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Eigenvalue asymptotics for Sturm–Liouville operators with singular potentials ☆

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Abstract

We derive eigenvalue asymptotics for Sturm–Liouville operators with singular complex-valued potentials from the space $W_2^{\alpha-1}(0,1)$, $\alpha \in [0,1]$, and Dirichlet or Neumann–Dirichlet boundary conditions. We also give application of the obtained results to the inverse spectral problem of recovering the potential from these two spectra.

Keywords: Eigenvalue asymptotics; Sturm–Liouville operators; Singular potentials

1. Introduction

In this paper we shall study eigenvalue asymptotics for Sturm–Liouville operators on the interval $[0,1]$ with distributional potentials. Namely, we assume that q is a complex-valued distribution from the Sobolev space $W_2^{\alpha-1}(0,1)$, $\alpha \in [0,1]$, and consider an operator T that (formally) corresponds to the differential expression

$$l(f) := -f'' + qf \tag{1.1}$$

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and, say, Dirichlet boundary conditions. Explicitly, the operator T is defined by the regularisation method that was suggested in [4] for the particular potential $q(x) = 1/x$ and was developed by Savchuk and Shkalikov in [30] for the class of distributional potentials from $W_2^{-1}(0,1)$. (Incidentally, in this situation the form-sum [1,10] and the generalised sum [20] methods yield the same operator.) We observe that the considered class of singular potentials include Dirac δ-type and Coulomb $1/x$-type interactions that are widely used in quantum mechanics and mathematical physics; see also [18] for other physical models leading to potentials from negative Sobolev spaces.

The regularisation method consists in rewriting (1.1) as

$$l(f) = l_\sigma(f) := -(f' - \sigma f)' - \sigma f', \tag{1.2}$$

where σ is any distributional primitive of q. We fix one such primitive $\sigma \in L_2(0,1)$ in what follows and call the expression $f' - \sigma f =: f^{[1]}$ the *quasi-derivative* of the function f. The natural L_2-domain of l_σ is

$$\operatorname{dom} l_\sigma = \bigl\{ f \in W_1^1(0,1) \mid f^{[1]} \in W_1^1(0,1),\ l_\sigma(f) \in L_2(0,1) \bigr\},$$

and we observe that for $f \in \operatorname{dom} l_\sigma$ the derivative $f' = \sigma f + f^{[1]}$ belongs to $L_2(0,1)$ (but need not be continuous), so that $\operatorname{dom} l_\sigma \subset W_2^1(0,1)$.

In the present paper, we shall only focus on Sturm–Liouville operators $T_{\mathrm{D}} = T_{\sigma,\mathrm{D}}$ and $T_{\mathrm{N}} = T_{\sigma,\mathrm{N}}$ that are generated by l_σ and the Dirichlet and the Neumann–Dirichlet boundary conditions, respectively, although other boundary conditions can also be treated in a similar manner (see, e.g., [18] for periodic and [31] for general regular boundary conditions). In other words, T_{D} and T_{N} are the restrictions of l_σ onto the domains

$$\begin{aligned} \operatorname{dom} T_{\mathrm{D}} &= \bigl\{ f \in \operatorname{dom} l_\sigma \mid f(0) = f(1) = 0 \bigr\}, \\ \operatorname{dom} T_{\mathrm{N}} &= \bigl\{ f \in \operatorname{dom} l_\sigma \mid f^{[1]}(0) = f(1) = 0 \bigr\}. \end{aligned} \tag{1.3}$$

It is known [30] that the operators T_{D} and T_{N} are closed, densely defined and have discrete spectra tending to $+\infty$. We denote by λ_n^2 (respectively, μ_n^2) the eigenvalues of T_{D} (respectively, T_{N}) counted with multiplicities and arranged by increasing of the real—and then, if equal, imaginary—parts of λ_n (respectively, μ_n). For definiteness, we shall always take λ_n and μ_n from the set

$$\Omega := \{ z \in \mathbb{C} \mid -\pi/2 < \arg z \leqslant \pi/2 \} \cup \{ z = 0 \}. \tag{1.4}$$

If $\alpha = 0$, i.e., if $\sigma \in L_2$, then the numbers λ_n and μ_n obey the asymptotics [15,29–31]

$$\lambda_n = \pi n + \tilde{\lambda}_n, \qquad \mu_n = \pi\left(n - \frac{1}{2}\right) + \tilde{\mu}_n, \tag{1.5}$$

where $(\tilde{\lambda}_n)_{n\in\mathbb{N}}$ and $(\tilde{\mu}_n)_{n\in\mathbb{N}}$ are some ℓ_2-sequences. It is reasonable to expect that if σ becomes smoother, then the remainders $\tilde{\lambda}_n$ and $\tilde{\mu}_n$ decay faster; for instance, if $\alpha = 1$, i.e., if $q \in L_2(0,1)$, then the classical result (see, e.g., [26, Theorem 3.4.1] or [28, Theorem 2.4]) states that $\tilde{\lambda}_n, \tilde{\mu}_n = \mathrm{O}(n^{-1})$. Thus the problem arises to characterise the decay of $\tilde{\lambda}_n$, $\tilde{\mu}_n$ depending on $\alpha \in [0,1]$.

Our interest in the above problem has stemmed from the inverse spectral theory for Sturm–Liouville operators with singular potentials. Namely, we proved in [16] that, as soon as the numbers λ_n^2 and μ_n^2 are real, strictly increase with n, interlace, and obey (1.5) with ℓ_2-sequences $(\tilde\lambda_n)$ and $(\tilde\mu_n)$, then there exists a unique real-valued $\sigma \in L_2(0,1)$ such that $\{\lambda_n^2\}$ and $\{\mu_n^2\}$ are spectra of the Sturm–Liouville operators $T_{\sigma,\mathrm{D}}$ and $T_{\sigma,\mathrm{N}}$, respectively, with distributional potential $q = \sigma' \in W_2^{-1}(0,1)$. It is reasonable to believe that if $\tilde\lambda_n$ and $\tilde\mu_n$ decay faster, then σ will be smoother. For example, the classical result of Marchenko [26, Theorem 3.4.1] claims that if, under the above assumptions, we have, in addition,

$$\tilde\lambda_n = \frac{A}{n} + \frac{\tilde\lambda_n'}{n}, \qquad \tilde\mu_n = \frac{A}{n} + \frac{\tilde\mu_n'}{n} \tag{1.6}$$

with real A and ℓ_2-sequences $(\tilde\lambda_n')$ and $(\tilde\mu_n')$, then the corresponding potential q is in $L_2(0,1)$ (thus $\alpha = 1$) and $A = \frac{1}{2\pi}\int_0^1 q$, cf. also [24, Theorem 3.3.1]. It would be desirable to "interpolate" between $\alpha = 1$ and $\alpha = 0$ and solve the inverse spectral problem for all intermediate $\alpha \in (0,1)$. The essential step towards such project is to derive eigenvalue asymptotics for Sturm–Liouville operators with potentials in $W_2^{\alpha-1}(0,1)$—i.e., to treat the direct spectral problem. And indeed, based on the results obtained here, we completely solve the inverse spectral problem for Sturm–Liouville operators with potentials in the scale $W_2^{\alpha-1}(0,1)$, $\alpha \in [0,1]$, in our paper [17].

Another motivation for this work is the recent papers [18,31], where similar questions are considered. In particular, Kappeler and Möhr in [18] found eigenvalue asymptotics for the Dirichlet and periodic Sturm–Liouville operators with complex-valued potentials that are *periodic* distributions from the space $W_2^{\alpha-1}(0,1)$, $\alpha \in (0,1]$. The Dirichlet eigenvalues λ_n^2 were proved there to obey the asymptotics

$$\lambda_n^2 = \pi^2 n^2 + \hat q(0) - \frac{\hat q(-2n) + \hat q(2n)}{2} + \nu_n, \tag{1.7}$$

where $\hat q(n)$ is the nth Fourier coefficient of q and the sequence (ν_n) belongs to $\ell_2^{2\alpha-1-\varepsilon}$ with $\varepsilon > 0$ arbitrary (the weighted ℓ_p^s spaces are defined at the end of Section 1); see [18] for more precise formulations. The authors performed the Fourier transform to work in the weighted ℓ_2 spaces rather than in the Sobolev spaces and then derived the estimates for the resolvent that yield the detailed localisation of the spectrum.

Savchuk and Shkalikov [31] considered Sturm–Liouville operators with complex-valued potentials q that are distributional derivatives of functions $u \in L_2(0,1)$. They generalised the notion of the Birkhoff regular boundary conditions to this singular case and, for Birkhoff regular boundary conditions, found eigenvalue and eigenfunction asymptotics by means of the *modified Prüfer substitution*. For the particular case of Dirichlet boundary conditions and the function u that is either

(i) of bounded variation over $[0,1]$, or
(ii) Lipschitz continuous on $[0,1]$ with exponent $\alpha \in (0,1)$, or
(iii) from $W_2^{\alpha}(0,1)$, $\alpha \in [0,1/2)$,

the authors found several terms of asymptotic expansions for eigenvalues and eigenfunctions. In particular, for $u \in W_2^\alpha(0,1)$, $\alpha \in (0, 1/2)$, they gave the formula

$$\lambda_n = \pi n - \int_0^1 u(t)\sin(2\pi nt)\,\mathrm{d}t + s_n \tag{1.8}$$

with $(s_n) \in \ell_p^{2\alpha}$ for all $p > 1$; the remainders s_n were further specified to be equal to

$$\begin{aligned} s_n := &-\frac{1}{2\pi n}\int_0^1 \sigma^2(t)\cos(2\pi nt)\,\mathrm{d}t + \frac{1}{2\pi n}\int_0^1 \sigma^2(t)\,\mathrm{d}t \\ &- 2\int_0^1\int_0^t u(t)u(s)\cos(2\pi nt)\sin(2\pi ns)\,\mathrm{d}s\,\mathrm{d}t + \rho_n \end{aligned} \tag{1.9}$$

with $(\rho_n) \in \ell_1^{2\alpha}$.

We also mention the papers [2,3,11–13,23,27,32], where the eigenvalue asymptotics were studied for several other types of singular potentials.

In the present paper, having in mind further concrete applications to the inverse spectral theory, we do not aim at the utmost possible generality. Instead, we confine ourselves to the Dirichlet and Neumann–Dirichlet boundary conditions and to potentials from the Sobolev space scale $W_2^{\alpha-1}(0,1)$, $\alpha \in [0,1]$, though the derived formula for the characteristic functions can be used to treat any generalised Birkhoff regular boundary conditions and the developed methods allow further application to potentials $q = \sigma'$ with σ of bounded variation or Lipschitz continuous as in [31].

For the Dirichlet eigenvalues, we generalise the related results of [18] in several ways: a wider class of potentials is treated (no periodicity is assumed and $\alpha = 0$ is included), ε is removed in the relation $(\nu_n) \in \ell_2^{2\alpha-1-\varepsilon}$ for ν_n as in (1.7) (see Remark 6.2), and more terms of asymptotic expansion are given. The asymptotic formulae for λ_n^2 derived here are basically the same as in [31] (see Remark 6.1); however, we allow the case $\alpha \geqslant 1/2$ and simultaneously treat the Neumann–Dirichlet case. We give a special representation of the remainders (required for the inverse analysis) as sine Fourier coefficients of some functions from the W_2^s-scale. This, however, does not yield an optimal result for the remainders s_n of (1.8) in terms of the ℓ_p^s-scale—roughly speaking, it only implies that $(s_n) \in \ell_2^{2\alpha}$ (cf. the above mentioned results of [31]). We extract from ρ_n of (1.9) one additional term falling into $\ell_1^{2\alpha}$ and show that the modified remainders $\tilde\rho_n$ form an ℓ_∞^γ-sequence with $\gamma = \min\{3\alpha, 1+\alpha\}$; observe that this result is incomparable to the inclusion $(\rho_n) \in \ell_1^{2\alpha}$ proved in [31]. Also [31] gives the eigenfunction asymptotics, which we do not study here (though the derived formula for the Cauchy matrix allows such an analysis).

Our main result describes the asymptotics of λ_n and μ_n in the following way. Here and hereafter, we denote by $s_n(f)$ and $c_n(f)$ the nth sine and cosine Fourier coefficient of a function f, i.e.,

$$s_n(f) = \int_0^1 f(x)\sin(\pi nx)\,\mathrm{d}x, \qquad c_n(g) = \int_0^1 f(x)\cos(\pi nx)\,\mathrm{d}x, \quad n \in \mathbb{Z}_+;$$

V and R will stand for the operators in $L_2(0,1)$ given by

$$(Vf)(x) = (1-2x)f(x), \qquad (Rf)(x) = f(1-x).$$

Theorem 1.1. *Assume that $q \in W_2^{\alpha-1}(0,1)$ for some $\alpha \in [0,1]$ and fix an arbitrary distributional primitive $\sigma \in W_2^{\alpha}(0,1)$ of q. Then there exists a function $\tilde\sigma \in W_2^{2\alpha}(0,1)$ such that*

$$\begin{aligned} \lambda_n &= \pi n - s_{2n}(\sigma) - s_{2n}(\tilde\sigma), \\ \mu_n &= \pi\left(n - \frac{1}{2}\right) + s_{2n-1}(\sigma) + s_{2n-1}(\tilde\sigma). \end{aligned} \tag{1.10}$$

A more detailed version of this theorem reads as follows.

Theorem 1.2. *Assume that $q \in W_2^{\alpha-1}(0,1)$ for some $\alpha \in [0,1]$ and fix an arbitrary distributional primitive $\sigma \in W_2^{\alpha}(0,1)$ of q. Then there exists a function $\omega \in W_2^{\gamma}(0,1)$, $\gamma := \min\{3\alpha, 1+\alpha\}$, such that*

$$\begin{aligned} \lambda_n &= \pi n + s_{2n}\big(\sigma^-\big) - s_{2n}(\sigma)c_{2n}(V\sigma) + s_{2n}(\omega), \\ \mu_n &= \pi\left(n - \frac{1}{2}\right) + s_{2n-1}\big(\sigma^+\big) - s_{2n-1}(\sigma)c_{2n-1}(V\sigma) + s_{2n-1}(\omega), \end{aligned} \tag{1.11}$$

where

$$\sigma^\pm(x) := \pm\sigma(x) \mp \int_0^x \sigma^2(t)\,\mathrm{d}t + \int_0^{1-x} \sigma(x+t)\sigma(t)\,\mathrm{d}t.$$

Theorem 1.2 and [31] imply the following corollary.

Corollary 1.3. *Under the assumptions* (*and in the notations*) *of Theorem* 1.2

$$\lambda_n = \pi n + s_{2n}\big(\sigma^-\big) - s_{2n}(\sigma)c_{2n}(V\sigma) + \mathrm{O}\big(n^{-\gamma}\big), \quad n \to \infty. \tag{1.12}$$

In rough terms, our main result states that if the potential q belongs to the Sobolev space $W_2^{\alpha-1}(0,1)$, then the remainders $\tilde\lambda_n$ and $\tilde\mu_n$ are sine Fourier coefficients of a function from $W_2^{\alpha}(0,1)$. This statement agrees with the earlier known results for $\alpha = 1$ and $\alpha = 0$ mentioned above (see also [32] for the case where q is the derivative of a function of bounded variation, i.e., where q is a signed measure and [2,3] for Sturm–Liouville operators in impedance form with impedance functions from $W_p^1(0,1)$). Notice that Theorem 1.1 implies the related results of the paper [18], though does not imply the results of [31] for the operator T_D and $\alpha \in [0,1/2)$. However, Corollary 1.3 gives better than in [31] estimates for the Dirichlet eigenvalues in the uniform norm (i.e., in the ℓ_∞^s-scale). Moreover, our approach is completely different from those of the works [18,31] and requires only minor effort to get the next terms in asymptotic eigenvalue expansions for small α.

The organisation of the paper is as follows. In Section 2 we derive an equivalent factorised form for the differential expression l_σ of (1.2) that is more convenient for our purposes. We use this factorised form in the next section to derive an integral representation for the characteristic functions of the operators T_{D} and T_{N}, and in Section 4 we show that the integrand in this representation possesses the desired smoothness. The problem is thus reduced to finding asymptotics of zeros of certain entire functions, which we establish in Section 5 and then prove the main results and some corollaries in Section 6. Several applications to the inverse spectral problem are given in Section 7. Finally, Appendix A provides some necessary facts from the interpolation theory.

Throughout the paper (in addition to the above-introduced notations $s_n(f)$, $c_n(f)$ for sine and cosine Fourier coefficients of a function f and V and R for the multiplication operator by $1-2x$ and the reflection operator about $x=1/2$, respectively) W_p^s with $p \in [1,\infty)$ and $s \in \mathbb{R}$ will be a shorthand notation for the Sobolev space $W_p^s(0,1)$; we shall also abbreviate $L_p(0,1)$ to L_p. The norm in W_2^s is denoted by $|\cdot|_s$ (see Appendix A). The space ℓ_p^s consists of sequences $x=(x_n)_{n\in\mathbb{N}}$ with

$$|x|_{\ell_p^s} := \left(\sum_{n\geqslant 1} n^{ps}|x_n|^p\right)^{1/p} < \infty$$

and is a Banach space (a Hilbert space for $p=2$) under the norm $|\cdot|_{\ell_p^s}$. For $p=\infty$ the norm above should be taken as $|x|_{\ell_\infty^s} := \sup_{n\geqslant 1}|x_n|n^s$.

2. Reduction of l_σ to the factorised form

We recall that, for a given potential $q \in W_2^{\alpha-1}$ with $\alpha \in [0,1]$, we have defined the Sturm–Liouville operator T_{D} (respectively, T_{N}) corresponding to the differential expression (1.1) and Dirichlet (respectively, Neumann–Dirichlet) boundary conditions as $T_{\mathrm{D}}f = l_\sigma(f)$ for $f \in \operatorname{dom} T_{\mathrm{D}}$ (respectively, as $T_{\mathrm{N}}f = l_\sigma(f)$ for $f \in \operatorname{dom} T_{\mathrm{N}}$). Here σ is a fixed distributional primitive of q, l_σ is given by

$$l_\sigma(f) := -(f'-\sigma f)' - \sigma f' = -\left(\frac{\mathrm{d}}{\mathrm{d}x}+\sigma\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}-\sigma\right)f - \sigma^2 f,$$

and $\operatorname{dom} T_{\mathrm{D}}$ and $\operatorname{dom} T_{\mathrm{N}}$ are described in (1.3).

In this section, we shall derive a representation for the differential expression l_σ in a slightly different form. Roughly speaking, the claim is that the last summand $(-\sigma^2 f)$ in the above formula can be removed by changing σ appropriately. More precisely, given $\tau \in L_2$, we denote by m_τ the differential expression

$$m_\tau(f) := -\left(\frac{\mathrm{d}}{\mathrm{d}x}+\tau\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}-\tau\right)f$$

considered on the natural L_2-domain

$$\operatorname{dom} m_\tau = \left\{ f \in W_1^1 \;\middle|\; f'-\tau f \in W_1^1,\ m_\tau(f) \in L_2 \right\}.$$

Our aim here is to show that (under some not very restrictive assumption) l_σ coincides with m_τ for a suitable choice of τ. See also [19] for similar results on the whole axis.

Denote by $\mathfrak{t}_{\mathrm{N}} = \mathfrak{t}_{\sigma,\mathrm{N}}$ the quadratic form of the operator $T_{\mathrm{N}} = T_{\sigma,\mathrm{N}}$. Integration by parts gives that, for all $f \in \operatorname{dom} T_{\mathrm{N}}$,

$$\mathfrak{t}_{\mathrm{N}}[f] := \int_0^1 l_\sigma(f)\overline{f} = \int_0^1 \bigl(|f'|^2 - \sigma f\overline{f'} - \sigma f'\overline{f}\bigr).$$

It is not difficult to show (see, e.g., [5,14]) that the quadratic form $\int_0^1 (\sigma f\overline{f'} - \sigma f'\overline{f})$ is relatively bounded with respect to the quadratic form $\int_0^1 (|f'|^2 + |f|^2)$ with relative bound zero; hence Theorem VI.1.33 of [21] implies that $\mathfrak{t}_{\mathrm{N}}$ is sectorial and that its domain is the same as for the unperturbed case $\sigma = 0$, i.e., that

$$\operatorname{dom} \mathfrak{t}_{\mathrm{N}} = \bigl\{ f \in W_2^1 \bigm| f(1) = 0 \bigr\}.$$

(As an aside we notice that, if we start from the quadratic form $\mathfrak{t}_{\mathrm{N}}$ and denote by S_{N} the corresponding sectorial operator, then $S_{\mathrm{N}} = T_{\mathrm{N}}$; thus the form-sum method and the regularisation method yield the same operator. Another consequence of this equality is that $\operatorname{dom} T_{\mathrm{N}} \subset \operatorname{dom} \mathfrak{t}_{\mathrm{N}} \subset W_2^1$; we also note that the inclusion $\operatorname{dom} T_{\mathrm{N}} \subset W_2^1$ follows from the relation $\operatorname{dom} l_\sigma \subset W_2^1$, which was explained in the Introduction. Finally, similar statements are also true for T_{D}.)

In the next proposition we assume that the quadratic form $\mathfrak{t}_{\mathrm{N}}$ is *strictly accretive*, i.e., that $\operatorname{Re} \mathfrak{t}_{\mathrm{N}}[f] > 0$ for all nonzero $f \in \operatorname{dom} \mathfrak{t}_{\mathrm{N}}$. Since $\mathfrak{t}_{\mathrm{N}}$ is sectorial, this situation can be achieved by adding to q a suitable positive constant and thus is not very restrictive.

Proposition 2.1. *Assume that $\sigma \in W_2^\alpha$, $\alpha \in [0,1]$, and that the quadratic form $\mathfrak{t}_{\mathrm{N}}$ is strictly accretive. Then there exists a function $\tau \in W_2^\alpha$ such that $\tau - \sigma \in W_1^1 \cap W_2^{2\alpha}$, $(\tau - \sigma)(0) = 0$, and $l_\sigma = m_\tau$. Moreover, the function*

$$\tilde{\phi}(x) := \tau(x) - \sigma(x) + \int_0^x \sigma^2(t)\,\mathrm{d}t, \qquad x \in [0,1],$$

belongs to W_2^γ with $\gamma = \min\{3\alpha, 1+\alpha\}$.

Proof. We shall take τ in the form u'/u, where u is any function satisfying the equation $l_\sigma(u) = 0$ and not vanishing anywhere in the interval $[0,1]$. After we have proved that such an u exists and that τ is of the required smoothness, verification of the equality $l_\sigma = m_\tau$ becomes an easy algebraic exercise (see below).

Denote by u a solution of the equation $l_\sigma(f) = 0$ satisfying the initial conditions $u(0) = 1$ and $u^{[1]}(0) = 0$. We recall that by definition the equality $l_\sigma(f) = 0$ is equivalent to the following first-order system:

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} f \\ f^{[1]} \end{pmatrix} = \begin{pmatrix} \sigma & 1 \\ -\sigma^2 & -\sigma \end{pmatrix}\begin{pmatrix} f \\ f^{[1]} \end{pmatrix}. \tag{2.1}$$

Since the entries of the 2×2 matrix in (2.1) are summable, this system enjoys the standard existence and uniqueness properties. In particular, the solution u with the stated initial conditions exists and is unique; moreover, both u and $u^{[1]}$ belong to W_1^1 and, a posteriori, $u \in W_2^1$.

We claim that u does not vanish on $[0,1]$. Assume the contrary, i.e., let there exist $x \in (0,1]$ such that $u(x)=0$. Then integration by parts gives

$$0=\int_0^x l_\sigma(u)\bar{u}=\int_0^x \bigl(|u'|^2-\sigma u\overline{u'}-\sigma u'\bar{u}\bigr).$$

We denote by v the function from W_2^1 that coincides with u on $[0,x]$ and equals zero on $[x,1]$ and observe that the above equation implies $\mathfrak{t}_{\mathrm{N}}[v]=0$. Recall that the quadratic form $\mathfrak{t}_{\mathrm{N}}$ is strictly accretive by assumption; henceforth we must have $v=0$, which is impossible in view of the equality $v(0)=u(0)=1$. The derived contradiction proves that $u(x)\neq 0$ for every $x\in[0,1]$.

Put now $\phi := u^{[1]}/u$ and $\tau := \phi+\sigma = u'/u$. The function ϕ is in W_1^1 and satisfies the equation $\phi' = -(\phi+\sigma)^2$ and the initial condition $\phi(0)=0$ (so that also $(\tau-\sigma)(0)=0$), and therefore

$$\phi(x)=-\int_0^x \phi^2(t)\,\mathrm{d}t-2\int_0^x \phi(t)\sigma(t)\,\mathrm{d}t-\int_0^x \sigma^2(t)\,\mathrm{d}t. \tag{2.2}$$

It follows from Lemma A.3 that $\phi\in W_2^\alpha$ and then repeated application of Lemma A.3 shows that the right-hand side of (2.2) belongs to $W_2^{2\alpha}$, so that $\phi\in W_2^{2\alpha}$. Next, Eq. (2.2) implies that

$$\tilde{\phi}(x)=-\int_0^x \phi^2(t)\,\mathrm{d}t-2\int_0^x \phi(t)\sigma(t)\,\mathrm{d}t,$$

and henceforth $\tilde{\phi}\in W_2^\gamma$ by Lemma A.3 as claimed.

Take now $f\in\operatorname{dom} l_\sigma$; then $f'-\tau f = f^{[1]}-\phi f\in W_1^1$ and, using the identity $\phi'=-(\phi+\sigma)^2$, we find that

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}x}+\tau\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}-\tau\right)f &= \bigl(f^{[1]}\bigr)'-\phi' f-\phi f'+\tau f^{[1]}-\tau\phi f\\ &= \bigl(f^{[1]}\bigr)'+(\phi+\sigma)^2 f-\phi f'+(\phi+\sigma)(f'-\sigma f)-(\phi+\sigma)\phi f\\ &= \bigl(f^{[1]}\bigr)'+\sigma f' = -l_\sigma(f).\end{aligned}$$

This shows that $l_\sigma\subset m_\tau$. The reverse inclusion is established analogously, and, as a result, we get $l_\sigma = m_\tau$. The proposition is proved. □

3. Integral representation of the characteristic functions

Assume that $\sigma\in L_2$ is such that the quadratic form $\mathfrak{t}_{\sigma,\mathrm{N}}$ is strictly accretive (see Section 2 for definitions). Then by Proposition 2.1 there exists $\tau\in L_2$ such that $\phi:=\tau-\sigma\in W_1^1$, $\phi(0)=0$, and $l_\sigma=m_\tau$. Consider the differential equation $m_\tau u=\lambda^2 u$, i.e.,

$$-\left(\frac{\mathrm{d}}{\mathrm{d}x}+\tau\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}-\tau\right)u=\lambda^2 u. \tag{3.1}$$

It can be written as a first order system

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \tau & 1 \\ -\lambda^2 & -\tau \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{3.2}$$

with $u_1 \equiv u$ and $u_2 \equiv u' - \tau u =: u_\tau^{[1]}$. For any $a, b \in \mathbb{C}$, there exists a unique solution $(u_1, u_2)^T$ of (3.2) subject to the initial conditions $u_1(0) = a$, $u_2(0) = b$, whence (3.1) has a unique solution v satisfying the initial conditions $v(0) = a$ and $v_\tau^{[1]}(0) = b$.

Denote by $s(\cdot, \lambda)$ and $c(\cdot, \lambda)$ the solutions of Eq. (3.1) obeying the initial conditions

$$s(0,\lambda) = c_\tau^{[1]}(0,\lambda) = 0, \qquad s_\tau^{[1]}(0,\lambda) = c(0,\lambda) = 1.$$

Observe that in view of the equality $\phi(0) = 0$ we have $u_\tau^{[1]}(0) = u^{[1]}(0) - \phi(0)u(0) = u^{[1]}(0)$ for any $u \in \operatorname{dom} l_\sigma = \operatorname{dom} m_\tau$. Therefore the numbers $\pm\lambda_n$ coincide with the zeros of the entire even function $s(1,\cdot)$, while $\pm\mu_n$ coincide with those of $c(1,\cdot)$. In both cases multiplicities are taken into account (i.e., if some λ^2 is an eigenvalue of T_{D} of algebraic multiplicity m, then λ is a zero of $s(1,\cdot)$ of order m, and similarly for T_{N}). The functions $c(1,\cdot)$ and $s(1,\cdot)$ are called the *characteristic functions* of the operators T_{N} and T_{D}, respectively.

Our aim in this section is to show that the characteristic functions $c(1,\cdot)$ and $s(1,\cdot)$ allow integral representations of a special form (see (3.10)). We do this by deriving first a special integral representation for the Cauchy matrix of system (3.2).

To begin with, we notice that the matrix-valued function

$$U(x) = U(x,\lambda) := \begin{pmatrix} c(x,\lambda) & s(x,\lambda) \\ c_\tau^{[1]}(x,\lambda) & s_\tau^{[1]}(x,\lambda) \end{pmatrix}$$

satisfies the initial condition $U(0) = I$ (with $I = \operatorname{diag}(1,1)$) and solves the equation

$$U' = (A + \tau J)U, \tag{3.3}$$

where

$$A = A_\lambda := \begin{pmatrix} 0 & 1 \\ -\lambda^2 & 0 \end{pmatrix}, \qquad J := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In other words, U is the Cauchy matrix of system (3.2). Since $\tau \in L_2$, Eq. (3.3) with the initial condition $U(0) = I$ is uniquely soluble and the solution U belongs to W_2^1 entrywise.

We shall, however, need a more explicit formula for the Cauchy matrix U. The standard variation of constants yields the equivalent integral equation for U in the form

$$U(x) = \mathrm{e}^{xA} + \int_0^x \mathrm{e}^{(x-t)A}\tau(t)JU(t)\,\mathrm{d}t, \tag{3.4}$$

where the exponent e^{xA} can be explicitly calculated as

$$\mathrm{e}^{xA} = \begin{pmatrix} \cos\lambda x & \frac{1}{\lambda}\sin\lambda x \\ -\lambda\sin\lambda x & \cos\lambda x \end{pmatrix}.$$

Integral equation (3.4) can be solved by the method of successive approximations; namely, with

$$U_0(x) := \mathrm{e}^{xA} \quad \text{and} \quad U_{n+1}(x) = \int_0^x \mathrm{e}^{(x-t)A}\tau(t)JU_n(t)\,\mathrm{d}t \quad \text{for } n \geqslant 0, \tag{3.5}$$

the solution formally equals $\sum_{n=0}^{\infty} U_n$. Our next aim is to show that this series converges in a suitable topology and that the sum is indeed the Cauchy matrix.

To this end we endow the space $\mathcal{M}_2 := M(2,\mathbb{C})$ of all 2×2 matrices with complex entries with the operator norm $|\cdot|$ of the Euclidean $\mathbb{C}^2$ space and denote by $W_2^1(\mathcal{M}_2)$ the Sobolev space of $\mathcal{M}_2$-valued functions on the interval $[0,1]$.

Lemma 3.1. *The series $\sum_{n=0}^{\infty} U_n$ with U_n given by* (3.5) *converges in $W_2^1(\mathcal{M}_2)$ to the Cauchy matrix U.*

Proof. Bearing in mind the identity $J\mathrm{e}^{xA} = \mathrm{e}^{-xA}J$ and using recurrent relations (3.5), we derive the formula

$$U_n(x) = \int_{\Pi_n(x)} \mathrm{e}^{(x-2\xi_n(\mathrm{t}))A}\,\tau(t_1)\ldots\tau(t_n)J^n\,\mathrm{d}t_1\ldots\mathrm{d}t_n, \tag{3.6}$$

in which we have put

$$\Pi_n(x) = \left\{\mathrm{t} := (t_1,\ldots,t_n) \in \mathbb{R}^n \;\middle|\; 0 \leqslant t_n \leqslant \cdots \leqslant t_1 \leqslant x\right\},$$
$$\xi_n(\mathrm{t}) = \sum_{l=1}^{n} (-1)^{l+1} t_l.$$

Observe that $0 \leqslant \xi_n(\mathrm{t}) \leqslant x$ for $\mathrm{t} \in \Pi_n(x)$; thus, denoting by C the maximum of $|\mathrm{e}^{xA}|$ over the interval $[-1,1]$, we get the estimate

$$\left|U_n(x)\right| \leqslant C \int_{\Pi_n(x)} \left|\tau(t_1)\right|\ldots\left|\tau(t_n)\right|\mathrm{d}t_1\ldots\mathrm{d}t_n = \frac{C}{n!}\left(\int_0^x |\tau|\right)^n.$$

Differentiating recurrent relations (3.5), we find that

$$U_n'(x) = AU_n(x) + \tau(x)JU_{n-1}(x), \tag{3.7}$$

and hence, with $C_1 := C(2|A|^2+3)^{1/2}$,

$$\|U_n\|_{W_2^1(\mathcal{M}_2)} := \left(\int_0^1 \left(|U_n'|^2 + |U_n|^2\right)\right)^{1/2} \leqslant \frac{C_1}{(n-1)!}\left(\int_0^1 |\tau|^2\right)^{1/2}\left(\int_0^1 |\tau|\right)^{(n-1)}.$$

This estimate justifies convergence of the series $\sum_{k=0}^{\infty} U_k$ in the $W_2^1(\mathcal{M}_2)$-topology to some $\mathcal{M}_2$-valued function V obeying the initial condition $V(0) = I$. Bearing in mind (3.7) and differentiating this series term-by-term, we see that V satisfies Eq. (3.3) and thus indeed equals the Cauchy matrix U. □

Our next aim is to get an integral representation for $U(1)$ of a special form. Upon change of variables $s = \xi_n(\mathrm{t})$, $y_l = t_{l+1}$, $l = 1, 2, \ldots, n-1$, we recast the integral in (3.6) for $x = 1$ as

$$U_n(1) = \int_0^1 \mathrm{e}^{(1-2s)A}\, \tau_n(s) J^n \,\mathrm{d}s.$$

Here $\tau_1 \equiv \tau(s)$ and, for all $n \in \mathbb{N}$,

$$\tau_{n+1}(s) = \int_{\Pi_n^*(s)} \tau\big(s + \xi_n(\mathrm{y})\big)\tau(y_1)\ldots\tau(y_n)\,\mathrm{d}y_1\ldots\mathrm{d}y_n \tag{3.8}$$

with

$$\Pi_n^*(s) = \big\{\mathrm{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n \;\big|\; 0 \leqslant y_n \leqslant y_{n-1} \leqslant \cdots \leqslant y_1 \leqslant s + \xi_n(\mathrm{y}) \leqslant 1\big\}. \tag{3.9}$$

Using the Cauchy–Schwarz inequality and Fubini's theorem, we find that for every $n \in \mathbb{N}$ the function τ_n belongs to L_2 and that

$$\begin{aligned}
|\tau_n|_0^2 &= \int_0^1 \big|\tau_n(s)\big|^2\,\mathrm{d}s \\
&\leqslant \frac{1}{(n-1)!}\int_0^1 \mathrm{d}s \int_{\Pi_{n-1}^*(s)} \big|\tau\big(s + \xi_{n-1}(\mathrm{y})\big)\tau(y_1)\ldots\tau(y_{n-1})\big|^2\,\mathrm{d}y_1\ldots\mathrm{d}y_{n-1} \\
&= \frac{1}{(n-1)!}\int_{\Pi_n(1)} \big|\tau(t_1)\big|^2\ldots\big|\tau(t_n)\big|^2\,\mathrm{d}t_1\ldots\mathrm{d}t_n = \frac{1}{(n-1)!\,n!}|\tau|_0^{2n}.
\end{aligned}$$

It follows that the series $\sum_{n=1}^{\infty}(\pm 1)^n \tau_n$ converges in L_2 to some function $\tau^{\pm}$; putting $K = \operatorname{diag}\{\tau^+, \tau^-\}$, we arrive at the desired representation for $U(1)$:

$$U(1) = \mathrm{e}^A + \int_0^1 \mathrm{e}^{(1-2s)A} K(s)\,\mathrm{d}s.$$

Spelling out the first row of this matrix equality, we get the following result.

Theorem 3.2. *Assume that $\sigma \in L_2$ is such that the quadratic form $\mathfrak{t}_{\sigma,\mathrm{N}}$ is strictly accretive and denote by $\tau \in L_2$ the function of Proposition 2.1, for which $l_\sigma = m_\tau$. Then the characteristic functions $c(1,\cdot)$ and $s(1,\cdot)$ of the operators T_N and T_D equal*

$$c(1,\lambda) = \cos\lambda + \int_0^1 \tau^+(s)\cos\lambda(1-2s)\,\mathrm{d}s,$$

$$s(1,\lambda) = \frac{\sin\lambda}{\lambda} + \int_0^1 \tau^-(s)\frac{\sin\lambda(1-2s)}{\lambda}\,\mathrm{d}s, \tag{3.10}$$

where the L_2-functions $\tau^\pm$ are defined by

$$\tau^\pm = \sum_{n=1}^{\infty}(\pm 1)^n \tau_n \tag{3.11}$$

with $\tau_1 \equiv \tau$ and τ_{n+1} given by (3.8) *for all $n \in \mathbb{N}$.*

In the case where σ (and thus τ) belongs to W_2^α with some positive α the functions τ_n are also smoother. We shall establish this fact in the following section.

4. Smoothness of the functions τ_n

The derivation of the integral representations for the characteristic functions $c(1,\cdot)$ and $s(1,\cdot)$ in Section 3 only used the fact that τ belongs to L_2. If, however, the potential q is a distribution from $W_2^{\alpha-1}$ with some $\alpha \in (0,1]$, then $\tau \in W_2^\alpha$ by Proposition 2.1, and we can expect that the functions τ_n of (3.8) also have some additional smoothness. The aim of this section is to make this statement precise.

Fix a natural $n \geqslant 2$ and consider an n-linear mapping $I_n : (L_2)^n \to L_2$ that acts according to the formula (cf. (3.8))

$$I_n(\mathbf{f})(s) = \int_{\Pi_{n-1}^*(s)} f_1\bigl(s + \xi_{n-1}(\mathrm{y})\bigr) f_2(y_1)\dots f_n(y_{n-1})\,\mathrm{d}y_1\dots\mathrm{d}y_{n-1},$$

$\mathbf{f} = (f_1,\dots,f_n) \in (L_2)^n$, the set $\Pi_{n-1}^*(s)$ is defined by (3.9), and $\xi_{n-1}(\mathrm{y}) := \sum_{l=1}^{n-1}(-1)^{l+1}y_l$. In particular, we see that

$$\tau_n = I_n(\tau,\dots,\tau).$$

First we show that, indeed, I_n maps $(L_2)^n$ into L_2.

Lemma 4.1. *For any $\mathbf{f} = (f_1,\dots,f_n) \in (L_2)^n$ the function $g := I_n(\mathbf{f})$ belongs to L_2 and*

$$|g|_0 \leqslant \frac{1}{\sqrt{(n-1)!}}\prod_{l=1}^n |f_l|_0.$$

Proof. Repeating the arguments of Section 3 used to prove that $\tau_n \in L_2$, we find that

$$\left|g(s)\right|^2 \leqslant \frac{1}{(n-1)!} \int\limits_{\Pi^*_{n-1}(s)} \left|f_1\bigl(s+\xi_{n-1}(\mathrm{y})\bigr)\right|^2 \left|f_2(y_1)\right|^2 \dots \left|f_n(y_{n-1})\right|^2 \mathrm{d}y_1 \dots \mathrm{d}y_{n-1}$$

and thus

$$|g|_0^2 \leqslant \frac{1}{(n-1)!} \int\limits_{\Pi_n(1)} \left|f_1(t_1)\right|^2 \left|f_2(t_2)\right|^2 \dots \left|f_n(t_n)\right|^2 \mathrm{d}t_1 \dots \mathrm{d}t_n \leqslant \frac{1}{(n-1)!} \prod_{l=1}^{n} |f_l|_0^2$$

as required. □

Now we give an equivalent formula for I_n. Let $g = I_n(\mathbf{f})$ and let h be an arbitrary function in L_2; then the L_2-scalar product $(g,h)_{L_2}$ of g and h can be recast as

$$(g,h)_{L_2} = \int\limits_{\Pi_n(1)} f_1(t_1)\dots f_n(t_n)\bar{h}\bigl(\xi_n(\mathrm{t})\bigr)\,\mathrm{d}t_1 \dots \mathrm{d}t_n. \tag{4.1}$$

Since such scalar products with h from a total set in L_2 completely determine the function g, we can use this formula to define the action of the mapping I_n.

The main result of this section is contained in Theorems 4.2 and 4.3 that show how I_n acts between the spaces W_2^α.

Theorem 4.2. *For $\alpha \in [0,1]$, let $\mathbf{f} \in (W_2^\alpha)^n$ and $g := I_n(\mathbf{f})$.*

(a) *For all $n \geqslant 2$ the function g belongs to $W_2^{2\alpha}$ and there exists a positive C independent of* $\mathbf{f}$ *and n such that, with $r := \max\{0, n-5\}$, we have*

$$|g|_{2\alpha} \leqslant \frac{C}{\sqrt{r!}} \prod_{l=1}^{n} |f_l|_\alpha.$$

(b) *For all $n \geqslant 4$ the function g belongs to $W_2^{3\alpha}$ and there exists a positive C independent of* $\mathbf{f}$ *and n such that, with $r := \max\{0, n-7\}$, we have*

$$|g|_{3\alpha} \leqslant \frac{C}{\sqrt{r!}} \prod_{l=1}^{n} |f_l|_\alpha.$$

Proof. Since the family $\{W_2^\alpha\}$, $\alpha \in \mathbb{R}$, constitutes a Hilbert scale, by virtue of Theorem A.2 and Lemma 4.1 it suffices to prove the theorem only for $\alpha = 1$. Observe also that statement (a) for $n = 2$ holds in view of Lemma A.4, so that we may assume that $n \geqslant 3$.

We shall divide the proof of the case $\alpha = 1$ and $n \geqslant 3$ into several steps and shall throughout denote by C_k positive constants independent of $\mathbf{f}$ and n.

Step 1. Let $n \geqslant 3$, $\mathbf{f} := (f_1, \dots, f_n) \in (L_2)^n$ and $f_n \in W_2^1$. We shall show that in this case the function g belongs to W_2^1 and

$$|g|_1 \leqslant \frac{C_1}{\sqrt{(n-3)!}} |f_n|_1 \prod_{j=1}^{n-1} |f_j|_0 \tag{4.2}$$

for some positive constant C_1.

Let ϕ be an arbitrary test function (i.e., a C^∞ function with support in $(0,1)$); then by the definition of the distributional derivative we have

$$(g', \phi) = -(g, \phi').$$

Integration by parts gives

$$\begin{aligned}
-\int_0^{t_{n-1}} f_n(t_n)\phi'\big(\xi_n(\mathrm{t})\big)\,\mathrm{d}t_n &= (-1)^n \int_0^{t_{n-1}} f_n(t_n)\phi'_{t_n}\big(\xi_n(\mathrm{t})\big)\,\mathrm{d}t_n \\
&= (-1)^{n+1} \int_0^{t_{n-1}} f_n'(t_n)\phi\big(\xi_n(\mathrm{t})\big)\,\mathrm{d}t_n \\
&\quad + (-1)^n f_n(t_{n-1})\phi\big(\xi_{n-2}(\tilde{\tilde{\mathrm{t}}})\big) + (-1)^{n+1} f_n(0)\phi\big(\xi_{n-1}(\tilde{\mathrm{t}})\big),
\end{aligned}$$

where we have put $\tilde{\mathrm{t}} = (t_1, \dots, t_{n-1})$ and $\tilde{\tilde{\mathrm{t}}} = (t_1, \dots, t_{n-2})$. It follows now that

$$\begin{aligned}
-(g, \phi') = (-1)^{n+1}&\big(I_n\big(f_1, \dots, f_{n-1}, f_n'\big), \phi\big) \\
&+ (-1)^{n+1} f_n(0)\big(I_{n-1}(f_1, \dots, f_{n-1}), \phi\big) \\
&+ (-1)^n \big(I_{n-2}(f_1, \dots, f_{n-3}, \tilde{f}_{n-2}), \phi\big),
\end{aligned}$$

where $\tilde{f}_{n-2} := A_+(f_{n-2}, f_{n-1}, f_n)$ and the multilinear mapping A_+ is given by

$$A_+(h_1, h_2, h_3)(x) := h_1(x) \int_0^x h_2(y)h_3(y)\,\mathrm{d}y. \tag{4.3}$$

We observe that A_+ acts boundedly from $(L_2)^3$ into L_2. In fact, if $h_1, h_2, h_3 \in L_2$, then $A_+(h_1, h_2, h_3)$ is a product of an L_2-function h_1 and a W_1^1-function $\int_0^x h_2 h_3$ and thus is in L_2; moreover,

$$\big|A_+(h_1, h_2, h_3)\big|_0 \leqslant |h_1|_0 \max_x \left|\int_0^x h_2 h_3\right| \leqslant |h_1|_0 |h_2|_0 |h_3|_0. \tag{4.4}$$

Hence

$$g' = (-1)^n I_n\big(f_1, \ldots, f_{n-1}, f_n'\big) + (-1)^n f_n(0) I_{n-1}(f_1, \ldots, f_{n-1}) \\ + (-1)^{n+1} I_{n-2}(f_1, \ldots, f_{n-3}, \tilde{f}_{n-2}) \tag{4.5}$$

in the sense of distributions. Since the right-hand side of the above equation belongs to L_2 by Lemma 4.1, we conclude that $g \in W_2^1$.

Recall that W_2^1 is continuously embedded into $C[0,1]$ and thus there is $C_2 > 0$ such that $\max_x |f| \leqslant C_2 |f|_1$ for all $f \in W_2^1$. Applying Lemma 4.1 to (4.5) and using the inequalities $|f_n(0)| \leqslant C_2 |f_n|_1$ and $|\tilde{f}_{n-2}|_0 \leqslant |f_{n-2}|_0 |f_{n-1}|_0 |f_n|_0$ (recall (4.4)), we arrive at estimate (4.2).

Step 2. Let $n \geqslant 3$, $\mathbf{f} := (f_1, \ldots, f_n) \in (L_2)^n$ and $f_1 \in W_2^1$. We shall show that then the function g belongs to W_2^1 and

$$|g|_1 \leqslant \frac{C_1}{\sqrt{(n-3)!}} |f_1|_1 \prod_{j=2}^{n} |f_j|_0 \tag{4.6}$$

with the same C_1 as in (4.2).

A direct verification shows that, with R being the reflection operator about $x = 1/2$,

$$I_n(f_1, \ldots, f_n) = I_n(Rf_n, \ldots, Rf_1)$$

if n is even and

$$I_n(f_1, \ldots, f_n) = R I_n(Rf_n, \ldots, Rf_1)$$

if n is odd. Since R is unitary in W_2^α for every $\alpha \in [0,1]$ (for the cases $\alpha = 0$ and $\alpha = 2$ this is evident and for intermediate values follows by interpolation, see Theorem A.1), the inclusion $g \in W_2^1$ and estimate (4.6) follow from the results of Step 1.

Step 3. Let $n \geqslant 3$, $\mathbf{f} := (f_1, \ldots, f_n) \in (L_2)^n$ and $f_1, f_n \in W_2^1$. Using relation (4.5) and bounds (4.2) and (4.6), we easily conclude that the function g belongs to W_2^2 and that

$$|g|_2 \leqslant \frac{C_3}{\sqrt{r!}} |f_1|_1 |f_n|_1 \prod_{l=2}^{n-1} |f_l|_0 \tag{4.7}$$

with some $C_3 > 0$ independent of $\mathbf{f}$ and $r := \max\{0, n-5\}$. The only thing to be justified is that for $n = 3$ the function $I_1(\tilde{f}_1) \equiv \tilde{f}_1 = A_+(f_1, f_2, f_3)$ belongs to W_2^1 and

$$|\tilde{f}_1|_1 \leqslant C_4 |f_1|_1 |f_2|_0 |f_3|_1.$$

This, however, easily follows from the formula

$$(\tilde{f}_1)'(x) = f_1'(x) \int_0^x f_2(t) f_3(t)\, \mathrm{d}t + f_1(x) f_2(x) f_3(x)$$

showing that $\tilde{f}_1'$ belongs to L_2 and providing the suitable estimate of its L_2-norm.

Formula (4.7) combined with the remarks made at the beginning of the proof and the obvious inequality $|h|_0 \leqslant |h|_1$ completes the proof of statement (a).

Step 4. Let $n \geqslant 4$, $\mathbf{f} := (f_1, \dots, f_n) \in (L_2)^n$ and $f_2 \in W_2^1$. We shall show that then the function g belongs to W_2^1 and the following identity holds:

$$\begin{aligned} g' = &-I_n\big(f_1, f_2', f_3, \dots, f_n\big) - I_{n-2}\big(A_+(f_1, f_2, f_3), f_4, \dots, f_n\big) \\ &+ I_{n-2}\big(A_-(f_3, f_2, f_1), f_4, \dots, f_n\big) + I_{n-2}\big(f_1, A_+(f_4, f_2, f_3), f_5, \dots, f_n\big), \end{aligned} \tag{4.8}$$

where A_+ is the mapping of (4.3) and A_- is given by

$$A_-(h_1, h_2, h_3)(x) := h_1(x) \int_x^1 h_2(y) h_3(y)\, \mathrm{d}y.$$

Given an arbitrary test function ϕ and integrating by parts, we get

$$\begin{aligned} -\int_{t_3}^{t_1} f_2(t_2)\phi'\big(\xi_n(\mathrm{t})\big)\,\mathrm{d}t_2 &= \int_{t_3}^{t_1} f_2(t_2)\phi'_{t_2}\big(\xi_n(\mathrm{t})\big)\,\mathrm{d}t_2 \\ &= -\int_{t_3}^{t_1} f_2'(t_2)\phi\big(\xi_n(\mathrm{t})\big)\,\mathrm{d}t_2 + f_2(t_1)\phi\big(\xi_{n-2}(\tilde{\mathrm{t}})\big) - f_2(t_3)\phi\big(\xi_{n-2}(\tilde{\tilde{\mathrm{t}}})\big), \end{aligned}$$

where $\tilde{\mathrm{t}} = (t_3, t_4, \dots, t_n)$ and $\tilde{\tilde{\mathrm{t}}} = (t_1, t_4, \dots, t_n)$. Substituting this relation into the expression for $-(g, \phi')$ (cf. (4.1)), after simple calculations we get

$$\begin{aligned} -(g, \phi') = &-\big(I_n\big(f_1, f_2', f_3, \dots, f_n\big), \phi\big) - \big(I_{n-2}\big(A_+(f_1, f_2, f_3), f_4, \dots, f_n\big), \phi\big) \\ &+ \big(I_{n-2}\big(A_-(f_3, f_2, f_1), f_4, \dots, f_n\big), \phi\big) \\ &+ \big(I_{n-2}\big(f_1, A_+(f_4, f_2, f_3), f_5, \dots, f_n\big), \phi\big), \end{aligned}$$

which yields (4.8).

Step 5. Assume that $n \geqslant 4$, $\mathbf{f} := (f_1, \dots, f_n) \in (W_2^1)^n$ and $g := I_n(\mathbf{f})$. Observe that the multilinear transformations A_+ and A_- map continuously $(W_2^1)^3$ into W_2^1, which can be verified by direct calculation or using Lemma A.3.

It follows from (4.8) by Step 3 of the proof that $g' \in W_2^2$, so that $g \in W_2^3$. The required norm estimate also follows from (4.7) and the continuity properties of A_+ and A_-. Thus statement (b) of the theorem is justified for the case $\alpha = 1$ and consequently, by the Interpolation theorem A.2, for all $\alpha \in [0, 1]$. The theorem is proved. □

For $n = 3$ we have a slightly worse result.

Theorem 4.3. *Assume that* $\alpha \in [0,1]$, $\gamma := \min\{3\alpha, 1+\alpha\}$, $\mathbf{f} := (f_1, f_2, f_3) \in (W_2^\alpha)^3$, *and* $g := I_3(\mathbf{f})$. *Then the function* g *belongs to* W_2^γ *and there is a constant* C *independent of* $\mathbf{f}$ *such that*

$$|g|_\gamma \leqslant C|f_1|_\alpha |f_2|_\alpha |f_3|_\alpha.$$

Proof. Using the definition of I_3 and changing the variables via $t_1 = y_1 - y_2$, $t_2 = y_2$, we arrive at the representation

$$\begin{aligned} g(s) &= \int\limits_{\Pi_2^*(s)} f_1(s+y_1-y_2) f_2(y_1) f_3(y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2 \\ &= \int\limits_0^s \mathrm{d}t_2\, f_3(t_2) \int\limits_0^{1-s} f_1(s+t_1) f_2(t_1+t_2)\,\mathrm{d}t_1. \end{aligned}$$

We now put

$$\begin{aligned} g_1(s) := \tilde{I}_3(\mathbf{f})(s) := &\int\limits_0^s \mathrm{d}t_2\, f_1(t_2) \int\limits_0^{t_2} f_2(t_1) f_3(t_1)\,\mathrm{d}t_1 \\ &- \int\limits_0^s \mathrm{d}t_2\, f_3(t_2) \int\limits_{t_2}^1 f_1(t_1) f_2(t_1)\,\mathrm{d}t_1 \end{aligned}$$

and

$$h(s) := J_3(\mathbf{f}) := I_3(\mathbf{f})(s) + \tilde{I}_3(\mathbf{f})(s)$$

and show that the function h belongs to $W_2^{3\alpha}$ and that, moreover,

$$|h|_{3\alpha} \leqslant C_1 |f_1|_\alpha |f_2|_\alpha |f_3|_\alpha \tag{4.9}$$

for some $C_1 > 0$ independent of $\mathbf{f}$. Since J_3 is a multilinear mapping, in view of Interpolation theorem A.2 and Lemma 4.1 it suffices to treat only the case $\alpha = 1$.

Assume therefore that $\mathbf{f} \in (W_2^1)^3$. Direct calculations show that

$$\begin{aligned} h'(s) &= g'(s) + g_1'(s) \\ &= f_1(s) \int\limits_0^s f_2(t) f_3(t)\,\mathrm{d}t - f_1(1) \int\limits_0^s f_2(1-s+t) f_3(t)\,\mathrm{d}t \\ &\quad + \int\limits_0^s \mathrm{d}t_2\, f_3(t_2) \int\limits_0^{1-s} f_1'(s+t_1) f_2(t_1+t_2)\,\mathrm{d}t_1. \end{aligned}$$

Integrating by parts in the last integral, we arrive at the relation

$$h'(s) = -I_3(f_1, f_2', f_3),$$

so that $h' \in W_2^2$ by Step 3 of the proof of Theorem 4.2. Also

$$|h'|_2 \leqslant C_2|f_1|_1|f_2'|_0|f_2|_1,$$

which yields estimate (4.9) for $\alpha = 1$. By interpolation, the results hold also for all intermediate $\alpha \in [0, 1]$.

Consider now the function g_1. Put

$$J(\psi_1, \psi_2) := \int_0^s \psi_1(t)\psi_2(t)\,\mathrm{d}t$$

for $\psi_1, \psi_2 \in L_2$; then direct calculations show that

$$g_1 = J(f_1, \psi_{23}) - J(f_3, R\psi_{12}),$$

where $\psi_{23} := J(f_2, f_3)$ and $\psi_{12} := J(Rf_1, Rf_2)$. Applying Lemma A.3 twice we find that $\psi_{12}, \psi_{23} \in W_2^{2\alpha}$, $g_1 \in W_2^{\alpha+\min\{2\alpha,1\}} = W_2^\gamma$ and that

$$|g_1|_\gamma \leqslant C_3|f_1|_\alpha|\psi_{23}|_{\min\{2\alpha,1\}} + C_3|f_3|_\alpha|\psi_{12}|_{\min\{2\alpha,1\}} \leqslant C_4|f_1|_\alpha|f_2|_\alpha|f_3|_\alpha.$$

The theorem is proved. $\square$

Remark 4.4. The statement of the previous theorem cannot be improved in the sense that the exponent $\gamma = \min\{3\alpha, 1+\alpha\}$ cannot be made larger. This follows from the fact that the results of Lemma A.3 are sharp. The same statement holds also for $\tau_3 = I_3(\tau, \tau, \tau)$.

Corollary 4.5. *Assume that $\alpha \in (0, 1]$, $\gamma := \min\{3\alpha, 1+\alpha\}$, $\tau \in W_2^\alpha$ and $\tau_n := I_n(\tau, \dots, \tau)$. Then for every $n \geqslant 3$ the function τ_n belongs to W_2^γ and, moreover,*

$$|\tau_n|_\gamma \leqslant \frac{C}{\sqrt{r!}}|\tau|_\alpha^n$$

with some constant $C > 0$ and $r := \max\{0, n-7\}$. In particular, the functions $\tau^\pm$ in Theorem 3.2 have the form

$$\tau^\pm = \pm\tau + \tau_2 + \phi^\pm$$

with some W_2^γ-functions $\phi^\pm$.

5. Asymptotics of zeros of some entire functions

As we have seen in the previous sections, the eigenvalue asymptotics is completely determined by the asymptotics of zeros for entire functions of a special form. The main result of this section shows how this asymptotics can be calculated.

Assume that f is an arbitrary function from W_2^α and put

$$F_c(\lambda) := \cos\lambda + \int_0^1 f(x)\cos\bigl[\lambda(1-2x)\bigr]\,dx,$$

$$F_s(\lambda) := \frac{\sin\lambda}{\lambda} + \int_0^1 f(x)\frac{\sin[\lambda(1-2x)]}{\lambda}\,dx.$$

These are even entire functions of λ; we denote by ξ_{2n-1} and ξ_{2n}, $n\in\mathbb{N}$, zeros of F_c and F_s, respectively, from the set Ω of (1.4). We repeat every zero $\lambda\neq 0$ according to its multiplicity, and if $\lambda=0$ is a zero of F_c or F_s of order $2m$, then we repeat it m times among ξ_{2n-1} or ξ_{2n}, respectively. We shall order ξ_k so that $\operatorname{Re}\xi_{2n+1} > \operatorname{Re}\xi_{2n-1}$ or $\operatorname{Re}\xi_{2n+1} = \operatorname{Re}\xi_{2n-1}$ and $\operatorname{Im}\xi_{2n+1} \geqslant \operatorname{Im}\xi_{2n-1}$ and similarly for ξ_{2n}.

It is known (cf. [26, Chapter 1.3] and [15]) that for $f\in L_2$ the numbers ξ_n have the form

$$\xi_n = \frac{\pi n}{2} + \tilde{\xi}_n$$

for some ℓ_2-sequence $(\tilde{\xi}_n)_{n\in\mathbb{N}}$ (in particular, the remainders $\tilde{\xi}_n$ are the Fourier coefficients of some L_2-function). It is reasonable to expect that if f is smoother (say, from W_2^α), then $\tilde{\xi}_n$ decay faster. This is precisely what the following theorem states.

Recall that for an L_2-function g we have denoted by $s_n(g)$ and $c_n(g)$ its sine and cosine Fourier coefficients, respectively, and by $V: L_2\to L_2$ the operator of multiplication by the function $(1-2x)$.

Theorem 5.1. *Assume that $\alpha\in(0,1]$, $\gamma=\min\{3\alpha, 1+\alpha\}$, $f\in W_2^\alpha$ and that the numbers $\tilde{\xi}_n$ are defined as above. Then there exists a function $g\in W_2^\gamma$ such that*

$$\tilde{\xi}_n = s_n(f) - s_n(f)c_n(Vf) + s_n(g), \quad n\in\mathbb{N}.$$

In particular, $(\tilde{\xi}_n)$ is a sequence of sine Fourier coefficients of some function from W_2^α.

Before proceeding with the proof of the theorem, we introduce the following spaces.

For any $g\in L_2$, we denote by $\mathbf{c}(g)$ and $\mathbf{s}(g)$ the sequences $(c_n(g))_{n=0}^\infty$ and $(s_n(g))_{n=0}^\infty$ of its cosine and sine Fourier coefficients, respectively, and put

$$\mathbf{C}_\alpha := \bigl\{\mathbf{c}(g) \bigm| g\in W_2^\alpha\bigr\}, \qquad \mathbf{S}_\alpha := \bigl\{\mathbf{s}(g) \bigm| g\in W_2^\alpha\bigr\}, \quad \alpha\in[0,2].$$

The lineals $\mathbf{C}_\alpha$ and $\mathbf{S}_\alpha$ are algebraically embedded into $\ell_2(\mathbb{Z}_+)$ and become Banach spaces under the induced norms

$$\bigl\|\mathbf{c}(g)\bigr\|_{\mathbf{C}_\alpha} := \|g\|_{W_2^\alpha}, \qquad \bigl\|\mathbf{s}(g)\bigr\|_{\mathbf{S}_\alpha} := \|g\|_{W_2^\alpha}.$$

For any $\mathbf{a}, \mathbf{b} \in \ell_2(\mathbb{Z}_+)$ we shall denote by $\mathbf{ab}$ the entrywise product of $\mathbf{a}$ and $\mathbf{b}$, i.e., the element of $\ell_2(\mathbb{Z}_+)$ with entries $(\mathbf{ab})_n := a_n b_n$.

To establish Theorem 5.1, we shall essentially rely on the following three lemmas. The first of them (Lemma 5.2) is proved in Appendix A, and the other two are simple corollaries of well-known facts and thus their proofs are omitted.

Lemma 5.2. *Suppose that* $\alpha, \beta \in [0,1]$ *and* $\mathbf{a} \in \mathbf{C}_\alpha$, $\mathbf{b} \in \mathbf{S}_\alpha$, $\tilde{\mathbf{a}} \in \mathbf{C}_\beta$, $\tilde{\mathbf{b}} \in \mathbf{S}_\beta$. *Then* $\mathbf{a}\tilde{\mathbf{a}} \in \mathbf{C}_{\alpha+\beta}$, $\mathbf{b}\tilde{\mathbf{b}} \in \mathbf{C}_{\alpha+\beta}$, $\mathbf{a}\tilde{\mathbf{b}} \in \mathbf{S}_{\alpha+\beta}$; *moreover, there exists a positive constant* $\rho > 0$ *such that*

$$\begin{aligned} \|\mathbf{a}\tilde{\mathbf{a}}\|_{\mathbf{C}_{\alpha+\beta}} &\leqslant \rho \|\mathbf{a}\|_{\mathbf{C}_\alpha} \|\tilde{\mathbf{a}}\|_{\mathbf{C}_\beta}, \\ \|\mathbf{b}\tilde{\mathbf{b}}\|_{\mathbf{C}_{\alpha+\beta}} &\leqslant \rho \|\mathbf{b}\|_{\mathbf{S}_\alpha} \|\tilde{\mathbf{b}}\|_{\mathbf{S}_\beta}, \\ \|\mathbf{a}\tilde{\mathbf{b}}\|_{\mathbf{S}_{\alpha+\beta}} &\leqslant \rho \|\mathbf{a}\|_{\mathbf{C}_\alpha} \|\tilde{\mathbf{b}}\|_{\mathbf{S}_\beta}. \end{aligned} \tag{5.1}$$

Lemma 5.3. *For every* $\alpha \in [0,1]$, *the operator* $Vf(x) = (1-2x)f(x)$ *acts boundedly in* W_2^α.

The claim follows directly from the Interpolation theorem A.1.

Lemma 5.4. *Suppose that* $\alpha \in [0, 1/2)$. *Then the Hilbert space*

$$H_\alpha := \left\{ \mathbf{a} = (a_n) \in \ell_2(\mathbb{Z}_+) \;\middle|\; a_0 = 0,\ \sum_{n=1}^{\infty} n^{2\alpha} |a_n|^2 < \infty \right\}$$

with norm $\|\mathbf{a}\|_{H_\alpha} := (\sum_{n=1}^\infty n^{2\alpha}|a_n|^2)^{1/2}$ *coincides with the space* $\mathbf{S}_\alpha$, *and the norms* $\|\cdot\|_{H_\alpha}$ *and* $\|\cdot\|_{\mathbf{S}_\alpha}$ *are equivalent.*

This is a corollary of a well-known fact about Fourier transforms of spaces W_2^α, see, e.g., [18, 25,31].

Proof of Theorem 5.1. Using the obvious relations

$$\begin{aligned} \cos\xi_{2n-1} &= (-1)^n \sin\tilde{\xi}_{2n-1}, \\ \sin\xi_{2n} &= (-1)^n \sin\tilde{\xi}_{2n}, \\ \cos\bigl[\xi_{2n-1}(1-2x)\bigr] &= (-1)^n \sin\bigl[\tilde{\xi}_{2n-1}(1-2x) - (2n-1)\pi x\bigr], \\ \sin\bigl[\xi_{2n}(1-2x)\bigr] &= (-1)^n \sin\bigl[\tilde{\xi}_{2n}(1-2x) - 2n\pi x\bigr] \end{aligned}$$

in the equalities $F_{\mathrm{c}}(\xi_{2n-1}) = 0$ and $F_{\mathrm{s}}(\xi_{2n}) = 0$, we find that

$$\sin\tilde{\xi}_n + \int_0^1 f(x)\sin\bigl[\tilde{\xi}_n(1-2x) - \pi n x\bigr]\,\mathrm{d}x = 0, \quad n \in \mathbb{N}. \tag{5.2}$$

Writing $\sin[\tilde{\xi}_n(1-2x) - \pi n x]$ as $\sin[\tilde{\xi}_n(1-2x)]\cos(\pi n x) - \sin(\pi n x)\cos[\tilde{\xi}_n(1-2x)]$, developing $\sin[\tilde{\xi}_n(1-2x)]$ and $\cos[\tilde{\xi}_n(1-2x)]$ into the Taylor series, and then changing summation and integration order (which is allowed in view of the absolute convergence of the Taylor series and the integrals), we represent (5.2) as

$$\sin\tilde{\xi}_n + \sum_{k=0}^{\infty}(-1)^k \frac{\tilde{\xi}_n^{2k+1}}{(2k+1)!} c_n\big(V^{2k+1}f\big) - \sum_{k=0}^{\infty}(-1)^k \frac{\tilde{\xi}_n^{2k}}{(2k)!} s_n\big(V^{2k}f\big) = 0. \tag{5.3}$$

Set $\tilde{\xi}_0 := 0$; then, as was mentioned above, the sequence $\mathbf{a} := (\tilde{\xi}_n)_{n\in\mathbb{Z}_+}$ belongs to $\ell_2(\mathbb{Z}_+)$, so that $\mathbf{a} \in \mathbf{S}_0$. Define the sequence $\mathbf{d} := (d_n)_{n\in\mathbb{Z}_+}$ through the relation

$$\mathbf{d} := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \mathbf{a}^{2k} \mathbf{s}\big(V^{2k}f\big) - \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \mathbf{a}^{2k+1} \mathbf{c}\big(V^{2k+1}f\big).$$

Using Lemmas 5.2 and 5.3 and denoting by ρ_1 the norm of the operator V in W_2^α, we find that

$$\big\|\mathbf{a}^{2k}\mathbf{s}\big(V^{2k}f\big)\big\|_{\mathbf{S}_\alpha} \leqslant (\rho\rho_1)^{2k}\|\mathbf{a}\|_{\mathbf{S}_0}^{2k}\|f\|_{W_2^\alpha},$$
$$\big\|\mathbf{a}^{2k+1}\mathbf{c}\big(V^{2k+1}f\big)\big\|_{\mathbf{S}_\alpha} \leqslant (\rho\rho_1)^{2k+1}\|\mathbf{a}\|_{\mathbf{S}_0}^{2k+1}\|f\|_{W_2^\alpha},$$

whence $\mathbf{d} \in \mathbf{S}_\alpha$.

Equation (5.3) implies that $\sin\tilde{\xi}_n = d_n$ for all $n \in \mathbb{Z}_+$; henceforth there exists $n_0 \in \mathbb{N}$ such that

$$|\tilde{\xi}_n| \leqslant 2|d_n| \tag{5.4}$$

for all $n \geqslant n_0$. Fix an arbitrary number $\beta \in (\alpha/3, \alpha/2)$; then $\beta < \alpha/2 \leqslant 1/2$ and $\mathbf{d} \in \mathbf{S}_\beta$. Lemma 5.4 and inequalities (5.4) now yield the inclusion $\mathbf{a} \in \mathbf{S}_\beta$.

Since the sequence $(\sin\tilde{\xi}_n)_{n\in\mathbb{Z}_+}$ can be written as

$$\sin\mathbf{a} := \sum_{k=0}^{\infty}(-1)^k \frac{\mathbf{a}^{2k+1}}{(2k+1)!}$$

and since by Lemma 5.2 and the inequality $2\beta < 1$ the series

$$\tilde{\mathbf{d}} := \sin\mathbf{a} - \mathbf{a} = \sum_{k=1}^{\infty}(-1)^k \frac{\mathbf{a}^{2k+1}}{(2k+1)!}$$

converges in $\mathbf{S}_{3\beta}$, we find that

$$\mathbf{a} = \mathbf{d} - \tilde{\mathbf{d}} \in \mathbf{S}_\alpha + \mathbf{S}_{3\beta} \subset \mathbf{S}_\alpha. \tag{5.5}$$

By the definitions of $\mathbf{d}$ and $\tilde{\mathbf{d}}$ equality (5.5) can be recast as

$$\mathbf{a} = \sum_{k=0}^{\infty}(-1)^k \frac{\mathbf{a}^{2k}}{(2k)!}\mathbf{s}\big(V^{2k}f\big) - \sum_{k=0}^{\infty}(-1)^k \frac{\mathbf{a}^{2k+1}}{(2k+1)!}\mathbf{c}\big(V^{2k+1}f\big) - \sum_{k=1}^{\infty}(-1)^k \frac{\mathbf{a}^{2k+1}}{(2k+1)!}.$$

Since in view of Lemmas 5.2 and 5.3 the sum

$$\sum_{k=1}^{\infty}(-1)^k\frac{\mathbf{a}^{2k}}{(2k)!}\mathbf{s}\big(V^{2k}f\big)-\sum_{k=1}^{\infty}(-1)^k\frac{\mathbf{a}^{2k+1}}{(2k+1)!}\mathbf{c}\big(V^{2k+1}f\big)-\sum_{k=1}^{\infty}(-1)^k\frac{\mathbf{a}^{2k+1}}{(2k+1)!}$$

falls into $\mathbf{S}_\gamma$, we conclude that

$$\mathbf{a}-\mathbf{s}(f)+\mathbf{ac}(Vf)\in\mathbf{S}_\gamma,$$

so that also

$$\big[\mathbf{a}-\mathbf{s}(f)\big]\mathbf{c}(Vf)\in\big[-\mathbf{ac}(Vf)+\mathbf{S}_\gamma\big]\mathbf{c}(Vf)\subset\mathbf{S}_\gamma.$$

This yields the desired result

$$\mathbf{a}-\mathbf{s}(f)+\mathbf{s}(f)\mathbf{c}(Vf)\in\mathbf{S}_\gamma,$$

and the theorem is proved. □

6. Proof of Theorems 1.1, 1.2 and Corollary 1.3

It suffices to establish only the refined asymptotics (1.11). Indeed, since by Lemmas A.3 and A.4 the functions $\int_0^x\sigma^2(t)\,\mathrm{d}t$ and $\int_0^{1-x}\sigma(x+t)\sigma(t)\,\mathrm{d}t$ belong to $W_2^{2\alpha}$ and by Lemmas 5.2 and 5.3 the numbers $s_n(\sigma)c_n(V\sigma)$ are nth sine Fourier coefficients of some function from $W_2^{2\alpha}$, formula (1.10) follows from (1.11).

Assume first that the potential $q\in W_2^{\alpha-1}$ is such that the associated quadratic form $\mathfrak{t}_\mathrm{N}$ is strictly accretive. Then by Proposition 2.1 we can find a function $\tau\in W_2^\alpha$ such that $\phi:=\tau-\sigma\in W_2^{2\alpha}\cap W_1^1$ and $l_\sigma=m_\tau$. By Corollary 4.5 the functions $\tau^\pm$ of (3.11) can be represented as

$$\tau^\pm=\pm\tau+\tau_2+\phi^\pm$$

with $\tau_2=I_2(\tau,\tau)$ and some W_2^γ-functions $\phi^\pm$. Taking into account Proposition 2.1 and Lemma A.4, we conclude that

$$\tau^\pm=\sigma^\pm+\tilde\phi^\pm,\tag{6.1}$$

where

$$\sigma^\pm(x):=\pm\sigma(x)\mp\int_0^x\sigma^2(t)\,\mathrm{d}t+I_2(\sigma,\sigma)(x)\tag{6.2}$$

and $\tilde\phi^\pm\in W_2^\gamma$. By virtue of Theorems 3.2 and 5.1 there exist functions $\psi^\pm\in W_2^\gamma$ such that

$$\begin{aligned}\tilde\lambda_n&=s_{2n}\big(\tau^-\big)-s_{2n}\big(\tau^-\big)c_{2n}\big(V\tau^-\big)+s_{2n}\big(\psi^-\big),\\ \tilde\mu_n&=s_{2n-1}\big(\tau^+\big)-s_{2n-1}\big(\tau^+\big)c_{2n-1}\big(V\tau^+\big)+s_{2n-1}\big(\psi^+\big).\end{aligned}\tag{6.3}$$

Since $(\tau^{\pm} \mp \sigma) \in W_2^{2\alpha}$, Eqs. (6.1)–(6.3) and Lemma 5.2 yield the representation

$$\begin{aligned}
\tilde{\lambda}_n &= s_{2n}(\sigma^-) - s_{2n}(\sigma)c_{2n}(V\sigma) + s_{2n}(\tilde{\psi}^-),\\
\tilde{\mu}_n &= s_{2n-1}(\sigma^+) - s_{2n-1}(\sigma)c_{2n-1}(V\sigma) + s_{2n-1}(\tilde{\psi}^+)
\end{aligned}\tag{6.4}$$

with some $\tilde{\psi}^{\pm} \in W_2^{\gamma}$. It remains to put $\omega := \frac{1}{2}(\tilde{\psi}^+ + R\tilde{\psi}^+ + \tilde{\psi}^- - R\tilde{\psi}^-)$ to get the required formula.

In a generic situation we add a suitable constant C to the potential q to get a potential $\hat{q} := q + C$ that falls into the above-considered case. (This can be done since the quadratic form $\mathfrak{t}_{\mathrm{N}}$ is bounded below, see Section 2.) Then the corresponding Dirichlet eigenvalues $\hat{\lambda}_n^2$ and Neumann–Dirichlet eigenvalues $\hat{\mu}_n^2$ have the form (6.4) with σ replaced by $\hat{\sigma} := \sigma + Ct$ and with $\hat{\sigma}^{\pm}$ calculated as in (6.2) for $\hat{\sigma}$ instead of σ. Since $\sigma - \hat{\sigma} \in W_2^2$, it is easily seen that $\sigma^{\pm} - \hat{\sigma}^{\pm} \in W_2^{\gamma}$. Calculating now the integrals, we arrive at the relations

$$\begin{aligned}
\hat{\lambda}_n &= \pi n + s_{2n}(\sigma^-) - s_{2n}(\sigma)c_{2n}(V\sigma) + s_{2n}(\hat{\psi}^-),\\
\hat{\mu}_n &= \pi(n - 1/2) + s_{2n-1}(\sigma^+) - s_{2n-1}(\sigma)c_{2n-1}(V\sigma) + s_{2n-1}(\hat{\psi}^+)
\end{aligned}$$

for some $\hat{\psi}^{\pm} \in W_2^{\gamma}$. Since $\hat{\lambda}_n = \pi n + a_n$ with $(a_n) \in \ell_2$, we find that

$$\hat{\lambda}_n - \lambda_n = \hat{\lambda}_n - \sqrt{\hat{\lambda}_n^2 - C} = \frac{C}{2\hat{\lambda}_n} + \mathrm{O}(\hat{\lambda}_n^{-3}) = \frac{C}{2\pi n} + \frac{b_n}{n^2}$$

for some $(b_n) \in \ell_2$, so that there exists a function $\chi \in W_2^2$ such that

$$s_{2n}(\chi) = \hat{\lambda}_n - \lambda_n, \quad n \in \mathbb{N}.$$

Thus

$$\tilde{\lambda}_n = s_{2n}(\sigma^-) - s_{2n}(\sigma)c_{2n}(V\sigma) + s_{2n}(\hat{\phi}^-)$$

with some $\hat{\phi}^- \in W_2^{\gamma}$. Similar arguments work for $\tilde{\mu}_n$ and yield the representation

$$\tilde{\mu}_n = s_{2n-1}(\sigma^+) - s_{2n-1}(\sigma)c_{2n-1}(V\sigma) + s_{2n-1}(\hat{\phi}^+)$$

with some $\hat{\phi}^+ \in W_2^{\gamma}$. It remains to put $\omega := \frac{1}{2}(\hat{\phi}^+ + R\hat{\phi}^+ + \hat{\phi}^- - R\hat{\phi}^-)$, and the proof of Theorem 1.2 is complete.

Remark 6.1. Straightforward calculations show that the asymptotic formula (1.11) established here corresponds to the asymptotic formulae (1.8), (1.9) of the paper [31] with $\rho_n = 2s_{2n}(\sigma)c_{2n}(t\sigma) + s_{2n}(\omega)$. Since $(\rho_n) \in \ell_2^{2\alpha}$ by the results of [31] and $s_{2n}(\sigma)c_{2n}(t\sigma)$ falls into $\ell_2^{2\alpha}$ for $\alpha \in [0, 1/2)$, one concludes that $(s_{2n}(\omega)) \in \ell_2^{2\alpha}$ for such α. Corollary 1.3 states that, moreover, $(s_{2n}(\omega)) \in \ell_\infty^{\gamma}$ for all $\alpha \in [0, 1]$.

Proof of Corollary 1.3. Assume that $\sigma \in W_2^\alpha(0,1)$ for some $\alpha \in [0,1]$ and $\omega \in W_2^\gamma$ is the function of Theorem 1.2. If $\alpha \in [0,1/3]$, then by Lemma A.5 one has

$$s_{2n}(\omega) = \mathrm{O}\big(n^{-3\alpha}\big), \quad n \to \infty,$$

and, in view of (1.11),

$$\lambda_n = \pi n + s_{2n}\big(\sigma^-\big) - s_{2n}(\sigma)c_{2n}(V\sigma) + \mathrm{O}\big(n^{-3\alpha}\big), \quad n \to \infty.$$

Let now $\alpha \in (1/3, 1]$. Since the function $\sigma \in W_2^\alpha$ satisfy then condition (vi) of Theorem 3.12 of [31], one has

$$\lambda_n = \pi n - s_{2n}(\sigma) + a_n$$

with $(a_n) \in \ell_1$. In view of (1.11) this yields

$$s_{2n}(\omega) = -s_{2n}(\psi) - b_n,$$

where

$$\psi(x) := \big(\sigma^- + \sigma\big)(x) = \int\limits_0^x \sigma^2(t)\,\mathrm{d}t + \int\limits_0^{1-x} \sigma(x+t)\sigma(t)\,\mathrm{d}t \tag{6.5}$$

and $(b_n) \in \ell_1$. By virtue of Lemmas A.3 and A.4 the function ψ belongs to the space $W_2^{2\alpha}$. Since, moreover, $\psi(0) = \psi(1)$, Lemma A.5 (see also Remark A.6) yields the inclusion $(s_{2n}(\psi))_{n\in\mathbb{N}} \in \ell_2^{2\alpha} \subset \ell_1$, i.e., $(s_{2n}(\omega))_{n\in\mathbb{N}} \in \ell_1$. Taking into account that $\gamma \geqslant 1$, we conclude that $\omega(0) = \omega(1)$. Lemma A.5 yields now the inclusion $(s_{2n}(\omega)) \in \ell_2^\gamma$ and the required relation (1.12) follows. □

Remark 6.2. Developing the above arguments, we can show that the function $\tilde\sigma$ of Theorem 1.1 is such that $(s_{2n}(\tilde\sigma)) \in \ell_2^{2\alpha}$. For $\alpha \in [0, 1/2)$ this is shown in [31]. Observe that

$$s_{2n}(\tilde\sigma) = s_{2n}(\omega) + s_{2n}(\sigma)c_{2n}(V\sigma) + s_{2n}(\psi)$$

with the function ω of Theorem 1.2 and ψ of (6.5). For $\alpha \in [1/2, 1]$ the proof of Corollary 1.3 establishes the inclusions $(s_{2n}(\omega)) \in \ell_2^\gamma \subset \ell_2^{2\alpha}$ and $(s_{2n}(\psi)) \in \ell_2^{2\alpha}$, while

$$\big(s_{2n}(\sigma)c_{2n}(V\sigma)\big) \in \ell_\infty^\alpha \cdot \ell_2^\alpha \subset \ell_2^{2\alpha}$$

by Lemma A.5.

If $q \in W_2^{\alpha-1}$ is periodic, then we find that

$$s_{2n}(\sigma) = -\frac{c_0(q)}{2\pi n} + \frac{c_{2n}(q)}{2\pi n}.$$

Inserting this into (1.10) and squaring, we get

$$\lambda_n^2 = \pi^2 n^2 + c_0(q) - c_{2n}(q) + d_n$$

with $(d_n) \in \ell_2^{2\alpha-1}$, which is to be compared to the corresponding result of [18], see (1.7). One can also show as in [18] that this estimate is uniform in q from bounded subsets of $W_2^{\alpha-1}$.

Remark 6.3. If $\alpha = 1$, then $\sigma(x) = h + \int_0^x q$ for some $h \in \mathbb{C}$, and formulae (1.11) give

$$\lambda_n = \pi n + \frac{\int_0^1 q}{2\pi n} + \frac{\tilde{\lambda}'_n}{n}, \qquad \mu_n = \pi\left(n - \frac{1}{2}\right) + \frac{2h + \int_0^1 q}{2\pi n} + \frac{\tilde{\mu}'_n}{n}$$

for some ℓ_2-sequences $(\tilde{\lambda}'_n)$ and $(\tilde{\mu}'_n)$. The choice $h = 0$ corresponds to the genuine Neumann boundary condition $y'(0) = 0$ at $x = 0$ for the operator T_{N}, while $h \neq 0$ produces the Robin boundary condition $y'(0) = hy(0)$, cf. (1.6) and below.

7. Application to inverse spectral problems

In a selfadjoint regular situation (i.e., for real-valued integrable q) the classical result of the inverse spectral theory states that the spectra λ_n^2 and μ_n^2 of Sturm–Liouville operators T_{D} and T_{N} completely determine the potential. In general, the reconstruction algorithm is quite nontrivial and requires solvability of the so-called Gelfand–Levitan–Marchenko equation. In this section we shall show how Theorem 1.1 can be used in the inverse spectral analysis for some singular (complex-valued) potentials.

Given the eigenvalues λ_n^2 and μ_n^2 of T_{D} and T_{N}, respectively, we denote by σ^* the function

$$\sigma^*(t) := 2\sum_{n=1}^{\infty}\left(\tilde{\mu}_n \sin\left[(2n-1)\pi t\right] - \tilde{\lambda}_n \sin[2\pi n t]\right). \tag{7.1}$$

Here, as usual, $\tilde{\lambda}_n$ and $\tilde{\mu}_n$ are defined through (1.5) and thus the series converges in L_2 as soon as $q \in W_2^{\alpha-1}$ with $\alpha \in [0, 1]$. We also observe that, according to Theorem 1.1, $\sigma^* = \sigma + \tilde{\sigma}$ for the function $\tilde{\sigma} \in W_2^{2\alpha}$ of that theorem.

Proposition 7.1. *Assume that $q \in W_2^{\alpha-1}$, $\alpha \in (0, 1)$, is such that the function σ^* of (7.1) belongs to W_2^{β} for some $\beta \in (0, 1]$, $\beta > \alpha$. Then $q \in W_2^{\beta-1}$.*

Proof. We use the so-called bootstrap method. Since $\sigma^* = \sigma + \tilde{\sigma}$ and $\tilde{\sigma} \in W_2^{2\alpha}$ in virtue of Theorem 1.1, we claim that σ, in fact, belongs to $W_2^{\alpha'}$ with $\alpha' = \min\{2\alpha, \beta\} > \alpha$. Thus the exponent α can either be taken equal to β or otherwise doubled. Repeating this procedure finitely many times, we reach the desired conclusion that $\sigma \in W_2^{\beta}$, i.e., that $q \in W_2^{\beta-1}$. $\square$

Developing the above arguments, we conclude that the function σ^* determines all principal singularities of the potential q. The meaning of this claim is that the functions σ and σ^* share all the singularities typical for W_2^{α}. We illustrate this issue by the following example.

Assume that q is such that $\sigma(x) = \int_0^x q(t)\,\mathrm{d}t$ has bounded variation over $[0, 1]$ (i.e., $\mathrm{d}\sigma$ is a finite Borel measure). Slightly abusing the terminology, we shall say that the potential q of the corresponding Sturm–Liouville operators $T_{\sigma,\mathrm{D}}$ and $T_{\sigma,\mathrm{N}}$ is a finite Borel measure; see also [6,8, 9,32] for precise definitions.

Proposition 7.2. *Assume that q is a finite Borel measure and σ^* is the function of* (7.1) *constructed through the corresponding Dirichlet λ_n^2 and Neumann–Dirichlet μ_n^2 eigenvalues. Then the discrete parts of the measures q and $\mathrm{d}\sigma^*$ coincide.*

Proof. Observe that $\sigma = \int q$, being of bounded variation over $[0, 1]$, belongs to $W_2^{1/2-\varepsilon}$ for all $\varepsilon \in (0, 1/2)$; henceforth $\tilde{\sigma} = \sigma^* - \sigma$ belongs to $W_2^{1/2+\varepsilon}$ for all $\varepsilon \in (0, 1/2)$ and thus is continuous. Therefore σ^* has the same (jump) discontinuities as σ, i.e., the measures $\mathrm{d}\sigma^*$ and $q = \mathrm{d}\sigma$ have the same discrete parts. □

As a final remark, we note the following. Roughly speaking, formula (1.11) implies that the Dirichlet spectrum determines the even part of the potential q, while the Neumann–Dirichlet spectrum the odd part of q. This explains why the unique reconstruction of the potential by the Dirichlet spectrum is possible if the odd part of q is prespecified, see [28].

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Appendix A. Interpolation and all that

We recall here some facts about interpolation between W_2^α spaces. For details, we refer the reader to [7,22,25].

By definition, the space W_2^0 coincides with L_2 and the norm $|\cdot|_0$ in W_2^0 is just the L_2-norm. The Sobolev space W_2^2 consists of all functions f in L_2, whose distributional derivatives f' and f'' also fall into L_2. Being endowed with the norm

$$|f|_2 := \left(|f|_0^2 + |f'|_0^2 + |f''|_0^2\right)^{1/2},$$

W_2^2 becomes the Hilbert space.

Now we interpolate between W_2^2 and W_2^0 to get the intermediate spaces W_2^s with norms $|\cdot|_s$ for $s \in (0, 2)$; namely, $W_2^{2s} := [W_2^2, W_2^0]_{1-s}$. The norms $|\cdot|_s$ are nondecreasing with $s \in [0, 2]$, i.e., if $s < t$ and $f \in W_2^t$, then $|f|_s \leqslant |f|_t$. The space W_2^1 consists of L_2 functions whose distributional derivative is again in L_2, and $(|f|_0^2 + |f'|_0^2)^{1/2}$ is the norm on W_2^1 that is equivalent to $|\cdot|_1$. Also, the Sobolev embedding theorem implies that W_2^α for $\alpha > 1/2$ is continuously embedded into $C[0, 1]$, the space of continuous functions.

The following result [7,22,25] provides a powerful tool in the study of mappings between the spaces W_2^s.

Theorem A.1 *(Interpolation theorem). Assume that $0 \leqslant s_0 < s_1$, $0 \leqslant r_0 < r_1$, and let T be a linear operator such that*

$$|Tf|_{r_0} \leqslant C_0 |f|_{s_0}, \qquad |Tg|_{r_1} \leqslant C_1 |g|_{s_1}$$

for all $f \in W_2^{s_0}$ and all $g \in W_2^{s_1}$. Put $s_t := (1-t)s_0 + ts_1$ and $r_t := (1-t)r_0 + tr_1$; then for every $t \in [0, 1]$ the operator T acts boundedly from $W_2^{s_t}$ to $W_2^{r_t}$ with norm not exceeding $C_0^{1-t} C_1^t$.

We formulate next one result on multilinear interpolation from [7] adapted to our purposes. It concerns analytic scales of Banach spaces (see also [22]), and we observe that the scales $\{W_2^{a\theta+b}\}_{\theta\in[0,1]}$ are analytic for any $a>0$ and $b\geqslant 0$.

Theorem A.2. *Assume that $\{E_\theta\}_{\theta\in[0,1]}$ and $\{G_\theta\}_{\theta\in[0,1]}$ are analytic scales of Banach spaces, $n\in\mathbb{N}$, and that $\mathcal{J}$ is a multilinear mapping from $(E_1)^n$ into G_1 satisfying the inequalities*

$$\big\|\mathcal{J}(\mathbf{g})\big\|_{G_0}\leqslant C_0\prod_{j=1}^{n}\|g_j\|_{E_0},\qquad \big\|\mathcal{J}(\mathbf{g})\big\|_{G_1}\leqslant C_1\prod_{j=1}^{n}\|g_j\|_{E_1}$$

for some positive constant C_0, C_1 and all $\mathbf{g}:=(g_1,g_2,\dots,g_n)\in(E_1)^n$. Then $\mathcal{J}$ can be uniquely extended to a multilinear mapping from $(E_\theta)^n$ into G_θ, $0\leqslant\theta\leqslant 1$, with norm not exceeding $C_0^{1-\theta}C_1^{\theta}$.

The following result is used in various places of the paper.

Lemma A.3. *Assume that $f\in W_2^\alpha$ and $g\in W_2^\beta$ for some $\alpha,\beta\in[0,1]$. Then the function $h(x):=\int_0^x(fg)$ belongs to $W_2^{\alpha+\beta}$ and there exists some constant C such that*

$$|h|_{\alpha+\beta}\leqslant C|f|_\alpha|g|_\beta. \tag{A.1}$$

Proof. For $\alpha=\beta=0$ the function h is absolutely continuous and $|h(x)|\leqslant|f|_0|g|_0$ by the Cauchy–Schwarz inequality, so that (A.1) holds with $C=1$. For $\alpha=0$ and $\beta=1$, we find that

$$|h'|_0=|fg|_0\leqslant\Big(\max_x\big|g(x)\big|\Big)|f|_0\leqslant C_1|f|_0|g|_1;$$

henceforth $h\in W_2^1$ and (A.1) is satisfied for some $C=C_2\geqslant 1$. To treat the case $\alpha=1$, $\beta=0$, just interchange f and g. For $\alpha=\beta=1$ we find that $h''=f'g+fg'\in L_2$ so that $h\in W_2^2$ with $|h|_2\leqslant C_3|f|_1|g|_1$.

Consider now a mapping $M_f:L_2\to L_2$ given by $M_fg(x)=\int_0^x(fg)$ with $f\in L_2$ fixed. Then, by the above, M_f acts boundedly in L_2 and W_2^1 and

$$\|M_f\|_{L_2\to L_2},\ \|M_f\|_{W_2^1\to W_2^1}\leqslant C_2|f|_0.$$

By interpolation, M_f is continuous in W_2^β for every $\beta\in[0,1]$ and its norm $\|M_f\|_{W_2^\beta\to W_2^\beta}$ is bounded by $C_2|f|_0$; in particular,

$$|h|_\beta\leqslant C_2|f|_0|g|_\beta.$$

Analogously, for a fixed $f\in W_2^1$ the operator M_f maps continuously W_2^β into $W_2^{1+\beta}$ and

$$|h|_{1+\beta}\leqslant\max\{C_2,C_3\}|f|_1|g|_\beta.$$

The two above-displayed formulae show that M_g for a fixed $g\in W_2^\beta$ is continuous as a mapping from L_2 into W_2^β and from W_2^1 into $W_2^{1+\beta}$. Interpolation now yields its continuity as a mapping from W_2^α into $W_2^{\alpha+\beta}$ for an arbitrary $\alpha \in [0,1]$ and establishes inequality (A.1). The lemma is proved. □

The next lemma establishes the required continuity of the mapping I_2, see Section 4.

Lemma A.4. *Assume that $f \in W_2^\alpha$ and $g \in W_2^\beta$ for some $\alpha, \beta \in [0,1]$. Then the function*

$$h(x) := \int_0^{1-x} f(x+t)g(t)\,\mathrm{d}t$$

belongs to $W_2^{\alpha+\beta}$ and there exists some constant C such that

$$|h|_{\alpha+\beta} \leqslant C|f|_\alpha |g|_\beta.$$

Proof. The proof is completely analogous to the one of Lemma A.3: one establishes first the statement for α and β equal to 0 or 1, and then interpolate. The only remark is that for $\beta = 1$ one should use a representation $h(x) = \int_x^1 f(s)g(s-x)\,\mathrm{d}s$ to be able to differentiate h. □

Lemma A.5.

(a) *Assume that $f \in W_2^\alpha$, $\alpha \in [0,1]$; then $s_{2n}(f) \in \ell_\infty^\alpha$ and $c_{2n}(f) \in \ell_2^\alpha$.*
(b) *Assume that $g \in W_2^\beta$, $\beta \in (1,2]$, and $g(0) = g(1)$; then $(s_{2n}(g)) \in \ell_2^\beta$.*

Proof. Part (a) follows by interpolation. Indeed, the mapping

$$S: f \mapsto \big(s_{2n}(f)\big)_{n\in\mathbb{N}}$$

is bounded from L_2 into ℓ_∞ and from W_2^1 into ℓ_∞^1. Since the spaces ℓ_∞^s form an analytic Banach scale [22], Interpolation theorem A.2 yields the result about $s_{2n}(f)$. Similar arguments justify the statement about $c_{2n}(f)$.

Part (b) requires only a slight modification. For an arbitrary $s \in [1,2]$ we put

$$\tilde W_2^s := \big\{h \in W_2^s \;\big|\; h(0) = h(1)\big\}.$$

Since the family $\{W_2^{\theta+1}\}_{\theta\in[0,1]}$ forms a Hilbert scale, by virtue of the general interpolation result from [25, Chapter 1, 13.4], the family $\{\tilde W_2^{\theta+1}\}_{\theta\in[0,1]}$ also is a Hilbert scale. Simple integration by parts shows that the above operator S maps continuously $\tilde W_2^1$ into ℓ_2^1 and $\tilde W_2^2$ into ℓ_2^2. By Interpolation theorem A.2 the operator S maps continuously $\tilde W_2^{\theta+1}$ into $\ell_2^{\theta+1}$ for all $\theta \in [0,1]$, and the proof is complete. □

Remark A.6. Part (b) of the lemma also holds true for $\beta \in (1/2, 1)$ (see, e.g., [18]).

For L_2-functions f and g we denote by $f * g$ their convolution, i.e.,

$$(f * g)(x) := \int_0^x f(x-t)g(t)\,\mathrm{d}t.$$

Recall also that R is the reflection operator about $x = 1/2$: $(Rf)(x) = f(1-x)$.

Lemma A.7. *Suppose that $f, g \in L_2$; then we have*

$$c_n(f)c_n(g) = c_n(h_1), \qquad s_n(f)s_n(g) = c_n(h_2), \qquad s_n(f)c_n(g) = s_n(h_3),$$

where the functions h_j, $j = 1, 2, 3$, are given by

$$\begin{aligned} h_1 &= \frac{1}{2}\bigl\{R(Rf * g + f * Rg) + f * g + Rf * Rg\bigr\}, \\ h_2 &= \frac{1}{2}\bigl\{R(Rf * g + f * Rg) - f * g - Rf * Rg\bigr\}, \\ h_3 &= \frac{1}{2}\bigl\{R(Rf * g - f * Rg) + f * g - Rf * Rg\bigr\}. \end{aligned} \tag{A.2}$$

Proof. We shall prove only the first equality, since the other ones are treated analogously. We have

$$2c_n(f)c_n(g) = \int_0^1 \int_0^1 f(x)g(t)\bigl[\cos \pi n(x-t) + \cos \pi n(x+t)\bigr]\,\mathrm{d}x\,\mathrm{d}t,$$

and simple calculations lead to

$$\begin{aligned} &\int_0^1 \int_0^1 f(x)g(t)\cos \pi n(x-t)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_0^1 \left(\int_0^{1-s} f(s+t)g(t)\,\mathrm{d}t + \int_0^{1-s} f(t)g(s+t)\,\mathrm{d}t \right) \cos \pi ns\,\mathrm{d}s, \\ &\int_0^1 \int_0^1 f(x)g(t)\cos \pi n(x+t)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_0^1 \left(\int_0^{s} f(s-t)g(t)\,\mathrm{d}t + \int_0^{s} f(1-t)g(1-s+t)\,\mathrm{d}t \right) \cos \pi ns\,\mathrm{d}s. \end{aligned}$$

Taking into account the relations

$$\int_0^{1-s} f(s+t)g(t)\,\mathrm{d}t = R(Rf*g)(s), \qquad \int_0^{s} f(1-t)g(1-s+t)\,\mathrm{d}t = Rf*Rg,$$

we get $c_n(f)c_n(g) = c_n(h_1)$ with h_1 as stated. The lemma is proved. $\square$

Proof of Lemma 5.2. By Lemma A.7 we have that $\mathbf{c}(f)\mathbf{c}(g) = \mathbf{c}(h_1)$, $\mathbf{s}(f)\mathbf{s}(g) = \mathbf{c}(h_2)$, and $\mathbf{s}(f)\mathbf{c}(g) = \mathbf{s}(h_3)$ for the functions h_1, h_2, and h_3 given by (A.2).

It is easily verified that, with the operator I_2 of Section 4, we have

$$I_2(f,g) = R(Rf*g).$$

Recalling that R is unitary in every W_2^α, we conclude by Lemma A.4 that the functions h_j of (A.2) are in $W_2^{\alpha+\beta}$ as soon as $f \in W_2^\alpha$ and $g \in W_2^\beta$ for some $\alpha, \beta \in [0,1]$ and that, moreover, with some $\rho > 0$ the inequality

$$|h_j|_{\alpha+\beta} \leqslant \rho |f|_\alpha |g|_\beta$$

holds. Recalling the definition of the norm in $\mathbf{S}_\alpha$ and $\mathbf{C}_\alpha$, we get the required estimates (5.1). $\square$

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