then $\bar{n} \in \bar{N}$ and $h \in H$ since of course $\bar{N}(A) \subseteq \bar{N}$ and $H(A) \subseteq H$ and one has $s(\kappa)^{-1}g = \bar{n}hn$. But then $g \in G_{(*)}$ by definition (see (3.2.1)). However, recalling (3.5.6) one has $m = m_0 \in H_0$. But then $g \in G_0{}^w$ by (3.5.7) and (3.5.33). Thus $G_0{}^w \cap G_{(*)}$ is not empty. Hence by definition $x_o = w_o \in (\mathfrak{h}_+)_0$ so that $(\mathfrak{h}_+)_0$ is not empty. Thus $k = 0$ by (3.5.24). The result then follows, as noted, by (3.5.25). Q.E.D.

3.6. The following is our main result in Section 3. Among other things it provides the structure for the integration of Hamilton's equation in Section 7.

THEOREM 3.6. *Let $\mathfrak{g}$ be a split real semi-simple Lie algebra. Let $\tilde{G}$ be the normalizer of $\mathfrak{g}$ in the complexified adjoint group of $\mathfrak{g}$. Let $l = \operatorname{rank} \mathfrak{g}$ and let $\mathcal{I}\colon \mathfrak{g} \to \mathbb{R}^l$ be the map defined by the l fundamental polynomial invariants on $\mathfrak{g}$. (See (2.3.1).)*

Now let $\mathfrak{h}_+$ be the open Weyl chamber of a split Cartan subalgebra $\mathfrak{h}$ defined *as in* (3.3.1). *Let* $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ *be the corresponding set of simple positive roots and let* $e_{\alpha_i}, e_{-\alpha_i}$ *be corresponding root vectors, normalized as in Section* 1.5. *Let* κ *be that element of the Weyl group* $W(\mathfrak{g}, \mathfrak{h})$ *such that* $\kappa\Pi = -\Pi$. *Let* $s(\kappa) \in \tilde{G}$ *be that unique element in the normalizer of* $\mathfrak{h}$ *such that* $s(\kappa)\, e_{\alpha_i} = e_{\kappa\alpha_i}$. *Let* $\mathfrak{n}$ *and* $\bar{\mathfrak{n}}$ *be the nilpotent subalgebras generated respectively by the* e_{α_i} *and the* $e_{-\alpha_i}$ *and let* $\bar{N}$, H, *and* N *be the subgroup of* $\tilde{G}$ *corresponding respectively to* $\bar{N}$, H, *and* N. *Also let* $f = \sum e_{-\alpha_i}$ *and let* Z, *as in Theorem* 3.2, *be the* $2l$*-dimensional manifold of all normalized Jacobi elements. That is, all elements* $y \in \mathfrak{g}$ *of the form* $y = f + x + \sum a_i e_{\alpha_i}$, *where* $x \in \mathfrak{h}$ *and* $a_i > 0$.

Now let $w_o \in \mathfrak{h}_+$ *and put* $w = f + w_o$. *Then for any* g *in the centralizer* $\tilde{G}^w$ *of* w *in* $\tilde{G}$ *there exist uniquely nonzero real numbers* $r_i(g)$, $i = 1, \ldots, l$, *such that* $ge_{\alpha_i} - r_i(g)\, e_{\alpha_i} \in \mathfrak{h} + \bar{\mathfrak{n}}$. *Put*

$$G_0^w = \{g \in \tilde{G}^w \mid \text{all } r_i(g) < 0\}. \tag{3.6.1}$$

This set may also be given by (3.5.7). *Then for any* $g \in G_0^w$ *there exist uniquely* $\bar{n}(g) \in N$, $h(g) \in H$, *and* $n(g) \in N$ *such that* $g = s(\kappa)\, \bar{n}(g)\, h(g)\, n(g)$. *Furthermore if* $\gamma = \mathscr{I}(w_o) \in \mathbb{R}^l$ *and* $Z(\gamma) = \mathscr{I}^{-1}(\gamma) \cap Z$ *then* $n(g)w \in Z(\gamma)$ *defining a map*

$$\beta_{(w)}\colon G_0^w \to Z(\gamma). \tag{3.6.2}$$

Moreover G_0^w *and* $Z(\gamma)$ *are manifolds and* (3.6.2) *is a diffeomorphism. In fact both manifolds are diffeomorphic to* $\mathbb{R}^l$. *Finally*

$$Z = \bigcup Z(\gamma) \tag{3.6.3}$$

is a disjoint union where, writing $\gamma = \mathscr{I}(w_o)$, *the union is over the open Weyl chamber* $\mathfrak{h}_+$ *as an index set.*

Proof. Now by (3.5.6) and (3.5.7) one has $\tilde{G}^w = \bar{n}_f(w)\, \tilde{M}H\bar{n}_f(w)^{-1}$. If $g \in \tilde{G}^w$ then by (3.5.8) there exist $0 \leqslant j \leqslant 2^l - 1$ and $h \in H$ such that $g = \bar{n}_f(w)\, m_j h \bar{n}_f(w)^{-1}$. Let $d_{ij} = \{0, 1\}$, $i = 1, \ldots, l$, be such that $m_j e_{\alpha_i} = (-1)^{d_{ij}}\, e_{\alpha_i}$. But since $\bar{n}e_{\alpha_i} - e_{\alpha_i} \in \mathfrak{h} + \bar{\mathfrak{n}} = \bar{\mathfrak{b}}$ it follows that for

$$r_i(g) = (-1)^{d_{ij}}\, h^{\alpha_i} \tag{3.6.4}$$

one has $ge_{\alpha_i} - r_i(g)\, e_{\alpha_i} \in \mathfrak{h} + \bar{\mathfrak{n}}$. This proves the first statement. Furthermore by (3.6.4) it follows that $r_i(g) < 0$ if and only if $d_{ij} = 1$ for $i = 1, \ldots, l$. But then by definition of $m = m_0$ this is the case if and only if $j = 0$. Thus G_0^w can be given by (3.6.1). Next the existence and uniqueness of the decomposition $g = s(\kappa)\, \bar{n}(g)\, h(g)\, n(g)$ for $g \in G_0^w$ are given by (3.5.9) and (3.4.14). But then $G_0^w \subseteq (G_{\mathbb{C}}^w)_{(*)}$, recalling (2.6.4), and hence (see (2.6.15)) the map $(\beta_{\mathbb{C}})_{(w)}$

is defined on G_0^w. By (3.5.15), recalling (3.5.24), one has $(\beta_{\mathbb{C}})_{(w)}(G_0^w) \subseteq Z(\gamma)$ so that (3.6.2) is defined where

$$\beta_{(w)} = (\beta_{\mathbb{C}})_{(w)} \mid G_0^w. \tag{3.6.5}$$

But then recalling that (3.5.12) is bijective it follows from (3.5.17), for $j = 0$, that (3.6.2) is also bijective. However, $Z(\gamma)$ is clearly an l-dimensional submanifold of $Z_{\mathbb{C}}(\gamma)$ by Theorem 3.2 and furthermore one has a diffeomorphism $Z(\gamma) \cong \mathbb{R}^l$ by Theorem 3.2. Also, G_0^w is clearly a submanifold of $(G_{\mathbb{C}}^w)_{(*)}$ and $G_0^w \cong \mathbb{R}^l$ by (3.5.7). Thus (3.6.2) is a diffeomorphism by (3.6.5) since $(\beta_{\mathbb{C}})_{(w)}$ is an algebraic isomorphism, recalling (2.6.15), and hence a diffeomorphism. The relation (3.6.3) now follows from (2.3.3), (3.5.26), and Proposition 3.3.1. Q.E.D.

As in the complex case, we will refer to the submanifolds $Z(\gamma)$ of Z (now in the real case) for $\gamma \in \mathscr{I}(\mathscr{h}_+)$, as the isospectral leaves of Z.

Remark 3.6. Let $w_o \in \mathscr{h}_+$, $w = f + w_o$, $\gamma = \mathscr{I}(w_o)$, and $g \in G_0^w$. Since $G_0^w \subseteq (G_{\mathbb{C}}^w)_{(*)}$ the diffeomorphism $\psi_g\colon (G_{\mathbb{C}}^w)_{(*)} \to (G_{\mathbb{C}}^y)_*$, where $y = n(g)w \in Z(\gamma)$ is defined on G_0^w. If $a \in (G_{\mathbb{C}}^w)_{(*)}$ we recall that $\psi_g(a) = n(g)\, g^{-1}an(g)^{-1}$. It follows then from Remark 3.5.2 that $\psi_g(G_0^w) = G_e^y$. Furthermore since, recalling Theorem 3.2 and its proof,

$$\beta_y = (\beta_{\mathbb{C}})_y \mid G_e^y \tag{3.6.6}$$

the commutative diagram (2.6.21) of algebraic isomorphisms, upon restriction, becomes a commutative diagram

$$\begin{tikzcd} G_0^w \arrow[r, "\beta_{(w)}"] \arrow[d, "\psi_g"'] & Z(\gamma) \\ G_e^y \arrow[ur, "\beta_y"'] & \end{tikzcd} \tag{3.6.7}$$

where all three maps are diffeomorphisms. If $g, a \in G_0^w$ and $\psi_g(a) = n(g)\, g^{-1}an(g)^{-1} = d \in G_e^y$ then by (2.6.13) one has

$$n(a) = n_d n(g), \tag{3.6.8}$$

where now $n(a)$, $n(g)$, and n_d are in N.

One consequence of (3.6.3), or rather its proof, is the answer to the following question (which perhaps is of independent interest): What is a necessary and sufficient condition that an element $x \in \mathscr{g}$ "can be put" in Jacobi form? The answer is: The element x should be a regular real semi-simple element. That is, PROPOSITION 3.6. *Let $x \in \mathfrak{g}$. A necessary and sufficient condition that x be G-conjugate to an element in Z is that $x \in R_+$. (See Section* 3.3.)

Proof. One has $Z \subseteq R_+$ by (3.3.4). Thus the condition is necessary. But if $x \in R_+$ then x is G-conjugate to an element $w_o \in \mathfrak{h}_+$ by Lemma 3.3.2. However, recalling (3.5.26) one has

$$\mathscr{I}(Z) = \mathscr{I}(\mathfrak{h}_+).$$

Thus there exists $y \in Z$ such that $\mathscr{I}(y) = \mathscr{I}(w_o)$. Hence y is G-conjugate to an element in $\mathfrak{h}_+$ by Lemma 3.3.2. This element must be w_o by Proposition 3.3.1. Thus x and y are G-conjugate. Q.E.D.

3.7. Finally, in this section we show that the space Z of normalized Jacobi elements may be smoothly parametrized by the product $H \times \mathfrak{h}_+$ of the split Cartan subgroup H and open Weyl chamber $\mathfrak{h}_+$.

THEOREM 3.7. *Let $\mathfrak{h}_+$, $\bar{N}$, H, N, $\tilde{G}$, and f be as in Theorem* 3.6. *Let $(g_o, w_o) \in H \times \mathfrak{h}_+$ and let $w = f + w_o$. Let $\bar{n}_f(w) \in \bar{N}$ be as in* (3.5.2) *so that $\bar{n}_f(w)\, w_o = w$. Let $m \in \tilde{G}$ be as in Section* 3.5 *so that m fixes H and $me_{\alpha_i} = -e_{\alpha_i}$, $i = 1,\ldots, l$. Put*

$$g = \bar{n}_f(w)\, mg_o \bar{n}_f(w)^{-1} \tag{3.7.1}$$

so that $g \in G_0{}^w$ by (3.5.7). *Let $n(g) \in N$ be as in Theorem* 3.6 *so that $n(g)w \in Z$ by Theorem* 3.6. *Now let*

$$H \times \mathfrak{h}_+ \to Z \tag{3.7.2}$$

be the map defined by $(g_o, w_o) \mapsto n(g)w$. Then (3.7.2) *is a diffeomorphism.*

Proof. It is obvious using Lemma 3.5.2 and (3.7.1) that the map

$$H \times \mathfrak{h}_+ \to G_0{}^w, \qquad (g_o, w_o) \mapsto g \tag{3.7.3}$$

is smooth. See (3.5.3). But then the smoothness of (3.4.15) implies that the map $H \times \mathfrak{h}_+ \to N$, $(g_o, w_o) \mapsto n(g)$ is smooth. It follows immediately then that (3.7.2) is smooth. But now if $\mathscr{I}(w_o) = \gamma$ then, by Theorem 3.6, the restriction of (3.7.2) to $H \times \{w_o\}$ maps $H \times \{w_o\}$ bijectively onto $Z(\gamma)$. But then (3.7.2) is a bijection since (3.6.3) is, by Theorem 3.6, a disjoint union. To prove Theorem 3.7 it therefore suffices to show that (3.7.2) is a local diffeomorphism. Let T, U, and V be the tangent space to $H \times \mathfrak{h}_+$, $H \times \{w_o\}$, and $\{g_o\} \times \mathfrak{h}_+$, respectively, at (g_o, w_o) so that $T = U \oplus V$. If σ denotes the map (3.7.2) it suffices then, by dimension, to show that its differential σ_* is injective on T. But now clearly the restriction $\sigma \mid H \times \{w_o\} = \beta_{(w)} \circ \tau$, where τ is the restriction of (3.7.3) to $H \times \{w_o\}$ and $\beta_{(w)}$ is given by (3.6.2). However, $\beta_{(w)}$ is a diffeomorphism by Theorem 3.6 and τ, given by group multiplication (see (3.7.1)), is obviously a diffeomorphism. Thus σ induces a diffeomorphism $H \times \{w_o\} \to Z(\gamma)$. However, $Z(\gamma)$ is an l-dimensional submanifold of Z by Theorem 3.2. Thus $\sigma_* \mid U$ is injective. Now clearly we may identify V with the tangent space to $\mathfrak{h}_+$ at w_o. Let $0 \neq v \in V$. Then by Proposition 3.3.1 there exists an invariant $I \in S(\mathfrak{g})^G$ such that $\langle v, dI \rangle \neq 0$. But since (3.6.2) is defined by the action of the adjoint group one has $\langle \sigma_* v, dI \rangle = \langle v, dI \rangle \neq 0$. This implies $\sigma_* v \notin \sigma_*(U)$ since clearly $\sigma_*(U)$ is the tangent space to $Z(\gamma)$ at $\sigma(g_o, w_o)$ and $I \mid Z(\gamma)$ is a constant function. Thus not only is $\sigma_* \mid V$ injective but $\sigma_*(V) \cap \sigma_*(U) = 0$. This implies that σ_* is injective on T. Q.E.D.

Given an element $y \in Z$ we will later refer to $g_o \in H$ and $w_o \in \mathfrak{h}_+$ as its $(H \times \mathfrak{h}_+)$ parameters in case (g_o, w_o) corresponds to y by (3.7.2).

Remark 3.7. If $y \in Z$ and (g_o, w_o) are its $H \times \mathfrak{h}_+$ parameters note that

$$y = f + z + \sum_{i=1}^{l} h(g)^{-\alpha_i} e_{\alpha_i}, \tag{3.7.4}$$

where $z \in \mathfrak{h}$ and g is given by (3.7.1). This is clear from (3.6.5) and the application of (2.6.6) to $\mathfrak{g}_{\mathbb{C}}$.

4. The Isospectral Leaf $Z(\gamma)$ as a Complete, Flat, Affinely Connected Manifold

4.1. In Section 4, F is either $\mathbb{R}$ or $\mathbb{C}$. As in Section 2.1 let $\mathfrak{g}$ be a semisimple Lie algebra over F which is split if $F = \mathbb{R}$ and where $l = \operatorname{rank} \mathfrak{g}$ let Z be the $2l$-dimensional manifold of normalized Jacobi elements defined as in Section 2.2. For any invariant $I \in S(\mathfrak{g})^G$ we recall that ξ_I is the vector field on Z defined so that $(\xi_I)_y = [y, (\delta_{\mathfrak{z}} I)(y)]$ for any $y \in Z$. See (2.2.19). Also, $\mathfrak{z}_y$ is the subspace of the tangent space to Z at y whose elements are all vectors of the form $(\xi_I)_y$, $I \in S(\mathfrak{g})^G$. See (2.2.21). Furthermore we note that by Theorem 2.2 the map $y \to \mathfrak{z}_y$ defines a smooth distribution (in the sense of E. Cartan) on Z of dimension l, which we denote by $\mathscr{Z}$. Of course any smooth involutory distribution on a manifold defines a foliation of the manifold by the family of maximal integral submanifolds. We refer to this foliation as the corresponding foliation. On the other hand for the case of Z one already has the disjoint union

$$Z = \bigcup_{\gamma \in \mathscr{S}(Z)} Z(\gamma), \tag{4.1.1}$$

where the isospectral leaves $Z(\gamma)$ by Theorems 2.5 and 3.6 are connected submanifolds of dimension l.

PROPOSITION 4.1. *The distribution $\mathscr{X}$ on Z is involutory and* (4.1.1) *is the corresponding foliation.*

Proof. As in Section 2.2 let

$$\varkappa = \{\xi_I \mid I \in S(\mathfrak{g})^G\}. \tag{4.1.2}$$

By Theorem 2.6, $\varkappa$ is a Lie algebra (in fact, a commutative Lie algebra). This proves that $\mathscr{X}$ is involutory.

Since $Z(\gamma)$ is a closed connected submanifold of dimension l for any $\gamma \in \mathscr{I}(Z)$ it then suffices only to prove

$$\varkappa_y = T_y(Z(\gamma)) \tag{4.1.3}$$

for any $y \in Z(\gamma)$. Here of course the right side of (4.1.3) is the tangent space to $Z(\gamma)$ at y. However, since $\dim Z = 2l$ this is an immediate consequence of (1.6.8), Proposition 2.3.1, and Theorem 2.2. Q.E.D.

4.2. Now let Λ be any smooth manifold. If Λ has an affine connection we can speak of covariant constant vector fields ξ (either global or local) on Λ. If the affine connection is flat (i.e., the curvature and torsion vanish) then in a sufficiently small neighborhood V of any point the space $\mathfrak{a}$ of such vector fields spans the tangent space at all points of V and is a commutative Lie algebra. On the other hand one easily sees that if $\mathfrak{a}$ is any commutative Lie algebra of vector fields on Λ which spans the tangent space at each point of Λ then there exists a unique flat affine connection such that the elements of $\mathfrak{a}$ are covariant constant.

Remark 4.2. One notes in particular that if Λ is an open subset of a Lie group A whose Lie algebra is Abelian then there is a unique flat affine connection on Λ such that any restriction Λ of a left invariant vector field on A is covariant constant.

Now if $I \in S(\mathfrak{g})^G$ and $\gamma \in \mathscr{I}(Z)$ then the vector field ξ_I is tangent to the leaf $Z(\gamma)$ by Proposition 4.1. Thus $\xi_I \mid Z(\gamma)$ is a vector field on $Z(\gamma)$.

PROPOSITION 4.2. *Let $\gamma \in \mathscr{I}(Z)$ be arbitrary. Then there exists a unique flat affine connection in the isospectral leaf $Z(\gamma)$ such that $\xi_I \mid Z(\gamma)$ is covariant constant for any invariant $I \in S(\mathfrak{g})^G$.*

Proof. Let $\varkappa(\gamma) = \{\xi_I$ restricted to $Z(\gamma) \mid I \in S(\mathfrak{g})^G\}$. By (4.1.3) it is clear that $\varkappa(\gamma)$ spans the tangent space at all points of $Z(\gamma)$. Thus it suffices to observe that $\varkappa(\gamma)$ is a commutative Lie algebra. But if $\varkappa$ is defined by (2.2.20) then $\varkappa$ is a commutative Lie algebra by Theorem 2.2. Thus $\varkappa(\gamma)$ is also a commutative Lie algebra since it is a homomorphic image of $\varkappa$. Q.E.D.

Henceforth any isospectral leaf $Z(\gamma)$ will also be regarded as having the structure of an affinely connected manifold where the affine connection is given by Proposition 4.2.

4.3. Now let $x \in \mathfrak{g}$ and $I \in S(\mathfrak{g})^G$. Recalling the definition of $\tilde{G}$ if $F = \mathbb{R}$ (see (3.4.4)), put $\tilde{G} = G$ if $F = \mathbb{C}$. Now $(\delta I)(x) \in \mathfrak{g}^x$ by (1.3.3) and hence I defines a (left invariant) vector field $L_I{}^x$ on the centralizer $\tilde{G}^x$ of x in $\tilde{G}$ such that if $g \in \tilde{G}^x$ and ψ is a smooth function on $\tilde{G}^x$ one has

$$(L_I{}^x\psi)(g) = \frac{d}{dt}\,\psi(g \exp t(\delta I)(x))\,|_{t=0}\,. \tag{4.3.1}$$

PROPOSITION 4.3. *Assume $x \in R$ (i.e., x is regular) and A is any open set in $\tilde{G}^x$. Then there exists a unique flat affine connection on A such that $L_I{}^x \mid A$ is covariant constant for any $I \in S(\mathfrak{g})^G$.*

Proof. Recall that $I_j \in S(\mathfrak{g})^G$, $j = 1,\ldots, l$, are the fundamental invariants. Since $x \in R$ then, as already noted in the proof of Proposition 2.1, the elements $(\delta I_j)(x)$ are linearly independent. But since $(\delta I)(x) \in \operatorname{cent} \mathfrak{g}^x$ by (1.3.3) for any $I \in S(\mathfrak{g})^G$ and $\dim \mathfrak{g}^x = l$ one recovers not only the well-known fact that $\mathfrak{g}^x$ is Abelian but also the fact that any element in $\mathfrak{g}^x$ is uniquely of the form $(\delta I)(x)$ for some $I \in S(\mathfrak{g})^G$. Thus any left (or equivalently right) invariant vector field on G^x is of the form $L_I{}^x$ for some $I \in S(\mathfrak{g})^G$. The result then follows immediately as noted in Remark 4.2. Q.E.D.

We now show that the isomorphisms of Theorems 2.4, 2.6, 3.2, and 3.6 are also isomorphisms of flat affinely connected manifolds.

THEOREM 4.3. *Let $\mathfrak{g}$ be a semi-simple Lie algebra over $F = \mathbb{R}$ or $\mathbb{C}$ which is split if $F = \mathbb{R}$. Let $l = \operatorname{rank} \mathfrak{g}$ and let $\mathcal{I}\colon \mathfrak{g} \to \mathbb{F}^l$ be the map defined as (2.3.1) by the fundamental invariant polynomials on $\mathfrak{g}$. Let $\mathfrak{h}$ be as in Section 2.1 so that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. (If $F = \mathbb{R}$ then $\mathfrak{h}$ is split and $\mathfrak{h}_+$ is an open Weyl chamber in $\mathfrak{h}$ defined as in (3.3.1).)*

Now let $w_0 \in \mathfrak{h}$, where $w_0 \in \mathfrak{h}_+$ if $F = \mathbb{R}$, and put $w = f + w_0$, where f is defined by (1.5.4). Let $\gamma = \mathcal{I}(w_0)$ and let $Z(\gamma)$ be the corresponding isospectral leaf of normalized Jacobi elements defined as in Section 2.3. Let $Z(\gamma)$ have the structure of a flat affinely connected manifold as in Proposition 4.2. Put $G_o{}^w = G^w_{(*)}$ if $F = \mathbb{C}$ and $G_o{}^w = G_0{}^w$ if $F = \mathbb{R}$ using the notation of (3.5.7) so that (since $w \in R$ and $G_o{}^w$ is open in $\tilde{G}^w$) $G_o{}^w$ has, as given by Remark 4.2, the structure of a flat affinely connected manifold. Let $\beta_{(w)}\colon G_o{}^w \to Z(\gamma)$ be the map defined as in (2.6.15) and (3.6.2) so that if $a \in G_o{}^w$ and $n(a)$ is defined by (2.6.2) and (3.4.14) then $\beta_{(w)}(a) = n(a)w$. Let $g \in G_o{}^w$ and let $y = \beta_{(w)}(g)$. Put $G_o{}^y = G^y_*$ if $F = \mathbb{C}$ and $G_o{}^y = G_e{}^y$ if $F = \mathbb{R}$ using the notation of (2.4.9) and (3.1.23) so that (since $y \in R$ and $G_o{}^y$ is open in $\tilde{G}^y$) $G_o{}^y$ also has, as given by Remark 4.2, the structure of a flat affinely connected manifold. Now for any $a \in G_o{}^w$ let $\psi_g(a) =$

$n(g)\,g^{-1}an(g)^{-1}$. Then $\psi_g(G_o{}^w) \subseteq G_o{}^y$. Finally let $\beta_y\colon G_o{}^y \to Z(\gamma)$ be the map given by (2.4.11) *and* (3.2.15) *so that if $d \in G_o{}^y$ and n_d is defined by* (2.4.6) *and* (3.2.12) *then $\beta_y(d) = n_d y$. Then one has a commutative diagram*

$$\begin{tikzcd} G_o{}^w \arrow[r, "\beta_{(w)}"] \arrow[d, "\psi_g"'] & Z(\gamma) \\ G_o{}^y \arrow[ur, "\beta_y"'] & \end{tikzcd} \tag{4.3.2}$$

Furthermore not only are the three maps in (4.3.2) *diffeomorphisms but they are isomorphisms of flat affinely connected manifolds. In fact for any invariant $I \in S(\mathfrak{g})^G$ one has for the corresponding mappings of vector fields, $\psi_g(L_I{}^w \mid G_o{}^w) = L_I{}^y \mid G_o{}^y$ and*

$$\beta_{(w)}(L_I{}^w \mid G_o{}^w) = \beta_y(L_I{}^y \mid G_o{}^y) = \xi_I \mid Z(\gamma), \tag{4.3.3}$$

where the vector fields $L_I{}^w$, $L_I{}^y$, and ξ_I are defined by (4.3.1) *and* (2.2.19).

Proof. The fact that w and y are in R, as noted previously, is a consequence of (2.1.7) since $w, y \in \mathcal{E}_f$. Now assume first that $F = \mathbb{C}$. Then $G^w_{(*)}$ and G^y_* are open respectively in $\tilde{G}^w$ and $\tilde{G}^y$ since in fact they are Zariski open. One has $\psi_g G^w_{(*)} \subseteq G^y_*$ by (2.6.11). The fact that the diagram (4.3.2) is commutative and all three maps are diffeomorphisms (in fact they are algebraic isomorphisms) was established in the proof of Theorem 2.6. See (2.6.21).

Now let $I \in S(\mathfrak{g})^G$. Then by (1.3.1) one has $n(g)(\delta I)(w) = (\delta I)(n(g)w) = (\delta I)(y)$. Thus since $n(g)\,\tilde{G}^w n(g)^{-1} = \tilde{G}^y$, conjugation by $n(g)$ carries $L_I{}^w$ to $L_I{}^y$. On the other hand $L_I{}^w$ is fixed by the left translation of $\tilde{G}^w$ by g^{-1}. Thus $\psi_g(L_I{}^w \mid G_o{}^w) = L_I{}^y \mid G_o{}^y$. Hence of course by the commutativity of (4.3.2) one has $\beta_{(w)}(L_I \mid G_o{}^w) = \beta_y(L_I{}^y \mid G_o{}^y)$. It suffices only to show (for the case where $F = \mathbb{C}$) that the vector field is $\xi_I \mid Z(\gamma)$. But now since g is an arbitrary element in $G_o{}^w$ it thus suffices to show

$$\beta_{(w)}((L_I{}^w)_g) = (\xi_I)_y\,. \tag{4.3.4}$$

Now let $g(t) = \exp t(\delta I)(w)$ for $t \in \mathbb{R}$. Put $a(t) = gg(t)$ so that $(L_I{}^w)_g$ is the tangent vector to $a(t)$ at $t = 0$. Let $d(t) = \psi_g(a(t)) = n(g)\,g(t)\,n(g)^{-1}$. Thus

$$d(t) = \exp t((\delta I)(y)) \tag{4.3.5}$$

since $n(g)w = y$. But if v is the left side of (4.3.4) then from the commutativity of (4.3.2) it follows that v is the tangent vector to $\beta_{(w)}(a(t)) = \beta_y(d(t))$ at $t = 0$. Now recalling (2.4.6) put $\bar{n}_{d(t)}h_{d(t)} = \bar{b}_{d(t)}$ so that

$$d(t) = \bar{b}_{d(t)}n_{d(t)}\,, \tag{4.3.6}$$

where we note that $\bar{b}_{d(t)} \in \bar{B}$. One notes also that $\bar{b}_{d(0)} = n_{d(0)} = 1$. Now let $x \in n$ be the tangent vector to $n_{d(t)}$ at $t = 0$. Then since $\beta_y(d(t)) = n_{d(t)} y$ it follows that

$$v = [x, y]. \tag{4.3.7}$$

On the other hand if $z \in \bar{b}$ is the tangent vector to $\bar{b}_{d(t)}$ at $t = 0$ then by (4.3.5) and (4.3.6) one has

$$(\delta I)(y) = z + x. \tag{4.3.8}$$

But then recalling (1.2.9) one must have $x = (\delta_n I)(y)$ so that $v = [(\delta_n I)(y), y]$. But then $v = [y, (\delta_{\bar{b}} I)(y)]$ by (1.3.3) so that $v = (\xi_I)_y$ by the definition of ξ_I. This proves (4.3.4) and hence the theorem is proved for $F = \mathbb{C}$.

Now assume $F = \mathbb{R}$. Then G_o^w and G_o^y are, by definition, the connected components G_0^w and G_e^y (recall that G_0^w is not the identity component), respectively, of $\tilde{G}^w$ and $\tilde{G}^y$. In particular they are, respectively, open subsets of $\tilde{G}^w$ and $\tilde{G}^y$.

Now the fact that the diagram (4.3.2) is commutative and that the three maps in question are diffeomorphisms has been established in the proof of Theorem 3.6. See Remark 3.6 and (3.6.7). In fact these statements were proved by first applying Theorem 2.6 (and its proof) to $\mathfrak{g}_\mathbb{C}$ and then by restricting our considerations of

$$(G_\mathbb{C}^w)_{(*)}, \qquad (G_\mathbb{C}^y)_*, \qquad \text{and} \qquad Z_\mathbb{C}(\gamma) \tag{4.3.9}$$

to the respective real submanifolds

$$G_0^w, \qquad G_e^y, \qquad \text{and} \qquad Z(\gamma). \tag{4.3.10}$$

But now again from what we have proved above in the complex case one has (4.3.3), where the vector fields in question are, in the obvious order, defined on the manifolds in (4.3.9). However, they are clearly tangent, in the same order, to the real submanifolds appearing in (4.3.10). Hence one has (4.3.3) for the case $F = \mathbb{R}$. Q.E.D.

Remark 4.3. We point out here that at least in one way there is a significant difference between the real and the complex cases of Theorem 4.3. Namely, it is only in the real case that the manifolds in question, i.e., in (4.3.10), are complete with respect to their flat affine connections. This completeness will be more meaningful when we deal with the integration of Hamilton's equations in Section 7.

5. Representations and the Function $\Phi_\lambda(g_0, w_0; t)$

5.1. Henceforth, that is, for the remainder of the paper, we assume $F = \mathbb{R}$ so that $\mathfrak{g}$ is a real split semi-simple Lie algebra and all of our previous results and notations for the case $F = \mathbb{R}$ apply.

Let $w_0 \in \mathfrak{h}_+$ and as usual put $w = f + w_0$. Let $a \in G_0^w$ so that $h(a) \in H$ (see (3.4.14)). Now recalling (2.1.1) one has $\mathfrak{h}' = \mathfrak{h}(H)$ so that $h(a)^\lambda$ is defined for any $\lambda \in \mathfrak{h}'$. Our main results (see Theorem 7.5) depend upon a certain formula for $h(a)^\lambda$. Obtaining this formula will be the main objective of Section 5. The formula will arise from the finite-dimensional representation theory of $\mathfrak{g}$—which we now consider.

Now the restriction $Q \mid \mathfrak{h}_{\mathbb{C}}$ induces a nonsingular bilinear form—also denoted by Q—on the dual space $\mathfrak{h}'_{\mathbb{C}}$. We may of course regard $\mathfrak{h}'$ as a real form of $\mathfrak{h}'_{\mathbb{C}}$ and note that Q is positive definite on $\mathfrak{h}'$. Now let Λ be the lattice in $\mathfrak{h}'$ defined by

$$\Lambda = \left\{\lambda \in \mathfrak{h}' \;\middle|\; \frac{2Q(\lambda, \alpha_i)}{Q(\alpha_i, \alpha_i)} \in \mathbb{Z},\ i = 1,..., l\right\}. \tag{5.1.1}$$

Now if $\nu_j \in \Lambda$, $j = 1,..., l$, are defined by the relation $2Q(\nu_j, \alpha_i)/Q(\alpha_i, \alpha_i) = \delta_{ij}$ then of course one has the direct sum

$$\Lambda = \sum_{i=1}^{l} \mathbb{Z}\nu_i . \tag{5.1.2}$$

Now let $G_{\mathbb{C}}^s$ be a fixed simply connected Lie group having $\mathfrak{g}_{\mathbb{C}}$ as its Lie algebra. The adjoint representation defines a homomorphism Ad: $G_{\mathbb{C}}^s \to G_{\mathbb{C}}$ and in fact it defines an exact sequence

$$(1) \to \text{cent } G_{\mathbb{C}}^s \to G_{\mathbb{C}}^s \xrightarrow{\text{Ad}} G_{\mathbb{C}} \to (1). \tag{5.1.3}$$

If $x \in \mathfrak{g}_{\mathbb{C}}$ then to avoid confusion we will write $\exp^s x$ for its exponential image in $G_{\mathbb{C}}^s$ and retain the previous notation $\exp x$ for its exponential image in the adjoint group $G_{\mathbb{C}}$. Also let $H_{\mathbb{C}}^s$, $N_{\mathbb{C}}^s$, and $\bar{N}_{\mathbb{C}}^s$ be the subgroups of $G_{\mathbb{C}}^s$ corresponding respectively to $\mathfrak{h}_{\mathbb{C}}$, $\mathfrak{n}_{\mathbb{C}}$, and $\bar{\mathfrak{n}}_{\mathbb{C}}$.

Now if $\mathfrak{h}'_{\mathbb{C}}(H_{\mathbb{C}}^s) \subseteq \mathfrak{h}'_{\mathbb{C}}$ is defined by (2.1.1), where $\mathfrak{h}'_{\mathbb{C}}$ replaces $\mathfrak{h}'$, $\mathfrak{h}_{\mathbb{C}}$ replaces $\mathfrak{h}$, and $\exp^s$ replaces exp, one knows that

$$\Lambda = \mathfrak{h}'_{\mathbb{C}}(H_{\mathbb{C}}^s) \tag{5.1.4}$$

so that h^ν is defined for any $\nu \in \Lambda$, $h \in H_{\mathbb{C}}^s$. The elements of Λ are often then referred to as the integral linear forms (on $H_{\mathbb{C}}^s$). The cone $D \subseteq \Lambda$ of dominant integral linear forms is defined by

$$D = \{\lambda \in \Lambda \mid Q(\lambda, \alpha_i) \geqslant 0,\ i = 1,..., l\}. \tag{5.1.5}$$

If $\mathbb{Z}_+$ is the set of nonnegative integers one clearly has

$$D = \sum_{i=1}^{l} \mathbb{Z}_+ \nu_i . \tag{5.1.6}$$

Now a complex vector space $V_\mathbb{C}$ will be called a $G_\mathbb{C}{}^s$ module if it is understood that there is a representation π: $G_\mathbb{C}{}^s \to \operatorname{Aut} V_\mathbb{C}$ of $G_\mathbb{C}{}^s$ on $V_\mathbb{C}$. In such a case we will write $gv \in V_\mathbb{C}$ for $\pi(g)v$, where $g \in G_\mathbb{C}{}^s$, $v \in V_\mathbb{C}$. Assume $V_\mathbb{C}$ is a $G_\mathbb{C}{}^s$ module. A vector $v \in V_\mathbb{C}$ is called a weight vector for $\mu \in \Lambda$ if $hv = h^\mu v$ for all $h \in H_\mathbb{C}{}^s$. For any $\mu \in \Lambda$ let $V_\mathbb{C}(\mu) \subseteq V_\mathbb{C}$ be the subspace of all weight vectors for μ. The element $\mu \in \Lambda$ is called a weight of $V_\mathbb{C}$ (or π) if $V_\mathbb{C}(\mu) \neq 0$. Now if μ is a weight of V then $0 \neq v \in V(\mu)$ is called a highest (resp. lowest) weight vector if $nv = v$ (resp. $\bar{n}v = v$) for all $n \in N_\mathbb{C}{}^s$ (resp. $\bar{n} \in \bar{N}_\mathbb{C}{}^s$). If $V_\mathbb{C}(\mu)$ contains a highest (resp. lowest) weight vector then μ is called a highest (resp. lowest) weight.

We now recall certain fundamental results of the Cartan–Weyl theory of representations. For each $\lambda \in D$ there exists an irreducible finite-dimensional (holomorphic) representation

$$\pi_\lambda \colon G_\mathbb{C}{}^s \to \operatorname{Aut} V_\mathbb{C}{}^\lambda \tag{5.1.7}$$

such that λ is a highest weight. Furthermore (1) as such $V_\mathbb{C}{}^\lambda$ is unique up to equivalence, (2) λ is the only highest weight of $V_\mathbb{C}{}^\lambda$, and (3) $\dim V_\mathbb{C}{}^\lambda(\lambda) = 1$.

Now for any $\lambda \in D$ let (5.1.7) be given and fixed. One knows then that if π is any finite-dimensional irreducible holomorphic representation of $G_\mathbb{C}{}^s$ then π is equivalent to π_λ for a unique $\lambda \in D$.

Now let $U = U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ over $\mathbb{R}$ and let $U_\mathbb{C}$, its complexification, be the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$ over $\mathbb{C}$. If the adjoint representation is extended the group $G_\mathbb{C}{}^s$ operates as a group of automorphisms of $U_\mathbb{C}$ and one has the direct sum

$$U_\mathbb{C} = \sum_{\mu \in \Lambda} U_\mathbb{C}(\mu), \tag{5.1.8}$$

where as above the $U_\mathbb{C}(\mu)$ are the weight spaces for this action. On the other hand given $\lambda \in D$ the representation π_λ induces a representation of $\mathfrak{g}_\mathbb{C}$ and hence of $U_\mathbb{C}$ on $V_\mathbb{C}{}^\lambda$, which we also denote by π_λ, so that $V_\mathbb{C}{}^\lambda$ becomes a $U_\mathbb{C}$ module and one has

$$U_\mathbb{C}(\mu)\, V_\mathbb{C}{}^\lambda(\nu) \subseteq V_\mathbb{C}{}^\lambda(\mu + \nu) \tag{5.1.9}$$

for any $\mu, \nu \in \Lambda$.

Now let G^s be the subgroup of $G_\mathbb{C}{}^s$ corresponding to $\mathfrak{g}$ and let N^s, H^s and

$\bar{N}^s$ be the subgroups of G^s corresponding to n, h, and $\bar{n}$. Note that since h^μ is real (in fact positive) for any $h \in H^s$ (or H) and $\mu \in \Lambda$ one has the direct sum

$$U = \sum U(\mu) \tag{5.1.10}$$

over Λ, where $U(\mu) = U \cap U_{\mathbb{C}}(\mu)$.

Now let $G_u{}^s$ be the maximal compact subgroup of $G_{\mathbb{C}}{}^s$ corresponding to the compact form g_u. See (3.1.6). One knows then (the "unitarian trick") that there exists a Hermitian (positive definite) inner product on $V_{\mathbb{C}}{}^\lambda$ which is invariant under the action of $G_u{}^s$. We will denote the inner product of v, $v' \in V_{\mathbb{C}}{}^\lambda$ by $\{v, v'\}$. It is chosen so that it is linear in v and conjugate linear in v'.

Now since $G_{\mathbb{C}}{}^s$ is simply connected the $*$-operation on $g_{\mathbb{C}}$ (see Section 3.1) induces a unique $*$-operation $a \mapsto a^*$ on $G_{\mathbb{C}}{}^s$ such that one has (3.1.8) for $a, b \in G_{\mathbb{C}}{}^s$ and (3.1.9) for $x \in g_{\mathbb{C}}$, where $\exp^s$ replaces exp. It is clear that the $*$-operations on $G_{\mathbb{C}}{}^s$ and $G_{\mathbb{C}}$ commute with Ad. It is also clear that the $*$-operation on $g_{\mathbb{C}}$ extends uniquely to $U_{\mathbb{C}}$ as a conjugate linear map such that $(u_1u_2)^* = u_2^*u_1^*$ for u_1, $u_2 \in U_{\mathbb{C}}$. One easily has

$$\{uv, v'\} = \{v, u^*v'\} \tag{5.1.11}$$

for any $v, v' \in V_{\mathbb{C}}{}^\lambda$ and u in $U_{\mathbb{C}}$ or $G_{\mathbb{C}}{}^s$.

Now fix once and for all a highest weight vector $v^\lambda \in V_{\mathbb{C}}{}^\lambda(\lambda)$ such that $\{v^\lambda, v^\lambda\} = 1$ and let V^λ be the $\mathbb{R}$-subspace of $V_{\mathbb{C}}{}^\lambda$ defined by

$$V^\lambda = Uv^\lambda. \tag{5.1.12}$$

Also for any $\mu \in \Lambda$ let $V^\lambda(\mu) = V^\lambda \cap V_{\mathbb{C}}{}^\lambda(\mu)$.

PROPOSITION 5.1. *The $\mathbb{R}$-subspace V^λ of $V_{\mathbb{C}}{}^\lambda$ is a real form of $V_{\mathbb{C}}{}^\lambda$. That is, $V_{\mathbb{C}}{}^\lambda = V^\lambda + iV^\lambda$ is a real direct sum. Furthermore if Q_*^λ is the restriction of the inner product $\{v, v'\}$ to V^λ then Q_*^λ is real valued (so that V^λ is a real Hilbert space). Furthermore V^λ is stable under the action of G^s and*

$$V^\lambda = \sum V^\lambda(\mu), \tag{5.1.13}$$

summed over the weights of π_λ, is an orthogonal direct sum with respect to Q_*^λ.

Proof. Now $V_{\mathbb{C}}{}^\lambda = U_{\mathbb{C}}v^\lambda$ since of course $V_{\mathbb{C}}{}^\lambda$ is $U_{\mathbb{C}}$ irreducible. Thus $V_{\mathbb{C}}{}^\lambda = V^\lambda + iV^\lambda$. To show V^λ is a real form it suffices to show $V^\lambda \cap iV^\lambda = 0$. For this it clearly is enough to show that Q_*^λ is real valued. Let T^s be the subgroup (maximal torus) of $G_u{}^s$ corresponding to ih. Now if $\mu_i \in \Lambda$, $i = 1, 2$, are distinct then the characters they define on T^s are distinct and hence one has

$$\{V_{\mathbb{C}}{}^\lambda(\mu_1), V_{\mathbb{C}}{}^\lambda(\mu_2)\} = 0 \tag{5.1.14}$$

since T^s operates as a group of unitary operators on $V_{\mathbb{C}}$. Now let $v_i \in V^\lambda$, $i = 1, 2$. Write $v_i = u_i v^\lambda$ for $u_i \in U$. Put $u = u_2^* u_1$. Then $u \in U$ since U is clearly stable under the $*$-operation. But now $\{v_1, v_2\} = \{u_1 v^\lambda, u_2 v^\lambda\} = \{u_2^* u_1 v^\lambda, v^\lambda\} = \{uv^\lambda, v^\lambda\}$. To show Q_*^λ is real valued it suffices then to show that $\{uv^\lambda, v^\lambda\} \in \mathbb{R}$. Let u_0 be the component of u in $U(0)$ (0 here denotes the zero weight) relative to (5.1.10). By (5.1.9) and (5.1.14) one then has $\{uv^\lambda, v^\lambda\} = \{u_0 v^\lambda, v^\lambda\}$. However, if $U(\mathfrak{h})$ is the enveloping algebra of $\mathfrak{h}$ over $\mathbb{R}$ then one knows

$$U(0) = U(\mathfrak{h}) \oplus ((U\mathfrak{n}) \cap U(0)) \tag{5.1.15}$$

is a direct sum, where $U\mathfrak{n}$ is the left ideal in U generated by $\mathfrak{n}$. In fact the validity of (5.1.15) follows from the validity of (5.1.15) for the complexification of the subspace involved. For the case over $\mathbb{C}$ see, e.g., Lemma 7.4.2 in [4, p. 230]. (In that case one depends on (5.1.15) to define the Harish-Chandra homomorphism.) But since v^λ is a highest weight vector one has

$$xv^\lambda = 0 \qquad \text{for all } x \in \mathfrak{n}_{\mathbb{C}} \tag{5.1.16}$$

and hence if u_1 is the component of u_0 in $U(\mathfrak{h})$ relative to (5.1.15) one has $\{u_0 v^\lambda, v^\lambda\} = \{u_1 v^\lambda, v^\lambda\}$. However, if $y \in \mathfrak{h}$ then $yv^\lambda = \langle \lambda, y \rangle v^\lambda$. But $\langle \lambda, y \rangle \in \mathbb{R}$ since $\lambda \in \mathfrak{h}'$. Thus $u_1 v^\lambda = rv^\lambda$ for some $r \in \mathbb{R}$ so that $\{u_1 v^\lambda, v^\lambda\} \in \mathbb{R}$. This proves that Q_*^λ is real valued and V^λ is a real form of $V_{\mathbb{C}}{}^\lambda$. But now $U(\mu - \lambda)\, v^\lambda \subseteq V^\lambda(\mu)$ by (5.1.9) and hence one has the sum (5.1.13) by (5.1.10). The sum is orthogonal by (5.1.14). The subspace V^λ is stable under G^s since it is clearly stable under $\mathfrak{g}$.

Q.E.D.

Remark 5.1. Note that $V^\lambda(\mu)$, for any $\mu \in \Lambda$ is a real form of the weight space $V_{\mathbb{C}}{}^\lambda(\mu)$. This follows obviously from (5.1.13) and (5.1.14).

We will refer to V^λ with respect to the action of G^s by $\pi_\lambda \mid G^s$ as a G^s-module.

5.2. Now let $G_*^s = \bar{N}^s H^s N^s$ so that G_*^s, using the Bruhat decomposition of $G_{\mathbb{C}}{}^s$, is an open connected subset of G^s and the map

$$\mathrm{Ad}\colon G_*^s \to G_*, \tag{5.2.1}$$

recalling (3.2.1) is a diffeomorphism.

Regarding Ad, as in (5.1.3), as a map from $G_{\mathbb{C}}{}^s$ to $G_{\mathbb{C}}$

LEMMA 5.2.1. *One has*

$$(\mathrm{Ad}^{-1} G_*) \cap G^s = (\mathrm{cent}\; G^s)\, G_*^s .$$

Furthermore if $c, c' \in$ cent G^s *are distinct then* cG_*^s *and* $c'G_*^s$ *are disjoint so that* *the connected components of* (cent G^s) G_*^s *are uniquely of the form* cG_*^s *for* $c \in \text{cent } G^s$.

Proof. The first statement is an immediate consequence of (5.1.3) and the surjectivity of (5.2.1). Now if $a, a' \in G_s^*$ and $ac = a'c'$, where $c, c' \in \text{cent } G^s$, then $\text{Ad } a = \text{Ad } ac = \text{Ad } a'c' = \text{Ad } a'$. But then $a = a'$ from the injectivity of (5.2.1) so that $c = c'$. This proves the lemma. Q.E.D.

Now let $w_o \in \mathfrak{h}_+$ and as usual let $w = f + w_o$. We recall that $\bar{n}_f(w) \in \bar{N}$ satisfies $\bar{n}_f(w)\, w_o = w$. See (3.5.2). Now let $m \in \tilde{G}$ be as in (3.5.5) so that $me_{\alpha_i} = -e_{\alpha_i}$, $i = 1,..., l$. Hence of course $me_{-\alpha_i} = -e_{-\alpha_i}$ so that $mf = -f$. But clearly $mw_o = w_o$. Thus if we put

$$\bar{n}_{-f}(w) = m^{-1}\bar{n}_f(w)m \tag{5.2.2}$$

then $\bar{n}_{-f}(w) \in \bar{N}$ and $\bar{n}_{-f}(w)\, w_o = -f + w_o$.

Identification 5.2. Henceforth to simplify notation we will identify G_*^s with G_* by the diffeomorphism (5.2.1). This means that $\bar{N}^s$ and N, H^s and H and also N^s and N are identified. The only possible confusion that can arise is with regard to multiplication of these elements. However, it should be clear from the context whether the multiplication is in $G_{\mathbb{C}}{}^s$ or $G_{\mathbb{C}}$.

Remark 5.2. One notes that both G_s^* and G_* are stable under the respective $*$-operations in $G_{\mathbb{C}}{}^s$ and $G_{\mathbb{C}}$ and that since Ad commutes with these operations no ambiguity with the $*$-operation is introduced by identifying G_*^s with G_*.

Let $s(\kappa) \in \tilde{G}$ be as in (3.4.12).

Lemma 5.2.2. *One has* $m^{-1}s(\kappa) \in G$. *Furthermore there exists a unique element* $s_o(\kappa) \in G^s$ *such that* (1) $\text{Ad } s_o(\kappa) = m^{-1}s_o(\kappa)$ *and such that* (2) *for any* $h \in H$ *and* $w_o \in \mathfrak{h}_+$ *one has the relation*

$$s_o(\kappa)^{-1}\, \bar{n}_{-f}(w)\, h(\bar{n}_f(w))^{-1} \in G_* \tag{5.2.3}$$

in G^s.

Proof. Let $w_o \in \mathfrak{h}_+$ and let $h \in H$. Then by (3.5.7) if $g \in \tilde{G}$ is defined by $g = \bar{n}_f(w)\, mh\bar{n}_f(w)^{-1}$ one has $g \in G_0{}^w$. Thus $s(\kappa)^{-1}g \in G_*$ by Proposition 3.5 (recalling 3.4.13). But $s(\kappa)^{-1}g = s(\kappa)^{-1}\, \bar{n}_f(w)\, mh\bar{n}_f(w)^{-1} = s(\kappa)^{-1}\, m\bar{n}_{-f}(w)\, h\bar{n}_f(w)^{-1}$. Thus in $\tilde{G}$ one has

$$s(\kappa)^{-1}\, m\bar{n}_{-f}(w)\, h\bar{n}_f(w)^{-1} \in G_*. \tag{5.2.4}$$

However, since $G_* \subseteq G$ and of course $\bar{n}_{-f}(w)\, h\bar{n}_f(w)^{-1} \in G$ one necessarily has $s(\kappa)^{-1}m \in G$. This proves the first statement. Now fix $s_1(\kappa) \in G^s$ so that $\operatorname{Ad} s_1(\kappa) = m^{-1}s(\kappa)$. Thus for any $c \in \operatorname{Cent} G^s$ one has

$$\operatorname{Ad}((cs_1(\kappa))^{-1}\, \bar{n}_{-f}(w)\, h\bar{n}_f(w)^{-1}) \in G_*\,, \tag{5.2.5}$$

where the multiplication in (5.2.5) is in G^s. But then by Lemma 5.2.1 there exists a unique $c_1 \in \operatorname{Cent} G^s$ such that if $s_o(\kappa) = c_1 s_1(\kappa)$ then $s_o(\kappa)^{-1}\, \bar{n}_{-f}(w)\, h\bar{n}_f(w)^{-1} \in G_*^s = G_*$.

Now consider the map

$$\mathscr{h}_+ \times H \to G^s, \tag{5.2.6}$$

where $(w_0', h') \mapsto s_o(\kappa)^{-1}\, \bar{n}_{-f}(w')\, h'\bar{n}_f(w')^{-1}$ and $w' = f + w_0'$. Now by (5.2.4), where w' replaces w and h' replaces h, and by Lemma 5.2.1 it follows that the image of (5.2.6) is in $(\operatorname{Cent} G^s)\, G_*^s$. However, by the connectivity of $\mathscr{h}_+ \times H$ the image of (5.2.6) must lie in one component of $(\operatorname{Cent} G^s)\, G_*^s$. Since (w_o, h) maps into G_*^s the entire image of (5.2.6) is in $G_*^s = G_*$. Except for the uniqueness of $s_o(\kappa)$ this proves the lemma. The ambiguity of $s_o(\kappa)$ satisfying (1) of the lemma is up to an element in $\operatorname{Cent} G^s$. This uniqueness then follows immediately from Lemma 5.2.1. Q.E.D.

Henceforth $s_o(\kappa)$ will denote the element of G^s given by Lemma 5.2.2. Let K^s be the (maximal compact) subgroup of G^s corresponding to $\mathscr{k} \subseteq \mathscr{g}$.

LEMMA 5.2.3. *One has* $s_o(\kappa) \in K^s$.

Proof. Now for $i = 1,..., l$ one has $s(\kappa)\, e_{\alpha_i} = e_{\kappa\alpha_i}$. However, since $\kappa^2 = 1$ in W one then has $s(\kappa)^2\, e_{\alpha_i} = r_i e_{\alpha_i}$ for some $r_i \in \mathbb{R}$. Thus since $Q(e_{\alpha_i}, e_{-\alpha_i}) = 1$ one has $r_i = Q(e_{-\alpha_i}, s(\kappa)^2\, e_{\alpha_i}) = Q(s(\kappa)^{-1}\, e_{-\alpha_i}, s(\kappa)\, e_{\alpha_i}) = Q(s(\kappa)^{-1}\, e_{-\alpha_i}, e_{\kappa\alpha_i})$ by the invariance of Q. But $-\kappa\alpha_i$ is also a simple positive root and hence $s(\kappa)\, e_{-\kappa\alpha_i} = e_{-\alpha_i}$. Thus $s(\kappa)^{-1}\, e_{-\alpha_i} = e_{-\kappa\alpha_i}$ so that $r_i = Q(e_{-\kappa\alpha_i}, e_{\kappa\alpha_i}) = 1$. This proves

$$s(\kappa)^2 = 1. \tag{5.2.7}$$

However, m operates as -1 on e_{α_i} and $e_{-\alpha_i}$ so that clearly $s(\kappa)$ and m commute and $m^2 = 1$. Hence $s_1(\kappa)^2 = 1$, where $s_1(\kappa) = m^{-1}s(\kappa)$. On the other hand for any $g \in G_{\mathbb{C}}$ and $x \in \mathscr{g}_{\mathbb{C}}$ one easily has by (3.1.8) and (3.1.9),

$$(g^*)^{-1}\, x^* = (gx)^*. \tag{5.2.8}$$

Thus $(s(\kappa)^{-1})^*\, e_{-\alpha_i} = e_{-\kappa\alpha_i}$ by (5.2.8) and (1.5.1). However, $s(\kappa)^{-1}\, e_{-\alpha_i} = e_{-\kappa\alpha_i}$. Similarly $s(\kappa)^{-1}$ $(=s(\kappa))$ and $(s(\kappa)^{-1})^*$ $(=s(\kappa)^*)$ agree on e_{α_i} and $\mathscr{h}$ so that $s(\kappa) = s(\kappa)^*$. Relation (5.2.8) also implies that $m = m^*$ so that $s_1(\kappa) = s_1(\kappa)^*$. But

$s_1(\kappa) \in G$ by Lemma 5.2.2 so that $s_1(\kappa) \in K$ by Lemma 3.1.2. However, as one knows, K^s is the inverse image of K in G^s under Ad. Thus $s_o(\kappa) \in K^s$ since $\mathrm{Ad}\, s_o(\kappa) = s_1(\kappa)$ by Lemma 5.2.2. Q.E.D.

Let $\lambda \in D$. Now recall we have fixed a highest weight vector $v^\lambda \in V^\lambda(\lambda)$. On the other hand one knows the set of weights of π_λ is stable under the Weyl group and in fact since $s_o(\kappa) \in G^s$ corresponds to κ one must have, for any $\mu \in \Lambda$

$$s_o(\kappa)\, V_{\mathbb{C}}^\lambda(\mu) = V_{\mathbb{C}}^\lambda(\kappa\mu). \tag{5.2.9}$$

Furthermore $\kappa\lambda$ is the lowest weight of $\pi\lambda$ and hence if we fix $v^{\kappa\lambda}$ by putting

$$v^{\kappa\lambda} = s_o(\kappa)\, v^\lambda \tag{5.2.10}$$

then $v^{\kappa\lambda}$ is a lowest weight vector. In fact one has

Proposition 5.2. *Let $\lambda \in D$; then $V^\lambda(\kappa\lambda)$ is one dimensional over $\mathbb{R}$ and $v^{\kappa\lambda} \in V^\lambda(\kappa\lambda)$, where*

$$\{v^{\kappa\lambda}, v^{\kappa\lambda}\} = 1. \tag{5.2.11}$$

Proof. One has $\dim V^\lambda(\kappa\lambda) = 1$ by (5.2.9) and Remark 5.1. But also one must have $s_o(\kappa)\, v^\lambda \in V^\lambda(\kappa\lambda)$ since, by Proposition 5.1, V^λ is stable under G^s. But now one has (5.3.3) by Lemma 5.2.3 since, clearly, $K^s \subseteq G_u{}^s$ so that Q_*^λ is certainly invariant under K^s. Q.E.D.

5.3. Now let $w_o \in \mathfrak{k}_+$ and conforming to our standard notation we put $w = f + w_o$. Our main concern here is with the determination of $h(a)^\lambda$ for $a \in G_0{}^w$ (see Section 3.5) and $\lambda \in \mathfrak{k}'$. We first note

Lemma 5.3. *For any $a \in G_0{}^w$ put $\rho_w(a) = m^{-1}\bar{n}_f(w)^{-1}\, a\bar{n}_f(w)$. Then $\rho_w(a) \in H$ and*

$$\rho_w\colon G_0{}^w \to H, \qquad a \mapsto \rho_w(a) \tag{5.3.1}$$

is a diffeomorphism.

Proof. This is obvious from (3.5.7), which asserts that $\bar{n}_f(w)\, mH\bar{n}_f(w)^{-1} = G_0{}^w$. Q.E.D.

We will generally conform to the notation of using the subscript o to denote the image in H of an element in $G_0{}^w$ under ρ_w. Thus $\rho_w(a) = a_o \in H$ for any $a \in G_0{}^w$.

Our determination of $h(a)^\lambda$ begins with

PROPOSITION 5.3. *Let $w_o \in \mathfrak{h}_+$ and let $\lambda \in D$. Put $w = f + w_o$. Then for any $a \in G_0^w$ one has (recalling* (3.4.14) *and* (3.5.9))

$$h(a)^\lambda = \{a_o \bar{n}_f(w)^{-1} v^\lambda, \bar{n}_{-f}(w)^* v^{\kappa\lambda}\}, \tag{5.3.2}$$

where $\bar{n}_f(w)$ and $\bar{n}_{-f}(w)$ are defined by (3.5.2) *and* (5.2.2) *and $a_o = \rho_w(a) \in H$, where ρ_w is defined by Lemma* 5.3. *Also the right side of* (5.3.2) *is a $\underset{\sim}{Q}_*^\lambda$ inner product (see Proposition* 5.1) *of vectors in the G^s-module V^λ.*

Proof. Let $a \in G_0^w$. *Then by* (3.4.14) *and* (3.5.9) *one has, in $\tilde{G}$,*

$$s(\kappa)^{-1} a = \bar{n}(a)\, h(a)\, n(a). \tag{5.3.3}$$

On the other hand if $a_o = \rho_w(a) \in H$ then clearly $a = \bar{n}_f(w)\, m a_o \bar{n}_f(w)^{-1} = m\bar{n}_{-f}(w)\, a_o \bar{n}_f(w)^{-1}$, recalling (5.2.2). Hence

$$s(\kappa)^{-1} a = (s(\kappa)^{-1} m)\, \bar{n}_{-f}(w)\, a_o \bar{n}_f(w)^{-1} \tag{5.3.4}$$

in $\tilde{G}$. But then by (5.3.3) one has

$$\bar{n}(a)\, h(a)\, n(a) = (s(\kappa)^{-1} m)\, \bar{n}_{-f}(w)\, a_o \bar{n}_f(w)^{-1}. \tag{5.3.5}$$

On the other hand, in G^s, one has, recalling Lemma 5.2.2, $s_o(\kappa)^{-1}\, \bar{n}_{-f}(w)\, a_o \bar{n}_f(w) \in G_* = G_*^s$. Thus by Lemma 5.2.2 one has

$$\bar{n}(a)\, h(a)\, n(a) = s_o(\kappa)^{-1}\, \bar{n}_{-f}(w)\, a_o \bar{n}_f(w)^{-1} \tag{5.3.6}$$

in G^s.

Now consider the G^s-module V^λ. Let $g \in G$ be the element given by (5.3.6). Now $n(a)\, v^\lambda = v^\lambda$ since v^λ is a highest weight vector of V^λ. But $(\bar{n}(a))^* \in N$ by (1.5.1) and (3.1.9) so that one also has $(\bar{n}(a))^* v^\lambda = v^\lambda$. Thus from the left side of (5.3.6) one has $\{g v^\lambda, v^\lambda\} = \{h(a)\, v^\lambda, v^\lambda\} = \{h(a)^\lambda v^\lambda, v^\lambda\} = h(a)^\lambda$. But now $s_o(\kappa) \in K^s$ by Lemma 5.2.3. But of course $K^s \subseteq G^s$ so that $s_o(\kappa)^{-1} = s_o(\kappa)^*$ and hence $(s_o(\kappa)^{-1})^* v^\lambda = s_o(\kappa)\, v^\lambda = v^{\kappa\lambda}$. Thus from the right side of (5.3.6) one has $\{g v^\lambda, v^\lambda\} = \{\bar{n}_{-f}(w)\, a_o \bar{n}_f(w)^{-1} v^\lambda, v^{\kappa\lambda}\} = \{a_o \bar{n}_f(w)^{-1} v^\lambda, \bar{n}_{-f}(w)^* v^{\kappa\lambda}\}$. This proves the lemma since $\{g v^\lambda, v^\lambda\} = h(a)^\lambda$. Q.E.D.

Remark 5.3. Since $\dim V^\lambda(\kappa\lambda) = 1$ there are exactly two vectors $v \in V^\lambda(\kappa\lambda)$ such that $\{v, v\} = 1$. Thus if one were to choose a normalized lowest weight vector there would be an ambiguity up to sign. But now by (5.2.11), $v^{\kappa\lambda}$ is a choice of one of these two vectors. We now observe that since $h(a)^\lambda > 0$ Proposition 5.3 implies that $v^{\kappa\lambda}$ is that choice such that $\{a_o \bar{n}_f(w)^{-1} v^\lambda, \bar{n}_{-f}(w)^* v^{\kappa\lambda}\} > 0$ for all $a_o \in H$ (see (5.3.1)) and all $w_o \in \mathfrak{h}_+$.

5.4. Now let $w_o \in \mathfrak{h}_+$, and, as usual, put $w = f + w_o$. Let $g_o \in H$ and put $g = \rho_w^{-1}(g_o) \in G_0{}^w$. See (5.3.1). Let $\bar{n}(g) \in \bar{N}$, $h(g) \in H$, and $\bar{n}(g) \in \bar{N}$ be defined by (3.4.14) recalling (3.5.9). Now since H is a vector group we can consider the square root $h(g)^{1/2}$ of $h(g)$ in H. Let $P^s = \exp^s \mathfrak{p}$. Using K^s and P^s we can, as one knows, take the polar decomposition of any element in G^s. Thus there exist unique elements $k(g_o, w_o) \in K^s$ and $p(g_o, w_o) \in P^s$ such that, in G^s,

$$h(g)^{1/2}\, n(g)\, \bar{n}_f(w) = k(g_o, w_o)\, p(g_o, w_o), \tag{5.4.1}$$

where $\bar{n}_f(w)$ as usual is given by (3.5.2).

Now recalling the set $\operatorname{Jac} \mathfrak{g}$ of Jacobi elements in $\mathfrak{g}$ (see (2.2.6)) let

$$\operatorname{Jac} \mathfrak{p} = \operatorname{Jac} \mathfrak{g} \cap \mathfrak{p} \tag{5.4.2}$$

so that $\operatorname{Jac} \mathfrak{p}$ is the set of all symmetric Jacobi elements. Using the notation (2.2.6) so that $y \in \operatorname{Jac} \mathfrak{g}$ if and only if y is of the form

$$y = x + \sum_{i=1}^{l} a_{-i} e_{-\alpha_i} + \sum_{i=1}^{l} a_i e_{\alpha_i}, \tag{5.4.3}$$

where $a_{-i} > 0$ and $a_i > 0$ and $x \in \mathfrak{h}$, one notes by (1.5.1) that $y \in \operatorname{Jac} \mathfrak{p}$ if and only if $a_{-i} = a_i$, $i = 1, \ldots, l$. It is clear then that $\operatorname{Jac} \mathfrak{p}$ is a closed connected two-dimensional submanifold of $\operatorname{Jac} \mathfrak{g}$.

Now for any $\gamma \in \mathscr{I}(\mathfrak{h}_+)$ (see (2.3.1)) let

$$(\operatorname{Jac} \mathfrak{p})(\gamma) = \mathscr{I}^{-1}(\gamma) \cap \operatorname{Jac} \mathfrak{p}$$

so that the sets $(\operatorname{Jac} \mathfrak{p})(\gamma)$, $\gamma \in \mathscr{I}(\mathfrak{h}_+)$, are the "isospectral" equivalence classes of $\operatorname{Jac} \mathfrak{p}$. The following is a corollary of Theorem 3.6 and hence could have been proved earlier. It is proved now instead for notational convenience.

THEOREM 5.4. *For any w_o in the open Weyl chamber $\mathfrak{h}_+$ (see (3.3.1)) let $w = f + w_o$, where f is given by (1.5.4). Let $G_0{}^w$ be defined by (3.5.7) and for any g_o in the split Cartan subgroup H let $\rho_w^{-1}(g_o) = g \in G_0{}^w$, where ρ_w is defined as in (5.3.1). Then if $p(g_o, w_o)$ is defined by (5.4.1) one has*

$$p(g_o, w_o) \in H. \tag{5.4.4}$$

On the other hand if $k(g_o, w_o) \in K^s$ is defined by (5.4.1) one has (regarding $\mathfrak{g}$ as a G^s-module using Ad)

$$k(g_o, w_o)\, w_o \in (\operatorname{Jac} \mathfrak{p})(\gamma), \tag{5.4.5}$$

where $\gamma = \mathscr{I}(w_o)$. *Moreover the map*

$$H \times \mathscr{h}_+ \to \operatorname{Jac} \mathscr{h}, \qquad (g_o, w_o) \mapsto k(g_o, w_o)\, w_o \tag{5.4.6}$$

is a diffeomorphism. Furthermore if $w_o \in \mathscr{h}_+$ *and* $\gamma = \mathscr{I}(w_o)$ *then* $(\operatorname{Jac} \mathscr{h})(\gamma)$ *is a closed connected submanifold of dimension* l *in* $\operatorname{Jac} \mathscr{h}$ *and*

$$H \to (\operatorname{Jac} \mathscr{h})(\gamma), \qquad g_o \to k(g_o, w_o)\, w_o \tag{5.4.7}$$

is a diffeomorphism so that as manifolds $(\operatorname{Jac} \mathscr{h})(\gamma) \cong \mathbb{R}^l$. *Finally*

$$\operatorname{Jac} \mathscr{h} = \bigcup_{w_o \in \mathscr{h}_+} (\operatorname{Jac} \mathscr{h})(\mathscr{I}(w_o)) \tag{5.4.8}$$

is a disjoint union and in fact (5.4.8) *is the decomposition of* $\operatorname{Jac} \mathscr{h}$ *as the union of leaves of a foliation.*

Proof. Let $Z \subsetneq \operatorname{Jac} \mathscr{g}$ be as in (2.2.4) (where of course $F = \mathbb{R}$). Let $y \in Z$ so that we may write $y = f + x + \sum_{i=1}^{l} r_i e_{\alpha_i}$. Let $a_y \in H$ be defined by the condition $(a_y)^{\alpha_i} = r_i^{-1/2}$, $i = 1, \ldots, l$. See (2.2.9). Obviously the map

$$Z \to H, \qquad y \mapsto a_y \tag{5.4.9}$$

is smooth. But clearly $a_y y \in \operatorname{Jac} \mathscr{h}$ and any element in $\operatorname{Jac} \mathscr{h}$ is of this form. On the other hand it is easy to see that the bijection (2.2.8) is a diffeomorphism and hence by considering the graph of (5.4.9) the map

$$Z \to \operatorname{Jac} \mathscr{h}, \qquad y \to a_y y \tag{5.4.10}$$

is a diffeomorphism. Obviously then (5.4.10) induces a diffeomorphism

$$Z(\gamma) \to (\operatorname{Jac} \mathscr{h})(\gamma) \tag{5.4.11}$$

for any $\gamma \in \mathscr{I}(\mathscr{h}_+)$. But then, by Theorem 3.2, $(\operatorname{Jac} \mathscr{h})(\gamma)$ is a connected, closed submanifold of dimension l of $\operatorname{Jac} \mathscr{h}$. The final statement of Theorem 5.4 and the fact that

$$(\operatorname{Jac} \mathscr{h})(\gamma) \cong \mathbb{R}^l \tag{5.4.12}$$

follow from (3.6.3), (3.2.14), and Proposition 4.1.

Now let $g_o \in H$ and $w_o \in \mathscr{h}_+$. Recall Theorem 3.7. Let $w = f + w_o$ so that $\bar{n}_f(w)\, w_o = w$ and let $g \in H$ be given by (3.7.1). Let $y = n(g)w$ so that $y \in Z(\gamma)$ where $\gamma = \mathscr{I}(w_o)$. But now $a_y = h(g)^{1/2}$ by (3.7.4). Thus if $z = h(g)^{1/2} y$ then

$z \in \operatorname{Jac} \mathfrak{p}$ by (5.4.9). But now $z = h(g)^{1/2} y = h(g)^{1/2} n(g) w = h(g)^{1/2} n(g) \bar{n}_f(w) w_o$. Thus

$$z = kpw_o \tag{5.4.13}$$

by (5.4.1), where we have written $k = k(g_o, w_o)$ and $p = p(g_o, w_o)$. Thus if $v = k^{-1}z$ then $pw_o = v$. But $v \in \mathfrak{p}$ since $z \in \mathfrak{p}$ and $k \in K^s$. Thus if one applies the $*$-operation it follows from (5.2.8), which is clearly valid if $G_{\mathbb{C}}{}^s$ is substituted for $G_{\mathbb{C}}$, that also $p^{-1}w_o = v$. Thus $p^2 w_o = w_o$, which clearly implies $pw_o = w_o$ (since $\operatorname{Ad} p$ is diagonalizable with positive eigenvalues). Thus $\operatorname{Ad} p \in G^{w_o} \cap P$. But $G^{w_o} \cap P = H$ by (3.1.27). However, $\operatorname{Ad} H = H$ by Identification 5.2 and as one knows the map $P^s \to P$ induced by Ad is bijective. Thus $p \in H$, proving (5.4.4). But also the relation $pw_o = w_o$ implies $kw_o = z$ by (5.4.13). This proves (5.4.5).

Now the map (5.4.7) is the composite of (5.3.1), (3.6.2), and (5.4.11). Since these three maps are diffeomorphisms it follows that (5.4.7) is a diffeomorphism. Furthermore the map (5.4.6) is a composite of (3.7.2) and the inverse to (5.4.10). Since these two maps are diffeomorphisms it follows also that (5.4.6) is a diffeomorphism. Q.E.D.

Remark 5.4. A familiar question when dealing with symmetric Jacobi matrices is the problem of diagonalization, even if one knows the spectrum of the matrix. In a more general setting, if $x \in \operatorname{Jac} \mathfrak{p}$ and $\lambda \in D$ find the eigenvectors of $\pi_\lambda(x)$ in terms of an orthonormal basis v_i^λ, $i = 1, \ldots, \dim V^\lambda$, of weight vectors in V^λ. Now let $(g_o, w_o) \in H \times \mathfrak{h}_+$ correspond to x under the bijection (5.4.6). Analogous to knowing the spectrum of a Jacobi matrix is the knowledge of the element w_o. Since the v_i^λ are eigenvectors of $\pi_\lambda(w_o)$ it is now clear from (5.4.5) that $\{\pi_\lambda(k(g_o, w_o)) v_i^\lambda\}$ is indeed an orthonormal basis of eigenvectors of $\pi_\lambda(x)$. The problem then reduces to determining $k(g_o, w_o)$. By (5.4.1) it is then a question of determining $h(g)$, $n(g)$, and $\bar{n}_f(w)$. The element $h(g)$ is given by (3.7.4). We will give a formula for $\bar{n}_f(w)^{-1}$ in Section 5.8. (The formula for $\bar{n}_f(w)$ is obtained from (5.8.7) by eliminating $(-1)^{(s)}$ and changing $\bar{s}$ to s.) Since $n(g)$ is unipotent one solves inductively for $n(g)$ by the relation $n(g)w = y$. See Section 7.8, where an example is worked out.

5.5. Let $\lambda \in D$. One notes that if v is a weight vector of λ then, by (5.1.9) and (5.1.14),

$$\{nv, v\} = \{\bar{n}v, v\} = \{v, v\} \tag{5.5.1}$$

for any $n \in N$ or $\bar{n} \in \bar{N}$.

PROPOSITION 5.5.1. *Let $g_o \in H$ and $w_o \in \mathfrak{h}_+$. Write $p = p(g_o, w_o)$, where the latter is given by* (5.4.1). *We recall $p \in H$ by* (5.4.4). *Then $g_o^{-1} p^2$ is independent* *of* g_o, *depending only on* w_o (*or equivalently* $w = f + w_o$) *so that we can define* $d(w) \in H$ *by*

$$g_o^{-1}p(g_o, w_o)^2 = d(w). \tag{5.5.2}$$

Furthermore if $\lambda \in D$ *then*

$$\{(d(w))v, (\bar{n}_f(w))^{-1} v^\lambda\} = \{\bar{n}_{-f}(w)v, v^{\kappa\lambda}\} \tag{5.5.3}$$

for any $v \in V^\lambda$, *where* $\bar{n}_f(w)$ *and* $\bar{n}_{-f}(w)$ *are defined by* (3.5.2) *and* (5.2.2). *Moreover one has*

$$d(w)^\lambda = \{\bar{n}_{-f}(w)\, v^\lambda, v^{\kappa\lambda}\} \tag{5.5.4}$$

and

$$d(w)^{-\kappa\lambda} = \{\bar{n}_f(w)^{-1}\, v^\lambda, v^{\kappa\lambda}\}. \tag{5.5.5}$$

Proof. Put $d = g_o^{-1}p^2$. Let $\lambda \in D$ and $v \in V^\lambda$ and put $C = \{dv, \bar{n}_f(w)^{-1} v^\lambda\}$. Then $C = \{pg_o^{-1}v, p(\bar{n}_f(w))^{-1} v^\lambda\}$ since g_o and p commute and $p = p^*$. But $p = k^{-1}h^{1/2}n\bar{n}_f(w)$ in G^s by (5.4.1), where we have written $k = k(g_o, w_o)$, $h = h(g)$, and $n = n(g)$. Then substituting for p and canceling out k^{-1} (since $(k^{-1})^* = k$) one has $C = \{h^{1/2}n\bar{n}_f(w)\, g_o^{-1}v, h^{1/2}nv^\lambda\}$. But $nv^\lambda = v^\lambda$. Hence by transposing $h^{1/2}$ one has

$$\begin{aligned} C &= \{hn\bar{n}_f(w)\, g_o^{-1}v, v^\lambda\} \\ &= \{\bar{n}hn\bar{n}_f(w)\, g_o^{-1}v, v^\lambda\}, \end{aligned} \tag{5.5.6}$$

where $\bar{n} = \bar{n}(g)$, since $\bar{n}^*v^\lambda = v^\lambda$ (recalling that $\bar{n}^* \in N$). But $\bar{n}hn = s_o(\kappa)^{-1}\,\bar{n}_{-f}(w)\, g_o\bar{n}_f(w)^{-1}$ by (5.3.6) in G^s. Thus $C = \{s_o(\kappa)^{-1}\,\bar{n}_{-f}(w)v, v^\lambda\} = \{\bar{n}_{-f}(w)v, v^{\kappa\lambda}\}$. That is,

$$\{dv, \bar{n}_f(w)^{-1}\, v^\lambda\} = \{\bar{n}_{-f}(w)v, v^{\kappa\lambda}\}. \tag{5.5.7}$$

But now if we choose $v = v^\lambda$ then $dv^\lambda = d^\lambda v^\lambda$. But $\{v^\lambda, \bar{n}_f(w)^{-1}\, v^\lambda\} = 1$ by (5.5.1). Thus

$$d^\lambda = \{\bar{n}_{-f}(w)\, v^\lambda, v^{\kappa\lambda}\}. \tag{5.5.8}$$

Obviously the right side of (5.5.8) depends only on w_o and not on g_o. Furthermore this is true for all $\lambda \in D$. But d is determined by d^{ν_i}, $i = 1,..., l$, where $\nu_i \in D$ is as in (5.1.2), since the ν_i are a basis of $\mathfrak{k}'$. Thus d depends only on w proving the first statement. Also, $d = d(w)$. But then (5.5.3) is just (5.5.7) and (5.5.4) is just (5.5.8). If we put $v = v^{\kappa\lambda}$ in (5.5.7) then since $\{\bar{n}_{-f}(w)\, v^{\kappa\lambda}, v^{\kappa\lambda}\} = 1$ by (5.5.1) and $d(w)\, v^{\kappa\lambda} = d(w)^{\kappa\lambda}\, v^{\kappa\lambda}$ one has $d(w)^{\kappa\lambda}\{v^{\kappa\lambda}, \bar{n}_f(w)^{-1}\, v^\lambda\} = 1$. But then (5.5.8) follows by inverting and reversing the order of the vectors. Q.E.D.

If $\mu, \nu \in \Lambda$ we will say that $\nu \geqslant \mu$ if $\nu - \mu$ is in the $\mathbb{Z}_+$-cone generated by the simple roots $\alpha_1, \ldots, \alpha_l$. If in addition $\nu \neq \mu$ we write $\nu > \mu$. One notes that

$$\nu > \mu \text{ implies } \langle \nu, w_o \rangle > \langle \mu, w_o \rangle \text{ for any } w_o \in \mathfrak{h}_+ . \tag{5.5.9}$$

Now for any $\lambda \in D$ let $q(\lambda) = \dim V^\lambda$. Let $\lambda \in D$. By the orthogonal direct sum (5.1.13) there exists a Q_*^λ-orthonormal basis v_i^λ, $i = 1, \ldots, q(\lambda)$, of V^λ such that the v_i^λ are weight vectors. We assume that such a basis is chosen once and for all where $v_1^\lambda = v^\lambda$ and $v_{q(\lambda)}^\lambda = v^{\kappa\lambda}$. See (5.2.10). Let $\lambda_i \in \Lambda$, $i = 1, \ldots, q(\lambda)$, be the weight corresponding to v_i^λ. Thus $\lambda_1 = \lambda$ and $\lambda_{q(\lambda)} = \kappa\lambda$. Furthermore since as one knows

$$V^\lambda = U(\bar{n})\, v^\lambda = U(n)\, v^{\kappa\lambda} \tag{5.5.10}$$

it follows from (5.1.9) that

$$\lambda > \lambda_i > \kappa\lambda \tag{5.5.11}$$

for all $1 < i < q(\lambda)$.

Now let $w_o \in \mathfrak{h}_+$ and let $w = f + w_o$. Also let $\bar{n}_f(w)$ and $\bar{n}_{-f}(w)$ be as in Proposition 5.5. Now put

$$b_i(\lambda, w) = \{\bar{n}_f(w)^{-1}\, v^\lambda, v_i^\lambda\}\{\bar{n}_{-f}(w)\, v_i^\lambda, v^{\kappa\lambda}\}. \tag{5.5.12}$$

Lemma 5.5.1. *One has $b_i(\lambda, w) \geqslant 0$ for $1 \leqslant i \leqslant q(\lambda)$. Furthermore one has strict positivity at the extremes. That is,*

$$b_1(\lambda, w) = \{\bar{n}_{-f}(w)\, v^\lambda, v^{\kappa\lambda}\} > 0 \tag{5.5.13}$$

and

$$b_{q(\lambda)}(\lambda, w) = \{\bar{n}_f(w)^{-1}\, v^\lambda, v^{\kappa\lambda}\} > 0. \tag{5.5.14}$$

Proof. Let $d(w) \in H$ be defined by (5.5.2). Then $\{d(w)\, v_i^\lambda, \bar{n}_f(w)^{-1}\, v^\lambda\} = d(w)^{\lambda_i}\{\bar{n}_f(w)^{-1}\, v^\lambda, v_i^\lambda\}$. But this equals $\{\bar{n}_{-f}(w)\, v_i^\lambda, v^{\kappa\lambda}\}$ by (5.5.3) and hence

$$d(w)^{\lambda_i}\, b_i(\lambda, w) = \{\bar{n}_{-f}(w)\, v_i^\lambda, v^{\kappa\lambda}\}^2. \tag{5.5.15}$$

But then (5.5.15) is nonnegative. Since $d(w)^{\lambda_i} > 0$ this proves $b_i(\lambda, w) \geqslant 0$. The equality in (5.5.13) and (5.5.14) is immediate from (5.5.1) and (5.5.12). The positivity then follows from the identities

$$b_1(\lambda, w) = d(w)^\lambda, \tag{5.5.16}$$

$$b_{q(\lambda)}(\lambda, w) = d(w)^{-\kappa\lambda}. \tag{5.5.17}$$

See (5.5.4) and (5.5.5). Q.E.D.

LEMMA 5.5.2. *Let* $a_o \in H$, $w_o \in \mathfrak{h}_+$ *and let* $\lambda \in D$. *Put* $a = \rho_w^{-1}(a_o) \in G_0{}^w$ *and* $w = f + w_o$. *Then*

$$h(a)^\lambda = \sum_{i=1}^{q(\lambda)} b_i(\lambda, w)\, a_o^{\lambda_i}. \tag{5.5.18}$$

Proof. By (5.3.2) one has $h(a)^\lambda = \{a_o \bar{n}_f(w)^{-1} v^\lambda, \bar{n}_{-f}(w)^* v^{\kappa\lambda}\}$. Hence if $c_i = \{a_o \bar{n}_f(w)^{-1} v^\lambda, v_i{}^\lambda\}$ and $c_i' = \{v_i{}^\lambda, \bar{n}_{-f}(w)^* v^{\kappa\lambda}\} = \{\bar{n}_{-f}(w)\, v_i{}^\lambda, v^{\kappa\lambda}\}$ one has

$$h(a)^\lambda = \sum_{i=1}^{q(\lambda)} c_i c_i'. \tag{5.5.19}$$

But $c_i = \{\bar{n}_f(w)^{-1} v^\lambda, a_o v_i{}^\lambda\} = a_o^{\lambda_i}\{\bar{n}_f(w)^{-1} v^\lambda, v_i{}^\lambda\}$ since $a_o = a_o^*$, recalling that $a_o \in H \subseteq P$. Thus $c_i c_i' = b_i(\lambda, w)\, a_o^{\lambda_i}$. But then (5.5.19) implies (5.5.18). Q.E.D.

PROPOSITION 5.5.2. *Let* $w_o \in \mathfrak{h}_+$ *and let* $w = f + w_o$. *Let* $G_0{}^w$ *be as in Theorem* 3.6. *Let* $g \in G_0{}^w$ *and put* $g_o = \rho_w(g) \in H$, *where* ρ_w *is defined by* (5.3.1). *Then for all* $t \in \mathbb{R}$ *one has* $g \exp tw \in G_0{}^w$ *and*

$$\rho_w(g \exp tw) = g_o \exp tw_o. \tag{5.5.20}$$

Now let $\lambda \in D$ *and let* $b_i(\lambda, w) \in \mathbb{R}$ *be defined by* (5.5.12), $i = 1,\ldots, q(\lambda) =$ $\dim V^\lambda$. *Then*

$$h(g \exp tw)^\lambda = \sum_{i=1}^{q(\lambda)} b_i(\lambda, w)\, g_o^{\lambda_i} e^{t\langle\lambda_i, w_o\rangle}, \tag{5.5.21}$$

where $h(a)$ *is defined by* (3.4.14). *See* (3.5.9).

Proof. Now, recalling (3.5.7) and (3.5.8), $G_1{}^w$ is the identity component of $\tilde{G}^w$. Thus $G_0{}^w G_1{}^w \subseteq G_0{}^w$. But of course $\exp tw \in G_1{}^w$. Thus $g \exp tw \in G_0{}^w$. Furthermore $\rho_w(g \exp tw) = m^{-1}\bar{n}_f(w)^{-1} g \exp tw \bar{n}_f(w) = (m^{-1}\bar{n}_f(w)^{-1} g \bar{n}_f(w))\, \bar{n}_f(w)^{-1} \exp tw \bar{n}_f(w) = \rho_w(g)\, \bar{n}_f(w)^{-1} \exp tw \bar{n}_f(w)$. But $\bar{n}_f(w)^{-1} \exp tw \bar{n}_f(w) = \exp tw_o$ since $\bar{n}_f(w)\, w_o = w$. Hence $\rho_w(g \exp tw) = \rho_w(g) \exp tw_o = g_o \exp tw_o$, proving (5.5.20). Thus if $a_o = g_o \exp tw_o$ then $a = g \exp tw$ using the notation of Lemma 5.5.2 and (5.5.20). But then $a_o^{\lambda_i} = g_o^{\lambda_i} e^{t\langle\lambda_i, w_o\rangle}$. Proposition 5.5.2 then follows from (5.5.18). Q.E.D.

The following result will later play the key role in determining the "phase" for scattering in the generalized Toda lattice.

We recall that κ is the element in the Weyl group W which maps the set of positive roots into the set of negative roots. Now we may regard W as operating on H as well as $\mathfrak{h}$. If $d \in H$ we will write $d^{-\kappa}$ for $\kappa(d^{-1})$. One notes that $(d^{-\kappa})^\lambda = d^{-\kappa\lambda}$ for any $\lambda \in \Lambda$.

THEOREM 5.5. *Let w_o be arbitrary in the open Weyl chamber $\mathfrak{h}_+$ and let g_o be arbitrary in the split Cartan subgroup H. Put $w = f + w_o$, where f is defined by* (1.5.4). *Let $G_0{}^w$ be defined by* (3.5.5–3.5.7) *and let $g = \rho_w^{-1}(g_o)$, where ρ_w is defined by* (5.3.1). *Now for any $t \in \mathbb{R}$ one has $g \exp tw \in G_0{}^w$ and $\rho_w(g \exp tw) = g_o \exp tw_o \in H$. Let $h(g \exp tw) \in H$ be defined by* (3.4.14). *See* (3.5.9), (3.4.13), *and* (3.2.1). *Then the curve $t \to h(g \exp tw)(g_o \exp tw_o)^{-1}$ in H converges as $t \to \infty$ and the curve $t \to h(\exp tw)(g_o \exp w_o)^{-\kappa}$ in H converges as $t \to -\infty$. Furthermore the limits are independent of g_o. In fact if $d(w) \in H$ is defined by* (5.5.2) *then, in H,*

$$\lim_{t\to\infty} h(g \exp tw)(g_o \exp tw_o)^{-1} = d(w) \tag{5.5.22}$$

and

$$\lim_{t\to-\infty} h(g \exp tw)(g_o \exp tw_o)^{-\kappa} = (d(w))^{-\kappa}. \tag{5.5.23}$$

Proof. Let $c(t) = h(g \exp tw)(g_o \exp tw_o)^{-1}$ and let $\tilde{c}(t) = h(g \exp tw)(g_o \exp tw_o)^{-\kappa}$. Now for any $\lambda \in D$ one has by (5.5.21)

$$c(t)^\lambda = b_1(\lambda, w) + \sum_{i=2}^{q(\lambda)} b_i(\lambda, w)\, g_o^{\lambda_i - \lambda} e^{t\langle \lambda_i - \lambda, w_o\rangle}$$

and

$$\tilde{c}(t)^\lambda = b_{q(\lambda)}(\lambda, w) + \sum_{i=1}^{q(\lambda)-1} b_i(\lambda, w)\, g_o^{\lambda_i - \kappa\lambda} e^{t\langle \lambda_i - \kappa\lambda, w_o\rangle}.$$

But now $\langle \lambda_i - \lambda, w_o\rangle < 0$ and $\langle \lambda_i - \kappa\lambda, w_o\rangle > 0$ by (5.5.11). Thus $\lim_{t\to\infty} c(t)^\lambda = b_1(\lambda, w)$ and $\lim_{t\to-\infty} \tilde{c}(t)^\lambda = b_{q(\lambda)}(\lambda, w)$. But then $\lim_{t\to\infty} c(t)^\lambda = d(w)^\lambda$ and $\lim_{t\to-\infty} \tilde{c}(t)^\lambda = (d(w))^{-\kappa\lambda}$ by (5.5.16) and (5.5.17).

On the other hand, recalling (5.1.2), the map

$$H \to (\mathbb{R}^*)^l, \qquad h \to (h^{\nu_1}, \ldots, h^{\nu_l}) \tag{5.5.24}$$

is a diffeomorphism since the elements ν_i are a basis of $\mathfrak{h}'$. Thus since we can put $\lambda = \nu_i$, $i = 1, \ldots, l$, this proves $\lim_{t\to\infty} c(t) = d(w)$ and $\lim_{t\to-\infty} \tilde{c}(t) = d(w)^{-\kappa}$. Q.E.D.

Related to the two maps $H \to H$, $g_o \mapsto h(g)\, g_o^{-1}$ and $g_o \mapsto h(g)\, g_o^{-\kappa}$ mentioned in Theorem 5.5 one has

PROPOSITION 5.5.3. *Let $\lambda \in D$ and let the notation be as in Theorem* 5.5. *Then*

$$(h(g)\, g_o{}^\kappa)^{-\lambda} = \{v^\lambda, n(g)\, v^{\kappa\lambda}\}. \tag{5.5.25}$$

Furthermore there exists a scalar $b \in \mathbb{R}$ *such that* $s(\kappa)^2 v_\lambda = bv_\lambda$ *and one has*

$$(h(g)\, g_o^{-1})^{-\lambda} = b\{\bar{n}(g)\, v^\lambda, v^{\kappa\lambda}\}. \tag{5.5.26}$$

Proof. By (5.3.6) one has $\bar{n}hn = s_o(\kappa)^{-1}\, \bar{n}_{-f} g_o \bar{n}_f^{-1}$, where we have written $\bar{n}$, h, and n for $\bar{n}(g)$, $h(g)$, and $n(g)$. Also, we have written $\bar{n}_{-f}$ and $\bar{n}_f^{-1}$ for $\bar{n}_{-f}(w)$ and $\bar{n}_f^{-1}(w)$. But then since $\bar{n}^* \in N$ and $h = h^*$ one has

$$\begin{aligned} h^\lambda\{v^\lambda, nv^{\kappa\lambda}\} &= \{v^\lambda, \bar{n}hnv^{\kappa\lambda}\} \\ &= \{v^\lambda, s_o(\kappa)^{-1}\, \bar{n}_{-f} g_o \bar{n}_f^{-1} v^{\kappa\lambda}\}. \end{aligned} \tag{5.5.27}$$

But clearly $\bar{n}_{-f} g_o \bar{n}_f^{-1} v^{\kappa\lambda} = g_o^{\kappa\lambda} v^{\kappa\lambda}$. Thus since $(s_o(\kappa)^{-1})^* = s_o(\kappa)$ by Lemma 5.2.3 one has $h^\lambda\{v^\lambda, nv^{\kappa\lambda}\} = g_o^{\kappa\lambda}\{v^{\kappa\lambda}, v^{\kappa\lambda}\}$ or $\{v^\lambda, nv^{\kappa\lambda}\} = h^{-\lambda} g_o^{-\kappa\lambda}$, proving (5.5.25). But now on the other hand $h^\lambda\{\bar{n}v^\lambda, v^{\kappa\lambda}\} = \{\bar{n}hnv^\lambda, v^{\kappa\lambda}\} = \{s_o(\kappa)^{-1}\, \bar{n}_{-f} g_o \bar{n}_f^{-1} v^\lambda, v^{\kappa\lambda}\}$. But one has $b \in \mathbb{R}$ such that $s_o(\kappa)\, v^{\kappa\lambda} = s_o(\kappa)^2\, v^\lambda = bv^\lambda$ since of course $\kappa^2 = 1$. But $\bar{n}_{-f}^*$ and $(\bar{n}_f^{-1})^*$ are in N and hence they fix v^λ. Also $g_o^* = g_o$. Thus $h^\lambda\{\bar{n}v^\lambda, v^{\kappa\lambda}\} = bg_o{}^\lambda$. This proves (5.5.26), since $b = \pm 1$ by Lemma 5.2.3. Q.E.D.

For any $\mu \in \Lambda$ let

$$o(\mu) = \langle \mu, x_o \rangle, \tag{5.5.28}$$

where $x_o \in \mathfrak{h}$ is defined as in Section 2.1.

Remark 5.5. Since $\lambda - \kappa\lambda$ is spanned by roots (using, e.g., (5.1.9)) one has $o(\lambda - \kappa\lambda) \in \mathbb{Z}$. We shall not use the following, but one can show that the scalar b in Proposition 5.5.3 is given by $b = (-1)^{o(\lambda - \kappa\lambda)}$.

5.6. In this section we wish to show the connection between the development here and the results and calculations of Moser in [19]. Let $x \in \operatorname{Jac} \mathfrak{p}$ so that x is a "symmetric" Jacobi element. By Theorem 5.4 there exist unique $w_o \in \mathfrak{h}_+$ and $g_o \in H$ such that $x = k(g_o, w_o)\, w_o$. Let g and w be defined as usual so that $w = f + w_o$ and $\rho_w^{-1}(g_o) \in G_0{}^w$. Now for any $\lambda \in D$ let, for $1 \leqslant i \leqslant q(\lambda) = \dim V^\lambda$,

$$r_i{}^\lambda(g_o, w_o) = \{kv_i{}^\lambda, v^{\kappa\lambda}\}, \tag{5.6.1}$$

where $k = k(g_o, w_o)$. We use the letter r here because of the connection, as will soon be seen, between $r_i{}^\lambda(g_o, w_o)$ and the coordinates r_i in [19]. Now as noted in Remark 5.4 the eigenvectors for $\pi_\lambda(x)$ are just $kv_i{}^\lambda$. The corresponding eigenvalues are clearly $\zeta_i = \langle \lambda_i, w_o \rangle$. See Section 5.5. Thus if ζ is indeterminate one has (as an entry in the resolvent)

$$\{(\zeta 1 - x)^{-1}\, v^{\kappa\lambda}, v^{\kappa\lambda}\} = \sum_{i=1}^{q(\lambda)} \frac{(r_i{}^\lambda)^2}{\zeta - \zeta_i}, \tag{5.6.2}$$

where 1 is the identity operator on V^λ and $r_i^\lambda = r_i^\lambda(g_o, w_o)$. Equation (5.6.2) follows immediately from the relation

$$(\zeta 1 - \pi_\lambda(x))^{-1} = \pi_\lambda(k)(\zeta 1 - \pi_\lambda(w_o))^{-1}\,\pi_\lambda(k^{-1}). \tag{5.6.3}$$

The r_i^λ may be given by

PROPOSITION 5.6. *One has*

$$(r_i^\lambda(g_o, w_o))^2 = h(g)^{\kappa\lambda}\, g_o^{-\lambda_i} b_i(\lambda, w), \tag{5.6.4}$$

where we recall $b_i(\lambda, w) = \{\bar{n}_f(w)^{-1} v^\lambda, v_i^\lambda\}\{\bar{n}_{-f}(w)\, v^{\lambda_i}, v^{\kappa\lambda}\}$. *See* (5.5.12).

Proof. If we put $h = h(g)$, $n = n(g)$, $\bar{n}_f = \bar{n}_f(w)$, and $p = p(g_o, w_o)$ then recalling (5.4.1) one has $k = h^{1/2} n \bar{n}_f\, p^{-1}$. But $r_i^\lambda = \{kv_i^\lambda, v^{\kappa\lambda}\}$. Thus since $(h^{1/2})^* = h^{1/2}$ and $n^* \in \bar{N}$ so that $n^* v^{\kappa\lambda} = v^{\kappa\lambda}$ one has

$$r_i^\lambda = h^{\kappa\lambda/2} p^{-\lambda_i}\{\bar{n}_f(w)\, v_i^\lambda, v^{\kappa\lambda}\}. \tag{5.6.5}$$

Now let $m \in H_{\mathbb{C}}$ be as in (3.5.5) and let $m_o \in H_{\mathbb{C}}{}^s$ be such that $\operatorname{Ad} m_o = m$. But then $m_o \bar{n}_f(w)(m_o)^{-1} = \bar{n}_{-f}(w)$ in $G_{\mathbb{C}}{}^s$ using (5.2.2), recalling also that $m^2 = 1$. But then if $c', c'' \in \mathbb{C}^*$ are defined by $m_o^{-1} v_i^\lambda = c' v_i^\lambda$ and $m_o^* v^{\kappa\lambda} = c'' v^{\kappa\lambda}$ one has $\{\bar{n}_{-f}(w)\, v_i^\lambda, v^{\kappa\lambda}\}c = \{\bar{n}_f(w)\, v_i^\lambda, v^{\kappa\lambda}\}$, where $c = c'\overline{c''}$. But $|c| = 1$ since m_o and m_o^* clearly have finite order. But then one must have

$$(\{\bar{n}_{-f}(w)\, v_i^\lambda, v^{\kappa\lambda}\})^2 = \{\bar{n}_f(w)\, v_i^\lambda, v^{\kappa\lambda}\}^2 \tag{5.6.6}$$

since c^2 times the left side of (5.6.6) is the right side of (5.6.6), both sides are nonnegative, and $|c^2| = 1$. (That is, if (5.6.6) is nonzero, one has $c^2 = 1$.)

But now if $d = d(w)$, recalling (5.5.2) one has $dv_i^\lambda = d^{\lambda_i} v_i^\lambda$ and hence $d^{\lambda_i}\{v_i^\lambda, (\bar{n}_f(w))^{-1} v^\lambda\} = \{\bar{n}_{-f}(w)\, v_i^\lambda, v^{\kappa\lambda}\}$. But then upon substituting for just one of the factors in the left side of (5.6.6) one has

$$d^{\lambda_i} b_i(\lambda, w) = \{\bar{n}_f(w)\, v_i^\lambda, v^{\kappa\lambda}\}^2. \tag{5.6.7}$$

But then by squaring (5.6.5) and substituting (5.6.7) one has

$$(r_i^\lambda)^2 = h^{\kappa\lambda} p^{-2\lambda_i} d^{\lambda_i} b_i(\lambda, w). \tag{5.6.8}$$

But $p^{-2} d = g_o^{-1}$. This proves (5.6.4). Q.E.D.

Now clearly one has

$$\sum_{i=1}^{q(\lambda)} (r_i^\lambda)^2 = 1 \tag{5.6.9}$$

by (5.2.11) and Lemma 5.2.3.

In [19] Moser considers the case where for us $G^s = Sl(n, \mathbb{R})$ and π_λ is the standard representation. That is, $V^\lambda = \mathbb{R}^n$. In that case if we write $r_i = r_i^\lambda$ then r_i^2 has the same meaning as in [19]. He then changes the definition of r_i^2 so that they become "homogeneous." That is, in our notation define $s_i^\lambda(g_o, w_o)$ by putting

$$s_i^\lambda(g_o, w_o) = h(g)^{-\kappa\lambda/2} r_i^\lambda(g_o, w_o) \tag{5.6.10}$$

so that by Proposition 5.6 one has, if $s_i^\lambda = s_i^\lambda(g_o, w_o)$,

$$(s_i^\lambda)^2 = g_o^{-\lambda_i} b_i(\lambda, w). \tag{5.6.11}$$

But then by (5.6.9) Eq. (5.6.2) becomes (see 3.9 in [19])

$$\{(\zeta 1 - x)^{-1} v^{\kappa\lambda}, v^{\kappa\lambda}\} = \Big(\sum_i \frac{(s_i^\lambda)^2}{\zeta - \zeta_i}\Big) \frac{1}{\sum_i (s_i^\lambda)^2}\cdot \tag{5.6.12}$$

Now the problem of solving the generalized Toda lattice, as we shall see in Section 7, rests with determining $h(g \exp tw)^\nu$ for suitable $\nu \in \mathfrak{h}$. For the standard Toda lattice one needs only the case $\nu = \alpha_i$, $i = 1, 2,\ldots, l$. However, if $x(t) = k(g_o \exp tw_o)\, w_o$ then the functions $h(g \exp tw)^{-\alpha_i/2}$ appear as matrix entries of $x(t)$. See (3.7.4) and (5.4.10). Furthermore as Moser notes, going back to Stieltjes, the left side of (5.6.12) with $x(t)$ replacing x is computed using a continued fraction expansion in the matrix entries. One notes that $v^{\kappa\lambda} = e_n$, using the notation of [19]. On the other hand (5.6.12) clearly becomes an equation for all $t \in \mathbb{R}$ if we substitute $(s_i^\lambda)^2 e^{-t\langle\lambda_i, w_o\rangle}$ for $(s_i^\lambda)^2$. These being simple exponential functions the method of [19] is then to use (5.6.12) to inductively solve for $h(g \exp tw)^{\alpha_i}$.

Now it is clear that a more direct and systematic approach to determine $h(g \exp tw)^\lambda$, for $\lambda \in D$ arises from (5.5.21). However, (5.5.21) is a formula for $h(g \exp tw)^\lambda$ only insofar as one can determine $\bar{n}_f(w)^{-1}$ and $\bar{n}_{-f}(w)$ (and their images under π_λ) explicitly. Furthermore the scattering phase change also will depend upon such a determination. We will solve this problem in Section 5.8 using the machinery of representation theory.

5.7. Let $\mathbb{R}[G^s]$ be the group algebra of G^s when the latter is regarded as an abstract group. Let $D(G^s) = \mathbb{R}[G^s] \otimes_{\mathbb{R}} U$. In this section only we will write $a \cdot u$ for au, where $a \in G^s$, $u \in U$, to denote the adjoint action of a on u. This is done to avoid confusion with the multiplication in $D(G^s)$, which will now be defined. One makes $D(G^s)$ into an algebra by retaining the given algebra structures in $\mathbb{R}[G^s]$ and U and putting $ana^{-1} = a \cdot u$ for $a \in G^s$, $u \in U$. One knows that we may identify $D(G^s)$ with the algebra (under convolution) of all distributions (in the sense of Schwartz) of finite support on G^s. Thus if $a \in G^s$ then $aU = Ua$ is the set of all distributions on G^s with support at a.

Now if V is a finite-dimensional real vector space then we will say that V is a smooth $D(G^s)$ module if it is a $D(G^s)$ module with respect to a representation $\pi: D(G^s) \to \operatorname{End} V$ such that $\pi \mid G^s$ is a representation of Lie groups and $\pi \mid \mathfrak{g}$ is the differential of $\pi \mid G^s$. Now a linear functional ψ on $D(G^s)$ will be called a representative functional if it is in the span of linear functionals of the form $\psi_{v,v'}$, where V is a smooth $D(G^s)$ module, $v \in V$, $v' \in V'$ (the dual to V), and $\psi_{v,v'}(a) = \langle av, v' \rangle$ for all $a \in D(G^s)$. Let $D(G^s)'$ be the space of all representative functionals on $D(G^s)$. Then as one knows, $D(G^s)$ and $D(G^s)'$ are non-singularly paired. (This is clear since (1) $\subseteq G^s \subseteq G_{\mathbb{C}}{}^s$ and $G_{\mathbb{C}}{}^s$ has a faithful finite-dimensional representation and (2) $D(G^s)'$ has the structure of an algebra—using the coalgebra structure on $D(G^s)$. Given distinct elements $g_i \in G^s$, $i = 1,..., k$, one easily then constructs $\psi \in D(G^s)'$, which vanishes to any given preassigned order at $g_2, ..., g_l$ and is such that $\psi(u) \neq 0$, where u is a given distribution with support at g_1.)

We now topologize $D(G^s)$ (the weak $*$ topology) so that $D(G^s)'$ is its continuous dual. One notes then, for example, that the series $\sum_{j=0}^{\infty} (x^j/j!)$ converges to $\exp^s x$ for any $x \in \mathfrak{g}$. Let $\hat{D}(G^s)$ be the completion of $D(G^s)$ with respect to this topology. If $a \in D(G^s)$ then left and right multiplication by a in $D(G^s)$ is clearly continuous and hence such an operator extends to $\hat{D}(G^s)$. This defines on $\hat{D}(G^s)$ the structure of a two-sided $D(G^s)$-module.

Now let $\mathscr{S}$ be the set of all finite sequences

$$s = (i_1, ..., i_k), \tag{5.7.1}$$

where $k \geqslant 0$ is arbitrary and i_j is an integer such that $1 \leqslant i_j \leqslant l$ for $j \leqslant k$. We will write $|s|$ for the length k of the element $s \in \mathscr{S}$.

Now let $U(\bar{n})$ and $U(n)$ be respectively the enveloping algebras of $\bar{n}$ and n. For convenience write $e_i = e_{\alpha_i}$ and $e_{-i} = e_{-\alpha_i}$ for $i = 1,..., l$. Furthermore if $s \in \mathscr{S}$ and s is given by (5.7.1) put

$$e_{-s} = e_{-i_k} \cdots e_{-i_2} e_{-i_1} \tag{5.7.2}$$

and

$$e_s = e_{i_1} e_{i_2} \cdots e_{i_k}. \tag{5.7.3}$$

If $|s| = 0$ then one puts $e_s = e_{-s} = 1$. In any case

$$e_s^* = e_{-s} \tag{5.7.4}$$

and since the e_i (resp. e_{-i}) generate n (resp. $\bar{n}$) it is clear that the e_s (resp. e_{-s}) for $s \in \mathscr{S}$ span $U(n)$ (resp. $U(\bar{n})$). It should be noted, however, that the e_{-s} (for example) are not linearly independent.

Now let $x_o \in h$ be as in Section 2.1 so that $[e_{-i}, x_o] = e_{-i}$. One thus has

$$[e_{-s}, x_o] = |s| e_{-s} \tag{5.7.5}$$

for any $s \in \mathscr{S}$. Now let $U(\bar{n})_k = \{u \in U(\bar{n}) \mid [u, x_0] = ku\}$ so that one has the direct sum

$$U(\bar{n}) = \sum_{k=0}^{\infty} U(\bar{n})_k \tag{5.7.6}$$

and $U(\bar{n})_k$ is spanned by all e_{-s}, $s \in \mathscr{S}$, where $|s| = k$.

Now let $D(\bar{N})$ be the subalgebra of $D(G^s)$ generated by $\bar{N}$ and $\bar{n}$ and let $\hat{D}(\bar{N})$ be the closure of $D(\bar{N})$ in $\hat{D}(G^s)$.

PROPOSITION 5.7.1. *For every $k \in \mathbb{Z}_+$ choose an arbitrary element $u_k \in (U(\bar{n}))_k$. Then the infinite sum $\sum_{k \in \mathbb{Z}_+} u_k$ converges in $\hat{D}(\bar{N})$. Furthermore any $u \in \hat{D}(\bar{N})$ can be uniquely written as an infinite sum*

$$u = \sum_{k \in \mathbb{Z}_+} u_k , \tag{5.7.7}$$

where $u_k \in U(\bar{n})_k$.

Proof. Let $U_+(\bar{n}) = \sum_{k=1}^{\infty} U(\bar{n})_k$ so that $U_+(\bar{n})$ is the augmentation ideal. Furthermore it is clear that the elements e_{-i}, $i = 1,\dots, l$, are a basis of $U(\bar{n})_1$ so that if $j \in \mathbb{Z}_+$ and $U_+(\bar{n})^j$ is the jth power of the augmentation ideal one has

$$U_+(\bar{n})^j = \sum_{k=j}^{\infty} U(\bar{n})_k . \tag{5.7.8}$$

Now let $D(\bar{N})'$ be the set of all restrictions $\psi \mid D(\bar{N})$, where $\psi \in D(G^s)'$. But if π is any faithful finite-dimensional holomorphic representation of $G_{\mathbb{C}}{}^s$ then $\pi(\bar{N})$ is a group of unipotent operators and $\pi \mid \bar{N}_{\mathbb{C}}$ is faithful. It follows easily then that if $\psi \in D(\bar{N})'$ then the restriction $\psi \mid U(\bar{n})$ vanishes on a power of $U_+(\bar{n})$ and that every such linear functional on $U(\bar{n})$ is uniquely of this form. Thus by (5.7.6) and (5.7.8) if $U(\bar{n})'_k$ is the dual space to $U(\bar{n})_k$ we may regard $U(\bar{n})'_k \subseteq D(\bar{N})'$, where $U(\bar{n})'_k$ is identified with the set of all elements in $D(\bar{N})'$ which vanish on $D(\bar{n})_j$ for $j \neq k$. One then has the direct sum

$$D(\bar{N})' = \sum_{k \in \mathbb{Z}_+} U(\bar{n})'_k . \tag{5.7.9}$$

Now let $y_k \in U(\bar{n})_k$ for all $k \in \mathbb{Z}_+$. Then for any $\psi \in D(\bar{N})'$ one has $\psi(y_k) = 0$ for k sufficiently large. This proves that the sum $\sum y_k$ converges. On the other hand if $\bar{n} \in \bar{N}$ then we may write $\bar{n} = \exp^s x$ for $x \in \bar{n}$. Thus $\bar{n} = \sum (x^j/j!)$. This (and in fact also (5.7.9)) implies that $U(\bar{n})$ is dense in $D(\bar{N})$. Now let $u \in \hat{D}(\bar{N})$ and let $u(j) \in U(\bar{n})$, $j = 1, 2,\dots$, be a sequence such that $u(j)$ converges to u. By (5.7.6) we may write $u(j) = \sum u(j)_k$, where $u(j)_k \in U(\bar{n})_k$. But now $u(j)_k$, as $j \to \infty$, must, by (5.7.9), have a limit $u_k \in U(\bar{n})_k$ and also $u = \sum u_k$. This is clear since $\psi(u(j)) = \psi(u(j)_k)$ converges for any $\psi \in U(\bar{n})'_k$. This proves the existence in Proposition 5.7. The uniqueness follows easily since u_k is characterized by the relation $\psi(u_k) = \psi(u)$ (which is of course defined) for any $\psi \in U(\bar{n})'_k$. Q.E.D.

PROPOSITION 5.7.2. *For any $s \in \mathscr{S}$ choose a scalar $b_s \in \mathbb{R}$. Then the infinite sum $\sum b_s e_{-s}$ relative to any simple ordering in $\mathscr{S}$ converges to an element in $\hat{D}(\bar{N})$ and this element is independent of the ordering. Furthermore any element in $\hat{D}(\bar{N})$ may be written as such a sum.*

Proof. Given any $\psi \in D(\bar{N})'$ it suffices in order to prove the first statement only to note that $\psi(e_{-s}) = 0$ for all $s \in \mathscr{S}$ such that $|s|$ is sufficiently large. But this is clear from (5.7.9) since $e_{-s} \in U(\bar{n})_{|s|}$. See (5.7.5). The second statement follows from Proposition 5.7.1 since $U(\bar{n})_k$ is spanned by all e_{-s} such that $|s| = k$. Q.E.D.

5.8. Now let $s \in \mathscr{S}$. Assume s is given by (5.7.1) so that in particular $|s| = k$. Now if $k \geqslant 1$ let $\varphi(s) \in \Lambda$ be the linear form on the Cartan subalgebra $\mathfrak{h}$ given by

$$\varphi(s) = \sum_{j=1}^{|s|} \alpha_{i_j}. \tag{5.8.1}$$

One puts $\varphi(s)$ equal to the constant function 1 on $\mathfrak{h}$ if $|s| = 0$.

On the other hand for any $0 \leqslant j \leqslant |s|$ let $s_j \in \mathscr{S}$ be the sequence obtained from s by "cutting off" the last j terms. That is,

$$s_j = (i_1, i_2, \dots, i_{|s|-j}). \tag{5.8.2}$$

One notes of course that $s_0 = s$ and

$$|s_j| = |s| - j. \tag{5.8.3}$$

We now observe that s defines a polynomial of degree $|s|$ on $\mathfrak{h}$ by putting for any $w_o \in \mathfrak{h}$

$$p(s, w_o) = \prod_{j=0}^{|s|} \langle \varphi(s_j), w_o \rangle. \tag{5.8.4}$$

Remark 5.8.1. It is useful to think of $p(s, w_o)$ as some sort of generalized "factorial" expression. Indeed putting $w_o = x_o$ (see Section 2.1) note that $p(s, x_o) = |s|!$

PROPOSITION 5.8.1. *If $w_o \in \mathfrak{h}_+$ then $p(s, w_o) > 0$ for any $s \in \mathscr{S}$.*

Proof. This is immediate from (5.8.1) and (5.8.2) since $\langle \alpha_i, w_o \rangle > 0$, $i = 1, \ldots, l$, for $w_o \in \mathfrak{h}_+$ by definition of $\mathfrak{h}_+$. Q.E.D.

Now for any $s \in \mathcal{S}$ let $\bar{s} \in \mathcal{S}$ be the sequence obtained from s by reversing the order. Thus if s is given by (5.7.1) then

$$\bar{s} = (i_k, i_{k-1}, \ldots, i_1). \tag{5.8.5}$$

Now recalling Lemma 3.5.2 we can determine the element $\bar{n}_f(w) \in \bar{N}$ and observe the dependence on $w_o \in \mathfrak{h}_+$.

PROPOSITION 5.8.2. *Let $w_o \in \mathfrak{h}_+$ and let $w = f + w_o$, where $f = \sum_{i=1}^{l} e_{-\alpha_i}$. Let $\bar{n}_f(w) \in \bar{N}$ be the element defined by* (3.5.2) *so that $\bar{n}_f(w) \cdot w_o = w$. Now let $\mathcal{S}$ be the set of all finite sequences $s = (i_1, \ldots, i_k)$ of integers where $1 \leqslant i_j \leqslant l$. For any $s \in \mathcal{S}$ let $p(s, w_o)$ be defined by* (5.8.4) *so that $p(s, w_o) > 0$ (recalling Proposition* 5.8.1). *Let $e_{-s} \in U(\bar{\mathfrak{n}})$ be defined by* (5.7.2). *Then recalling* (5.2.2) *(and also Proposition* 5.7.2) *one has*

$$\bar{n}_{-f}(w) = \sum_{s \in \mathcal{S}} (-1)^{|s|} \frac{e_{-s}}{p(s, w_o)}. \tag{5.8.6}$$

Furthermore recalling (5.8.5) *one has*

$$\bar{n}_f(w)^{-1} = \sum_{s \in \mathcal{S}} (-1)^{|s|} \frac{e_{-\bar{s}}}{p(s, w_o)}. \tag{5.8.7}$$

Proof. Now let $s \in S$. Then if $\{t^1, \ldots, t^l\}$ is the set of all $t \in S$ such that $t_1 = s$ (using the notation of (5.8.2)) one clearly has

$$fe_{-s} = \sum_{i=1}^{l} e_{-t^i}. \tag{5.8.8}$$

On the other hand one also has

$$e_{-\bar{s}}f = \sum_{i=1}^{l} e_{-\bar{t}^i}. \tag{5.8.9}$$

Now let $v \in \hat{D}(\bar{N})$ be the element (see Proposition 5.7.2) defined by the right side of (5.8.6). Then by (5.8.8) one notes that

$$-fv = \sum_{\substack{s \in \mathcal{S} \\ |s| \geqslant 1}} (-1)^{|s|} \frac{e_{-s}}{p(s_1, w_o)}. \tag{5.8.10}$$

On the other hand since $[e_{-i}, w_o] = \langle \alpha_i, w_o \rangle e_{-i}$ one certainly has

$$[e_{-s}, w_o] = \langle \varphi(s), w_o \rangle e_{-s}. \tag{5.8.11}$$

However, note that

$$\frac{\langle \varphi(s), w_o \rangle}{p(s, w_o)} = \frac{1}{p(s_1, w_o)}. \tag{5.8.12}$$

But then by (5.8.10) one has in $\hat{D}(G^s)$,

$$[v, w_o] = \sum_{\substack{s \in \mathscr{S} \\ |s| \geqslant 1}} (-1)^{|s|} \frac{e_{-\bar{s}}}{p(s_1, w_o)}. \tag{5.8.13}$$

The term for $|s| = 0$ does not occur in (5.8.13) since $[1, w_o] = 0$. Thus by (5.8.10) one has (in $\hat{D}(G^s)$)

$$[v, w_o] = -fv. \tag{5.8.14}$$

But in $\hat{D}(G^s)$ one has the relation (see the statement following (5.2.2)) $\bar{n}_{-f}(w)\, w_o(\bar{n}_{-f}(w))^{-1} - w_o = -f$. Substituting then for $-f$ in (5.8.14) one has $[v, w_o] = \bar{n}_{-f}(w)\, w_o(\bar{n}_{-f}(w))^{-1}v - w_o v$. Thus $vw_o = \bar{n}_{-f}(w)\, w_o(\bar{n}_{-f}(w))^{-1}v$ and hence if we put $u = \bar{n}_{-f}(w)^{-1}v$ then $uw_o = w_o u$. That is, $u \in \hat{D}(\bar{N})$ and u commutes with w_o. Now let $u_k \in U(\bar{n})_k$ be defined by (5.7.7) for any $k \in \mathbb{Z}_+$. Since $[w_o, u] = 0$ and since $U(\bar{n})_k$ is stable under $\operatorname{ad} w_o$ it follows from the uniqueness of (5.7.7) that $[w_o, u_k] = 0$ for all k. However, since $\operatorname{ad} w_o$ is diagonalizable with a strictly negative spectrum in $\bar{n}$ it follows that the same is true in $U(\bar{n})_k$ for any $k > 0$. Thus $u_k = 0$ for all $k > 0$. On the other hand the leading term of v is 1, using (5.7.7), since $p(s, w_o) = 1$ if $|s| = 0$. By considering the exponential series the same is then true for $(\bar{n}_{-f}(w))^{-1}$. Thus $u_o = 1$. Indeed if $s, s' \in \mathscr{S}$ then $e_{-s}e_{-s'} = e_{-s''}$ for some $s'' \in \mathscr{S}$ where $|s''| = |s| + |s'|$ so that using the expansions given by (5.7.7) one easily see that u_o must be the product of the constant terms of $\bar{n}_f(w)^{-1}$ and v. Thus $u = 1$ so that $\bar{n}_{-f}(w) = v$. This proves (5.8.6).

Now let y be the right side of (5.8.7). Then by (5.8.9) one has

$$yf = -\sum_{\substack{s \in \mathscr{S} \\ |s| \geqslant 1}} (-1)^{|s|} \frac{e_{-\bar{s}}}{p(s_1, w_o)}. \tag{5.8.15}$$

On the other hand $[w_o, y]$ is also given by the right side of (5.8.15) using (5.8.11) and (5.8.12). Thus

$$[w_o, y] = yf. \tag{5.8.16}$$

But $\bar{n}_f(w)\, w_o(\bar{n}_f(w))^{-1} - w_o = f$. Hence if we substitute for f in (5.8.16) we obtain $y\bar{n}_f(w) = 1$ in a manner similar to the proof of (5.8.6). That is, $y = (\bar{n}_f(w))^{-1}$. Q.E.D.

Now let $D(N)$ be the subalgebra of $D(G^s)$ generated by N and n and let $\hat{D}(N)$ be the closure of $D(N)$ in $\hat{D}(G^s)$. It is obvious that Proposition 5.7.2 is valid if N replaces $\bar{N}$ and e_s replaces e_{-s} for all $s \in \mathscr{S}$.

But now $(\bar{n}_{-f}(w))^* \in N$ by (1.5.1) and (3.1.9).

PROPOSITION 5.8.3. *Let the notation be as in Proposition* 5.8.2. *See also* (5.7.3). *Then*

$$(\bar{n}_{-f}(w))^* = \sum_{s \in \mathscr{S}} (-1)^{|s|} \frac{e_s}{p(s, w_o)}. \tag{5.8.17}$$

Proof. Let $a \in G^s$ and $x_j \in U$, $j = 1, 2, \ldots$. Assume that x_j converges to a in $D(G^s)$. We then assert that x_j^* converges to a^*. Indeed recalling the definition of representative functional in Section 5.7 and the particular representative functional $\psi_{v,v'}$ one notes that $\psi_{v,v'}(x_j^*)$ converges to $\psi_{v,v'}(a^*)$ in case $V = V^\lambda$ by (5.1.11). But then the result follows easily for all V using the complete reducibility of smooth $D(G^s)$ modules. Thus x_j^* converges to a^*. But then (5.8.17) follows from (5.8.6) and (5.7.4). Q.E.D.

Remark 5.8.2. One major advantage of formulas (5.8.6), (5.8.7), and (5.8.17) is that for any given smooth $D(G^s)$ module we need consider only a finite number of summands.

5.9. Now let $\lambda \in D$ and let

$$\mathscr{S}^\lambda = \{s \in \mathscr{S} \mid \varphi(s) = \lambda - \kappa\lambda\}. \tag{5.9.1}$$

Now $\lambda - \kappa\lambda$ is the difference between the highest and lowest weights of V^λ and hence we can write $\lambda - \kappa\lambda = \sum_{i=1}^{l} m_i \alpha_i$, where $m_i \in \mathbb{Z}_+$. If $o(\lambda - \kappa\lambda)$ is defined as in (5.5.28) note that $o(\lambda - \kappa\lambda) = \sum m_i$ and hence the cardinality of $\mathscr{S}^\lambda$ is given by

$$|\mathscr{S}^\lambda| = \frac{o(\lambda - \kappa\lambda)!}{m_1! \cdots m_l!}. \tag{5.9.2}$$

One notes also that $\mathscr{S}^\lambda$ is stable under the map $\mathscr{S} \to \mathscr{S}$, $s \to \bar{s}$.

Now for any $s \in \mathscr{S}^\lambda$ clearly e_{-s} carries v^λ into a multiple of $v^{\kappa\lambda}$. Thus there exists $c_{s,\lambda} \in \mathbb{R}$ such that

$$e_{-s} v^\lambda = c_{s,\lambda} v^{\kappa\lambda}. \tag{5.9.3}$$

PROPOSITION 5.9.1. *Let* $\lambda \in D$ *and* $w_o \in h_+$. *Then if* $p(s, w_o)$ *is defined by* (5.8.4) *for any* $s \in \mathscr{S}$ *one has*

$$\begin{aligned}(-1)^{o(\lambda-\kappa\lambda)} \sum_{s \in \mathscr{S}^\lambda} \frac{c_{s,\lambda}}{p(s, w_o)} &= \{\bar{n}_{-f}(w)\, v^\lambda, v^{\kappa\lambda}\} \\ &= d(w)^\lambda,\end{aligned} \tag{5.9.4}$$

where we recall that $d(w) \in H$ *is defined by* (5.5.2). *Furthermore*

$$(-1)^{o(\lambda-\kappa\lambda)} \sum_{s\in\mathcal{S}^\lambda} \frac{c_{s,\lambda}}{p(\bar{s}, w_o)} = \{\bar{n}_f(w)^{-1} v^\lambda, v^{\kappa\lambda}\} = d(w)^{-\kappa\lambda}. \tag{5.9.5}$$

In particular the numbers given by the sums in (5.9.4) *and* (5.9.5) *are positive.*

Proof. If $s \in \mathcal{S}$ then by (5.1.14) and (5.2.11) clearly $\{e_{-s}v^\lambda, v^{\kappa\lambda}\} = 0$ if $s \notin \mathcal{S}^\lambda$ and $\{e_{-s}v^\lambda, v^{\kappa\lambda}\} = c_{s,\lambda}$ if $s \in \mathcal{S}$. But then (5.9.4) and (5.9.5) follow from (5.8.6), (5.8.7), (5.5.4), and (5.5.5). Note that we have interchanged s and $\bar{s}$ in (5.8.7). Q.E.D.

Since the numbers $c_{s,\lambda}$, $s \in \mathcal{S}^\lambda$ play such an important role in our solution of the generalized Toda lattice we will give another expression for them purely in terms of U. For any $s, t \in \mathcal{S}^\lambda$ the element $e_t e_{-s}$ clearly commutes with $\mathfrak{h}$. That is, in the notation of (5.1.10) one has $e_t e_{-s} \in U(0)$. Now by (5.1.15) there exists a unique element $u_{t,s} \in U(\mathfrak{h})$ such that $e_t e_{-s} - u_{t,s}$ is in the left ideal $U\mathfrak{n}$. On the other hand we can regard $U(\mathfrak{h})$ as the algebra of real-valued polynomial functions on $\mathfrak{h}'$ so that in particular if $u \in U(\mathfrak{h})$ then $uv^\lambda = u(\lambda)\, v^\lambda$.

Proposition 5.9.2. *Let* $\lambda \in D$. *Then the* $o(\lambda - \kappa\lambda) \times o(\lambda - \kappa\lambda)$ *matrix* $\{u_{t,s}(\lambda)\}$, *indexed by* $t, s \in \mathcal{S}^\lambda$ *has rank* 1 *and hence there exists a unique, up to sign, vector* $\{c_{s,\lambda}\}$ *in* $\mathbb{R}^{o(\lambda-\kappa\lambda)}$ *such that*

$$c_{s,\lambda} c_{t,\lambda} = u_{t,s}(\lambda) \tag{5.9.6}$$

for all s, t. *The sign is, however, determined by the relation*

$$(-1)^{o(\lambda-\kappa\lambda)} \sum_{s\in\mathcal{S}^\lambda} c_{s,\lambda} > 0. \tag{5.9.7}$$

Proof. Now clearly $\{e_{-s}v^\lambda, e_{-t}v^\lambda\} = \{e_t e_{-s}v^\lambda, v^\lambda\} = \{u_{s,t}v^\lambda, v^\lambda\} = u_{s,t}(\lambda)$ since $U\mathfrak{n}$ annihilates v^λ. But $c_{s,\lambda}c_{t,\lambda} = \{e_{-s}v^\lambda, e_{-t}v\}$. Thus one has (5.9.6) and hence $\{u_{t,s}(\lambda)\}$ has rank 1 or 0. Now by Remark 5.8.1 choosing $w_o = x_o$ one has $p(s, x_o) = |\, s \,|!$ Thus, by (5.9.4), the left side of (5.9.7) is just $o(\lambda - \kappa\lambda)!\, d(x)^\lambda$, where $x = f + x_o$. This proves (5.9.7) and also that the rank of $\{u_{t,s}(\lambda)\}$ cannot be zero. Q.E.D.

5.10. Now given $\lambda \in D$ let $\Delta^\lambda \subseteq \Lambda$ be the set of weights of V^λ. Now for any $\nu \in \Delta^\lambda$ let

$$\mathcal{S}^\lambda(\nu) = \{s \in \mathcal{S}^\lambda \mid \varphi(s_{o(\nu-\kappa\lambda)}) = \lambda - \nu\}, \tag{5.10.1}$$

where we use the notation of (5.8.1) and (5.8.2). That is, if $s \in \mathscr{S}^\lambda$ is given by (5.7.1) then $s \in \mathscr{S}^\lambda(\nu)$ in case $\sum_{j=1}^{o(\lambda-\nu)} \alpha_{i_j} = \lambda - \nu$.

Now for any $s \in \mathscr{S}$ and integer j, where $0 \leqslant j \leqslant |s|$, let $p_j(s, w_o)$ be the polynomial of degree $|s|$ in w_o defined by putting

$$p_j(s, w_o) = p((\overline{s_{|s|-j}}), w_o)\, p(((\bar{s})_j), w_o), \tag{5.10.2}$$

using the notation of (5.8.2) and (5.8.5). Explicitly if $s = (i_1, ..., i_{|s|})$ then

$$p_j(s, w_o) = \left(\prod_{k=0}^{j} \left\langle \sum_{p=k+1}^{j} \alpha_{i_p}, w_o \right\rangle\right)\left(\prod_{m=j}^{|s|} \left\langle \sum_{q=j+1}^{m} \alpha_{i_q}, w_o \right\rangle\right). \tag{5.10.3}$$

It is clear of course that $p_j(s, w_o) > 0$.

Remark 5.10. Recall Remark 5.8.1, where it was suggested that $p(s, w_o)$ should be regarded as a sort of factorial "function." In the same sense one should perhaps think of $p_j(s, w_o)$ in terms of binomial coefficients. Indeed if $w = x_o$, recalling Section 2.1, one easily has

$$p_j(s, x_o) = j!(|s| - j)!$$

For the two extreme cases one clearly has

$$p_0(s, w_o) = p(s, w_o) \tag{5.10.4}$$

and

$$p_{|s|}(s, w_o) = p(\bar{s}, w_o). \tag{5.10.5}$$

Now for any $g_o \in H$, $w_o \in \mathfrak{h}_+$, $\lambda \in D$, and $t \in \mathbb{R}$ let

$$\Phi_\lambda(g_o, w_o; t) = (-1)^{o(\lambda-\kappa\lambda)} \sum_{\nu \in \Delta^\lambda} \left(\sum_{s \in \mathscr{S}^\lambda(\nu)} \frac{c_{s,\lambda}}{p_{o(\lambda-\nu)}(s, w_o)}\right) g_o^{\,\nu} e^{-t\langle \nu, w_o\rangle}, \tag{5.10.6}$$

where $c_{s,\lambda}$ is defined by (5.9.3), Δ^λ is the set of weights of V^λ and $o(\mu)$ is defined by (5.5.28).

THEOREM 5.10. *Let $\mathfrak{g}$ be a real split semi-simple Lie algebra. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a split Cartan subalgebra and let $\mathfrak{h}_+ \subseteq \mathfrak{h}$ be an open Weyl chamber. Let G be the adjoint group and let $H \subseteq G$ be the subgroup corresponding to h. Let $g_o \in H$, $w_o \in \mathfrak{h}_+$ and let λ be the highest weight of an irreducible representation of $\mathfrak{g}$ and for any $t \in \mathbb{R}$ let $\Phi_\lambda(g_o, w_o; t)$ be defined by* (5.10.6) *where $c_{s,\lambda}$ is defined by* (5.9.3) *and $p_j(s, w_o)$ is defined by* (5.10.3), *recalling that $\alpha_1, ..., \alpha_l$ are the simple positive roots. Now let $w = \sum_{i=1}^{l} e_{-\alpha_i} + w_o$ for any fixed choice of negative simple root vectors $e_{-\alpha_i}$. Let $G_0^{\,w}$ be the connected component of the centralizer of w in G*

defined by (3.5.5)–(3.5.7) *so that* $g \exp tw \in G_0{}^w$, *where* $g = \rho_w^{-1}(g_o)$ *and* ρ_w *is defined by* (5.3.1). *Now let* $h(g \exp tw) \in H$ *be defined by* (2.6.2) *using* (2.4.5), (2.6.1), *and* (3.5.9). *Then one has*

$$h(g \exp(-t)w)^\lambda = \Phi_\lambda(g_o, w_o; t). \tag{5.10.7}$$

Proof. For the statement about the arbitrariness of the $e_{-\alpha_i}$ see Remark 1.5.1. Now for any $\nu \in \Delta^\lambda$ let P_ν: $V^\lambda \to V^\lambda(\mu)$ be the orthogonal projection relative to Q_*^λ. Now if $b_i(\lambda, w_o)$ is defined by (5.5.12) then clearly

$$\{P_\nu \bar{n}_f(w)^{-1} v^\lambda, P_\nu \bar{n}_{-f}(w)^* v^{\kappa\lambda}\} = \sum_{\lambda_i = \nu} b_i(\lambda, w_o) \tag{5.10.8}$$

and hence by (5.5.21) if $b_\nu(\lambda, w_o)$ is the left side of (5.10.8) then

$$h(g \exp tw)^\lambda = \sum_{\nu \in \Delta^\lambda} b_\nu(\lambda, w_o)\, g_o{}^\nu e^{t\langle \nu, w_o\rangle}. \tag{5.10.9}$$

Now for any $\mu \in \mathfrak{h}'$ let $\mathscr{S}^{(\mu)} = \{s \in \mathscr{S} \mid \varphi(s) = \mu\}$. Then by (5.8.7) and (5.8.17) for any $\nu \in \Delta^\lambda$ one has upon interchanging s' and $\bar{s}'$

$$P_\nu \bar{n}_f(w)^{-1} v^\lambda = \sum_{s' \in \mathscr{S}^{(\lambda-\nu)}} (-1)^{|s'|} \frac{e_{-s'} v^\lambda}{p(\bar{s}', w_o)} \tag{5.10.10}$$

and

$$P_\nu \bar{n}_{-f}(w)^* v^{\kappa\lambda} = \sum_{s'' \in \mathscr{S}^{(\nu-\kappa\lambda)}} (-1)^{|s''|} \frac{e_{s''} v^{\kappa\lambda}}{p(s'', w_o)}. \tag{5.10.11}$$

Now composition of sequences clearly induces a bijection

$$\mathscr{S}^{(\lambda-\nu)} \times \mathscr{S}^{(\nu-\mu\lambda)} \to \mathscr{S}^\lambda(\nu), \qquad (s', s'') \mapsto s, \tag{5.10.12}$$

where we note that $e_{-s} = e_{-s''} e_{-s'}$. One then has $\{e_{-s'} v^\lambda, e_{s''} v^{\kappa\lambda}\} = \{e_{-s} v^\lambda, v^{\kappa\lambda}\} = c_{s,\lambda}$. Furthermore $p(\bar{s}', w_o) = p(\overline{(s_{o(\nu-\kappa\lambda)})}, w_o)$ and $p(s'', w_o) = p(\overline{((\bar{s})_{o(\lambda-\nu)})}, w_o)$ so that

$$p(\bar{s}', w_o)\, p(s'', w_o) = p_{o(\lambda-\nu)}(s, w_o). \tag{5.10.13}$$

But also $|s'| + |s''| = o(\lambda - \kappa\lambda)$ so that

$$b_\nu(\lambda, w) = (-1)^{o(\lambda-\nu\lambda)} \sum_{s \in \mathscr{S}^\lambda(\nu)} \frac{c_{s,\lambda}}{p_{o(\lambda-\nu)}(s, w_o)}. \tag{5.10.14}$$

But then (5.10.7) follows from (5.10.9) and (5.10.14). Q.E.D.

5.11. Formula (5.10.7) expresses $h(g \exp(-t)w)^\lambda$ as a finite sum of exponentials with explicit but clearly complicated coefficients. On the other hand the asymptotic values of $h(g\exp(-t)w)^\lambda$, or rather $\log h(g\exp(-t)w)^\lambda$, as $t \to \pm\infty$ may be given by simple linear formulas. We may also replace $\lambda \in D$ by any $\mu \in \mathscr{h}'$.

THEOREM 5.11. *Let the notation be as in Theorem* 5.10 *and let κ be the Weyl group element which takes positive roots to negative roots. See Section* 2.4. *Let $\mu \in \mathscr{h}'$ and let $d(w) \in H$ be defined by* (5.5.2). (*See also* (5.5.4) *and* (5.5.5).) *Then if $R_\pm(g_o, w_o; t)$ are the real-valued remainder functions defined by*

$$\begin{aligned}\log h(g\exp(-t)w)^\mu &= \langle -\kappa\mu, w_o\rangle t + \log(g_o d(w)^{-1})^{\kappa\mu} + R_+(g_o, w_o; t)\\ &= \langle -\mu, w_o\rangle t + \log(g_o d(w))^\mu + R_-(g_o, w_o; t)\end{aligned}$$

one has $\lim_{t\to+\infty} R_+(g_o, w_o; t) = \lim_{t\to-\infty} R_-(g_o, w_o; t) = 0$.

Proof. By definition, one has with exponentiation

$$\frac{h(g\exp(-t)w)^\mu}{e^{t\langle -\kappa\mu, w_o\rangle}(g_o\, d(w)^{-1})^{\kappa\mu}} = e^{R_+(g_o, w_o; t)} \tag{5.11.1}$$

and

$$\frac{h(g\exp(-t)w)^\mu}{e^{t\langle -\mu, w_o\rangle}(g_o\, d(w))^{\mu}} = e^{R_-(g_o, w_o; t)}. \tag{5.11.2}$$

However, the denominators in (5.11.1) and (5.11.2), respectively, are $(g_o\exp(-t)\, w_o d(w)^{-1})^{\kappa\mu}$ and $(g_o\exp(-t)\, w_o d(w))^\mu$. But then the left sides of (5.11.1) and (5.11.2), respectively, approach 1 as $t \to +\infty$ and $t \to -\infty$ by (5.5.23) and (5.5.22). This proves the theorem. Q.E.D.

6. THE SYMPLECTIC STRUCTURE OF (Z, ω_Z) AND THE INTEGRATION OF ξ_I, $I \in S(\mathscr{g})^G$

6.1. We recall some aspects of the theory of symplectic manifolds. See Chapter 4 in [15] for more details. Let (X, ω_X), or more simply X, if ω_X is understood, be a symplectic manifold of dimension $2n$. That is, X is a smooth (i.e., C^∞) manifold of dimension $2n$ and ω_X is a closed nonsingular smooth differential 2-form on X. Thus if $C^\infty(X)$ is the space of all smooth functions on X and Der $C^\infty(X)$ is the Lie algebra of all smooth vector fields on X then $\omega_X(\xi, \eta) \in C^\infty(X)$ for any $\xi, \eta \in$ Der $C^\infty(X)$ and $\omega_X(\xi, \eta)$ is alternating in ξ and η.

Now for any $\varphi \in C^\infty(X)$ one defines a (Hamiltonian) vector field ξ_φ on X by the relation

$$\omega_X(\xi_\varphi, \eta) = \eta\varphi \tag{6.1.1}$$

for all $\eta \in \operatorname{Der} C^\infty(X)$. Furthermore $C^\infty(X)$ inherits a Poisson structure (see Section 1.1) where for any $\varphi, \psi \in C^\infty(X)$ one defines $[\varphi, \psi] = \xi_\varphi \psi$. Moreover if $\operatorname{Ham} X = \{\xi_\varphi \mid \varphi \in C^\infty(X)\}$ then $\operatorname{Ham} X$ is a Lie subalgebra of $\operatorname{Der} C^\infty(X)$ and (assuming X is connected)

$$0 \to \mathbb{R} \xrightarrow{\text{inj}} C^\infty(X) \xrightarrow{\varphi \to \xi_\varphi} \operatorname{Ham} X \to 0 \tag{6.1.2}$$

is an exact sequence of Lie algebras defining $C^\infty(X)$ as a central extension of $\operatorname{Ham} X$. We have identified $\mathbb{R}$ here with the constant functions on X.

As one knows, one may express ω_X in terms of the Poisson structure.

PROPOSITION 6.1. *Assume that $\varphi_1, \ldots, \varphi_{2n}$ is a global coordinate system on X. Let $\Psi_{ij} \in C^\infty(X)$ be defined by $\Psi_{ij} = [\varphi_i, \varphi_j]$. Then the $2n \times 2n$ matrix $\{\Psi_{ij}\}$ is invertible at all points and if $\{\Psi^{ij}\}$ is the inverse matrix of functions one has*

$$\omega_X = \tfrac{1}{2} \sum_{i,j}^{2n} \Psi^{ij}\, d\varphi_i \wedge d\varphi_j\,. \tag{6.1.3}$$

Proof. That $\{\Psi_{ij}\} = \{\xi_{\varphi_i} \varphi_j\}$ is invertible is obvious from the tangent space–cotangent space isomorphism induced by ω_X. Now for any $\varphi, \psi \in C^\infty(X)$ one easily has

$$\omega_X(\xi_\psi, \xi_\varphi) = [\varphi, \psi] \tag{6.1.4}$$

from (6.1.1). In particular $\omega_X(\xi_{\varphi_p}, \xi_{\varphi_q}) = [\varphi_q, \varphi_p]$. On the other hand it is immediate that if ω is the right side of (6.1.3) one also has $\omega(\xi_{\varphi_p}, \xi_{\varphi_q}) = [\varphi_q, \varphi_p]$. This proves $\omega = \omega_X$. Q.E.D.

6.2. Now symplectic manifolds arise in a number of ways. One of these is from the coadjoint orbits of Lie groups. We recall some of the details. See Chapter 5 in [15] for a more complete account. Let A be a connected Lie group and let $\mathfrak{a}$ be its Lie algebra so that $\mathfrak{a}$ is an A-module with respect to the adjoint representation. Also $\mathfrak{a}'$, the dual to $\mathfrak{a}$, is an A-module with respect to the coadjoint representation. The latter is defined by contragradience so that if $a \in A$, $x \in \mathfrak{a}$, and $g \in \mathfrak{g}'$

$$\langle g, x \rangle = \langle ag, ax \rangle. \tag{6.2.1}$$

Now let $O \subseteq \mathfrak{a}'$ be an orbit of A in $\mathfrak{a}'$. That is, O is a homogeneous space for A of the form $O = Ah$ for some $h \in \mathfrak{a}'$. Now for any $y \in \mathfrak{a}$ let $\varphi^y \in C^\infty(O)$ be the function defined by $\varphi^y(g) = \langle g, y \rangle$ for any $g \in O$. Also for any $x \in \mathfrak{a}$ let $\xi_x \in \operatorname{Der} C^\infty(O)$ be the vector field on O corresponding to the action of the one-parameter group $\exp(-t)x$ on O. The vector field ξ_x is characterized by the relation

$$\xi_x \varphi^y = \varphi^{[x,y]} \tag{6.2.2}$$

for any $x, y \in \mathfrak{a}$.

PROPOSITION 6.2.1. *There exists a unique symplectic structure ω_O on O (defining a symplectic manifold (O, ω_O)) such that*

$$[\varphi^x, \varphi^y] = \varphi^{[x,y]} \tag{6.2.3}$$

for any $x, y \in \mathfrak{a}$. Furthermore if ξ_{φ^x} is the Hamiltonian vector field corresponding to φ^x then

$$\xi_{\varphi^x} = \xi_x \,. \tag{6.2.4}$$

Proof. The existence of a symplectic structure ω_O on O where (6.2.3) is satisfied is established by Theorem 5.3.1 in [15]. The uniqueness follows from Proposition 6.1 since locally the set of functions $\{\varphi^x\}$, $x \in \mathfrak{a}$, contains a coordinate system. The relation (6.2.4) follows from the equality of the left sides of (6.2.2) and (6.2.3) for all $y \in \mathfrak{a}$. Q.E.D.

Henceforth (O, ω_O), for any coadjoint orbit O, will be the symplectic manifold given by Proposition 6.2.1.

Now let the notation be as in Section 1 so the symmetric algebra $S(\mathfrak{a})$ (here $F = \mathbb{R}$) has a Poisson structure. Also, g_i, x_j are respectively a basis and a dual basis of $\mathfrak{a}'$ and $\mathfrak{a}$. Furthermore $S(\mathfrak{a})$ is regarded as the algebra of polynomial functions on $\mathfrak{a}'$. Now for any $u \in S(\mathfrak{a})$ let φ^u be the restriction $u \mid O$.

PROPOSITION 6.2.2. *For any $u \in S(\mathfrak{a})$, coadjoint orbit $O \subseteq \mathfrak{a}'$ and $g \in O$, one has*

$$(\xi_{\varphi^u})_g = \sum_i ((\partial(g_i)u)(g))(\xi_{x_i})_g \,. \tag{6.2.5}$$

Furthermore for $u, v \in S(\mathfrak{a})$ one has

$$[\varphi^u, \varphi^v] = \varphi^{[u,v]} \tag{6.2.6}$$

with respect to the Poisson structure on $S(\mathfrak{a})$ defined by Proposition 6.2.1.

Proof. It is immediate from (1.1.2) that

$$(d\varphi^u)_g = \sum_i ((\partial(g_i)u)(g))(d\varphi^{x_i})_g \,. \tag{6.2.7}$$

But now (6.2.5) follows from (6.2.4) and (6.2.7). But (6.2.5) and (6.2.7) (for v instead of u) implies

$$[\varphi^u, \varphi^v] = \sum_{i,j} (\partial(g_i)u)(\partial(g_j)v)[\varphi^{x_i}, \varphi^{x_j}]. \tag{6.2.8}$$

But $[\varphi^{x_i}, \varphi^{x_j}] = \varphi^{[x_i, x_j]}$ by (6.2.3) and $\varphi^{[x_i, x_j]}(g) = \langle g, [x_i, x_j]\rangle$. But now substituting (1.1.2) in (1.1.3) and then comparing (1.1.3) with (6.2.8) one has $\varphi^{[u,v]}(g) = [u, v](g) = [\varphi^u, \varphi^v](g)$. This proves (6.2.6). Q.E.D.

6.3. Now let the notation be as in Section 1.2 where $F = \mathbb{R}$ so that $\mathfrak{g}$ is a real semi-simple Lie algebra with a fixed Cartan decomposition, and Q is an invariant bilinear form which on any simple component is a positive multiple of the Killing form. Furthermore $\mathfrak{a}$ is an arbitrary subalgebra of $\mathfrak{g}$. Using Q, the inner product Q_*, and the Cartan decomposition we have associated to $\mathfrak{a}$ three subspaces $\mathfrak{a}^o$, $\mathfrak{a}^\perp$ and $\mathfrak{a}^*$ of $\mathfrak{g}$. We recall that $\mathfrak{a}^o$ is the Q-orthogonal subspace to $\mathfrak{a}$, $\mathfrak{a}^\perp$ is Q_*-orthocomplement to $\mathfrak{a}$ and $\mathfrak{a}^*$ is non-singularly paired to $\mathfrak{a}$ by Q. Now let $f \in \mathfrak{g}$ be arbitrary and let $(\mathfrak{a}^*)_f = f + \mathfrak{a}^*$ using the notation of Section 1.6. Even though $(\mathfrak{a}^*)_f$ is only an affine subspace and not necessarily a linear subspace we note that

$$\mathfrak{g} = (\mathfrak{a}^*)_f \oplus \mathfrak{a}^o \tag{6.3.1}$$

is still a direct sum. (That is, any element in $\mathfrak{g}$ has a unique sum decomposition relative to the summands in (6.3.1).) This is clear since $\mathfrak{a}^o = (\mathfrak{a}^*)^\perp$ by (1.2.5) and of course $f + \mathfrak{g} = \mathfrak{g}$. Let

$$P = \mathfrak{g} \to (\mathfrak{a}^*)_f \tag{6.3.2}$$

be the projection on $(\mathfrak{a}^*)_f$ according to the decomposition (6.3.1). Now it is immediate from (6.3.1) that if $g \in \mathfrak{a}'$ there exists a unique element $y_g \in (\mathfrak{a}^*)_f$ such that

$$Q(y_g, x) = \langle g, x\rangle \tag{6.3.3}$$

for all $x \in \mathfrak{a}$ and the map

$$T: \mathfrak{a}' \to (\mathfrak{a}^*)_f, \qquad g \mapsto y_g \tag{6.3.4}$$

is bijective and is in fact, clearly, a diffeomorphism.

Now let $A \subseteq G$ be the subgroup corresponding to $\mathfrak{a}$. For any $a \in A$ and $y \in (\mathfrak{a}^*)_f$ let $a \cdot y \in (\mathfrak{a}^*)_f$ be defined by

$$a \cdot y = Pay. \tag{6.3.5}$$

PROPOSITION 6.3. *Equation* (6.3.5) *defines an action of A on $(\mathfrak{a}^*)_f$. Furthermore regarding $\mathfrak{a}'$ as an A-manifold (i.e., a manifold on which A operates) with respect to the coadjoint representation the map* (6.3.4) *is an isomorphism of A-manifolds.*

Proof. Let $a \in A$, $g \in \mathfrak{a}'$, and $x \in \mathfrak{a}$. Then substituting $a^{-1}x$ for x one has from the invariance of Q that $Q(ay_g, x) = \langle ag, x\rangle = Q(y_{ag}, x)$. But now $ay_g - a \cdot y_g \in \mathfrak{a}^0$ so that $Q(a \cdot y_g, x) = Q(y_{ag}, x)$. But then $a \cdot y_g = y_{ag}$ from the injectivity of (6.3.4). This proves both statements of the proposition. Q.E.D.

Proposition 6.3 enables one to carry over the coadjoint–symplectic theory of A from $\mathfrak{a}'$ to $(\mathfrak{a}^*)_f$. Thus any orbit Y of A in $(\mathfrak{a}^*)_f$ is a symplectic where the symplectic structure ω_Y is defined by

$$\omega_Y = \Gamma(\omega_{\Gamma^{-1}Y}), \tag{6.3.6}$$

where of course the action of the diffeomorphism Γ on differential forms is defined in the usual differential-geometric way.

6.4. We apply Proposition 6.3 to the case of the example of Sections 1.5 and 2–5, where $\mathfrak{g}$ is a real split semi-simple Lie algebra. The notation is as in, say, Section 1.5, so that $\mathfrak{a} = \bar{\mathfrak{b}}$, $A = \bar{B}$, $\mathfrak{a}^* = \mathfrak{b}$ and f is given by (1.5.4). Then $(\mathfrak{a}^*)_f = \mathfrak{b}_f = f + \mathfrak{b}$. However, we are interested in only one orbit of $\bar{B}$ in $\mathfrak{b}_f$.

Let h_i, $i = 1,\ldots, l$, be the basis of $\mathfrak{h}$ defined by putting

$$h_i = [e_{\alpha_i}, e_{-\alpha_i}]. \tag{6.4.1}$$

Now recalling the definition of the $2l$-dimensional submanifold $Z \subseteq \mathfrak{b}_f$ (see (2.2.3) and (2.2.4)) one has a global coordinate system $\rho_i, \gamma_j \in C^\infty(Z)$, $i, j = 1,\ldots, l$, such that $y \in Z$ if and only if y is of the form

$$y = f + \sum_{i=1}^{l} \rho_i(y)\, h_i + \sum_{j=1}^{l} \gamma_j(y)\, e_{\alpha_j}, \tag{6.4.2}$$

where $\rho_i(y) \in \mathbb{R}$ and $\gamma_j(y) \in \mathbb{R}^*$ are arbitrary. Of course since the γ_j are positive valued they are of course invertible. We recall also that for any invariant $I \in S(\mathfrak{g})^G$ we defined a vector field ξ_I on Z by (2.2.18) and (2.2.19) so that for any $y \in Z$ one has $(\xi_I)_y = [y, (\delta_{\bar{\mathfrak{b}}} I)(y)]$ where $\delta_{\bar{\mathfrak{b}}} I$ is defined by (1.2.7). See (1.2.10).

PROPOSITION 6.4. *Z is an orbit of $\bar{B}$ in $\mathfrak{b}_f$. Furthermore the corresponding symplectic structure on Z is given by*

$$\omega_Z = \sum_{i=1}^{l} d\rho_i \wedge \gamma_i^{-1}\, d\gamma_i\,. \tag{6.4.3}$$

Moreover if $\xi_{I|Z}$ is the Hamiltonian vector field corresponding to $I \mid Z \subseteq C^\infty(Z)$ for any $I \in S(\mathfrak{g})^G$ then one has

$$\xi_{I|Z} = \xi_I . \tag{6.4.4}$$

Proof. Let $\bar{b} \in \bar{\ell}$. We may write $\bar{b} = h \exp x$, where $x \in \bar{n}$ and $h \in H$. Let $c_i \in \mathbb{R}$ be defined by $x + \sum c_i e_{-\alpha_i} \in [\bar{n}, \bar{n}]$. Now let $y = f + e$, where $e = \sum_{i=1}^{l} e_{\alpha_i}$. Then $y \in Z \subseteq \ell_f$. Furthermore if $z = f + \sum_{i=1}^{l} c_i h_i + \sum a_i e_{\alpha_i}$, where h_i is given by (6.4.1) and $a_i = h^{\alpha_i}$, then $z \in Z$. But one notes that

$$\bar{b} y - z \in \bar{n}. \tag{6.4.5}$$

On the other hand $\bar{n} = (\bar{\ell})^o$ so that $\bar{b} \cdot y = z$ using the notation of (6.3.5). However, it is clear that z is an arbitrary element of Z and hence $Z = \bar{B} \cdot y$. This proves that Z is a $\bar{B}$ orbit in ℓ_f.

Now let $\Gamma: \bar{\ell}' \to \ell_f$ be the map (6.3.4) where of course $\bar{\ell} = \alpha$. Thus if $O = \Gamma^{-1}(Z)$ then O is a coadjoint orbit of $\bar{B}$ in $\bar{\ell}'$. For any $u \in S(\bar{\ell})$ let $\varphi^u \in C^\infty(O)$ be defined as in Section 6.1 and let $\psi^u \in C^\infty(Z)$ be defined by putting $\psi^u = u \mid Z$. It is clear from (6.3.3) that $\psi^u \circ \Gamma = \varphi^u$ and hence, using (6.2.3), with regard to the Poisson structure in $S(\bar{\ell})$ and $C^\infty(Z)$ one has

$$[\psi^u, \psi^v] = \psi^{[u,v]} \tag{6.4.6}$$

for $u, v \in S(\bar{\ell})$.

Now let $z_j \in h, j = 1,..., l$, be the basis of h such that

$$Q(z_j, h_i) = \delta_{ij} . \tag{6.4.7}$$

But from (1.5.2) it follows that

$$Q(x, h_i) = \langle \alpha_i , x \rangle \tag{6.4.8}$$

for any $x \in h$. Thus one must have

$$[e_{-\alpha_i}, z_j] = \delta_{ij} e_{-\alpha_i} . \tag{6.4.9}$$

On the other hand it follows from (6.4.7) and (6.4.2) that

$$\psi^{z_j} = \rho_j . \tag{6.4.10}$$

Furthermore using (1.5.2) one notes that

$$\psi^{e_{-\alpha_i}} = \gamma_i . \tag{6.4.11}$$

Thus by (6.4.6) and (6.4.9) one has, in $C^\infty(Z)$,

$$[\gamma_i, \rho_j] = \delta_{ij}\gamma_i . \tag{6.4.12}$$

On the other hand since clearly $\psi^z = 0$ for any $z \in [\bar{n}, \bar{n}]$ and since h is commutative it follows from (6.4.10) and (6.4.11) that for all i, j

$$[\gamma_i, \gamma_j] = [\rho_i, \rho_j] = 0. \tag{6.4.13}$$

On the other hand the γ_i together with the ρ_j define a coordinate system in Z. We may then apply Proposition 6.1 to compute ω_Z in terms of $d\gamma_i$ and $d\rho_j$, using of course (6.4.12) and (6.4.13). In fact the computation is immediate and yields (6.4.3).

Now for any $x \in \bar{b}$ let ζ_x be the vector field in Z which corresponds to ξ_x by the isomorphism Γ. See (6.3.4). Thus by (6.2.2) one has

$$\zeta_x \psi^z = \psi^{[x,z]} \tag{6.4.14}$$

for $x, z \in \bar{b}$. Now let x_i be a basis of $\bar{b}$ and y_j be a basis of b such that $Q(x_i, y_j) = \delta_{ij}$. Now for any $\varphi \in C^\infty(Z)$ let ξ_φ here be the Hamiltonian vector field on Z corresponding to φ. Now for any $u \in S(g)$ and $z \in g$ let $i(z)u \in S(g)$ be defined as in Section 1.2. For any $y \in Z$ we now assert that

$$(\xi_{u|Z})_y = \sum_j (i(y_j)u)(y)(\zeta_{x_j})_y . \tag{6.4.15}$$

Indeed for any $u \in S(g)$ it is clear from (6.3.1) that there exists a unique element $\bar{u} \in S(\bar{b})$ such that $\bar{u} \mid b_f = u \mid b_f$. Furthermore since b_f is stable under translation by elements in b one has upon differentiation, $\overline{i(z)u} = i(z)\bar{u}$ for $z \in b$. But then both sides of (6.4.15) do not change if $\bar{u}$ is substituted for u. However, for $\bar{u}$ one has (6.4.15) by (6.2.5). One recalls (6.3.3) and (6.3.6). This proves (6.4.15).

But now we may identify the tangent space to Z at y with $h + d_1 \subseteq b$ as in (2.2.10). But by (6.2.2) $(\zeta_x\psi^z)(y) = Q(y, [x, z]) = Q([y, x], z)$. Thus if $P_b: g \to b$ is the Q_*-orthogonal projection (i.e., Ker $P_b = \bar{n} = b^o$) then clearly

$$(\zeta_x)_y = P_b[y, x] \in h + d_1 . \tag{6.4.16}$$

Now recalling the definition of $\delta_b^- u$ in Section 1.2 one then has

$$(\xi_{u|Z})_y = P_b[y, (\delta_b^- u)(y)] \tag{6.4.17}$$

by (1.2.10). However, if $u = I \in S(g)^G$ then $(\xi_I)_y = [y, (\delta_{\bar{b}} I)(y)] \in b$ by (2.2.22). Thus

$$\xi_I = \xi_{I|Z} . \qquad \text{Q.E.D.}$$

6.5. Now using the isomorphism (3.5.27) which maps $\mathfrak{h}$ to $\mathfrak{h}'$ the bilinear form $Q \mid \mathfrak{h}$, as one knows, induces a positive definite bilinear form $Q \mid \mathfrak{h}'$ on $\mathfrak{h}'$. For $i, j = 1,..., l$ let

$$b_{ij} = Q(\alpha_i, \alpha_j). \tag{6.5.1}$$

Remark 6.5. Clearly b_{ij} depends upon the choice of Q. However, if C is the $l \times l$ matrix defined by $C_{ij} = b_{ij}/b_{ii}$ then C is of course the Cartan matrix and is independent of Q.

Now among the fundamental invariants $I_j \in S(\mathfrak{g})^G$, $j = 1,..., l$, we may clearly fix I_1 so that

$$I_1(x) = \tfrac{1}{2}Q(x, x) \tag{6.5.2}$$

for any $x \in \mathfrak{g}$.

PROPOSITION 6.5. *Let $\rho_i, \gamma_j \in C^\infty(Z)$ be defined by* (6.4.2). *Since $\gamma_j(Z) \subseteq \mathbb{R}^*$ we may define $\varphi_j \in C^\infty(Z)$ by putting $\varphi_j = \log \gamma_j$. Then ρ_i, φ_j, $i, j = 1,..., l$, define a coordinate system on Z. In fact the map*

$$Z \to \mathbb{R}^{2l} \tag{6.5.3}$$

given by $y \mapsto (\rho_1(y),..., \rho_l(y), \varphi_1(y),..., \varphi_l(y))$ is a diffeomorphism. Moreover

$$\omega_Z = \sum_{i=1}^{l} d\rho_i \wedge d\varphi_i . \tag{6.5.4}$$

Furthermore if b_{ij} is defined by (6.5.1) *then*

$$I_1 \mid Z = \tfrac{1}{2} \sum_{i,j=1}^{l} b_{ij}\rho_i\rho_j + \sum_{i=1}^{l} e^{\varphi_i}. \tag{6.5.5}$$

Proof. The first statement is immediate from the definition of Z. See (2.2.3) and (2.2.4). But now $d\varphi_i = \gamma_j^{-1} d\gamma_i$ and hence (6.5.4) follows from (6.4.3). On the other hand if $y \in Z$ is given by (6.4.2) then $I_1(y) = \tfrac{1}{2}Q(y, y) = Q(f, \sum_i \gamma_i(y)\, e_{\alpha_i}) + \tfrac{1}{2}\sum_{i,j} \rho_i(y)\, \rho_j(y)\, Q(h_i, h_j)$. But $Q(f, e_{\alpha_i}) = 1$ and hence $Q(f, \sum_i \gamma_i(y)\, e_{\alpha_i}) = \sum_i \gamma_i(y) = \sum_i e^{\varphi_i(y)}$. On the other hand $h_i \to \alpha_i$ by (3.5.27), recalling (3.5.28), (3.1.14), and (6.4.1) so that $b_{ij} = Q(h_i, h_j)$. One thus obtains (6.5.5). Q.E.D.

6.6. Now let (X, ω_X) be a symplectic manifold. Let $2n = \dim X$. Let $T_p(X)$, for any $p \in X$, be the tangent space to X at p. Assume that for any $p \in X$ one has an n-dimensional subspace $F_p \subseteq T_p(Z)$ and that the map given by $p \mapsto F_p$ is a smooth involutory distribution F (in the sense of Cartan) on X.

By a leaf of F we mean a maximal connected integral submanifold $M \subseteq X$. Thus F is called a real polarization of (X, ω_X) in case $\omega_X \mid M = 0$ for any leaf M of F. That is, the leaves of F are Lagrangian submanifolds of X.

Now assume that F is a real polarization of (X, ω_X). For any open subset $U \subseteq X$ let $C_F^\infty(U) = \{\varphi \in C^\infty(U) \mid v\varphi = 0 \text{ for all } p \in U \text{ and } v \in F_p\}$. It follows easily from (6.1.1) and the maximal isotropic property of F_p that, using the notation of Section 6.1, one has

$$(\xi_\varphi)_p \in F_p \text{ for any } p \in U \text{ and } \varphi \in C_F^\infty(U). \tag{6.6.1}$$

Thus if M is any leaf of F and $U \subseteq X$ is any open subset then the restriction

$$\xi_\varphi \mid M \cap U \text{ is tangent to } M \cap U \text{ for any } \varphi \in C_F^\infty(U). \tag{6.6.2}$$

But then (6.6.1) implies $[\varphi, \psi] = 0$, and hence

$$[\xi_\varphi, \xi_\psi] = 0 \text{ for any } \varphi, \psi \in C_F^\infty(U). \tag{6.6.3}$$

But now if M is a leaf of F then it is an easy consequence of (6.6.2) and (6.6.3) that there exists a unique flat affine connection on M such that $\xi_\varphi \mid M \cap U$ is covariant constant for any open set $U \subseteq X$ and $\varphi \in C_F^\infty(U)$. See Section 4.2. We will refer to this as the affine connection on M induced by F. The real polarization F will be said to be complete in case the affine connection induced by F on M is complete for every leaf of F. That is, all parametrized geodesics on M are defined for all values of the parameter.

Remark 6.6.1. In case (X, ω_X) is a coadjoint orbit of a simply connected exponential solvable Lie group A and F is an A-invariant real polarization the condition of completeness has been called the Pukansky condition. A theorem of Pukansky (see [21]) asserts that the unitary representation of A associated to F is irreducible if and only if F is complete.

An important property of completeness is that the vector field ξ_φ for any $\varphi \in C_F^\infty(X)$, can be globally integrated.

PROPOSITION 6.6. *Assume that F is a complete, real polarization of a symplectic manifold (X, ω_X). Then for any $\varphi \in C_F^\infty(X)$ there exists a one-parameter group,* $\exp t\xi_\varphi$, $t \in \mathbb{R}$, *of symplectic diffeomorphisms of X such that for any $\psi \in C^\infty(X)$ and $p \in X$*

$$\frac{d}{dt} \psi(\exp(-t)\, \xi_\varphi \cdot p) \big|_{t=0} = (\xi_\varphi \psi)(p), \tag{6.6.4}$$

where the dot denotes the diffeomorphism action on X.

Proof. Let $\varphi \in C_F{}^\infty(X)$ and let $p \in X$. Let M be the leaf of F containing p. Let $c(t) \in M$, for all $t \in \mathbb{R}$, be the geodesic on M whose tangent vector at $t = 0$ is $(-\xi_\varphi)_p$. Then $(\exp t\xi_\varphi) \cdot p$ is defined so that $c(t) = (\exp t\xi_\varphi) \cdot p$. It is immediate then that $\exp t\xi_\varphi$ is a one-parameter group of symplectic diffeomorphisms satisfying (6.6.4). Q.E.D.

Remark 6.6.2. In general if ξ is a globally integrable vector field on a manifold M then the corresponding one parameter group of diffeomorphisms will be written as $\exp t\xi$. The direction of the trajectories $\exp t\xi \cdot p$ of ξ will also be taken in the reverse direction of ξ. That is, if $\psi \in C^\infty(M)$ one has $(d/dt)\, \psi(\exp(-t)\xi \cdot p)\,|_{t=0} = (\xi\psi)(p)$ for any $p \in M$. This of course guarantees the correct functorial properties for a Lie group of diffeomorphisms. Furthermore it produces the correct signs for Hamilton's equations. See (7.1.5).

6.7. Now recalling Section 4.1, $\mathscr{X}$ is an involutory distribution of $\dim l$ on Z. To conform to the notation of Section 6.6 we write $F = \mathscr{X}$ and recall then that for any $y \in Z$ (see (2.2.21))

$$F_y = \{(\xi_I)_y \mid I \in S(\mathfrak{g})^G\}, \tag{6.7.1}$$

where we recall that (see (2.2.22))

$$(\xi_I)_y = [y, (\delta_{\bar{\mathfrak{g}}} I)(y)], \tag{6.7.2}$$

where $\delta_{\mathfrak{g}} I$ is defined by (1.2.7).

Furthermore by Proposition 4.1 and (3.5.26) the leaves of F are all the submanifolds of the form $Z(\gamma)$, $\gamma \in \mathscr{I}(\mathfrak{h}_+)$, and any such leaf has been given (see end of Section 4.2) the structure of a flat affinely connected manifold.

THEOREM 6.7.1. *Let $\mathfrak{g}$ be a split semi-simple Lie algebra. Let $l = \operatorname{rank} \mathfrak{g}$ and let $Z \subseteq \mathfrak{g}$, defined as in Section* 2.2, *be the $2l$-dimensional submanifold of normalized Jacobi elements. Let ω_Z be the symplectic structure on Z defined as in Section* 6.3 *and given explicitly by* (6.4.3) *and let F be the l-dimensional involutory distribution on Z defined by* (6.7.1). *Let $\mathscr{I}\colon \mathfrak{g} \to \mathbb{R}^l$ be the map defined as in* (2.3.1) *and let $\mathfrak{h}_+$ be the open Weyl chamber defined by* (3.3.1). *For each $\gamma \in \mathscr{I}(\mathfrak{h}_+)$ let $Z(\gamma)$ be defined by* (2.3.2) *so that by Proposition* 4.2 *and* (3.5.26) *the $Z(\gamma)$ are flat affinely connected manifolds and are the leaves of F.*

Then F is a complete real polarization of (Z, ω_Z). Furthermore the affine connec- on $Z(\gamma)$ is the same as the one induced by F. Moreover for any invariant $I \in S(\mathfrak{g})^G$ one has $I \mid Z \in C_F{}^\infty(Z)$ and ξ_I is the corresponding Hamiltonian vector field. In particular ξ_I is globally integrable. In fact let $y \in Z$ and let $I \in S(\mathfrak{g})^G$. Let $\gamma = \mathscr{I}(y)$ so that by (3.5.26) *and Proposition* 3.3.1 *there exists a unique $w_o \in \mathfrak{h}_+$ such that $\mathscr{I}(w_o) = \gamma$. Let $w = f + w_o$, where f is given by* (1.5.4), *and let $z = (\delta I)(w)$*

so that $z \in \mathfrak{g}^w$ *using the notation of Section* 1.2 *and recalling* (1.3.3). *Let* $G_0{}^w$ *be as* (3.5.5)–(3.5.7) *and let* $\beta_{(w)}: G_0{}^w \to Z(\gamma)$ *be the isomorphism* (3.6.2). *Let* $g = \beta_{(w)}^{-1}(y) \in G_0{}^w$. *Then for all* $t \in \mathbb{R}$ *one has* $g \exp(-t)z \in G_0{}^w$ *and*

$$\beta_{(w)}(g \exp(-t)z) = \exp t\xi_I \cdot y, \tag{6.7.3}$$

where the dot is the same as in Proposition 6.6.

Proof. For any $I \in S(\mathfrak{g})^G$ we recall that $I^f \in S(\mathscr{b})$ has been defined by the relation $I^f(x) = I(f + x)$ for $x \in \mathscr{b}$. See Section 1.4. For any $u \in S(\mathscr{b})$ let, as in the proof of Proposition 6.4, $\psi^u \in C^\infty(Z)$ be defined by $\psi^u = u \mid Z$. We assert that

$$\psi^{I^f} = I \mid Z. \tag{6.7.4}$$

Indeed if $y \in Z$ we may write $y = f + x$, where $x \in \mathscr{b}$. Then $I(y) = I(f + x) = I^f(x) = I^f(f + x) = I^f(y)$ using (1.2.6) since $f \in (\mathscr{b})^o = \bar{n}$. This proves (6.7.4). But now if $I, J \in S(\mathfrak{g})^G$ and we put $u = I^f$, $v = J^f$ then $\xi_{\psi^u} = \xi_I$ and $\xi_{\psi^v} = \xi_J$ by (6.4.4). But then by (6.1.4) and (6.4.6)

$$\omega_Z(\xi_I, \xi_J) = [\psi^v, \psi^u] = \psi^{[v,u]}. \tag{6.7.5}$$

But $[v, u] = [J^f, I^f] = 0$ by Theorem 1.4. Thus (6.7.5) vanishes and hence $Z(\gamma)$, recalling Proposition 4.1, is Lagrangian for all $\gamma \in \mathscr{I}(\mathscr{h}_+)$. Thus F is a real polarization. Furthermore by definition $I \mid Z$ is constant on $Z(\gamma)$ so that $I \mid Z \in C_F{}^\infty(Z)$ and hence ξ_I is covariant constant with respect to the affine connection induced by F. Recalling the definition (see end of Section 4.2) of the given affine connection on $Z(\gamma)$ this proves that both affine connections are the same.

Now let $y \in Z$. Put $\gamma = \mathscr{I}(y)$. By (3.5.26) and Proposition 3.3.1 there exists a unique $w_o \in \mathscr{h}_+$ such that $\mathscr{I}(w_o) = \gamma$. Then $Z(\gamma)$ is the leaf of F containing y and by Theorem 4.3

$$\beta_{(w)}: G_0{}^w \to Z(\gamma) \tag{6.7.6}$$

is an isomorphism of flat affinely connected manifolds where the affine connection on $G_0{}^w \subseteq \tilde{G}^w$ is defined by the Abelian group structure on $\tilde{G}^w$. In particular the geodesics in $G_0{}^w$ are translates of one-parameter groups $\exp tx$, where $x \in \mathfrak{g}^w$. Since $G_0{}^w$ is a connected component of $\tilde{G}^w$ it is obviously complete. Thus $Z(\gamma)$ is complete and hence F is complete.

Now let $I \in S(\mathfrak{g})^G$ and let $g \in G_0{}^w$ be such that $\beta_{(w)}(g) = y$. Let $L_I{}^w$ be the vector field on $\tilde{G}^w$, defined by (4.3.1), so that if $\exp tL_I{}^w$ is the one-parameter group of diffeomorphisms of $\tilde{G}^w$ defined by $L_I{}^w$, one has $\exp tL_I{}^w \cdot a =$

$a \exp(-t)(\delta I)(w)$ for any $a \in \tilde{G}^w$. But $\beta_{(w)}(L_I^w \mid G_0^w) = \xi_I \mid Z(\gamma)$ by (4.3.3). This proves (6.7.3) since $z = (\delta I)(w)$. Q.E.D.

One also would know the flow defined by ξ_I if one knew the coordinate values of $\exp t\xi_I \cdot y$. For "half" the coordinates this is given explicitly by

THEOREM 6.7.2. *Let $y \in Z$ and let $w_0 \in \mathfrak{h}_+$, as in Theorem 6.7.1, be such that $\mathscr{I}(w_0) = \mathscr{I}(y)$. Also as in Theorem 6.7.1 let $w = f + w_0$ and let $g \in G_0^w$ be such that $\beta_{(w)}(g) = y$. Now let H be the split Cartan subgroup defined as in Section 1.5 and also as in Section 1.5 let $\alpha_1, \ldots, \alpha_l$ be a corresponding set of simple positive roots. For any $a \in G_0^w$ let $h(a) \in H$ be defined by (3.4.14). Let $I \in S(\mathfrak{g})^G$ and let $(\delta I)(w) = z \in \mathfrak{g}^w$. Finally let $\varphi_1, \ldots, \varphi_l \in C^\infty(Z)$ be "half" the coordinates on Z defined as in Proposition 6.5. Then (see Theorem 6.7.1)*

$$\varphi_i(\exp t\xi_I \cdot y) = \log h(g \exp(-t)z)^{-\alpha_i}. \tag{6.7.7}$$

Proof. Now, recalling Theorem 3.6, if $a \in G_0^w$ then we may write $a = s(\kappa)\, \bar{n}(a)\, h(a)\, n(a)$, where the four factors are defined as in the statement of Theorem 3.6. Furthermore $\beta_{(w)}(a) = n(a)w \in Z$ by Theorem 3.6 and hence we can write $n(a)w = f + x + \sum_{i=1}^{l} c_i e_{\alpha_i}$, where $x \in \mathfrak{h}$. But by (3.7.4) one has in fact

$$c_i = h(a)^{-\alpha_i}. \tag{6.7.8}$$

But now if $\gamma_i \in C^\infty(Z)$ is defined by (6.4.2) then (6.7.8) implies $\gamma_i(\beta_{(w)}(a)) = h(a)^{-\alpha_i}$. Thus one has

$$\varphi_i(\beta_{(w)}(a)) = \log h(a)^{-\alpha_i}. \tag{6.7.9}$$

But then (6.7.7) follows from (6.7.3) by putting $a = g \exp(-t)z$. Q.E.D.

6.8. We are mainly interested in applying Theorems 6.7.1 and 6.7.2 for the case where I is the quadratic invariant I_1. See (6.5.2). In that case the element z becomes w itself.

Our solution to the generalized Toda lattice will depend on the following theorem together with the explicit formula for $h(g \exp(-t)w)^\lambda$ given in Theorem 5.10. See (5.10.6) and (5.10.7).

THEOREM 6.8.1. *Let $\{\rho_i, \varphi_j\}$, $i, j = 1, \ldots, l$, be the coordinate system on Z given in Proposition 6.5. Let ω_Z be the symplectic structure on Z defined in Section 6.3 and given explicitly (see (6.5.4)) by*

$$\omega_Z = \sum_{i=1}^{l} d\rho_i \wedge d\varphi_i. \tag{6.8.1}$$

Let $I_1 \in S(\mathfrak{g})^G$ be defined by (6.5.2) *so that $I_1 \mid Z$ has been given explicitly* (*see* (6.5.5)) *by*

$$I_1 \mid Z = \tfrac{1}{2} \sum_{i,j=1}^{l} b_{ij}\rho_i\rho_j + \sum_{i=1}^{l} e^{\varphi_i}, \tag{6.8.2}$$

where $b_{ij} = Q(\alpha_i, \alpha_j)$. Let ξ be the Hamiltonian vector field on Z corresponding to $I_1 \mid Z$. Then ξ is globally integrable. Let $\exp t\xi$ *be the corresponding flow.* (*See Theorem* 6.7.1.) *Let $y \in Z$ and let $w_o \in \mathfrak{h}_+$ be such that $\mathscr{I}(y) = \mathscr{I}(w_o)$. Let $w = f + w_o$ and let $g \in G_0{}^w$ be such that $\beta_{(w)}(g) = y$.* (*We are using the notation of Theorem* 6.7.1.) *Then*

$$\exp t\xi \cdot y = \beta_{(w)}(g \exp(-t)w). \tag{6.8.3}$$

Furthermore

$$\varphi_i(\exp t\xi \cdot y) = \log h(g \exp(-t)w)^{-\alpha_i}, \tag{6.8.4}$$

$i = 1,..., l$.

Proof. Theorem 6.8.1 follows immediately from Theorems 6.7.1 and 6.7.2 as soon as one shows that $(\delta I_1)(w) = w$. Let $\{x_i\}$ and $\{y_i\}$ both be bases of $\mathfrak{g}$ such that $Q(x_i, y_j) = \delta_{ij}$. Then clearly

$$I_1 = \tfrac{1}{2} \sum_{i=1}^{\dim \mathfrak{g}} x_i y_i . \tag{6.8.5}$$

But then for any $z \in \mathfrak{g}$ one has

$$(\delta I_1)(z) = \tfrac{1}{2} \Big(\sum_i Q(x_i, z)\, y_i + \sum_i Q(y_i, z)\, x_i\Big) \tag{6.8.6}$$

by (1.2.1) and (1.2.2). But clearly both sums in (6.8.6) are equal to z. Thus $(\delta I_1)(z) = z$. Q.E.D.

Using the diagram (3.6.7) one may give a conceptually simpler expression (in that it only involves y itself) for the trajectory $\exp t\xi \cdot y$. However, it does not seem to lead to an explicit formula.

THEOREM 6.8.2. *Let the notation be as in Theorem* 6.8.1. *Then there exist for all $t \in \mathbb{R}$ unique elements $\bar{n}_t \in \bar{N}$, $h_t \in H$, and $n_t \in N$, where $\bar{N}$, H, and N are as in Section* 3.2, *such that*

$$\exp ty = \bar{n}_t h_t n_t . \tag{6.8.7}$$

Moreover with respect to the adjoint action one has $n_{-t}y \in Z$ and in fact

$$\exp t\xi \cdot y = n_{-t}y. \tag{6.8.8}$$

Proof. Recalling the notation and statement of Theorem 3.6 one has $\beta_{(w)}(g \exp tw) = \beta_y(\psi_g(g \exp tw))$. See (3.6.7). But $\psi_g(g \exp tw) = n(g) \exp twn(g)^{-1}$. However, $n(g)w = \beta_{(w)}(g) = y$. Thus $\psi_g(g \exp tw) = \exp ty$ and hence

$$\beta_{(w)}(g \exp tw) = \beta_y(\exp ty). \tag{6.8.9}$$

But now if we write $y(t) = \exp ty$ then $y(t) \in G_e^y$ and hence by Theorem 3.2 one has (6.8.7), where $\bar{n}_t = \bar{n}_{y(t)}$, $h_t = h_{y(t)}$, and $n_t = n_{y(t)}$. But for any $d \in G_e^y$ one has $\beta_y(d) = n_d y$. Thus $\beta_y(\exp ty) = n_t y$. But then (6.8.8) follows from (6.8.9) and (6.8.3). Q.E.D.

Remark 6.8. The solution to the generalized Toda lattice will be based on (6.8.4) and formula (5.10.7) for $h(g \exp(-t)w)^\lambda$. Given the "initial condition" y what is needed is only the "spectrum" of y, namely, w_o. The "input" is w_o together with the constants (the exponential action angle coordinates of the initial condition y) g_o^ν for $\nu \in \Lambda$. This, however, is determined from (5.5.25). That is, one easily determines $n(g)$ inductively by the relation $n(g)w = y$. See, e.g., (7.8.25). But then (5.5.25) yields $g_o^{\nu_i}$ and hence g_o^ν for any $\nu \in \Lambda$. One notes (see (3.7.4)) that y itself has $h(g)^{-\alpha_i}$ are coordinates so that $h(g)^{-\lambda}$ is known in (5.5.25). Also we remark that $n(g)$ is naturally determined by y. That is, one has

$$n(g)\, \bar{n}_f(w)\, w_o = y \tag{6.8.10}$$

and one sees easily from the injectivity of (2.4.5) that $n(g)$ and $\bar{n}_f(w)$ are the unique elements $n \in N$ and $\bar{n} \in \bar{N}$ such that $n\bar{n}w_o = y$. Finally it is then to be noted that $\bar{n}$ depends only on the isospectral leaf containing y whereas n depends on the "action angle" coordinates of y.

7. Dénouement; the Formula for $q_i(x(t))$

7.1. Let $(\mathbb{R}^{2n}, \omega)$ be the classical $2n$-dimensional phase space. That is, one has linear (canonical) coordinates $p_i, q_j \in C^\infty(\mathbb{R}^{2n})$, $i, j = 1,..., n$, and the symplectic structure is given by

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i. \tag{7.1.1}$$

One easily sees that if $H \in C^\infty(\mathbb{R}^{2n})$ then the corresponding Hamiltonian vector field ξ_H (see (6.1.1)) is given explicitly by

$$\xi_H = \sum_{i=1}^{l} \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i}. \tag{7.1.2}$$

Now recall certain aspects of the Hamilton–Jacobi theory. We envision a mechanical system consisting of n particles moving on a line. The space $\mathbb{R}^{2n}$ is the set of all classical states, p_i and q_i are respectively the momentum and position of the ith particle and $H = H(p_1, \ldots, p_n, q_1, \ldots, q_n) \in C^\infty(\mathbb{R}^{2n})$ is the total energy of the system. Assume ξ_H is globally integrable so that one has the action on $\mathbb{R}^{2n}$ of the one-parameter group $\exp t\xi_H$ of symplectic diffeomorphisms where if $\psi \in C^\infty(\mathbb{R}^{2n})$ and $z \in \mathbb{R}^{2n}$ then

$$\begin{aligned}[\psi, H](z) &= -(\xi_H \psi)(z) \\ &= \frac{d}{dt} \psi(\exp t\xi_H \cdot z)\,\big|_{t=0}.\end{aligned} \tag{7.1.3}$$

That is, the map $t \to z(t) = \exp t\xi_H \cdot z$ is a trajectory of $-\xi_H$. In physical terms $z(t)$ is the state the mechanical system would occupy at time t if it occupied the state z at time $t = 0$. Now it is immediate from (6.1.1) and (7.1.1) that

$$\xi_{q_i} = \frac{\partial}{\partial p_i} \quad \text{and} \quad \xi_{p_i} = -\frac{\partial}{\partial q_i}. \tag{7.1.4}$$

Now also regard p_i and q_i, respectively, as the functions $t \to p_i(z(t))$, $t \to q_i(z(t)$, of t. It then follows from (7.1.3), recognizing that $[\psi, H]$ is also $\xi_\psi H$, that

$$\frac{\partial H}{\partial p_i}(z(t)) = \frac{\partial q_i}{dt}, \qquad \frac{\partial H}{\partial q_i}(z(t)) = -\frac{\partial p_i}{dt}. \tag{7.1.5}$$

These relations are of course just Hamilton's equations.

Now let m_i be the mass of the ith particle. We assume that H takes the usual form as the sum of the kinetic and potential energies. That is, where $V \in C^\infty(\mathbb{R}^{2n})$ depends only on the q_i (i.e., $\partial V/\partial p_i = 0$), we assume that

$$H = \sum_{i=1}^{n} \frac{p_i^2}{2m_i} + V. \tag{7.1.6}$$

It follows then from (7.1.6) that

$$p_i(z(t)) = m_i \frac{dq_i}{dt} \tag{7.1.7}$$

so that the trajectory $z(t)$ is very simply determined as soon as one knows the velocities dq_i/dt.

Now let $\mathscr{R} \subseteq C^\infty(\mathbb{R}^{2n})$ be the $(2n+1)$-dimensional space of functions spanned by the p_i, q_j, and the constant function 1. One defines an alternating bilinear form B_ω on $\mathscr{R}$ by the relation

$$[r, s] = B_\omega(r, s)1 \tag{7.1.8}$$

for $r, s \in \mathscr{R}$. Let $\mathscr{P}, \mathscr{Q} \subseteq \mathscr{R}$ be respectively the n-dimensional subspaces spanned by the p_i and q_j. It is clear that $\mathscr{P}$ and $\mathscr{Q}$ are totally singular subspaces which, however, are non-singularly paired with respect to B_ω. In fact one readily has

$$B_\omega(q_i, p_j) = \delta_{ij}. \tag{7.1.9}$$

We now assume that the potential function V takes a certain special form. Let $l \leqslant n$ and assume that $\psi_i \in \mathscr{Q}$, $i = 1,..., l$, are linearly independent. That is, there exists an $l \times n$ matrix $A = (a_{ij})$ of rank l such that

$$\psi_i = \sum_{j=1}^{n} a_{ij} g_j, \qquad i = 1,..., l. \tag{7.1.10}$$

Let $r_i \in \mathbb{R}^*$, $i = 1,..., l$, be some positive constants. We assume that $V = \sum_i r_i e^{\psi_i}$ so that the Hamiltonian H has the form

$$H = \sum_{i=1}^{n} \frac{p_i^2}{2m_i} + \sum_{i=1}^{l} r_i e^{\psi_i}. \tag{7.1.11}$$

Let $z \in \mathbb{R}^{2n}$ be arbitrary. We will consider the question of determining the trajectory $z(t) = \exp t\xi_H \cdot z$ when the ψ_i satisfy a certain property—to be stated below.

Now $[q, H]$ is contained in $\mathscr{P}$ for any $q \in \mathscr{Q}$. In fact $[q_i, H] = p_i/m_i$ by (7.1.9) and hence

$$\beta_H: \mathscr{Q} \to \mathscr{P}$$

is a linear isomorphism where $\beta_H(q) = [q, H]$. One then defines a bilinear form B_H on $\mathscr{Q}$ by putting

$$B_H(q, q') = B_\omega(q, \beta_H q'). \tag{7.1.12}$$

That is,

$$B_H(q, q')1 = [q, [q', H]]. \tag{7.1.13}$$

It is clear then that B_H is symmetric and positive definite. In fact one easily has

$$B_H(q_i, q_j) = \delta_{ij}/m_i. \tag{7.1.14}$$

Now let $\mathscr{Q}_1 \subseteq \mathscr{Q}$ be the l-dimensional subspace spanned by the ψ_i and put $\mathscr{P}_1 = \beta_H \mathscr{Q}_1$. Since β_H is an isomorphism one has $\dim \mathscr{P}_1 = l$. Furthermore since B_H is positive definite it follows from (7.1.12) that $\mathscr{Q}_1$ and $\mathscr{P}_1$ are non-singularly paired by B_ω. It follows that there exist uniquely $p_i' \in \mathscr{P}_1$, $i = 1,..., l$, such that $B_\omega(\psi_i, p_j') = \delta_{ij}$. Thus if we define $q_i' \in \mathscr{R}$, $i = 1,..., l$, by putting

$$q_i' = \psi_i + \log r_i 1 \tag{7.1.15}$$

then for $i, j = 1,..., l$

$$[q_i', p_j'] = \delta_{ij} 1. \tag{7.1.16}$$

Now let $\mathscr{Q}_2$ be the orthocomplement of $\mathscr{Q}_1$ in $\mathscr{Q}$ with respect to B_H and put $\mathscr{P}_2 = \beta_H \mathscr{Q}_2$. Obviously one has the direct sum

$$\mathscr{P} = \mathscr{P}_1 + \mathscr{P}_2. \tag{7.1.17}$$

Now let q_i', $i = l + 1,..., n$, be a B_H orthonormal basis of $\mathscr{Q}_2$. Again, $\mathscr{P}_2$ is non-singularly paired to $\mathscr{Q}_2$ by B_ω and hence there exist uniquely $p_i' \in \mathscr{P}_2$, $i = l + 1,..., n$, such that $[q_i', p_j'] = \delta_{ij} 1$. But now clearly $\mathscr{P}_2$ and $\mathscr{Q}_1$ are each other's orthocomplements with respect to B_ω. By symmetry the same statement is true for $\mathscr{P}_1$ and $\mathscr{Q}_2$. Thus p_i', q_j', $i, j = 1,..., n$, is a coordinate system in $\mathbb{R}^{2n}$ and

$$\omega = \sum_{i=1}^{l} dp_i' \wedge dq_i' + \sum_{j=l+1}^{n} dp_j' \wedge dq_j'. \tag{7.1.18}$$

Now the symmetric $l \times l$ matrix $B = (b_{ij}')$ defined by

$$b_{ij}' = B_H(\psi_i, \psi_j) \tag{7.1.19}$$

will play an important role for us. We first observe

PROPOSITION 7.1. *Let H be the Hamiltonian given by* (7.1.11). *Then with respect to the coordinates p_i', q_j', $i, j = 1,..., n$, one has*

$$H = H_1 + \sum_{i=l+1}^{n} \frac{(p_i')^2}{2}, \tag{7.1.20}$$

where

$$H_1 = \tfrac{1}{2} \sum_{i,j=1}^{l} b'_{ij} p'_i p'_j + \sum_{i=1}^{l} e^{q'_i}. \tag{7.1.21}$$

Proof. Obviously $e^{q'_i} = r_i e^{\psi_i}$, $i = 1,..., l$. It suffices then to prove that if

$$K = \tfrac{1}{2} \sum_{i,j=1}^{l} b'_{ij} p'_i p'_j + \sum_{i=l+1}^{n} \frac{(p'_i)^2}{2} \tag{7.1.22}$$

then K is the kinetic energy in (7.1.11). Since both K and the kinetic energy are in the symmetric subspace $S^2(\mathscr{P})$ it suffices to show that $B_K = B_H$, where B_K is the symmetric bilinear form on $\mathscr{Q}$ given by $B_K(q, q')1 = [q, [q', K]]$. But now evidently $B_K(\psi_i, \psi_j) = b'_{ij} = B_H(\psi_i, \psi_j)$ by (7.1.15) and (7.1.16). Thus B_K and B_H agree on $\mathscr{Q}_1$. Also $\mathscr{Q}_1$ and $\mathscr{Q}_2$ are B_K orthogonal and q'_i, $i = l + 1,..., n$, are B_K orthonormal. Thus $B_H = B_K$. Q.E.D.

7.2. Now put $Z' = \mathbb{R}^{2l}$ and let $\rho'_i, \varphi'_j \in C^\infty(Z')$, where $i, j = 1,..., l$, be a linear coordinate system. Then $(Z', \omega_{Z'})$ is just $2l$ phase space with the ρ'_i and φ'_j as canonical coordinates if we put

$$\omega_{Z'} = \sum_{i=1}^{l} d\rho'_i \wedge d\varphi'_i. \tag{7.2.1}$$

Now let $I'_1 \in C^\infty(Z')$ be the function defined by putting

$$I'_1 = \tfrac{1}{2} \sum_{i,j=1}^{l} b'_{ij} \rho'_i \rho'_j + \sum_{i=1}^{l} e^{\varphi'_i}, \tag{7.2.2}$$

where b'_{ij} is given by (7.1.18). Also let

$$\delta\colon \mathbb{R}^{2n} \to Z' \tag{7.2.3}$$

be the unique smooth map such that $\rho'_i \circ \delta = p'_i$ and $\varphi'_i \circ \delta = q'_i$, $i = 1,..., l$. Finally let

$$\Gamma_2\colon \mathscr{P} \to \mathscr{P}_2 \tag{7.2.4}$$

be the projection defined by (7.1.17) so that $\mathscr{P}_1 = \operatorname{Ker} \Gamma_2$.

Now, recalling the notation of Section 6.1, ξ_H and $\xi_{I'_1}$ are respectively the Hamiltonian vector fields on $\mathbb{R}^{2n}$ and Z' corresponding to the functions $H \in C^\infty(\mathbb{R}^{2n})$ and $I'_1 \in C^\infty(Z')$. We recall also that A is the $l \times n$ matrix (a_{ij}) defined in (7.1.10). In addition one notes that since the ψ_i are linearly independent the $l \times l$ matrix B defined by (7.1.19) is invertible and hence if A^* is the transpose of A then A^*B^{-1} is an $n \times l$ matrix.

PROPOSITION 7.2. *Let H be the Hamiltonian function on $\mathbb{R}^{2n}$ given by* (7.1.11) *and let $I_1' \in C^\infty(Z')$ be given by* (7.2.2). *Then ξ_H is globally integrable on $\mathbb{R}^{2n}$ in case $\xi_{I_1'}$ is globally integrable on Z'. Furthermore in such a case if $x \in \mathbb{R}^{2n}$ is an arbitrary initial point and, for notational convenience, we put $q_j(t) = q_j(\exp t\xi_H \cdot x)$, $j = 1,..., n$, then*

$$q_j(t) - q_j(0) = \frac{1}{m_j}\left(c_j t + \sum_{k=1}^{l} (A^*B^{-1})_{jk}\,(\varphi_k'(t) - \varphi_k'(0))\right), \qquad (7.2.5)$$

where c_j is the constant $(\Gamma_2 p_j)(x)$ (see (7.2.4)) *and letting $y' \in Z'$ be defined by $\delta x = y'$ (see* (7.2.3)) *one defines $\varphi_k'(t)$ by putting $\varphi_k'(t) = \varphi_k'(\exp t\xi_{I_1'} \cdot y')$.*

Proof. Assume $\xi_{I_1'}$ is globally integrable. Let $y(t) = \exp t\xi_{I_1'} \cdot y'$ and let $\mathbb{R} \to \mathbb{R}^{2n}$, $t \to x(t)$ be the curve in $\mathbb{R}^{2n}$ defined by the relation $p_i'(x(t)) = \rho_i'(y(t))$, $q_i'(x(t)) = \varphi_i'(y(t))$, $i = 1,..., l$, $p_k'(x(t)) = p_k'(x)$, and $q_k'(x(t)) = q_k'(x)$, $k = l + 1,..., n$. But now recalling (7.2.3), (7.1.21), and (7.2.2), one obviously has $I_1' \circ \delta = H_1$. But since Hamilton's equations are satisfied for the function I_1' along the curve $y(t)$ with respect to the φ_i' and ρ_j' it follows immediately that Hamilton's equations are satisfied for the function $H_1 \in C^\infty(\mathbb{R}^{2n})$ along the curve $x(t)$ with respect to the p_i' and q_j'. It follows then that ξ_{H_1} is globally integrable and

$$x(t) = \exp t\xi_{H_1} \cdot x. \qquad (7.2.6)$$

Now let $H_2 = \sum_{i=l+1}^{n} ((p_i')^2/2)$. It is obvious that ξ_{H_2} is globally integrable where $p_i'(\exp t\xi_{H_2} \cdot x) = p_i'(x)$, $i = 1,..., n$, $q_i'(\exp t\xi_{H_2} \cdot x) = q_i'(x)$ for $i \leqslant l$ and $q_i'(\exp t\xi_{H_2} \cdot x) = q_i'(x) + tp_i'(x)$, $i = l + 1,..., n$. But $H = H_1 + H_2$ and clearly $[H_1, H_2] = 0$. Thus the vector fields ξ_{H_1} and ξ_{H_2} commute and since $\xi_H = \xi_{H_1} + \xi_{H_2}$ it follows, as one knows, that ξ_H is globally integrable where

$$\exp t\xi_H \cdot x = \exp t\xi_{H_2} \cdot (\exp t\xi_{H_1} \cdot x). \qquad (7.2.7)$$

Now for any $f \in C^\infty(\mathbb{R}^{2n})$ and $t \in \mathbb{R}$ put $f(t) = f(\exp t\xi_H \cdot x)$. From (7.1.7) one then clearly has the integral

$$q_j(t) - q_j(0) = \frac{1}{m_j} \int_0^t p_j(s)\, ds. \qquad (7.2.8)$$

Now for any $p \in \mathscr{P}_2$ one has $[p, \psi_i] = 0$, $i = 1,..., l$, since $\mathscr{P}_2$ is B_ω orthogonal to $\mathscr{Q}_1$. Thus clearly $[H, p] = \xi_H p = 0$. Thus $p(t)$ is a constant function of t. But $\Gamma_2 p_j \in \mathscr{P}_2$. Hence

$$\frac{1}{m_j} \int_0^t (\Gamma_2 p_j)(s)\, ds = \frac{c_j t}{m_j}, \qquad (7.2.9)$$

where $c_j = (\Gamma_2 p_j)(x)$.

For any $g \in C^\infty(Z')$ and $t \in \mathbb{R}$ let $g(t) = g(\exp t\xi_{l_1'} \cdot y')$, where recalling (7.2.3) one puts $\delta x = y' \in Z'$.

Now let $\Gamma_1 = I - \Gamma_2$, where I is the identity operator on $\mathscr{P}$ so that Γ_1: $\mathscr{P} \to \mathscr{P}_1$ is the projection whose kernel is $\mathscr{P}_2$. But $p_j = \Gamma_1 p_j + \Gamma_2 p_j$. Thus to prove (7.2.5) it suffices by (7.2.8) and (7.2.9) to show that

$$\sum_{k=1}^{l} (A^* B^{-1})_{jk} (\varphi_k'(t) - \varphi_k'(0)) = \int_0^t (\Gamma_1 p_j)(s)\, ds. \tag{7.2.10}$$

Now complete the $l \times n$ matrix A to an $n \times n$ matrix (a_{ij}), where for $i > l$ one defines a_{ij} by the relation $\sum_{j=1}^n a_{ij} q_j = q_i'$. Now recalling (7.1.18) one then easily has (taking the transpose with respect to B_ω) the relation

$$\sum_{i=1}^{n} a_{ij} p_i' = p_j \tag{7.2.11}$$

for all $1 \leqslant i \leqslant n$. But now $\Gamma_1 p_i' = 0$ for $i > l$ and $\Gamma_1 p_i' = p_i'$ for $i \leqslant l$. Thus applying Γ_1 to both sides of (7.2.11) one just replaces n by l. That is,

$$\sum_{i=1}^{l} a_{ij} p_i' = \Gamma_1 p_j. \tag{7.2.12}$$

Now let μ_i, $i = 1,..., l$, be the basis of $\mathscr{P}_1$ defined by putting $\mu_i = \beta_H(\psi_i) = [\psi_i, H] = [q_i', H]$. Thus $\mu_i = -\xi_H q_i'$. But now, recalling the first paragraph in this proof, one has $q_i'(t) = \varphi_i'(t)$. Thus

$$\varphi_i'(t) - \varphi_i'(0) = \int_0^t \mu_i(s)\, ds. \tag{7.2.13}$$

But now the μ_i and p_j', where $i, j = 1,..., l$, are both bases of $\mathscr{P}_1$. We assert that

$$\mu_i = \sum_{j=1}^{l} b_{ij}' p_j'. \tag{7.2.14}$$

Indeed if s_{ij} is defined by the relation $\mu_i = \sum_j s_{ij} p_j'$ then $B_\omega(\psi_j, \mu_i) = s_{ij}$ since $B_\omega(\psi_j, p_i') = \delta_{ij}$. However, $\mu_i = \beta_H \psi_i$. Thus $s_{ij} = B_\omega(\psi_j, \beta_H \psi_i) = B_H(\psi_j, \psi_i) = b_{ij}'$. This proves (7.2.14). But then since B is the $l \times l$ matrix (b_{ij}') one has $p_i' = \sum_{k=1}^l (B^{-1})_{ik} \mu_k$. Substituting in (7.2.12) one then has

$$\sum_{k=1}^{l} (A^* B^{-1})_{jk}\, \mu_k = \Gamma_1 p_j. \tag{7.2.15}$$

But now evaluating both sides along the curve $\exp s\xi_H \cdot x$ and integrating from 0 to t one obtains (7.2.10) from (7.2.13). Q.E.D.

7.3. We now recall the definition of a Cartan matrix. (The definition will be made for the semi-simple and not just for the simple case.) See [11, Chap. 11] for more details. We remark first that the Cartan matrices classify the complex semi-simple Lie algebras or the compact semi-simple Lie algebra or, as we shall take it here, the real split semi-simple Lie algebras. Assume that $\mathfrak{g}$ is a real semi-simple Lie algebra, say, of rank l, which admits a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ such that the eigenvalues of $\operatorname{ad} x$ are real for all $x \in \mathfrak{h}$. Then $\mathfrak{g}$ is a real split semi-simple Lie algebra and $\mathfrak{h}$ is a split Cartan subalgebra. This in fact has been our assumption in earlier sections. Let $\mathfrak{h}'$ be the dual space to $\mathfrak{h}$ and let (μ, ν) be a Weyl group invariant positive definite inner product on $\mathfrak{h}'$. Let $\alpha_i \in \mathfrak{h}'$, $i = 1,..., l$, be the set of simple positive roots relative to some lexicographical ordering in $\mathfrak{h}'$. Then the $l \times l$ matrix C defined by

$$C_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \tag{7.3.1}$$

is called a Cartan matrix. One knows that the matrix is independent of the bilinear form (as long as it is Weyl group invariant) and the entries C_{ij} are integers. Furthermore the matrix completely determines the structure of $\mathfrak{g}$. Now assume that V is some finite-dimensional real vector space with a positive definite inner product P. Let $v_1,..., v_l \in V$ be linearly independent vectors. Then if $2P(v_i, v_j)/P(v_j, v_j) = C'_{ij}$ is a Cartan matrix there thus exist uniquely up to isomorphism a split semi-simple Lie algebra $\mathfrak{g}$ of rank l, a split Cartan subalgebra $\mathfrak{h}$, and simple roots α_i such that one has $C_{ij} = C'_{ij}$. Also if C'_{ij} is a Cartan matrix one may introduce a diagram, the Dynkin diagram, to describe the angles and relative lengths of the vectors v_i. The Dynkin diagram is based on the fact that if v_i and v_j are not orthogonal and $P(v_i, v_i) \geqslant P(v_j, v_j)$ then

$$P(v_i, v_i)/P(v_j, v_j) = 1, 2, \text{ or } 3 \tag{7.3.2}$$

and accordingly the angle between v_i and v_j is 120, 135, or 150°. (See, e.g., [5].) With regard to terminology in this paper, where the situation is warranted we will either say that C'_{ij} is a Cartan matrix or that v_i, $i = 1,..., l$, defines a Dynkin diagram. In either case we will refer to $\mathfrak{g}$, $\mathfrak{h}$, and the α_i as a corresponding split semi-simple Lie algebra, a split Cartan subalgebra, and a set of simple positive roots.

We recall in Section 1.2 we permitted some flexibility in the definition of the invariant bilinear form Q on $\mathfrak{g}$. This may now be normalized according to P. As usual Q is "carried" over to $\mathfrak{h}'$ using the isomorphism $\mathfrak{h} \to \mathfrak{h}'$ defined by $Q \mid \mathfrak{h}$.

PROPOSITION 7.3. *Assume that $v_1,..., v_l \in V$ defines a Dynkin diagram with respect to some positive definite inner product P on V. Let $\mathfrak{g}$, $\mathfrak{h}$, and $\alpha_i,..., \alpha_l$*

be a corresponding split semi-simple Lie algebra, a Cartan subalgebra, and a set of simple positive roots. Then there exists a unique invariant bilinear form Q on $\mathfrak{g}$ which on each simple component of $\mathfrak{g}$ is a positive multiple of the Killing form such that

$$Q(\alpha_i, \alpha_j) = P(v_i, v_j) \tag{7.3.3}$$

for $i, j = 1,..., l$.

Proof. Recalling the notion of connectedness of a Dynkin diagram (or Coxeter graphs—see, e.g., Section 11.3 in [11]) one notes from (7.3.2) that the ratios $r_{ij} = P(v_i, v_i)/P(v_j, v_j)$ are uniquely determined in case α_i and α_j are in the same connected component. In such a case one then clearly has $r_{ij} = Q(\alpha_i, \alpha_i)/Q(\alpha_j, \alpha_j)$ for any choice of Q. However, as one knows, the connected components of the diagram correspond to the simple components of $\mathfrak{g}$. We may thus uniquely normalize Q so that $Q(\alpha_i, \alpha_i) = P(\alpha_i, \alpha_i)$ for any i. One notes here that replacing Q on a simple component of $\mathfrak{g}$ by a positive multiple λQ is equivalent to replacing Q on the corresponding subspace of $\mathfrak{h}'$ by the positive multiple Q/λ.

One then has (7.3.3) from the equality $C'_{ij} = 2Q(\alpha_i, \alpha_j)/Q(\alpha_j, \alpha_j)$. Q.E.D.

7.4. Now returning to the Hamiltonian H of (7.1.11) we now assume that the $\psi_i \in \mathscr{Q}$, $i = 1,..., l$, define a Dynkin diagram with respect to B_H. Before proceeding we will give some examples where this is the case.

(1) $l = n - 1$. All the masses are equal and $\psi_i = q_i - q_{i+1}$, $i = 1,..., n-1$. Then H is the Hamiltonian of the usual nonperiodic Toda lattice. The potential here is that of "nearest-neighbor" particle interaction. The Dynkin diagram is that of A_{n-1}, using standard notation so that $\mathfrak{g}$ can be taken to be the Lie algebra of $Sl(n, \mathbb{R})$ and $\mathfrak{h}$ is the space of traceless real diagonal matrices.

(2) $l = n$. All the masses are equal and $\psi_i = q_i - q_{i+1}$, $i = 1,..., n-1$. But $\psi_n = q_n$. The potential is similar to the one above except that the last particle may also be regarded as interacting with a fixed mass. The Dynkin diagram is of B_n so that $\mathfrak{g}$ is the Lie algebra of $SO(n, n+1)$.

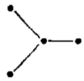
(3) Similar to (2) except that $\psi_n = 2q_n$. The last particle interacts even more strongly with a fixed mass. Hence the Dynkin diagram is that of C_n and $\mathfrak{g}$ is the Lie algebra of the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$. In Section 7.8 we will apply Theorem 7.5 below to solve the three-body problem (the case $n = 3$ here) given by

$$H = \sum_{i=1}^{3} \frac{p_i^2}{2} + e^{q_1 - q_2} + e^{q_2 - q_3} + e^{2q_3}. \tag{7.4.1}$$

(4) $l = n$. All the masses are equal. The case is that of (1) except that in

addition $\psi_n = q_{n-1} + q_n$. Thus the potential is similar to (1) except that the center of mass of the last two particles interacts with a fixed mass. Here the Dynkin diagram is that of D_n and $\mathfrak{g}$ can be taken to be the Lie algebra of $SO(n, n)$.

We wish to emphasize that there are of course an infinite number of different dynamical systems which correspond to the same Dynkin diagram. In (1)–(4) above we have just given one system for each of the classical Lie algebras. A particularly interesting Dynkin diagram is that of D_4. The diagram is



(7.4.2)

and as the figure somewhat suggests one may give a system (different from that in (4) above) where $l = n = 4$, so that there are four particles, three of which are interacting with a fixed body and whose center of gravity is interacting with the fourth particle. The Hamiltonian is

$$H = \sum_{i=1}^{4} \frac{p_i^2}{2} + e^{q_1} + e^{q_2} + e^{q_3} + e^{(q_4 - q_1 - q_2 - q_3)/2}. \tag{7.4.3}$$

Remark 7.4.1. For a classical Lie algebra one may solve the system by considering the usual Toda lattice for a larger number of variables. This is no longer true for the exceptional Lie algebras F_4, E_6, E_7, and E_8. In these cases if $z \in \mathfrak{g}$ is a Jacobi element there is no representation π such that $\pi(z)$ is a Jacobi operator in the usual sense, as one may show. An example of a dynamical system corresponding to F_4 is the four-body problem given by

$$H = \sum_{i=1}^{4} \frac{p_i^2}{2} + e^{q_1 - q_2} + e^{q_2 - q_3} + e^{(q_4 - q_1 - q_2 - q_3)/2}. \tag{7.4.4}$$

Among all the others then Theorem 7.5 below will thus describe the trajectories of the systems (7.4.3) and (7.4.4) (in terms of the four fundamental representations of D_4 and F_4, respectively).

Remark 7.4.2. Even though our assumption concerning the ψ's is very special we wish to note that there is some latitude in satisfying the assumption. Namely, given the ψ_i one varies the inner product B_H within an n-dimensional variety of such inner products by varying the masses m_i. This raises the possibility of satisfying the assumption for a certain choice of the masses m_i. More precisely recalling that $b'_{ij} = B_H(\psi_i, \psi_j)$ our assumption about the ψ_i is that C' should be a Cartan matrix where $C'_{ij} = b'_{ij}/b'_{ij}$. This is of course just a statement about the matrix B where, as in Proposition 7.2, $B = (b'_{ij})$. Thus given the ψ_i, that is, given the $l \times n$ matrix A (see (7.1.10)) the following proposition describes how B varies with the masses m_i.

PROPOSITION 7.4.1. *Let A be respectively the $l \times n$ matrix defined as in*

(7.1.10) *and let* A^* *be the transpose of* A. *Let* M *be the* $n \times n$ *diagonal matrix where* $M_{ii} = 1/m_i$ *and let* B, *as in Proposition* 7.2, *be the* $l \times l$ *matrix defined by putting* $B_{ij} = B_H(\psi_i, \psi_j)$. *Then*

$$B = AMA^*. \tag{7.4.5}$$

Proof. One has for $i, j = 1,..., l$,

$$B_H(\psi_i, \psi_j) = B_H\left(\sum_{r=1}^{n} a_{ir}q_r, \sum_{s-1}^{l} a_{js}q_s\right). \tag{7.4.6}$$

But then (7.4.5) follows from the relation $B_H(q_r, q_s) = \delta_{rs}/m_r$. See (7.1.14). Q.E.D.

Now let $\mathfrak{g}$, $\mathfrak{h}$, and α_i, $i = 1,..., l$, correspond to the Dynkin diagram of the ψ_i. Thus $\mathfrak{g}$ is real split semi-simple Lie algebra, $\mathfrak{h}$ is a split Cartan subalgebra, and $\alpha_i \in \mathfrak{h}'$ are the simple roots for a fixed system Δ_+ of positive roots relative to $(\mathfrak{g}, \mathfrak{h})$. We use the notation and results of Sections 6 and 7.1.3. By Proposition 7.3 we may fix the bilinear form Q in $\mathfrak{g}$ so that

$$Q(\alpha_i, \alpha_j) = B_H(\psi_i, \psi_j) \tag{7.4.7}$$

for $i, j = 1,..., l$. That is, recalling (6.5.1) and (7.1.19) one has $b'_{ij} = b_{ij}$ and hence the matrix $B = (b'_{ij})$ is just given by $B = (b_{ij}) = (Q(\alpha_i, \alpha_j))$.

Now recall (see Sections 6.3 and 6.4) that (Z, ω_Z) is a $2l$-dimensional symplectic manifold where $Z \subseteq \mathfrak{g}$ is the manifold of normalized Jacobi elements on $\mathfrak{g}$ and ω_Z is defined in Section 6.4. We recall also (see Proposition 6.5) that ρ_i, φ_j, $i, j = 1,..., l$, are a global coordinate system on Z and ω_Z is explicitly given by (6.5.4). In addition we recall that the restriction $I_1 \mid Z$ of the fundamental invariant $I_1 \in S(\mathfrak{g})^G$ is given by (6.5.5). See (6.5.2).

Now let $(Z', \omega_{Z'})$, ρ'_i, φ'_j, and I'_1 be as in Section 7.2.

PROPOSITION 7.4.2. *There exists a unique symplectic isomorphism*

$$\sigma: Z' \to Z \tag{7.4.8}$$

of $(Z', \omega_{Z'})$ *and* (Z, ω_Z) *such that* $\rho_i \circ \sigma = \rho'_i$, $\varphi_j \circ \sigma = \varphi'_j$. *Moreover one has* $I_1 \circ \sigma = I'_1$.

Proof. Since $b_{ij} = b'_{ij}$ this is immediate from a comparison of (7.2.1) and (7.2.2) with (6.8.1) and (6.8.2). Q.E.D.

Now let A, B, and M be respectively the $l \times n$, $l \times l$, and $n \times n$ matrices defined, say, as in (7.1.10), (7.1.19), and Proposition 7.4.1. Obviously B is invertible (since the ψ_i are linearly independent). Put

$$D = B^{-1}AM. \tag{7.4.9}$$

PROPOSITION 7.4.3. *The $l \times n$ matrix D is characterized by the properties* (1)

$$DA^* = 1_l, \tag{7.4.10}$$

where 1_l is the $l \times l$ identity matrix and (2) *A^*D is the transpose of the $n \times n$ matrix corresponding to the projection $\Gamma_1: \mathscr{P} \to \mathscr{P}_1$, relative to the basis p_i of $\mathscr{P}$. That is, if $\Gamma_2: \mathscr{P} \to \mathscr{P}_2$ is the complementary projection, as in* (7.2.4), *then for $j = 1,..., n$*

$$p_j - \sum_{k=1}^{n} (A^*D)_{jk}\, p_k = \Gamma_2 p_j. \tag{7.4.11}$$

Proof. One has (7.4.10) immediately from (7.4.5). Now recall (7.1.10). We recall also (see the proof of Proposition 7.2) that $\mu_k = [\psi_k, H] = [\sum_{r=1}^{n} a_{kr} q_r, H]$. But $[q_r, H] = p_r/m_r$. Thus

$$\mu_k = \sum_{r=1}^{n} a_{kr} \frac{p_r}{m_r}. \tag{7.4.12}$$

Now substituting (7.4.12) in (7.2.15) one has $\Gamma_1 p_j = \sum_{r=1}^{n} (A^*B^{-1}AM)_{jr}\, p_r$. But $B^{-1}AM = D$. This proves (7.4.11). Since A^*D is then given the $l \times n$ matrix D is uniquely characterized, as one knows, from the additional relation $DA^* = 1_l$. Q.E.D.

Now Proposition 7.2 enables us now to put into effect Theorem 6.8.1 to assert that ξ_H is globally integrable and to determine the trajectory $\exp t\xi_H \cdot x$ for any x in phase space $\mathbb{R}^{2n}$. What of course is needed is the Jacobi element (see (7.4.8)) $\sigma\delta x = y \in Z$. This plays a critical role in more than one way. One notes that the corresponding trajectory in Z is a curve on the isospectral leaf $Z(\gamma) \subseteq Z$, where $\gamma = \mathscr{I}(y)$. (See (6.8.3) and (3.6.2).) The following gives an explicit formula for y in terms of the position $q_i(x)$ and momentum $p_j(x)$ coordinates of x.

PROPOSITION 7.4.1. *Let $x \in \mathbb{R}^{2n}$ and let $Z \subseteq \mathfrak{g}$ be as in* (2.2.3) *and* (2.2.4) *the space of normalized Jacobi elements. Let $\delta: \mathbb{R}^{2n} \to Z'$ and let $\sigma: Z' \to Z$ be defined as in* (7.2.3) *and* (7.4.8), *respectively. Put $y = \sigma\delta x$. Let $h_i \in \mathfrak{h}$ be the basis of $\mathfrak{h}$ defined as in* (6.4.1) *and let $f \in \mathfrak{g}$ be as in* (1.5.4). *Let e_{α_i}, $i = 1,..., l$, be the simple positive root vectors defined as in Section* 1.5 *and let $\rho_i, \varphi_j, i, j =$*

$1,\dots,l$, be the coordinate system on Z defined as in Proposition 6.5 so that y can be written

$$y = f + \sum_{i=1}^{l} \rho_i(y)\, h_i + \sum_{j=1}^{l} e^{\varphi_j(y)} e_{\alpha_j}. \tag{7.4.13}$$

Then if $A = (a_{ij})$ and $D = (d_{ij})$ are the $l \times n$ matrices defined by (7.1.10) *and* (7.4.9), *respectively, and r_j is defined as in* (7.1.11) *one has*

$$\varphi_j(y) = \sum_{k=1}^{n} a_{jk} q_k(x) + \log r_j \tag{7.4.14}$$

and

$$\rho_i(y) = \sum_{k=1}^{n} d_{ik} p_k(x). \tag{7.4.15}$$

Proof. Recalling (7.2.3) and (7.4.8) one has $\varphi_j \circ \sigma\delta = q_j'$ so that $\varphi_j(y) = q_j'(x)$. But $q_j' = \psi_j + \log r_j$. The formula (7.4.14) then follows from (7.1.10). Now by (7.2.3) and (7.4.8) one has $p_i' = \rho_i \circ \sigma\delta$. Thus $p_i'(x) = \rho_i(y)$. But now by (7.2.14) one has $p_i' = \sum_{k=1}^{l} (B^{-1})_{ik}\, \mu_k$. Recalling (7.4.12) one then has $p_i' = \sum_{k=1}^{n} d_{ik} p_k$ since $D = B^{-1}AM$. This proves 7.4.15. Q.E.D.

7.5. Now recalling Section 3.3 let $\mathfrak{h}_+ \subseteq \mathfrak{h}$ be the open Weyl chamber defined as in (3.3.1) and let H be the Cartan subgroup of $G = \mathrm{Ad}\, \mathfrak{g}$ corresponding to $\mathfrak{h}$. Also let $D \subseteq \mathfrak{h}'$ be defined in (5.1.5) so that D parametrizes the equivalence classes of the finite-dimensional irreducible holomorphic representations of $\mathfrak{g}_{\mathbb{C}}$ or $G_{\mathbb{C}}{}^s$. We recall that π_λ, for $\lambda \in D$, is a representation corresponding to λ. See (5.1.7). Now let $\nu_k \in D$, $k = 1,\dots, l$, be defined as in Section 5.1 so that $D = \sum_k \mathbb{Z}_+ \nu_k$. See (5.1.6). The representations π_{ν_k}, $k = 1,\dots, l$, are called the fundamental representations of $\mathfrak{g}_{\mathbb{C}}$ or $G_{\mathbb{C}}{}^s$. We will write π_k for π_{ν_k}, and in much of the earlier notation we will replace λ by k when $\lambda = \nu_k \in D$.

Let $g_o \in H$ and $w_o \in \mathfrak{h}_+$. For any $\lambda \in D$ and $t \in \mathbb{R}$ we defined the function $\Phi_\lambda(g_o, w_o; t)$ in Section 5.10. See (5.10.6). We write $\Phi_k(g_o, w_o; t)$ for this function when $\lambda = \nu_k$. If $\Delta^k \subseteq \mathfrak{h}'$ is the set of weights of the fundamental representation π_k then by (5.10.6)

$$\Phi_k(g_o, w_o; t) = (-1)^{o(\nu_k - \kappa\nu_k)} \sum_{\nu \in \Delta^k} r_\nu(w_o)\, g_o{}^\nu e^{-t\langle \nu, w_o\rangle}, \tag{7.5.1}$$

where r_ν is the rational function of w_o given by

$$r_\nu(w_o) = \sum_{s \in \mathcal{S}^k(\nu)} \frac{c_{s,k}}{p_{o(\nu_k - \nu)}(s, w_o)}, \tag{7.5.2}$$

where $\mathcal{S}^k(\nu) = \mathcal{S}^\lambda(\nu)$ for $\lambda = \nu_k$ and $\mathcal{S}^\lambda(\nu)$ is defined in (5.10.1). Also, $c_{s,k} = c_{s,\nu_k}$, where the latter is defined by (5.9.3). Furthermore $p_j(s, w_o)$ is defined in (5.10.2). See also (5.10.3). The expression $o(\mu)$ is defined in (5.5.28). For later use we put $\mathcal{S}^k = \mathcal{S}^\lambda$ when $\lambda = \nu_k$ and $\mathcal{S}^\lambda$ is defined in Section 5.9.

Now as in (7.3.1) let C be the Cartan matrix defined by $C_{ij} = 2Q(\alpha_i, \alpha_j)/Q(\alpha_j, \alpha_j)$. If R is the $l \times l$ diagonal matrix defined by putting $R_{jj} = 2/Q(\alpha_j, \alpha_j)$ one has

$$C = BR, \tag{7.5.3}$$

recalling (7.1.19). But also recalling (see Section 5.1) the definition of the highest weights ν_k of the fundamental representations one notes that the matrix expressing the simple roots α_j in terms of the ν_k is just the Cartan matrix. That is,

$$\alpha_j = \sum C_{jk}\nu_k \tag{7.5.4}$$

since $2Q(\nu_i, \alpha_j)/Q(\alpha_j, \alpha_j) = \delta_{ij}$.

Remark 7.5.1. With regard to formula (7.5.2) one first of all has $\mathcal{S}^k(\nu) \subseteq \mathcal{S}^k$. The Cartan matrix C (or rather C^{-1}) which plays a critical role in describing the geometry of the ψ_i in our Hamiltonian H also may be used to describe the set $\mathcal{S}^k$ of sequences

$$s = (i_1, i_2, \ldots, i_{o(\nu_k - \kappa\nu_k)}), \tag{7.5.5}$$

where $1 \leqslant i_j \leqslant l$ and, if $d_k = o(\nu_k - \kappa\nu_k)$, $\sum_{j=1}^{d_k} \alpha_{ij} = \nu_k - \kappa\nu_k$. In fact inverting (7.5.4) one has $\nu_k = \sum (C^{-1})_{kj}\, \alpha_j$. But $-\kappa\alpha_j$ is again a simple positive root and if $1 \leqslant \bar{j} \leqslant l$ is defined by

$$\alpha_{\bar{j}} = -\kappa\alpha_j \tag{7.5.6}$$

one has

$$\nu_k - \kappa\nu_k = \sum_{j=1}^{l} ((C^{-1})_{kj} + (C^{-1})_{k\bar{j}})\, \alpha_j\,. \tag{7.5.7}$$

Thus $(C^{-1})_{kj} + (C^{-1})_{k\bar{j}}$ is a nonnegative integer and

$$o(\nu_k - \kappa\nu_k) = \sum_{j=1}^{l} (C^{-1})_{kj} + (C^{-1})_{k\bar{j}}\,. \tag{7.5.8}$$

Also,

$$\operatorname{card} \mathcal{S}^k = \frac{o(\nu_k - \kappa\nu_k)!}{\prod_{j=1}^{l} ((C^{-1})_{kj} + C^{-1}_{k\bar{j}})!}. \tag{7.5.9}$$

In fact $\mathcal{S}^k$ is the set of all sequences s such that any $1 \leqslant j \leqslant l$ occurs $(C^{-1})_{kj} + (C^{-1})_{kJ}$ times in s.

The following theorem is one of the main results of the paper. It determines the trajectories of the Hamiltonian vector field ξ_H. In the theorem which follows the next theorem we give the expression for the asymptotic, or scattering, behavior of the system. First, however, we introduce certain constants. Recalling Proposition 7.4.3 let E be the $n \times n$ projection matrix defined by putting $E = A^*D$. It reduces to the identity if $l = n$. By Proposition 7.4.3, E may be given, in general, by

$$\Gamma_1 p_i = \sum_{j=1}^{n} E_{ij} p_j, \qquad i = 1,\ldots, n. \tag{7.5.10}$$

For any $x \in \mathbb{R}^{2n}$ and $i = 1,\ldots, n$ let

$$\bar{q}_i(x) = q_i(x) - \frac{1}{m_i} \sum_{j=1}^{n} E_{ij} m_j q_j(x) \tag{7.5.11}$$

and

$$\bar{p}_i(x) = p_i(x) - \sum_{j=1}^{n} E_{ij} p_j(x). \tag{7.5.12}$$

Also, where $r_j > 0$, $j = 1,\ldots, l$, are the coefficients occurring in (7.1.11) let

$$b_i = -\sum_{j=1}^{l} (A^*B^{-1})_{ij} \log r_j, \tag{7.5.13}$$

using the notation of Proposition 7.4.1, for $i = 1,\ldots, n$.

Remark 7.5.2. One notes that $\bar{q}_i(x)$ and $\bar{p}_i(x)$ vanish if $l = n$. Also,

$$b_i = 0 \qquad \text{if all} \quad r_k = 1 \tag{7.5.14}$$

as in the usual Toda lattice and in the examples of Section 7.4.

THEOREM 7.5. *Let* $(\mathbb{R}^{2n}, \omega)$ *be a classical phase space with canonical coordinates* $p_i, q_j \in C^\infty(\mathbb{R}^{2n})$, $i, j = 1,\ldots, n$. *Let* $H \in C^\infty(\mathbb{R}^{2n})$ *be a Hamiltonian function of the form*

$$H = \sum_{j=1}^{n} \frac{p_j^2}{2m_j} + r_1 e^{\psi_1} + \cdots + r_l e^{\psi_l}, \tag{7.5.15}$$

where (the masses) $m_j > 0$, $r_i > 0$, *and* $l \leqslant n$,

$$\psi_i = \sum_{j=1}^{n} a_{ij} q_j, \qquad i = 1,\ldots, l, \tag{7.5.16}$$

are linearly independent linear combinations of the q_j's which define a Dynkin diagram with respect to the bilinear form B_H given by (7.1.12). *See* (7.1.13) *and* (7.1.14). *Then there exist, uniquely up to isomorphism, a real split semi-simple Lie algebra $\mathfrak{g}$ of rank l, a split Cartan subalgebra $\mathfrak{h}$, simple positive roots $\alpha_1, \ldots, \alpha_l$ in the dual $\mathfrak{h}'$, a nonsingular invariant bilinear form Q on $\mathfrak{g}$ and correspondingly on $\mathfrak{h}'$ such that*

$$B_H(\psi_i, \psi_j) = Q(\alpha_i, \alpha_j). \tag{7.5.17}$$

Let $\mathfrak{h}_+ \subseteq \mathfrak{h}$ be the open Weyl chamber corresponding to the α_i. Let G be the adjoint group of $\mathfrak{g}$ and let $H \subseteq G$ be the subgroup corresponding to $\mathfrak{h}$.

Now let ξ_H be the Hamiltonian vector field on $\mathbb{R}^{2n}$ corresponding to the Hamiltonian function H. See (7.1.2). *Then ξ_H is globally integrable on $\mathbb{R}^{2n}$. Let $x \in \mathbb{R}^{2n}$ be arbitrary and let $x(t) = \exp \xi_H \cdot x$ be the trajectory of ξ_H, where $x(0) = x$. In particular then $q_i(x(t))$ is the position of the ith particle at time t when the initial state of the system is $(q_1(x), \ldots, q_n(x), p_1(x), \ldots, p_n(x))$. Let $y \in \mathfrak{g}$ be the normalized Jacobi element defined by* (7.4.13)–(7.4.15) *and let $(g_o, w_o) \in H \times \mathfrak{h}_+$ be the "parameters" corresponding to y in the sense of Section* 3.7. *That is, $w_o \in \mathfrak{h}_+$ (the "diagonal" representation of y) is defined by the relation $\mathscr{I}(w_o) = \mathscr{I}(y)$ (see* (2.3.1)) *and $g_o \in H$ is defined so that $\rho(g) = g_o$, where $g \in G_0{}^w$ satisfies $\beta_{(w)}(g) = y$, using the notation of Theorem* 3.6. *Then if $\Phi_k(g_o, w_o; t)$ is the finite sum of exponentials defined by* (7.5.1) *(and hence given in terms of the fundamental representation of $\mathfrak{g}_{\mathbb{C}}$) one has for $i = 1, \ldots, n$*

$$q_i(x(t)) = \bar{q}_i(x) + \frac{1}{m_i}\left(b_i + \bar{p}_i(x)t - 2\sum_{k=1}^{l} \frac{a_{ki}}{Q(\alpha_k, \alpha_k)} \log \Phi_k(g_o, w_o; t)\right), \tag{7.5.18}$$

where m_i is given in (7.5.15), *a_{ki} is given in* (7.5.16) *and $\bar{q}_i(x)$, $\bar{p}_i(x)$, and b_i are given respectively by* (7.5.11), (7.5.12), *and* (7.5.13).

Proof. Recalling the notation and statement of Theorem 6.8.1 the Hamiltonian vector field ξ on Z corresponding to $I_1 \mid Z$ is globally integrable. Furthermore if, for notational convenience, $\varphi_k(t) = \varphi_k(\exp t\xi \cdot y)$ then, by (6.8.4),

$$\varphi_k(t) = \log h(g \exp(-t)w)^{-\alpha_k}, \tag{7.5.19}$$

where, as usual, $w = f + w_o$. But then by Proposition 7.4.2 the Hamiltonian vector field $\xi_{I_1'}$ on Z' is globally integrable and if one puts $\varphi_k'(t) = \varphi_k'(\exp t\xi_{I_1'} \cdot y')$, where $\delta x = y'$, one has

$$\varphi_k'(t) = \varphi_k(t), \tag{7.5.20}$$

where we choose $y = \sigma y'$. One notes also that, by Proposition 7.4.4, y is given by (7.4.13)–(7.4.15). But now ξ_H is globally integrable by Proposition 7.2

and we may apply (7.2.5) to determine $q_i(t)$, which we have, more properly, written here as $q_i(x(t))$. But now by (7.1.15), $\varphi'_k(0) = q'_k(x) = \psi_k(x) + \log r_k$, recalling the definition (see (7.2.3)) of δ. But $\psi_k = \sum_{j=1}^n a_{kj} q_j$. Thus for $s = 1,\ldots, l$

$$\begin{aligned}\sum_{k=1}^{l} (B^{-1})_{sk}\, \varphi'_k(0) &= \sum_{j=1}^{n} (B^{-1}A)_{sj}\, q_j(x) + \sum_{k=1}^{l} (B^{-1})_{sk} \log r_k \\ &= \sum_{j=1}^{n} D_{sj} m_j q_j(x) + \sum_{k=1}^{l} (B^{-1})_{sk} \log r_k\end{aligned} \tag{7.5.21}$$

since one has $D = B^{-1}AM$ and hence $B^{-1}A = DM^{-1}$. See (7.4.9). But then for $i = 1,\ldots, n$, since $E = A^*D$, one has, by (7.5.13) upon applying A^* to (7.5.21),

$$\sum_{k=1}^{l} (A^*B^{-1})_{ik} \varphi'_k(0) = \sum_{j=1}^{n} E_{ij} m_j q_j(x) - b_i\,.$$

But $q_i(t)$ equals $q_i(x)$ for $t = 0$ and hence by (7.2.5) and (7.5.11) one has

$$q_i(t) = \bar{q}_i(x) + \frac{1}{m_i}\left(c_i t + b_i + \sum_{k=1}^{l} (A^*B^{-1})_{ik}\, \varphi'_k(t)\right), \tag{7.5.22}$$

where $c_i = (\Gamma_2 p_i)(x)$. But then by (7.5.10) and (7.5.12)

$$c_i = \bar{p}_i(x). \tag{7.5.23}$$

Now $\varphi'_j(t) = \log h(g \exp(-t)w)^{-\alpha_j}$ for $j = 1,\ldots, l$, by (7.5.19) and (7.5.20) and hence for $k = 1,\ldots, l$

$$\sum_{j=1}^{l} (B^{-1})_{kj}\, \varphi'_j(t) = \sum_{j=1}^{l} (B^{-1})_{kj} \log h(g \exp(-t)w)^{-\alpha_j}. \tag{7.5.24}$$

But $B^{-1} = RC^{-1}$ by (7.5.3) and hence $\sum_{j=1}^{l} (B^{-1})_{kj}\, \alpha_j = 2\nu_k/Q(\alpha_k, \alpha_k)$ by (7.5.4). But then if we put $(B^{-1})_{jk}$ in the exponential on the right side of (7.5.24) one has

$$\begin{aligned}\sum_{j=1}^{l} (B^{-1})_{kj}\, \varphi'_j(t) &= -\frac{2}{Q(\alpha_k, \alpha_k)} \log h(g \exp(-t)w)^{\nu_k} \\ &= -\frac{2}{Q(\alpha_k, \alpha_k)} \log \Phi_k(g_o, w_o; t)\end{aligned}$$

by (5.10.7) for $\lambda = \nu_k$. Hence applying A^* one has for $i = 1,\ldots, n$

$$\sum_{j=1}^{l} (A^*B^{-1})_{ij}\, \varphi_j'(t) = -2 \sum_{k=1}^{l} \frac{a_{ki}}{Q(\alpha_k, \alpha_k)} \log \Phi_k(g_o, w_o; t). \quad (7.5.25)$$

The result (7.5.18) then follows from (7.5.22) and (7.5.23), recalling that $q_i(t) = q_i(x(t))$. Q.E.D.

Remark 7.5.3. One notes that knowing the $q_i(x(t))$ determines the entire trajectory $x(t)$ in $\mathbb{R}^{2n}$. This is clear from Hamilton's equations, (7.1.5), which for the case at hand asserts that

$$p_i(x(t)) = m_i \left(\frac{dq_i}{dt}\right)(t).$$

7.6. We now determine the asymptotic behavior of $q_i(x(t))$ as $t \to +\infty$ and $-\infty$. Since $q_i(x(t))$, by (7.5.18), is linear in t plus the log of a finite sum of exponentials it follows easily that $q_i(x(t))$ is asymptotic to straight lines as $t \to \pm\infty$. We wish to determine both of these straight lines.

For any $i = 1,\ldots, n$ let $\mu_i^- \in \mathfrak{h}'$ be the linear form on the Cartan subalgebra $\mathfrak{h}$ defined by

$$\mu_i^- = 2 \sum_{k=1}^{l} \frac{a_{ki}\nu_k}{Q(\alpha_k, \alpha_k)} \quad (7.6.1)$$

and put $\mu_i^+ = \kappa\mu_i^-$, where κ, as usual, is the element of the Weyl group which interchanges the positive and negative roots. Now let $x \in \mathbb{R}^{2n}$ and let $w_o \in \mathfrak{h}_+$ be as in Theorem 7.5. Put

$$v_i^+(x) = \frac{1}{m_i} (\bar{p}_i(x) + \langle \mu_i^+, w_o \rangle) \quad (7.6.2)$$

and

$$v_i^-(x) = \frac{1}{m_i} (\bar{p}_i(x) + \langle \mu_i^-, w_o \rangle). \quad (7.6.3)$$

Also let $d(w)$ be the element of the Cartan subgroup H defined in (5.5.2). It depends only on the isospectral leaf $Z(\gamma) \subseteq Z$ containing $y = \sigma\delta x$. As usual $w = w_o + f$. Also, $\gamma = \mathscr{I}(y)$. For $i = 1,\ldots, n$ and where $g_o \in H$ is defined as in Theorem 7.5 put

$$u_i^+(x) = \bar{q}_i(x) + \frac{1}{m_i} (b_i + \log((g_o^{-1}\, d(w))^{\mu_i^+})) \quad (7.6.4)$$

and

$$u_i^-(x) = \bar{q}_i(x) + \frac{1}{m_i} (b_i + \log((g_o^{-1}\, d(w)^{-1})^{\mu_i^-})). \quad (7.6.5)$$

Remark 7.6. Recall that $g_o^{\nu_k}$ and $d(w)^{\nu_k}$ are given in more explicit terms by (5.5.25) and (5.5.4). With regard to (5.5.25) one uses (3.7.4) to determine $h(g)^\lambda$. On the other hand note that, by (7.6.1),

$$\log((g_o^{-1}\, d(w)^{-1})^{\mu_i^-}) = -2 \sum_{k=1}^{l} \frac{a_{ki}}{Q(\alpha_k, \alpha_k)} (\log(g_o^{\nu_k}) + \log(d(w)^{\nu_k})). \tag{7.6.6}$$

On the other hand if $1 \leqslant \bar{k} \leqslant l$ is defined by (7.5.6) then

$$\log((g_o^{-1}\, d(w))^{\mu_i^+}) = -2 \sum_{k=1}^{l} \frac{a_{\bar{k}i}}{Q(\alpha_k, \alpha_k)} (-\log(g_o^{\nu_k}) + \log(d(w)^{\nu_k})) \tag{7.6.7}$$

since $Q(\alpha_k, \alpha_k) = Q(\alpha_{\bar{k}}, \alpha_{\bar{k}})$ and hence $-\kappa\nu_k = \nu_{\bar{k}}$.

The point is of course that the two asymptotic lines are $v_i^+(x)t + u_i^+(x)$ and $v_i^-(x)t + u_i^-(x)$. They are thus given by the highest ν_k, and lowest weights, $\kappa\nu_k$, of the fundamental representations of $\mathfrak{g}_{\mathbb{C}}$.

THEOREM 7.6. *Let the notation be as in Theorem* 7.5 *and as in* (7.6.2), (7.6.3) *and* (7.6.4), (7.6.5). *Then for* $i = 1, \ldots, n$

$$\lim_{t\to+\infty} ((v_i^+(x)t + u_i^+(x)) - q_i(x(t)) = 0 \tag{7.6.8}$$

and

$$\lim_{t\to-\infty} ((v_i^-(x)t + u_i^-(x)) - q_i(x(t)) = 0 \tag{7.6.9}$$

Proof. By (7.6.1) and (5.10.7) one has

$$\log(h(g\exp(-t)w)^{\mu_i^-}) = 2 \sum_{k=1}^{l} \frac{a_{ki}}{Q(\alpha_k, \alpha_k)} \log \Phi_k(g_o, w_o; t) \tag{7.6.10}$$

so that by (7.5.18)

$$q_i(x(t)) = \bar{q}_i(x) + \frac{1}{m_i}(b_i + \bar{p}_i(x)t - \log(h(g\exp(-t)w)^{\mu_i^-})). \tag{7.6.11}$$

But then if $\mu = \mu_i^-$ in Theorem 5.11 one easily has

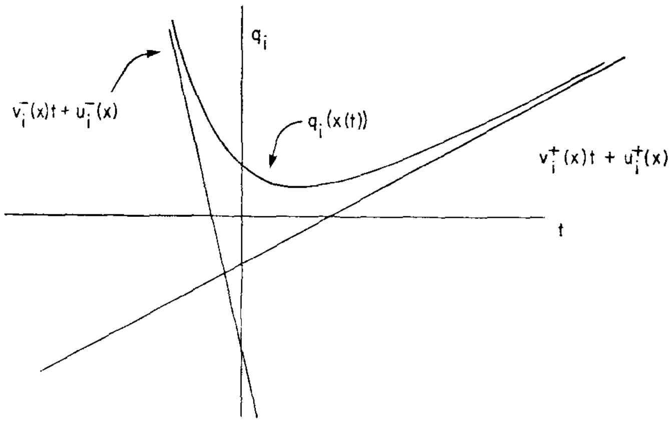
$$q_i(x(t)) - (v_i^+(x)t + u_i^+(x)) = -R_+(g_o, w_o; t) \tag{7.6.12}$$

and

$$q_i(x(t)) - (v_i^-(x)t + u_i^-(x)) = -R_-(g_o, w_o; t). \tag{7.6.13}$$

However, $\lim_{t\to+\infty} R_+(g_o, w_o; t) = \lim_{t\to-\infty} R_-(g_o, w_o; t) = 0$ by Theorem 5.11. This proves (7.6.8) and (7.6.9). Q.E.D.

We refer respectively to $v_i^+(x)$ and $v_i^-(x)$ as the $+\infty$ and $-\infty$ limiting velocities of the ith particle when it occupies the classical state $x \in \mathbb{R}^{2n}$ at time $t = 0$. Also (with J. Moser) under the same conditions we refer to $u_i^+(x)$ and $u_i^-(x)$, respectively, as the $+\infty$ and $-\infty$ limiting phases of the ith particle. Graphically plotting the position q_i of the ith particle against time one has



Of the two "parameters" (g_o, w_o) of $y = \sigma\delta x \in Z$ (see Section 3.7) we recall that $w_o \in \mathscr{h}_+$ picks out the isospectral leaf $Z(\gamma)$, $\gamma = \mathscr{I}(y)$, containing y and g_o (with its action angle coordinates $\log g_o^\nu$) picks out y in that leaf. See Theorems 3.7 and 4.3.

Proposition 7.6.1. *Let the notation be as in Theorem* 7.6.1. *Then for any* $x \in \mathbb{R}^{2n}$ *the difference* $v_i^+(x) - v_i^-(x)$ *of the limiting velocities of the ith particle depends only on the "spectrum" of the Jacobi element* $y = \sigma\delta x \in \mathfrak{g}$. *In fact one has*

$$v_i^+(x) - v_i^-(x) = \langle \kappa\mu_i^- - \mu_i^-, w_o \rangle. \tag{7.6.14}$$

The phase change, on the other hand, depends upon both g_o *and* w_o. *In fact recalling that* $d(w)$ *depends only on* w_o *one has*

$$u_i^+(x) - u_i^-(x) = \log g_o^{\mu_i^- - \kappa\mu_i^-} + \log d(w)^{\mu_i^- + \kappa\mu_i^-}. \tag{7.6.15}$$

Proof. This is immediate from definitions (7.6.2), (7.6.3) and (7.6.4), (7.6.5). Q.E.D.

Proposition 7.6.2. *Let the notation be as in Theorem* 7.6.1. *Assume that every one of the simple components of* $\mathfrak{g}$ *is of type* A_1, B_l, C_l, D_{2l}, G_2, F_4,

E_7 or E_8. Then the phase change depends only on the action angle coordinates. That is,

$$\mu_i^+(x) - \mu_i^-(x) = 2 \log g_o^{\mu_i^-}. \tag{7.6.16}$$

Moreover if in addition $l = n$ then one has a reversal of velocities. That is,

$$v_i^+(x) = -v_i^-(x). \tag{7.6.17}$$

Proof. If $\mathfrak{g}$ satisfies the condition as stated then one knows that κ equals minus the identity on $\mathfrak{h}'$. (See, e.g., Theorem 0.16 in [7, p. 335].) Thus $\mu_i^-(x) = -\mu_i^+(x)$. But then (7.6.15) implies (7.6.16). On the other hand if $l = n$ then as noted in Remark 7.5.2 one has $\bar{p}_i(x) = 0$. Thus (7.6.17) follows from definition (7.6.2), (7.6.3). Q.E.D.

7.7. For the usual Toda lattice $l = n - 1$ and $\psi_i = q_i - q_{i+1}$, $i = 1,..., n - 1$. Also, $r_i = m_i = 1$, $i = 1,..., n$. The corresponding Dynkin diagram is then of type A_l so that we can take $\mathfrak{g}$ to be the space of all real $n \times n$ matrices of trace zero and $\mathfrak{h}$ to be the space of all diagonal matrices in $\mathfrak{g}$. If $w_o \in \mathfrak{h}$ and

$$w_o = \operatorname{diag}(w_1, w_2, ..., w_n), \qquad \text{where} \quad w_i \in \mathbb{R}, \quad i \geqslant 1, \tag{7.7.1}$$

then we may choose the simple roots α_i, $i = 1,..., l$, so that

$$\langle \alpha_i, w_o \rangle = w_i - w_{i+1}. \tag{7.7.2}$$

One then has

$$w_o \in \mathfrak{h}_+ \qquad \text{if and only if} \quad w_i > w_{i+1} \tag{7.7.3}$$

for all $i = 1,..., l$. Also, the Weyl group element κ is given by

$$\kappa w_o = \operatorname{diag}(w_n, w_{n-1}, ..., w_1). \tag{7.7.4}$$

One notes that (recalling that the masses m_i all equal 1) $B_H(\psi_i, \psi_i) = 2$ so that

$$Q(\alpha_i, \alpha_i) = 2 \tag{7.7.5}$$

for the invariant bilinear form Q. It follows easily then that Q is given by

$$Q(u, v) = \operatorname{tr} uv \tag{7.7.6}$$

for $u, v \in \mathfrak{g}$, where the multiplication uv is as operators on $\mathbb{R}^n$.

Now the action of $\mathfrak{g}$ on $\mathbb{R}^n$ extends to an action of $\mathfrak{g}$ as a Lie algebra of derivations of the exterior algebra $\Lambda\mathbb{R}^n$. The homogeneous subspace $\Lambda^k\mathbb{R}^n$ is stable under $\mathfrak{g}$ and as one knows, the fundamental representations π_k, $k = 1,\ldots, n-1$, may be taken so that

$$\pi_k\colon \mathfrak{g} \to \operatorname{End} \Lambda^k\mathbb{R}^n \tag{7.7.7}$$

and (7.7.7) is the restriction of this action of $\mathfrak{g}$ to $\Lambda^k\mathbb{R}^n$. The set Δ^k of weights of π_k is in natural correspondence with the set of all $\binom{n}{k}$ subsets $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$ of k integers from 1 to n. In fact using such subsets as parameters, $\nu = \nu(i_1,\ldots, i_k) \in \Delta^k$, the correspondence is given by

$$\langle \nu, w_o\rangle = w_{i_1} + w_{i_2} + \cdots + w_{i_k} \tag{7.7.8}$$

for $w_o \in \mathfrak{h}$. Now also write $r_{i_1,\ldots,i_k}(w_o) = r_\nu(w_o)$, using the notation of (7.5.2) so that $r_{i_1,\ldots,i_k}(w_o)$ is a rational function of the eigenvalues w_i of w_o. Now H can be taken to be the group of all diagonal positive matrices of determinant 1. Thus if $g_o \in H$ then we can write for $g_i > 0$, $i = 1,\ldots, n$,

$$g_o = \operatorname{diag}(g_1,\ldots, g_n). \tag{7.7.9}$$

We then note that for $g_o \in H$, $w_o \in \mathfrak{h}_+$ the function $\Phi_k(g_o, w_o; t)$ takes the form

$$\Phi_k(g_o, w_o; t) = c \sum_{1\leqslant i_1<\cdots<i_k\leqslant n} r_{i_1,\ldots,i_k}(w_o)\, g_{i_1} \cdots g_{i_k} \exp(-t(w_{i_1} + \cdots + w_{i_k})), \tag{7.7.10}$$

where $c = (-1)^{o(\nu_k-\kappa\nu_k)}$. See (7.5.1).

Remark 7.7.1. Recall that $\Phi_k(g_o, w_o; t) = h(g \exp(-t)w)^{\nu_k}$. For computational purposes, by (7.7.7), one may then determine $\Phi_k(g_o, w_o; t)$ by taking the $k \times k$ principal minor of the standard representation on $\mathbb{R}^n$ of the element $s_o(\kappa)^{-1}\, \bar{n}_{-f}(w)\, g_o \exp(-t)\, w_o(\bar{n}_f(w))^{-1} = a(t)$, (see (5.3.6)), that is, the $k \times k$ principal minor of $\pi_1(a(t))$. The matrices $\pi_1(s_o(\kappa)^{-1})$ and $\pi_1(g_o \exp(-t)\, w_o)$ are easy to write down. The matrices $\pi_1(\bar{n}_{-f}(w))$ and $\pi_1(\bar{n}_f(w)^{-1})$ are computable from (5.8.6) and (5.8.7). In fact the matrix $\pi_1(\bar{n}_{-f}(w))$, using Proposition 5.8.2, is written down in Proposition 7.7.3.

If one does use (7.7.10) to determine $\Phi_k(g_o, w_o; t)$ then first of all with regard to the constant c.

PROPOSITION 7.7.1. *One has*

$$o(\nu_k - \kappa\nu_k) = k(n-k) \tag{7.7.11}$$

so that if n is odd then $c = 1$ for all k and if n is even then c alternates in sign with k.

Proof. It is clear from (7.7.2) and (7.7.6) that if $w_o \in \mathfrak{h}$ is arbitrary and is given by (7.7.1) then

$$\langle \nu_k , w_o \rangle = w_1 + w_2 + \cdots + w_k . \tag{7.7.12}$$

Furthermore by (7.7.4) one has

$$-\kappa \nu_k = \nu_{n-k} . \tag{7.7.13}$$

But since $w_1 + \cdots + w_n = 0$ one then easily has

$$\langle \nu_k - \kappa \nu_k , w_o \rangle = (w_1 + \cdots + w_k) - (w_{n-k} + w_{n-k+1} + \cdots + w_n). \tag{7.7.14}$$

On the other hand if $x_o \in \mathfrak{h}$ is defined as in Section 2.1, then, by (7.7.2),

$$x_o = \operatorname{diag}\left(\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{1-n}{2}\right). \tag{7.7.15}$$

But $o(\nu_k - \kappa\nu_k) = \langle \nu_k - \kappa\nu_k , x_o \rangle$. But then by (7.7.14) and (7.7.15) one has $o(\nu_k - \kappa\nu_k) = (n-1) + (n-3) + \cdots + (n-2k+1) = k(n-k)$. Q.E.D.

Now by (1.5.2) and (7.7.6) we may clearly choose root vectors $e_{\alpha_i}, e_{-\alpha_i} \in \mathfrak{g}$ so that in terms of the usual matrix units e_{ij} one has, for $i = 1,\ldots, n-1 = l$,

$$e_{\alpha_i} = e_{i\, i+1} , \qquad e_{-\alpha_i} = e_{i+1\, i} . \tag{7.7.16}$$

It follows then from (1.5.1) (see Remark 1.5.1) that with regard to the Cartan decomposition of $\mathfrak{g}$ one has that $\mathfrak{k}$ is the Lie algebra of all $n \times n$ skew-symmetric matrices. The inner product (see Section 5.1) in $V^{\nu_k} = \Lambda^k \mathbb{R}^n$, using the notation of Section 5.1, may then be taken to be the standard inner product on $\Lambda^k \mathbb{R}^n$. In particular if $\epsilon_i \in \mathbb{R}^n$, $i = 1,\ldots, n$, is the standard basis of $\mathbb{R}^n$ (i.e., the jth coordinate of ϵ_i is δ_{ij}), then the elements $\{\epsilon_{i_1} \wedge \cdots \wedge \epsilon_{i_k}\}$, where $1 \leqslant i_1 < \cdots < i_k \leqslant n$, are an orthonormal weight basis of $\Lambda^k \mathbb{R}^n$. In particular the highest weight vector, now written $v^k = v^{\nu_k}$, can be taken to be given as

$$v^k = \epsilon_1 \wedge \cdots \wedge \epsilon_k . \tag{7.7.17}$$

Recall that $\mathscr{S}$ for the present case is the set of all finite sequences

$$s = (i_1 ,\ldots, i_d), \tag{7.7.18}$$

where $1 \leqslant i_j \leqslant l = n-1$.

LEMMA 7.7. *Let* $1 \leqslant j \leqslant n$. *Then for any* $s \in \mathscr{S}$ *one has* $e_{-s}\epsilon_j = 0$ *unless* *either s is trivial or is of the form $s = (j, j+1, j+2, \ldots, i-1)$, where $j < i \leqslant n$, in which case $e_{-s}\epsilon_j = \epsilon_i$.*

Proof. This is immediate from (7.7.16). Indeed by (7.7.16) $e_{-\alpha_k}\epsilon_i = 0$ unless $k = i$ in which case $e_{-\alpha_k}\epsilon_i = \epsilon_{i+1}$. Q.E.D.

Let $m_{ki} \in \mathbb{Z}_+$ be defined so that $\nu_k - \kappa\nu_k = \sum_{i=1}^{l} m_{ki}\alpha_i$.

Remark 7.7.2. Note that $\nu_k - \kappa\nu_k = \nu_{n-k} - \kappa\nu_{n-k}$ by (7.7.13) so that to determine the m_{ki} one need consider only the case where $k \leqslant [n/2]$. In this case by (7.7.14) and (7.7.2) one easily has that $m_{ki} = i$ for $i \leqslant k$, $m_{ki} = k$ for $k \leqslant i \leqslant n-k$ and $m_{ki} = n - i$ for $i \geqslant n - k$.

By definition $\mathscr{S}^k$ is the set of all $s \in \mathscr{S}$ so that if s is given by (7.7.18) then, by (7.7.11), $d = k(n-k)$ and i occurs m_{ki} times among the i_j. The definition of $r_{i_1 \ldots i_k}(w_o)$ in (7.7.10) depends upon a sum over a subset of $\mathscr{S}^k$. By (7.5.2) what is needed then, to use (7.7.10) to determine $\Phi_k(g_o, w_o; t)$, is a knowledge of $c_{s,k}$ for $s \in \mathscr{S}^k$.

PROPOSITION 7.7.2. *One has $c_{s,k} = 0$ or $(-1)^{k(n-k)}$ for any $s \in \mathscr{S}^k$. Furthermore for any $s \in \mathscr{S}$ and $1 \leqslant i_1 < \cdots < i_k \leqslant n$ one has $e_{-s}\epsilon_{i_1} \wedge \cdots \wedge \epsilon_{i_k} = 0$ or there exists $1 \leqslant j_1 < \cdots < j_k \leqslant n$ so that*

$$e_{-s}\epsilon_{i_1} \wedge \cdots \wedge \epsilon_{i_k} = e_{j_1} \wedge \cdots \wedge e_{j_k}. \tag{7.7.19}$$

Finally using the notation of (5.2.10) *the lowest weight vector $v^{\kappa\nu_k}$ is given by*

$$v^{\kappa\nu_k} = (-1)^{k(n-k)}\, \epsilon_{n-k} \wedge \epsilon_{n-k+1} \wedge \cdots \wedge \epsilon_n. \tag{7.7.20}$$

Proof. For any $1 \leqslant i \leqslant l$ note that $e_{-\alpha_i}\epsilon_{i_1} \wedge \cdots \wedge \epsilon_{i_k} = 0$ unless there exists a j such that $i = i_j$ and $i_{j+1} \geqslant 2 + i_j$ (putting $i_{k+1} = n + 1$), in which case

$$e_{-\alpha_i}\epsilon_{i_i} \wedge \cdots \wedge \epsilon_{i_j} \wedge \cdots \wedge \epsilon_{i_k} = \epsilon_{i_1} \wedge \cdots \wedge \epsilon_{i_j+1} \wedge \cdots \wedge \epsilon_{i_k}. \tag{7.7.21}$$

This proves the second statement and (7.7.19) in particular. Thus if $s \in \mathscr{S}^k$ one has either $e_{-s}\epsilon_1 \wedge \cdots \wedge \epsilon_k = 0$ or

$$e_{-s}\epsilon_1 \wedge \cdots \wedge \epsilon_k = \epsilon_{n-k} \wedge \cdots \wedge \epsilon_n. \tag{7.7.22}$$

Let $\mathscr{S}^k_*$ be the set of all $s \in \mathscr{S}^k$ satisfying (7.7.22). But now since $v^{\kappa\nu_k}$ has unit length it follows that $v^{\kappa\nu_k} = a\epsilon_{n-k} \wedge \cdots \wedge \epsilon_n$, where a is either 1 or -1. But then by (7.7.22) one has $c_{s,k} = 0$ if $s \notin \mathscr{S}^k_*$ and $c_{s,k} = a$ if $s \in \mathscr{S}^k_*$. But then $a = (-1)^{k(n-k)}$ by (5.9.7) and (7.7.11). Q.E.D.

Remark 7.7.3. The question as to whether $c_{s,k} = (-1)^{k(n-k)}$ or 0, i.e., whether $s \in \mathscr{S}_*^k$ or not, is of course a readily resolved combinatorial question using (7.7.21).

For the standard Toda lattice Theorem 7.5 becomes

THEOREM 7.7.1. *Let the notation be as in Theorem* 7.5, *where* $l = n - 1$, $\psi_i = q_i - q_{i+1}$, *and* $r_j = m_j = 1$ *for* $j = 1,..., n$ *so that we can take* $\mathfrak{g}$ *to be the set of all real* $n \times n$ *matrices of trace zero,* $\mathfrak{h}$ *to be the set of diagonal matrices in* $\mathfrak{g}$, $\alpha_1 ,..., \alpha_l$ *to be the set of simple roots given by* (7.7.2), *and* Q *equal to the bilinear form on* $\mathfrak{g}$ *given by* (7.7.6). *Then for any* x *in phase space* $\mathbb{R}^{2n}$ *one has*

$$q_i(x(t)) = \frac{1}{n}\left(\sum_{j=1}^{n} (q_j(x) + tp_j(x))\right)$$
$$+ \log \Phi_{i-1}(g_o , w_o ; t) - \log \Phi_i(g_o , w_o ; t), \tag{7.7.23}$$

where $\log \Phi_j(g_o , w_o ; t) = 0$ *for* $j = 0$ *or* n *and is otherwise given by taking the* log *of* (7.7.10).

Proof. Applying Theorem 7.5 for the present case we note first of all that $b_i = 0$ by (7.5.14). Next recalling (7.1.17) one easily has that $\mathscr{P}_2$ is the one-dimensional space spanned by $p_1 + \cdots + p_n$ and $\mathscr{P}_1$ is the $(n - 1)$-dimensional space of all $p \in \mathscr{P}$ of the form $p = \sum_{i=1}^{n} c_i p_i$, where $\sum c_i = 0$. It follows easily that $\Gamma_2 p_j = (1/n) \sum_{i=1}^{n} p_i$ for any j. But then by (7.5.11) and (7.5.12) one has $\bar{q}_i(x) = (1/n) \sum_{j=1}^{n} q_j(x)$ and $\bar{p}_i(x) = (1/n) \sum_{j=}^{n} p_j(x)$. Next recalling (7.1.10) one has $a_{ij} = 0$ unless $j = i, i + 1$, and

$$a_{ii} = 1, \qquad a_{i\, i+1} = -1. \tag{7.7.24}$$

The result (7.7.23) then follows from (7.5.18) and (7.7.5). Q.E.D.

We now wish to recover the results of Moser (see 4.3 in [19, p. 481]) on the scattering of the Toda lattice.

By (7.6.1) and (7.7.24) one has

$$\mu_i^- = \nu_i - \nu_{i-1} \tag{7.7.25}$$

for $i = 1,..., n - 1$, where we put $\nu_0 = 0$. But $-\kappa\nu_i = \nu_{n-i}$ by (7.7.13) and hence

$$\mu_i^+ = \nu_{n+1-i} - \nu_{n-i}, \tag{7.7.26}$$

where $\nu_n = 0$.

PROPOSITION 7.7.3. *Let the notation be as in Proposition* 5.8.2 *and above* *where w_o is given by* (7.7.1). *Let $m = \{m_{ij}\}$ be the $n \times n$ matrix $m = \pi_1(\bar{n}_{-f}(w))$. Then $m_{ij} = 0$ for $i < j$, $m_{ii} = 1$ and for $i > j$ one has*

$$m_{ij} = (-1)^{i+j}((w_j - w_{j+1})(w_j - w_{j+2}) \cdots (w_j - w_i))^{-1}. \qquad (7.7.27)$$

Proof. As in Lemma 7.7 let $s \in \mathscr{S}$ be of the form $s = (j, j+1, \ldots, i-1)$, where $1 \leqslant j < i \leqslant n$. For such a sequence s one clearly has

$$p(s, w_o) = (w_j - w_{j+1})(w_j - w_{j+2}) \cdots (w_j - w_i), \qquad (7.7.28)$$

recalling (5.8.4) and (7.7.2). By Lemma 7.7 one has $e_{-s}\epsilon_j = \epsilon_i$ if s is of this form and $e_{-s}\epsilon_j = 0$ if s is not of this form. The result then follows immediately from the computation of $\pi_1(\bar{n}_{-f}(w))\,\epsilon_j$ using (5.8.6). Q.E.D.

Now let $\delta_k(w_o)$ be the product of the determinant of the $k \times k$ minor in the lower left-hand corner of the matrix m with $(-1)^{k(n-k)}$. That is, where m is given by Proposition 7.7.3 put

$$\delta_k(w_o) = (-1)^{k(n-k)} \det_{i,j=1,\ldots,k} m_{n-k+i\;j}\,. \qquad (7.7.29)$$

Remark 7.7.4. One may show inductively that

$$\delta_k(w_o) = \frac{\prod_{i<j\leqslant k} w_i - w_j}{\prod_{i<s\leqslant n, i\leqslant k} w_i - w_s}\,. \qquad (7.7.30)$$

The agreement of the following application of Theorem 7.7.1 with the results of Moser in [19] on the scattering of the Toda lattice will be clarified in Remark 7.7.5.

THEOREM 7.7.2. *Let the notation be as in Theorem* 7.6 *and as above. Thus for any point x in phase space $v_i^+(x)t + u_i^+(x)$ and $v_i^-(x)t + u_i^-(x)$ respectively, are the asymptotic lines of the position curve $(t, q_i(x(t))$, as $t \to +\infty$ and $t \to -\infty$, of the ith particle in the standard Toda lattice when the system occupies x at $t = 0$. Then if w_o is defined as in Theorem* 7.5 (*so that w_o picks out the isospectral leaf of the Jacobi matrix $y = \sigma\delta x$*) *and is described as a diagonal matrix by* (7.7.1) *one has, in terms of the eigenvalues of w_o,*

$$v_k^+(x) = w_{n+1-k} + \bar{p}(x) \qquad (7.7.31)$$

and

$$v_k^-(x) = w_k + \bar{p}(x), \qquad (7.7.32)$$

where $\bar{p}(x) = (1/n)\sum_{i=1}^{n} p_i(x)$. *In particular then*

$$v_k^+(x) = v_{n+1-k}^-(x). \tag{7.7.33}$$

Furthermore the phase difference $u_{n+1-i}^+ - u_i^-$ *also depends only on* w_o. *In fact if* $\delta_k(w_o)$ *is defined by* (7.7.29) *one has*

$$u_{n+1-k}^+(x) - u_k^-(x) = \log \delta_k(w_o)^2 - \log \delta_{k-1}(w_o)^2. \tag{7.7.34}$$

Proof. Recalling (7.7.12) one has

$$\langle \nu_k , w_o \rangle = w_1 + w_2 + \cdots + w_k .$$

Thus $\langle \mu_k^+, w_o\rangle = w_{n+1-k}$ and $\langle \mu_k^-, w_o\rangle = w_k$ by (7.7.25) and (7.7.26). But then (7.7.31) and (7.7.32) follow from (7.6.2) and (7.6.3).

But now computing $u_{n+1-k}^+(x) - u_k^-(x)$ using (7.6.4) and (7.6.5) note that $\bar{q}_k(x)$, b_k/m_k, and also the dependence on g_o cancel out. That is, by (7.6.4) and (7.6.5)

$$u_{n+1-k}^+(x) - u_k^-(x) = \log d(w)^{2(\nu_k - \nu_{k-1})}. \tag{7.7.35}$$

But now by (5.5.4) one has $d(w)^{\nu_k} = \{\bar{n}_{-f}(w)\, v^{\nu_k}, v^{\kappa\nu_k}\}$. But then recalling (7.7.17) and (7.7.20), $d(w)^{\nu_k}$ is just $(-1)^{k(n-k)}$ times the determinant of the $k \times k$ minor in the lower left-hand corner of $\pi_1(\bar{n}_{-f}(w))$. That is,

$$d(w)^{\nu_k} = \delta_k(w_o). \tag{7.7.36}$$

Then (7.7.34) follows from (7.7.35) and (7.7.36). Q.E.D.

Remark 7.7.5. We now align the notation here with that in [19]. Let L be the symmetric Jacobi matrix given in [19, p. 473]. Now let D be the diagonal matrix

$$D = \mathrm{diag}(e^{x_1/2}, \ldots, e^{x_n/2}). \tag{7.7.37}$$

Then

$$2\left(DLD^{-1} - \frac{1}{n}(\mathrm{tr}\, L)I\right) = y, \tag{7.7.38}$$

where y is the normalized Jacobi element defined in Theorem 7.5 of this paper and I is the $n \times n$ identity matrix. Since $2b_k = -y_k$ by 2.1 in the notation of [19] (y_k in [19] is p_k here) the initial state of the system u has the negative of the momentum considered in this paper. Thus if $x_k(t)$ is defined as in [19] and $q_k(x(t))$ is defined as in this paper one has

$$x_k(-t) = q_k(x(t)). \tag{7.7.39}$$

In particular

$$2b_k = p_k(x), \tag{7.7.40}$$

where in the notation of Theorem 7.7.2 here $p_k(x)$ is the momentum of the kth particle in the initial state x. Now the eigenvalues of y in decreasing order are given here, by (7.7.1) and (7.7.3), as $w_1, \ldots, w_n$. The eigenvalues of L in increasing order are given as $\lambda_1, \ldots, \lambda_n$ in [19]. Thus by (7.7.38) one has

$$2\lambda_{n-k+1} = \frac{2 \operatorname{tr} L}{n} + w_k. \tag{7.7.41}$$

The asymptotic lines in [19] for $+\infty$ and $-\infty$ are written as $\alpha_k^+ t + \beta_k^+$ and $\alpha_k^- t + \beta_k^-$. However, by (7.7.39) one must have

$$\alpha_k^+ = -v_k^-(x), \qquad \alpha_k^- = -v_k^+(x). \tag{7.7.42}$$

Also,

$$\beta_k^+ = u_k^-(x), \qquad \beta_k^- = u_k^+(x). \tag{7.7.43}$$

But now (see Section 4 in [19]) one has $\alpha_k^+ = -2\lambda_{n-k+1}$ and $\alpha_k^- = -2\lambda_k$. Noting (7.7.41) this checks with our results (7.7.31) and (7.7.32) since, by (7.7.40), $\bar{p}(x) = (2/n) \operatorname{tr} L$. Also, the result 4.2 in [19] is given here as (7.7.33). Now by (7.7.43)

$$\beta_{n-k+1}^+ - \beta_k^- = u_{n-k+1}^-(x) - u_k^+(x). \tag{7.7.44}$$

By (7.7.34) the right side of (7.7.44) is given as $\log \delta_{n-k}(w_o)^2 - \log \delta_{n-k+1}(w_o)^2$. However, noting Remark 7.7.4 this yields

$$\beta_{n-k+1}^+ - \beta_k^- = 2 \log \frac{\prod_{j=1}^{k-1} (w_{n-k+1} - w_{n-j+1})}{\prod_{i=k+1}^{n} (w_{n-i+1} - w_{n-k+1})}. \tag{7.7.45}$$

But by (7.7.31), (7.7.32), and (7.7.42)

$$w_{n-k+1} - w_{n-r+1} = \alpha_r^- - \alpha_k^-.$$

Thus

$$\beta_{n-k+1}^+ - \beta_k^- = 2 \log \frac{\prod_{j=1}^{k-1} (\alpha_j^- - \alpha_k^-)}{\prod_{i=k+1}^{n} (\alpha_k^- - \alpha_i^-)}. \tag{7.7.46}$$

But this is exactly the result 4.3 in [19, p. 481].

7.8. As a further application of Theorem 7.5 consider the three-body problem mentioned in (7.4.1). Here $n = 3$ and $l = 3$ and we recall the Hamiltonian H given by

$$H = \sum_{i=1}^{3} \frac{p_i^2}{2} + e^{q_1-q_2} + e^{q_2-q_3} + e^{2q_3}. \tag{7.8.1}$$

This then differs from the usual Toda lattice in that the particle whose position is given by q_3 is interacting also exponentially with some fixed mass. The Dynkin diagram in question, as one easily sees, is so that $\mathfrak{g}_{\mathbb{C}}$ is of type C_3 and hence $\mathfrak{g}$ is isomorphic to the Lie algebra of Sp(6, $\mathbb{R}$).

We shall be dealing here with 6×6 matrices u and we will often write

$$u = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{7.8.2}$$

where $A = A(u)$, $B = B(u)$, $C = C(u)$, and $D = D(u)$ are 3×3 matrices. If E is a 3×3 matrix let E' be the 3×3 matrix obtained from E by transposing with respect to the diagonal which runs from the lower left-hand corner to the upper right-hand corner. Thus $(E')_{ij} = E_{4-i\,4-j}$. We find it particularly convenient to identify $\mathfrak{g}$ with the set of all matrices u in (7.8.2), where

$$D = -A', \qquad C = C', \qquad \text{and} \qquad B = B'. \tag{7.8.3}$$

Remark 7.8.1. It is hoped that the reader is not mislead by the fact that the 6 in question describing the sizes of the matrices is also the dimension of the phase space of our system. If the e^{2q_3} term in (7.8.1) were replaced by e^{q_3} then $\mathfrak{g} \cong SO(3, 4)$ and we would be dealing with 7×7 matrices. The present case was chosen to simplify the computations.

The Cartan subalgebra $\mathfrak{h}$ is the set of all $w_o \in \mathfrak{g}$ such that $C(w_o) = B(w_o) = 0$ and

$$A(w_o) = \begin{pmatrix} w_1 & 0 & 0 \\ 0 & w_2 & 0 \\ 0 & 0 & w_3 \end{pmatrix}. \tag{7.8.4}$$

The open Weyl chamber $\mathfrak{h}_+ \subseteq \mathfrak{h}$ may then be defined by the condition that $w_o \in \mathfrak{h}_+$ if and only if $w_1 > w_2 > w_3 > 0$. One notes that the simple roots are given by

$$\langle \alpha_1, w_o \rangle = w_1 - w_2, \qquad \langle \alpha_2, w_o \rangle = w_2 - w_3, \qquad \langle \alpha_3, w_o \rangle = 2w_3. \tag{7.8.5}$$

The bilinear form Q on $\mathfrak{g}$ is also easily seen to be given by $Q(y, z) = \frac{1}{2} \operatorname{tr} yz$. The condition that $\mathfrak{k}$ is the set of all skew-symmetric elements in $\mathfrak{g}$ then normalizes the root vectors up to sign. The choice of sign for the simple root vectors will be made so that if $w_o \in h_+$, where $A(w_o)$ is given by (7.8.4), then

$$w = \begin{pmatrix} w_1 & 0 & 0 & 0 & 0 & 0 \\ 1 & w_2 & 0 & 0 & 0 & 0 \\ 0 & 1 & w_3 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & -w_3 & 0 & 0 \\ 0 & 0 & 0 & -1 & -w_2 & 0 \\ 0 & 0 & 0 & 0 & -1 & -w_1 \end{pmatrix}, \tag{7.8.6}$$

recalling that $w = f + w_o$ and f is given by (1.5.4).

Now the Lie group G^s of Section 5.1 may be taken to be the subgroup of $Gl(6, \mathbb{R})$ corresponding to $\mathfrak{g}$. The unipotent subgroups $\bar{N}$, and N of G^s are easily seen to be respectively groups of lower and upper triangular matrices and $H \subseteq G^s$ is a group of diagonal matrices. The element $\bar{n}_f(w) \in \bar{N}$ we recall is uniquely characterized by the conditions that $\bar{n}_f(w)\, w_o \bar{n}_f(w)^{-1} = w$. Using (5.8.7), as in the case of Proposition 7.7.3, one easily determines $\bar{n}_f(w)^{-1}$. In fact, explicitly

$$A(\bar{n}_f(w)^{-1}) = \begin{pmatrix} 1 & 0 & 0 \\ \dfrac{1}{w_2 - w_1} & 1 & 0 \\ \dfrac{1}{(w_1 - w_3)(w_2 - w_3)} & \dfrac{1}{w_3 - w_2} & 1 \end{pmatrix}, \tag{7.8.7}$$

$$D(\bar{n}_f(w)^{-1}) = \begin{pmatrix} 1 & 0 & 0 \\ \dfrac{1}{w_2 - w_3} & 1 & 0 \\ \dfrac{1}{(w_1 - w_2)(w_1 - w_3)} & \dfrac{1}{w_1 - w_2} & 1 \end{pmatrix}. \tag{7.8.8}$$

Of course $B(\bar{n}_f(w)^{-1}) = 0$. However, $(\sqrt{2}/2)\, C(\bar{n}_f(w)^{-1})$ equals

$$\begin{bmatrix} \dfrac{-1}{2w_3(w_2 + w_3) \times (w_1 + w_3)} & \dfrac{1}{2w_3(w_2 + w_3)} & \dfrac{-1}{2w_3} \\ \dfrac{1}{(w_2 - w_3)(w_2 + w_3) \times 2w_2(w_1 + w_2)} & \dfrac{-1}{(w_2 - w_3)(w_2 + w_3) \times 2w_2} & \dfrac{1}{(w_2 - w_3)(w_2 + w_3)} \\ \dfrac{-1}{(w_1 - w_2)(w_1 - w_3) \times (w_1 + w_3)(w_1 + w_2)\, 2w_1} & \dfrac{1}{(w_1 - w_2)(w_1 - w_3) \times (w_1 + w_3)(w_1 + w_2)} & \dfrac{-1}{(w_1 - w_2)(w_1 - w_3) \times (w_1 + w_3)} \end{bmatrix}.$$

Remark 7.8.2. One recognizes the product of roots occurring in the denominators. One knows that the most general root φ is given by $\langle \varphi, w_o \rangle = \pm w_i \pm w_j$, $1 \leqslant i \leqslant j \leqslant 3$.

The element $\bar{n}_{-f}(w) \in \bar{N}$ is given by (5.8.6). One has $A(\bar{n}_{-f}(w))_{ij} = (D(\bar{n}_f(w)^{-1})')_{ij}(-1)^{i+j}$. The same is true if A and D are interchanged. Furthermore

$$C(\bar{n}_{-f}(w)) = C(\bar{n}_f(w)^{-1})'. \tag{7.8.10}$$

One can now be very explicit about the element $d(w) \in H$ which we recall enters into the scattering of the mechanical system. An element $g_o \in H$ is determined by $A(g_o)$ since one easily has

$$D(g_o) = (A(g_o)^{-1})'. \tag{7.8.11}$$

PROPOSITION 7.8.1. *One has*

$$A(d(w)) = \begin{bmatrix} \dfrac{\sqrt{2}}{(w_1^2-w_2^2)(w_1^2-w_3^2)\,2w_1} & 0 & 0 \\ 0 & \dfrac{\sqrt{2}(w_1^2-w_2^2)}{(w_2^2-w_3^2)(w_1+w_2)^2\,2w_2} & 0 \\ 0 & 0 & \dfrac{\sqrt{2}(w_1^2-w_3^2)(w_2^2-w_3^2)}{(w_1+w_3)^2(w_2+w_3)^2\,2w_3} \end{bmatrix}.$$

Proof. One sees easily that the highest weight ν_i, $i = 1, 2, 3$, of the fundamental representation π_i is given by

$$\langle \nu_i, w_o \rangle = \sum_{j=1}^{i} w_j. \tag{7.8.13}$$

It follows then that if we extend the action of G^s on $\mathbb{R}^6$ to the exterior algebra $\Lambda\mathbb{R}^6$, such that G^s operates as a group of automorphisms then we can take V^{ν_i} to be the subspace of $\Lambda^i\mathbb{R}^6$ spanned by $G^s\epsilon_1 \wedge \cdots \wedge \epsilon_i$. We are using the notation of Section 7.7. Also, the inner product on V^{ν_i} is just the restriction to V^{ν_i} of the inner product on $\Lambda^i\mathbb{R}^l$ defined in Section 7.7. This is clear since $\mathfrak{k}$ is just the space of skew-symmetric elements in $\mathfrak{g}$. One then has

$$v^{\nu_i} = \epsilon_1 \wedge \cdots \wedge \epsilon_i \tag{7.8.14}$$

and $v^{\kappa\nu_i} = \pm\epsilon_{7-i} \wedge \cdots \wedge \epsilon_6$. But now by (5.5.4)

$$d(w)^{\nu_i} = \{\bar{n}_{-f}(w)\, v^{\nu_i}, v^{\kappa\nu_i}\}.$$

Thus $d(w)^{\nu_i}$ is given by the absolute value (since it is necessarily positive) of the determinant of the $i \times i$ minor in the lower left-hand corner of $C(\bar{n}_f(w)^{-1})$. See (7.8.9). This fixes

$$v^{\kappa\nu_i} = -\epsilon_{7-i} \wedge \cdots \wedge \epsilon_6 \tag{7.8.15}$$

and one has

$$\begin{aligned} d(w)^{\nu_1} &= \frac{\sqrt{2}}{(w_1^2 - w_2^2)(w_1^2 - w_3^2)\, 2w_1}, \\ d(w)^{\nu_2} &= \frac{1}{(w_1^2 - w_3^2)(w_2^2 - w_3^2)\, w_1 2w_2}, \\ d(w)^{\nu_3} &= \frac{\sqrt{2}}{(w_1 + w_2)^2(w_1 + w_3)^2(w_2 + w_3)^2\, 2w_2 2w_3}. \end{aligned} \tag{7.8.16}$$

But then by (7.8.14) the diagonal entries in $A(d(w))$ are just the relative quotients $d(w)^{\nu_1}$, $d(w)^{\nu_2-\nu_1}$, and $d(w)^{\nu_3-\nu_2}$. But then (7.8.12) follows from (7.8.16).
Q.E.D.

Now let $g_o \in H$ and write

$$A(g_o) = \operatorname{diag}(g_1, g_2, g_3). \tag{7.8.17}$$

Thus

$$A(g_o \exp tw_o) = \begin{pmatrix} g_1 e^{tw_1} & 0 & 0 \\ 0 & g_2 e^{tw_2} & 0 \\ 0 & 0 & g_3 e^{tw_3} \end{pmatrix}. \tag{7.8.18}$$

Now by (5.2.10), (7.8.14), and (7.8.15) the element $s_o(\kappa)^{-1} \in G$ is given by $A(s_o(\kappa)^{-1}) = 0 = D(s_o(\kappa)^{-1})$,

$$C(s_o(\kappa)^{-1}) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \tag{7.8.19}$$

and $B(s_o(\kappa)^{-1}) = -C(s_o(\kappa)^{-1})$. Put $a = s_o(\kappa)^{-1}\, \bar{n}_{-f}(w)(g_o \exp tw_o)\, \bar{n}_f(w)^{-1}$ so that recalling (5.3.6), a can be written

$$a = \bar{n}hn \tag{7.8.20}$$

for $\bar{n} \in \bar{N}$, $h \in H$, $n \in N$. In fact in the notation of Section 5.3, where g is the element in $G_0^w \subseteq \operatorname{Ad} \mathscr{g}_{\mathbb{C}}$ corresponding to g_o by (5.3.1), one has $\bar{n} = \bar{n}(g \exp tw)$, $h = h(g \exp tw)$, and $n = n(g \exp tw)$. We are particularly interested in

$h(g \exp tw)$ since, recalling Theorem 7.5 and (5.10.7), this determines the solution of the mechanical system.

Now let M_1 be the 3×3 matrix given by putting

$$\begin{aligned} M_1 = {} & C(\bar{n}_{-f}(w))\, A(g_o \exp tw_o)\, A(\bar{n}_f(w)^{-1}) \\ & + D(\bar{n}_{-f}(w))\, A((g_o \exp tw_o)^{-1})'\, C(\bar{n}_f(w))^{-1} \end{aligned} \tag{7.8.21}$$

and put

$$M(g_o, w_o; t) = FM_1, \tag{7.8.22}$$

where

$$F = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Remark 7.8.3. Note that the 3×3 matrix $M(g_o, w_o; t)$ has been determined here since all the components in (7.8.21) have been written down.

PROPOSITION 7.8.2. *For any $g_o \in H$, $w_o \in \mathfrak{h}_+$ and $t \in \mathbb{R}$, $h(g \exp tw)^{\nu_i}$ for $i = 1, 2, 3$ is equal to the determinant of the principal $i \times i$ minor of $M(g_o, w_o; t)$.*

Proof. By (7.8.20) one has

$$h(g \exp tw)^{\nu_i} = \{av^{\nu_i}, v^{\nu_i}\}. \tag{7.8.23}$$

But then the proposition follows from (7.8.14) and the definition of a. Q.E.D.

Now let x be a point in the phase space $\mathbb{R}^6$ and regard x as the initial point for the time development of the system whose Hamiltonian is given by (7.8.1). If we put $b_i = p_i(x)$, $i = 1, 2, 3$, and $a_j = \exp(q_j(x) - q_{j+1}(x))$, $j = 1, 2$, and $a_3 = \exp(2q_3(x))$ then the Jacobi element $y \in \mathfrak{g}$ corresponding to x as defined in Theorem 7.5 is clearly given by

$$y = \begin{pmatrix} b_1 & a_1 & 0 & 0 & 0 & 0 \\ 1 & b_2 & a_2 & 0 & 0 & 0 \\ 0 & 1 & b_3 & \sqrt{2}a_3 & 0 & 0 \\ 0 & 0 & \sqrt{2} & -b_3 & -a_2 & 0 \\ 0 & 0 & 0 & -1 & -b_2 & -a_2 \\ 0 & 0 & 0 & 0 & -1 & -b_1 \end{pmatrix}. \tag{7.8.24}$$

Now let $g_o \in H$ and $w_o \in \mathfrak{h}_+$ be the "parameters" for y as defined at the end of Section 3.7, recalling Theorem 3.7. Thus if w_o is given by (7.8.4) then the eigenvalues of y are $\pm w_i$, $i = 1, 2, 3$, where $w_1 > w_2 > w_3 > 0$. To determine g_o first let $n \in N$ be the unique element (see Proposition 2.3.2) such that

$nwn^{-1} = y$. In the notation of Theorem 3.7 one has $n = n(g)$. Thus w and y are explicitly given by (7.8.6) and (7.8.24), respectively, and n is the unique upper triangular unipotent 6×6 matrix such that

$$nw = yn. \tag{7.8.25}$$

Since n is unipotent one inductively solves for n_{ij}. In fact what is needed here is the 3×3 matrix $B(n)$. Then one finds

$$B(n)_{11} = \frac{a_1}{\sqrt{2}}(2b_1 + b_2 - w_1 - w_2 - w_3) + \frac{(b_1 - w_1)(b_1 - w_2)(b_1 - w_3)}{\sqrt{2}}$$

$$\begin{aligned} B(n)_{21} = \frac{1}{\sqrt{2}} &((b_2 - w_3)(b_1 - w_1) + (b_2 - w_3)(b_2 - w_2) \\ &+ (b_1 - w_1)(b_1 - w_2) + a_1 + a_2), \end{aligned}$$

$$B(n)_{31} = \frac{1}{\sqrt{2}}((b_1 - w_1) + (b_2 - w_2) + (b_3 - w_3)),$$

$$\begin{aligned} B(n)_{12} = -\frac{1}{\sqrt{2}} &(a_1[(b_1 + w_3)(2b_1 + b_2 - w_1 - w_2 - w_3) \\ &+ (b_2 - w_3)(b_1 + b_2 - w_1 - w_2) \\ &+ (b_1 - w_1)(b_1 - w_2) + a_1 + a_2] \\ &+ (b_1^2 - w_1^2)(b_1 - w_2)(b_1 - w_3)), \end{aligned}$$

$$\begin{aligned} B(n)_{22} = -\frac{1}{\sqrt{2}} &(a_1(2b_1 + 2b_2 - w_1 - w_2) + a_2(b_1 + 2b_2 + b_3 - w_1 - w_2) \\ &+ (b_1 - w_1)(b_1 - w_2)(b_1 - w_3) \\ &+ (b_2^2 - w_3^2)(b_1 + b_2 - w_1 - w_2) \\ &+ (b_2 + w_3)(b_1 - w_1)\, b_1 - w_2), \end{aligned}$$

$$\begin{aligned} B(n)_{32} = -\frac{1}{\sqrt{2}} &(a_1 + a_2 + 2a_3 + (b_2 - w_3)(b_1 + b_2 - w_1 - w_2) \\ &+ (b_1 - w_1)(b_1 - w_2) \\ &+ (b_3 + w_3)(b_1 + b_2 + b_3 - w_1 - w_2 - w_3)), \end{aligned}$$

$$\begin{aligned} (B(n))_{13} = -\frac{1}{\sqrt{2}} &(a_1a_2(w_1 - b_3) + 2a_1a_3(w_1 + b_2) + a_2^2(w_1 - b_1) \\ &- a_1(w_1^2 - b_3^2)(w_1 + b_2) - a_2(w_1 - b_1)(w_1 + b_2)(w_1 + b_3) \\ &- 2a_3((w_1 - b_1)(w_1^2 - b_2^2) \\ &+ (w_1 - b_1)(w_1^2 - b_2^2)(w_1^2 - b_3^2)), \end{aligned}$$

$$B(n)_{23} = -\frac{1}{\sqrt{2}}(-2a_1a_3 + a_1(w_1^2 - b_3^2) + a_2(w_1 - b_1)(w_1 + b_3) + 2a_3(w_1 - b_1)(w_1 - b_2) - (w_1 - b_1)(w_1 - b_2)(w_1^2 - b_3^2)),$$

$$B(n)_{33} = -\frac{1}{\sqrt{2}}(-a_1(w_1 - b_3) - a_2(w_2 - b_2) + (w_1 - b_1)(w_1 - b_2)(w_1 - b_3)).$$

As a consequence one has a formula for g_o in terms of y and w_o.

PROPOSITION 7.8.3. *Let $B_i(n)$, for $i = 1, 2, 3$, be the $i \times i$ minor of the 3×3 matrix $B(n)$. (The matrix $B(n)$ is explicitly written down above.) Then if g_i, $i = 1, 2, 3$, are the diagonal entries of $A(g_o)$ as in* (7.8.17) *one has*

$$\begin{aligned} g_1 &= a_1a_2a_3^{1/2}(-\det B_1(n)),\\ g_2 &= a_2a_3^{1/2}\det B_2(n)(\det B_1(n))^{-1},\\ g_3 &= a_3^{1/2}\det B_3(n)\det(B_2(n))^{-1}. \end{aligned} \tag{7.8.26}$$

Proof. Since $\mathfrak{g}$ is of type C_3 one knows that the Weyl group element κ is given by

$$\kappa = -1, \tag{7.8.27}$$

where 1 here is the identity element. Thus in the notation of Theorem 5.5 one has $g_o^{-\kappa} = g_o$. But then by (5.5.25)

$$\begin{aligned} (h(g)^{-1}g_o)^{\nu_i} &= \{v^{\nu_i}, n(g)\, v^{\kappa\nu_i}\}\\ &= -\det B_i(n) \end{aligned} \tag{7.8.28}$$

by (7.8.14) and (7.8.16) since $n(g) = n$. This implies that $B_i(n)$ is invertible. But now by (7.8.5) and (7.8.13) one has

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3/2 &= \nu_1,\\ \alpha_1 + 2\alpha_2 + \alpha_3 &= \nu_2,\\ \alpha_1 + 2\alpha_2 + 3\alpha_3/2 &= \nu_3. \end{aligned} \tag{7.8.29}$$

On the other hand by (3.7.4) one has $a_i = h(g)^{-\alpha_i}$, $i = 1, 2, 3$. Thus by (7.8.28) and (7.8.29)

$$\begin{aligned} g_o^{\nu_1} &= a_1a_2a_3^{1/2}(-\det B_1(n)),\\ g_o^{\nu_2} &= a_1a_2^2a_3(-\det B_2(n)),\\ g_o^{\nu_3} &= a_1a_2a_3^{3/2}(-\det B_3(n)). \end{aligned}$$

But $g_1 = g_o^{\nu_1}$, $g_2 = g_o^{\nu_2-\nu_1}$, and $g_3 = g_o^{\nu_3-\nu_2}$. This proves (7.8.26). Q.E.D.

We can now integrate Hamilton's equations for the Hamiltonian (7.8.1). The only terms entering into the solution below which we have not written down here are the eigenvalues $\pm w_i$, $i = 1, 2, 3$, for the matrix y. This of course reduces to solving a cubic equation.

THEOREM 7.8.1. *Let x be a point in the phase space $\mathbb{R}^6$. Let $q_i(x(t))$, $i = 1, 2, 3$, be the position of the ith particle at time t of the mechanical system whose Hamiltonian is*

$$H = \sum_{i=1}^{3} \frac{p_i^2}{2} + e^{q_1 - q_2} + e^{q_2 - q_3} + e^{2q_3}$$

and which at $t = 0$ occupies the state x. Let y be the 6×6 matrix given by (7.8.24), *where $b_i = p_i(x)$, $i = 1, 2, 3$, and $a_i = \exp(q_i(x) - q_{i+1}(x))$, $i = 1, 2$, and $a_3 = e^{2q_3(x)}$. Then there uniquely exists $w_1 > w_2 > w_3 > 0$ such that $\pm w_i$, $i = 1, 2, 3$, are the eigenvalues of y. Let w_o be the corresponding 6×6 diagonal matrix defined by* (7.8.3) *and* (7.8.4) *and let g_o be the 6×6 diagonal matrix given by* (7.8.11), (7.8.17), *and* (7.8.26). *Let $M(g_o, w_o; t)$ be the 3×3 matrix defined by* (7.8.21) *and* (7.8.22) *and for $i = 1, 2, 3$, let $M_i(g_o, w_o; t)$ be the principal $i \times i$ minor of $M(g_o, w_o; t)$. Then* $\det M_i(g_o, w_o; t) > 0$ *and one has*

$$\begin{aligned} q_1(x(t)) &= -\log \det M_1(g_o, w_o; -t), \\ q_2(x(t)) &= \log(\det M_1(g_o, w_o; -t)/\det M_2(g_o, w_o; -t)), \end{aligned}$$

and

$$q_3(x(t)) = \log(\det M_2(g_o, w_o; -t)/\det M_3(g_o, w_o; -t)).$$

Proof. The statement concerning the eigenvalues of y follows from (7.8.5) and (3.3.4). We use formula (7.5.18) for $q_i(x(t))$. Since clearly $\mathscr{P}_1 = \mathscr{P}$ in the present case one has $\bar{q}_i(x) = \bar{p}_i(x) = 0$. Also, all b_i (as defined in (7.5.13) and not as above) vanish by (7.5.14) since all $r_i = 1$). But now the matrix $A = (a_{ij})$ defined as in (7.1.10) is just

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}. \qquad (7.8.30)$$

On the other hand $Q(\alpha_i, \alpha_i) = 2$, $i = 1, 2$, and $Q(\alpha_3, \alpha_3) = 4$. Thus by (7.5.18)

$$\begin{aligned} q_1(x(t)) &= -\log \Phi_1(g_o, w_o; t), \\ q_2(x(t)) &= \log(\Phi_1(g_o, w_o; t)/\Phi_2(g_o, w_o; t)), \\ q_3(x(t)) &= \log(\Phi_2(g_o, w_o; t)/\Phi_3(g_o, w_o; t)). \end{aligned}$$

But now recalling (5.10.7) and the notation of Section 7.5, $\Phi_i(g_o, w_o; t) = h(g\exp(-t)w)^{\nu_i}$. But then the result follows from Proposition (7.8.2). Q.E.D.

The scattering of our mechanical system is explicitly given in

THEOREM 7.8.2. *For $i = 1, 2, 3$ let $v_i^+(x)t + u_i^+(x)$ and let $v_i^-(x)t + u_i^-(x)$ be the asymptotic lines defined by $(t, q_i(x(t))$ as $t \to +\infty$ and $t \to -\infty$, respectively. Then the asymptotic velocities are given by $v_i^+(x) = -w_i$ and $v_i^-(x) = w_i$. The corresponding phases are given by*

$$u_1^+(x) = \log g_1 + \log \frac{(w_1^2 - w_2^2)(w_1^2 - w_3^2)\, 2w_1}{\sqrt{2}},$$

$$u_2^+(x) = \log g_2 + \log \frac{(w_2^2 - w_3^2)(w_1 + w_2)\, 2w_2}{\sqrt{2}(w_1 - w_2)}, \tag{7.8.31}$$

$$u_3^+(x) = \log g_3 + \log \frac{(w_1 + w_3)(w_2 + w_3)\, 2w_3}{\sqrt{2}(w_1 - w_3)(w_2 - w_3)},$$

where g_1, g_2, and g_3 are given by (7.8.26) *and one has*

$$u_i^-(x) = u_i^+(x) - 2\log g_i \tag{7.8.32}$$

for $i = 1, 2, 3$.

Proof. We use Theorem 7.6 and formulas (7.6.1)–(7.6.3). By (7.8.30) and the relations which follow it one has

$$\mu_1^- = \nu_1, \qquad \mu_2^- = \nu_2 - \nu_1, \qquad \mu_3^- = \nu_3 - \nu_2 \tag{7.8.33}$$

and hence

$$\mu_1^+ = -\nu_1, \qquad \mu_2^+ = \nu_1 - \nu_2, \qquad \mu_3^+ = \nu_3 - \nu_2. \tag{7.8.34}$$

But then $\langle \mu_i^-, w_o \rangle = w_i$ and $\langle \mu_i^+, w_o \rangle = -w_i$. The relation $v_i^+(x) = -w_i$ and $v_i^-(x) = w_i$ then follow from (7.6.2) and (7.6.3) since as noted in the proof of Theorem 7.8.1 one has $\bar{p}_i(x) = 0$. But also, as noted there, $\bar{q}_i(x) = b_i = 0$. Thus

$$u_i^+(x) = \log \left(\frac{d(w)}{g_o}\right)^{\mu_i^+} \tag{7.8.35}$$

and

$$u_i^-(x) = \log \left(\frac{1}{g_o\, d(w)}\right)^{\mu_i^-}. \tag{7.8.36}$$

But then if $d(w)_i$ are the diagonal entries of $A(d(w))$, $i = 1, 2, 3$, as defined by (7.8.16) one has, by (7.8.5), (7.8.33), and (7.8.34)

$$u_i^+(x) = \log\left(\frac{g_i}{d(w)_i}\right) \tag{7.8.37}$$

and

$$u_i^-(x) = \log\left(\frac{1}{g_i\, d(w)_i}\right). \tag{7.8.38}$$

But then (7.8.31) follows from Proposition 7.8.1 and (7.8.37). Relation (7.8.32) is immediate from (7.8.37) and (7.8.38). Q.E.D.

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References

1. M. Adler, Completely integrable systems and symplectic actions, preprint.
2. L. Auslander and B. Kostant, Polarization and unitary representations of solvable Lie groups, *Invent. Math.* **14** (1971), 255–354.
3. O. I. Bogoyavlensky, On perturbations of the periodic Toda lattice, *Comm. Math. Phys.* **51** (1976), 201–209.
4. J. Dixmier, "Algèbres enveloppantes," Cahiers scientifiques no. 27, Gauthier–Villars, Paris, 1974.
5. E. B. Dynkin, The structure of semi-simple Lie algebras, *Amer. Math. Soc. Transl. Ser.* 1, No. 17 (1950).
6. E. B. Dynkin, Semi-simple subalgebras of semi-simple Lie algebras, *Amer. Math. Soc. Transl. Ser.* 2 **6** (1957), 111–244.
7. E. B. Dynkin, The maximal subgroups of the classical groups, *Amer. Math. Soc. Transl. Ser.* 2 **6** (1957), 245–378.
8. H. Flaschka, On the Toda lattice, I, *Phys. Rev. B* **9** (1974), 1924–1925; II, *Progr. Theor. Phys.* **51** (1974), 703–716.
9. I. M. Gelfand and D. A. Kazhdan, Representations of the group $Gl(n, K)$ where K is a local field, *in* "Lie Groups and Their Representations," pp. 95–118, Wiley, New York, 1975.
10. R. Hermann, "Toda Lattices, Cosymplectic Manifolds, Backlund Transformations, Kinks"; Part A, 15, Part B, 18, "Interdisciplinary Mathematics," Math. Sci. Press.
11. J. Humphreys, "Introduction to Lie Algebras and Representation Theory," Graduate Texts in Mathematics No. 9, Springer-Verlag, New York/Berlin, 1972.

12. B. KOSTANT, The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group, *Amer. J. Math.* **81** (1959), 973–1032.
13. B. KOSTANT, Lie algebra cohomology and the generalized Borel–Weil theorem, *Ann. of Math.* **74** (1961), 329–387.
14. B. KOSTANT, Lie group representations on polynomial rings, *Amer. J. Math.* **85** (1963), 327–404.
15. B. KOSTANT, Quantization and unitary representations, *in* "Lecture Notes in Mathematics No. 170," pp. 87–208, Springer-Verlag, New York/Berlin, 1970.
16. B. KOSTANT, On convexity, the Weyl group and the Iwasawa decomposition, *Ann. Sci. École Normale Supérieure* **6** (1973), 413–455.
17. B. KOSTANT, On Whittaker vectors and representation theory, *Invent. Math.* **48** (1978), 101–184.
18. B. KOSTANT AND S. RALLIS, Orbits and representations associated with symmetric spaces, *Amer. J. Math.* **93** (1971), 753–809.
19. J. MOSER, Finitely many mass points on the line under the influence of an exponential potential—an integrable system, *in* "Battelle Rencontres Summer Lectures 1974," pp. 467–487, Lecture Notes in Mathematics, Springer-Verlag, New York/Berlin.
20. M. A. OLSHANETZKI AND A. M. PERELOMOV, Completely integrable Hamiltonian systems connected with semi-simple Lie algebras, *Invent. Math.* **37** (1976), 93–108.
21. L. PUKANSZKY, On the theory of exponential groups, *Trans. Amer. Math. Soc.* **126** (1967), 487–507.
22. I. R. SHAFAREVICH, "Basic Algebraic Geometry," Springer-Verlag, New York/Berlin, 1977.
23. M. SUGUIRA, Conjugate classes of Cartan subalgebras in real semi-simple Lie algebras, *J. Math. Soc. Japan* **11** (1959), 374–434.
24. M. TODA, Studies of a non-linear lattice, *Phys. Rep. C, Phys. Lett.* **8** (1975), 1–125.
25. P. VAN MOERBEKE, The spectrum of Jacobi matrices, *Invent. Math.* **37** (1976), 45–81.
26. V. S. VARADARAJAN, "Lie Groups, Lie Algebras and Their Representations," Prentice–Hall, Englewood Cliffs, N.J., 1974.
27. M. ADLER, On a trace functional for formal pseudodifferential operators and the symplectic structure for the Korteweg–de Vries type equations, *Invent. Math.* **50** (1979), 219–248.
28. M. A. OLSHANETZKI AND M. A. PERELOMOV, Explicit solution of the classical generalized Toda models, preprint, 1978.
29. A. N. LEZNOV AND M. V. SAVELIEV, Representation of zero curvature for the system of nonlinear partial differential equations $x_{a,z\bar{z}} = \exp(kx)_a$ and its integrability, preprint, 1979, Serpurchev, USSR.