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PSEUDO-OPEN FUNCTIONS ARE MONOTONIC DECREASING FOR ORDINAL AND CARDINAL INVARIANTS

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A new property of pseudo-open functions is presented in this paper. Pseudo-open functions are monotonic decreasing with respect to ordinal and cardinal invariants defined by compact and sequential closures. A weak form of continuity, called k-continuous, is defined, characterized and used in the proof of the monotonicity properties of pseudo-open mappings. The relationships between the classes of k-continuous and sequentially continuous mappings in the category of all topological spaces and the continuous mappings in the subcategories of k-spaces and sequential spaces are presented

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1. Introduction

The main results of this study provide new insight about the nature of pseudo-open functions. In particular, it is shown that the pseudo-open functions are ordinal monotonic decreasing with respect to the ordinal invariants κ (compact order) and σ (sequential order) introduced by Arhangelskii and Franklin [2] and cardinal monotonic decreasing for cardinal invariants which are uniquely related to κ and σ.

Since these invariants are defined in the subcategories of k-spaces and sequential spaces, this study also contains a presentation of a weak form of continuity, called k-continuous, which is the compact analogy of sequential continuity. Characterizations of these weaker forms of continuity are used in the proof of the monotonic properties.

The definition and characterizations of k-continuous mappings are given in Section 3. The monotonicity properties are established in Section 4, and summarized in Conclusion 4.4. The relationships between the properties of these functions in the category of all topological spaces and in the subcategories of k-spaces and sequential spaces are presented in Section 5. Examples are presented in Section 6. All spaces are assumed to be Hausdorff and the positive integers are denoted by N.

2. Definitions and preliminaries

The ordinal invariants for topological spaces which are considered here are the compact order and sequential order of a space. These notions were introduced by Arhangelskii and Franklin [2] and their definitions, with some notational changes, are repeated here for completeness. We will denote by $\mathrm{kcl}(A)$ ($\mathrm{scl}(A)$) the set of all points p such that there exists a compact set K such that $p \in \mathrm{cl}_K(K \cap A)$ (there exists a sequence $\{p_i\}$ in A such that $p_i \to p$). As to whether A^α denotes the αth k-closure or sequential closure of A will be clear from the context.

Definitions 2.1 [2] Let A be a subset of a topological space X. Let $A^0 = A$.

(a,) For each non-limit ordinal $\alpha = \beta + 1$, let $A^\alpha = \mathrm{kcl}(A^\beta)$. For each limit ordinal α let $A^\alpha = \bigcup\{A^\beta : \beta < \alpha\}$. The *compact order* of X is defined as $\kappa(X) = \inf\{\alpha : A^\alpha = \mathrm{cl}_X(A)$, for all $A \subset X\}$, if this inf exists.

(b,) For each non-limit ordinal $a = \beta + 1$, let $A^\alpha = \mathrm{scl}(A^\beta)$. For each limit ordinal α, let $A^\alpha = \bigcup\{A^\beta : \beta < a\}$. The *sequential order* of X is defined as $\sigma(X) = \inf\{\alpha : A^\alpha = \mathrm{cl}_X(A)$, for all $A \subset X\}$, if this inf exists.

The following theorems indicate the intrinsic relationship between these ordinal invariants and the subcategories of k-spaces and sequential spaces. Part (a,) of the following is amended to include a recent result from [4]. The tightness of X is denoted $t(X)$.

Theorems 2.2 [2]. (a,) *X is a k-space if and only if $\kappa(X)$ exists. In this case $\kappa(X) \leq \alpha$ where α is the least ordinal of cardinality $t(X)^+$.*

(b) *X is a sequential space if and only if $\sigma(X)$ exists. In this case $\sigma(X) \leq \omega_1$.*

Recently cardinal invariants which are related to κ and σ in a natural way have been introduced and studied in [4].

Definitions 2.3 [4]. Let X be a topological space. The *pointwise k-cardinal* of X is defined as $PK(X) = \sup\{PK(A) : A \subset X\}$, where $PK(A)$ is the least cardinal m such that if $q \in \mathrm{cl}(A)$ and α is the least ordinal such that $q \in A^\alpha$, then $\mathrm{card}(\alpha) \leq m$. The *$k$-cardinal* of X is defined as $K(X) = \sup\{K(A) : A \subset X\}$, where $K(A)$ is the cardinality of the least ordinal α such that $\mathrm{cl}(A) = A^\alpha$. $PS(X)$, the *pointwise sequential cardinal*, and $S(X)$, the *sequential cardinal*, and defined similarly. These cardinal invariants are defined only when the sup exists.

Some of the results in [4] relating to this study are the following.

Lemma 2.3 [4]. *If X is a topological space and the following cardinals exist, then*
(a) $PK(X) \leq K(X) \leq PK(X)^+$, *and*
(b) $PS(X) \leq S(X) \leq PS(X)^+$.

Theorem 2.4 [4]. (a) *X is k-space if and only if PK(X) exists. In this case* $PK(X) \leq t(X)$.
(b) *X is a k-space if and only if K(X) exists. In this case* $K(X) \leq t(X)^+$.
(c) *X is a sequential space if and only if PS(X) exists. In this case,* $PS(X) \leq \aleph_0$.
(d) *X is a sequential space if and only if S(X) exists. In this case,* $S(X) \leq \aleph_1$.

The following definition will allow the results of the theorems in Section 4 to be stated in terms of monotonicity of a class of mappings.

Definitions 2.4. Let $\mathscr{C}$ be a category of spaces and let $\mathscr{F}$ be a class of functions defined on the spaces in $\mathscr{C}$ such that if $f \in \mathscr{F}$ and f is defined on $X \in \mathscr{C}$, then $f(X) \in \mathscr{C}$.

(a) If η is an ordinal invariant defined on $\mathscr{C}$ and for each $f \in \mathscr{F}$ and for each $X \in \mathscr{C}$ such that f is defined on X, $\eta(X) \geq \eta(f(X))$, then the class $\mathscr{F}$ is said to be *ordinal monotonic decreasing* (or *η-monotonic*) on $\mathscr{C}$.

(b) If M is a cardinal invariant defined on $\mathscr{C}$ and for each $f \in \mathscr{F}$ and for each $X \in \mathscr{C}$ such that f is defined on X, $M(X) \geq M(f(X))$, then the class $\mathscr{F}$ is said to be *cardinal monotonic decreasing* (or *M-monotonic*) on $\mathscr{C}$.

3. k-continuous mappings

It is the purpose of this section to properly define and characterize a form of continuity which is uniquely suited to the subcategory of k-spaces. A characterization of k-continuous mappings is used in the next section on monotonicity of pseudo-open functions. Two additional characterizations of sequential continuity follow from the results of this section.

The sequentially continuous functions are those for which the image of a convergent sequence is a convergent sequence. This viewpoint of sequential continuity does not indicate a nearness preserving analogy for k-continuity. In particular, "the image of a compact set is compact" is satisfied by $f:[0,1] \to R$, defined by $f(x) = 0$ for rational x and $f(x) = 1$ for irrational x. Another equivalent way of viewing sequential continuity is: $f: X \to Y$ is sequentially continuous if and only if $f_S: S \to Y$ is continuous, for each convergent sequence $S = \{p\} \cup \{p_i : i \in N\}$, where $p_i \to p$. Accordingly the following definition is given.

Definition 3.1. A mapping $f: X \to Y$ is said to be *k-continuous*, if $f_K: K \to Y$ is continuous for each compact set $K \subset X$.

From the definition, the k-continuous image of a compact set is compact. A useful characterization of k-continuity is: $f: X \to Y$ is k-continuous if and only if for each compact set $K \subset X$ and for each net $\{x_\alpha\}$ in K, if x is a cluster point of $\{x_\alpha\}$, then $f(x)$ is a cluster point of $\{f(x_\alpha)\}$ in $f(K)$. A set $A \subset X$ is *k-closed* (*sequentially closed*) provided: $K \cap A$ is closed in K for each compact set $K \subset X$ (if $\{p_i\}$ is a sequence in A where $p_i \to p$, then $p \in A$). Recall in the following propositions, that the k-closure (sequential closure) of a set is not necessarily k-closed (sequentially closed).

Proposition 3.2. *Let $f: X \to Y$. The following are equivalent.*

(a) *f is k-continuous.*

(b) *$f^{-1}(H)$ is k-closed in X, for each k-closed set $H \subset Y$.*

(c) *$f^{-1}(G)$ is k-open in X, for each k-open set $G \subset Y$.*

(d) $f(\mathrm{kcl}(A)) \subset \mathrm{kcl}(f(A))$, *for each $A \subset X$.*

(e) $\mathrm{kcl}(f^{-1}(B)) \subset f^{-1}(\mathrm{kcl}(B))$, *for each $B \subset Y$.*

In view of the characterizations, (d) and (e), of this proposition, two additional characterizations of sequential continuity, (e) and (f), should be added to the list in Theorem 3.1 of [5], as follows:

Proposition 3.3. *The following are equivalent for a mapping $f: X \to Y$.*

(a) *f is sequentially continuous.*

(b) *$f^{-1}(U)$ is sequentially open, for each sequentially open set $U \subset Y$.*

(c) *$f^{-1}(H)$ is sequentially closed, for each sequentially closed set $H \subset Y$.*

(d) *If H is countable and sequentially closed in Y, then $f^{-1}(H)$ is sequentially closed.*

(e) $f(\mathrm{scl}(A)) \subset \mathrm{scl}(fA))$, *for each $A \subset X$.*

(f) $\mathrm{scl}(f^{-1}(B)) \subset f^{-1}(\mathrm{scl}(B))$, *for each $B \subset Y$.*

The following theorem and corollary constitute the extension of these characterizations to arbitrary ordinals.

Theorem 3.4. *A function $f: X \to Y$ is a k-continuous (sequentially continuous) if and only if $(f^{-1}(A))^\alpha \subset f^{-1}(A^\alpha)$, for each ordinal α and for each $A \subset Y$.*

Proof. (The proof of the k-continuous case is given.) Let $A \subset Y$, and let α be an ordinal. Then $(f^{-1}(A))^0 = f^{-1}(A^0)$ by definition and $(f^{-1}(A))^1 \subset f^{-1}(A^1)$ by Proposition 3.2(e). Suppose for each $\beta < \alpha$, $(f^{-1}(A))^\beta \subset f^{-1}(A^\beta)$. Let α be a non-limit ordinal, say $\alpha = \beta + 1$. Then $(f^{-1}(A))^\beta \subset f^{-1}(A^\beta)$. By Proposition 3.2(e),

$$(f^{-1}(A))^\alpha = \mathrm{kcl}((f^{-1}(A))^\beta \subset \mathrm{kcl}(f^{-1}(A^\beta)) \subset f^{-1}(\mathrm{kcl}(A^\beta)) = f^{-1}(A^\alpha).$$

Thus, in this case, $(f^{-1}(A))^\alpha \subset f^{-1}(A^\alpha)$. Now suppose α is a limit ordinal. Since

$(f^{-1}(A))^\beta \subset f^{-1}(A^\beta)$, for each $\beta < \alpha$,

$$(f^{-1}(A))^\alpha = \bigcup\{(f^{-1}(A))^\beta : \beta < \alpha\} \subset \bigcup\{f^{-1}(A^\beta) : \beta < \alpha\}$$
$$= f^{-1}(\bigcup\{A^\beta : \beta < \alpha\} = f^{-1}(A^\alpha).$$

Hence, in this case also, $(f^{-1}(A))^\alpha \subset f^{-1}(A^\alpha)$. This completes the proof.

Corollary 3.5. *A function $f: X \to Y$ is a k-continuous (sequentially continuous) if and only if $f(A^\eta) \subset (f(A))^\eta$, for each ordinal η and for each $A \subset X$.*

Proof. From Theorem 3.4, $(f^{-1}(f(A)))^\eta \subset f^{-1}((f(A))^\eta)$. Then, $A^\eta \subset f^{-1}((f(A))^\eta)$ and $f(A^\eta) \subset (f(A))^\eta$. This completes the proof.

This section is concluded by a final comment on the relationship between these two weak forms of continuity. Examples of 4.3 and 4.4 in [3] show that k-quotient and sequentially quotient are independent notions. In Example 6.3 it is shown that sequential continuity does not imply k-continuity. However we have the following:

Theorem 3.6. *Every k-continuous mapping is sequentially continuous.*

Proof. Let $f: X \to Y$ be a k-continuous mapping, and let $\{p_i\}$ be a convergent sequence in X, say $p_i \to p$. Let $K = \{p\} \cup \{p_i : i \in N\}$. Then $f_K : K \to f(K)$ is k-continuous. By Theorem 5.2, f_K is continuous. Hence, since $p_i \to p$ in K, $f(p_i) \to f(p)$ in $f(K)$. Accordingly, f is sequentially continuous. This completes the proof.

4. Pseudo-open mappings are monotonic decreasing

The notion of a pseudo-open mapping was introduced by Arhangelskii [1]. He defines a mapping $f: X \to Y$ to be *pseudo-open* if for each $y \in Y$ and for each open set $U \subset X$ such that $f^{-1}(y) \subset U$, then $y \in \operatorname{Int}(f(U))$. The equivalent statement, $f: X \to Y$ is pseudo-open if and only if $p \in \operatorname{cl}_Y(B)$ implies $f^{-1}(p) \cap \operatorname{cl}_X(f^{-1}(B)) \neq \emptyset$, is used in the proof of the following theorem.

Theorem 4.1. *If f is a pseudo-open k-continuous mapping from a k-space X onto a space Y, then $\kappa(X) \geq \kappa(Y)$.*

Proof. A pseudo-open k-continuous surjection defined on a k-space is a quotient mapping. Thus, Y is a k-space and $\kappa(Y)$ exists. Let $\kappa(X) = \eta$, and let $B \subset Y$. Consider any point $p \in \operatorname{cl}_X(f^{-1}(B^\eta))$. Then

$$f(p) \in f(\operatorname{cl}_X(f^{-1}(B^\eta)) \subset \operatorname{cl}_Y(B^\eta) = \operatorname{cl}_Y(B).$$

Since $f(p) \in \operatorname{cl}_Y(B)$ and f is pseudo-open, $f^{-1}(f(p)) \cap \operatorname{cl}_X(f^{-1}(B)) \neq \emptyset$. Since

$$(f^{-1}(B))^\eta = \operatorname{cl}_X(f^{-1}(B)) \quad \text{and} \quad (f^{-1}(B))^\eta \subset f^{-1}(B^\eta).$$

$f^{-1}(f(p)) \cap f^{-1}(B^{\eta}) \neq \emptyset$. Hence, $f(p) \cap B^{\eta} \neq \emptyset$ and $p \in f^{-1}(B^{\eta})$. Thus $\mathrm{cl}_X(f^{-1}(B^{\eta})) \subset f^{-1}(B^{\eta})$ or $f^{-1}(B^{\eta})$ is closed. Since f is a quotient mapping, B^{η} is closed in Y. Thus, for each $B \subset Y$, B^{η} is closed. Accordingly $\kappa(Y) = \inf\{\alpha : B^{\alpha} = \mathrm{cl}_Y(B), B \subset Y\} \leq \eta = \kappa(X)$. This completes the proof.

Theorem 4.2. *If f is a pseudo-open sequentially continuous mapping from a sequential space X onto a space Y, then $\sigma(X) \geq \sigma(Y)$.*

Theorems 4.1 and 4.2 cannot be improved by replacing pseudo-open with quotient, as is shown in Example 6.4. Also, in Example 6.5 it is shown that monotonicity of a quotient mapping does not imply the mapping is pseudo-open.

Let KTOP and STOP be the subcategories of k-spaces and sequential spaces respectively. From the preceding theorems, and the definitions of the cardinal invariants PK, K, PS and S, the following specific conclusions can be stated.

Conclusions 4.3. (a) *The class of pseudo-open k-continuous functions in* KTOP *are κ-monotonic, PK and K-monotonic.*

(b) *The class of pseudo-open sequentially continuous functions in* STOP *are σ-monotonic, PS and S-monotonic.*

The conclusions in 4.3 are specific in the sense that they are made in the restricted subcategories KTOP and STOP. These conclusions can easily be extended to the category of all topological spaces, TOP, by letting kX and sX be the k-extension and sequential extension of a space X in TOP. The k-extension of a space X is the set X retopologized by letting each k-open set be an open set in the larger topology. Define sX in the same manner. Now extend the domain of definition of κ and σ, as Arhangelskii and Franklin [2] indicate, as follows: for each X in TOP let $\kappa(X) = \kappa(kX)$ and $\sigma(X) = \sigma(sX)$. By this method the following more general conclusion may be stated.

Conclusions 4.4. (a) *The class of pseudo-open k-continuous functions in* TOP *are κ-monotonic, PK and K-monotonic.*

(b) *The class of pseudo-open sequentially continuous functions in* TOP *are σ-monotonic, PS and S-monotonic.*

5. Properties of classes of mappings

In this section a functional characterization of k-spaces is presented. From this and other known characterizations the relationships between the weaker forms of continuous (and quotient) mappings in TOP and the continuity (and quotient) properties of these mappings in KTOP and STOP can be observed. The intrinsic connection between sequential spaces and sequentially continuous mappings has been shown in [6].

Theorem 5.1 [6]. *A space is sequential if and only if each sequentially continuous mapping on X is continuous.*

Thus, the following characterization of k-spaces comes as no surprise. It can be proven easily by supposing X is not a k-space. Then the identity mapping $e : X \to kX$ is k-continuous but not continuous.

Theorem 5.2. *A space X is a k-space if and only if each k-continuous mapping on X is continuous.*

Corollaries 5.3. (a) $f : X \to Y$ *is k-continuous if and only if* $f : kX \to kY$ *is continuous.*
(b) $f : X \to Y$ *is sequentially continuous if and only if* $f : sX \to sY$ *is continuous.*

That is, the class of functions in TOP which are k-continuous (sequentially continuous) is precisely the class of functions which are continuous in KTOP (STOP).

In regard to the behavior of quotient mappings in the subcategories KTOP and STOP, we refer to the classes of k-quotient [3] and sequentially quotient [5] mappings. A mapping $f : X \to Y$ is *k-quotient* (*sequentially quotient*) provided: H is k-closed (sequentially closed) in Y if and only if $f^{-1}(H)$ is k-closed (sequentially closed) in X. Clearly, every k-quotient (sequentially quotient) mapping is k-continuous (sequentially continuous). From Theorems 3.1 and 3.2 of [3] and Theorems 5.2 and 5.3 in [5], the following statements are valid.

Theorem 5.4. *If X is k-space (sequential space), then every quotient mapping defined on X is k-quotient (sequentially quotient).*

Theorem 5.5. *Y is a k-space (sequential space) if and only if every k-quotient (sequentially quotient) continuous mapping onto Y is quotient.*

Corollaries 5.6. (a) $f : X \to Y$ *is k-quotient if and only if* $f : kX \to kY$ *is quotient.*
(b) $f : X \to Y$ *is sequentially quotient if and only if* $f : sX \to sY$ *is quotient.*

That is, the class of functions in TOP which are k-quotient (sequentially quotient) is precisely the class of functions which are in quotients in KTOP (STOP).

6. Examples

Example 6.1. *The continuous one to one compact k-quotient sequentially quotient image of a sequential space is not necessarily a k-space.*

Consider Arens' space, $A=\{(0,0)\}\cup\{(1/n,1/m): n, m\in N\}$ topologized as follows: each $\{(1/n,1/m)\}$ is open and a basic open n hood of $(0,0)$ is any set of the form

$$\{(0,0)\}\cup\{(1/n,1/m): n\geq n_0, m\geq m_n\}.$$

The set $\{(0,0)\}$ is k-open, but not open. Thus, A is not a k-space. Let A^* be the set A with the discrete topology. Let $e: A^*\to A$ be the identity mapping. The compact sets in both A and A^* are finite. Clearly e is continuous one to one compact and both k-quotient and sequentially quotient, but not quotient as $e^{-1}(\{(0,0)\})$ is open but $\{(0,0)\}$ is not open in A. Note that $sA = kA = A^*$.

Example 6.2. *There is a continuous open countable to one mapping f from a sequential space onto a convergent sequence such that* $\text{kcl}(f^{-1}(A))\neq f^{-1}(\text{kcl}(A))$ *and* $\text{scl}(f^{-1}(A))\neq f^{-1}(\text{scl}(A))$.

Consider the space

$$S_2=\{(0,0)\}\cup\{(1/n,0): n\in N\}\cup\{(1/n,1/m): n, m\in N\}$$

in [2] and [4], and let f be the horizontal projection onto $X=\{(-1,0)\}\cup\{(-1,1/m): m\in N\}$ with the usual relative topology. Then

$$(0,0)\in f^{-1}(\text{kcl}(\{(-1,1/m): m\in N\})),$$

but

$$(0,0)\notin \text{kcl}(f^{-1}(\{(-1,1/m): m\in N\})).$$

Also,

$$\kappa(S_2)=\sigma(S_2)=2>1=\kappa(f(S_2))=\sigma(f(S_2))$$

and

$$K(S_2)=S(S_2)=2>1=K(f(S_2))=S(f(S_2)).$$

Example 6.3. *There is a sequentially continuous one to one open perfect (not continuous) mapping defined on a compact space, which is not k-continuous.*

Let Y be the ordinal space $[0,\Omega]$, where Ω is the first uncountable ordinal. The sequential extension of Y, sY, has the same open sets with the singular exception that $\{\Omega\}$ is open in sY. The identity mapping $e: Y\to sY$ has all the properties mentioned above. But e is not k-continuous, because $\{\Omega\}$ is k-open in sY and $\{\Omega\}=e^{-1}(\{\Omega\})$ is sequentially open in Y but not k-open in Y.

Example 6.4. *There is a quotient mapping defined on a countable locally compact metric space which is not monotonic. The mapping is also compact covering* [7] *and sequence covering* [8].

Consider S_2 as the quotient space of the disjoint topological union, X, of the spaces $\{0\} \cup \{1/j : j \in N\}$. Then the quotient mapping has the properties mentioned and $\kappa(X) = \sigma(X) = 1 < 2 = \kappa(S_2) = \sigma(S_2)$.

It should be noted that since compact covering and sequence covering mappings are not necessarily monotonic, these compact and sequential analogs of pseudo-open mappings do not suffice to insure monotonicity. They are analogous to pseudo-open mappings in the sense that they distinguish points in the closure of a set versus distinguishing non-closed sets as the quotient mappings do.

Example 6.5. *Monotonicity of a quotient mapping from a locally compact countably compact space onto a compact space does not imply that the mapping is pseudo-open.*

Let $Y = [0, \Omega]$ be the ordinal space of Example 6.3. Let $X_1 = Y - \{\Omega\}$ and let X_2 be the subspace of Y consisting of all limit ordinals. Let X be the disjoint topological union of X_1 and X_2. The mapping $f : X \to Y$ defined by $f(\alpha) = \alpha$ is a quotient mapping which is not pseudo-open and f is monotonic because $\kappa(X) = 1 = \kappa(Y)$.

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