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Density of monodromy actions on non-abelian cohomology

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Abstract

In this paper we study the monodromy action on the first Betti and de Rham non-Abelian cohomology arising from a family of smooth curves. We describe sufficient conditions for the existence of a Zariski-dense monodromy orbit. In particular, we show that for a Lefschetz pencil of sufficiently high degree the monodromy action is dense.

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1. Introduction

We work in the category of quasi-projective schemes over \mathbb{C} . Let $f : X \rightarrow B$ be a smooth projective morphism with connected fibers of dimension one and genus at least two. Fix a base point $o \in B$ and let X_o be the corresponding fiber of f . In this paper we study the monodromy action of $\pi_1(B, o)$ on the degree one non-Abelian Betti and de Rham cohomology of X_o . Let $\pi_1(X_o)$ denote the abstract fundamental group of X_o and let

$$\text{mon} : \pi_1(B, o) \rightarrow \text{Out}(\pi_1(X_o))$$

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be the geometric monodromy representation of the family $f: X \rightarrow B$. For any positive integer n let

$$\text{mon}_B^n: \pi_1(B, o) \rightarrow \text{Aut}(H_B^1(X_o, \text{GL}(n, \mathbb{C})))$$

be the induced monodromy action on the non-Abelian Betti cohomology with coefficients in $\text{GL}(n, \mathbb{C})$.

The de Rham object which corresponds to mon_B^n is the Gauss–Manin connection ∇_{DR}^n on the relative de Rham cohomology stack $H_{\text{DR}}^1(X/B, \text{GL}(n, \mathbb{C}))$. While the non-Abelian Betti and de Rham cohomology are most naturally viewed as stacks, for the purposes of the present paper it suffices to work with the corresponding coarse moduli spaces. To indicate that we will write $M_B(X_o, n)$ and $M_{\text{DR}}(X/B, n)$ rather than $H_B^1(X_o, \text{GL}(n, \mathbb{C}))$ and $H_{\text{DR}}^1(X/B, \text{GL}(n, \mathbb{C}))$. Concretely, $M_B(X_o, n)$ denotes the moduli space of (semisimplifications of) representations of $\pi_1(X_o)$ in $\text{GL}(n, \mathbb{C})$ and $M_{\text{DR}}(X_o, n)$ denotes the moduli space of rank n algebraic local systems of complex vector spaces on X_o . The total space $M_{\text{DR}}(X/B, n) \rightarrow B$ is a quasi-projective variety over B whose fiber over the point o is $M_{\text{DR}}(X_o, n)$.

For a loop $\gamma \in \pi_1(B, o)$ the action of $\text{mon}_B^n(\gamma)$ on $M_B(X_o, n)$ is given by composing a n -dimensional representation of $\pi_1(X_o)$ with some lift of the outer automorphism $\text{mon}(\gamma)$ of $\pi_1(X_o)$. This gives a well-defined action on conjugacy classes of representations of $\pi_1(X_o)$ and results in an algebraic automorphism $\text{mon}_B^n(\gamma): M_B(X_o, n) \rightarrow M_B(X_o, n)$.

There is an analytic action mon_{DR}^n of $\pi_1(B, o)$ on $M_{\text{DR}}(X_o, n)$ which is most naturally described through the Riemann–Hilbert correspondence (see e.g. [2, 17, Section 7]). It is shown in [17, Section 7] that the passage from a local system to its monodromy representation induces an isomorphism of analytic spaces

$$\tau: M_{\text{DR}}(X_o, n)^{\text{an}} \xrightarrow{\sim} M_B(X_o, n)^{\text{an}}.$$

Now given γ we can define an analytic automorphism of $\text{mon}_{\text{DR}}^n(\gamma)$ of $M_{\text{DR}}(X_o, n)$ by putting $\text{mon}_{\text{DR}}^n(\gamma) = \tau^{-1} \circ \text{mon}_B^n(\gamma) \circ \tau$. This analytic action is the monodromy of the algebraic non-Abelian Gauss–Manin connection ∇_{DR}^n on the total space $M_{\text{DR}}(X/B, n)$ [17, Section 8].

It is natural to try to understand the complexity of the algebraic (respectively analytic) action of $\pi_1(B, o)$ on $M_B(X_o, n)$ (respectively $M_{\text{DR}}(X_o, n)$) by measuring in some way the size of the $\pi_1(B, o)$ -orbits on $M_B(X_o, n)$ and $M_{\text{DR}}(X_o, n)$. Analogous questions concerning the monodromy action of $\pi_1(B, o)$ on spaces of special representations of $\pi_1(X_o)$ (e.g. real representations, representations with compact image, projective structures, etc.) have been the focus of active research in the recent years [4, 5, 12]. In that direction the result most relevant to our setup is a theorem of Goldman who showed in [5] that the mapping class group acts ergodically on the space of all representations of $\pi_1(X_o)$ into $\text{SU}(2)$. Unfortunately, Goldman’s proof does not generalize to the case of representations into $\text{SU}(n)$ for $n > 2$ and we do not know whether the mapping class group of X_o still acts ergodically on the space of such representations. Instead of pursuing the ergodicity question in its full generality we chose to work with non-Abelian cohomology with complex coefficients. This

allows us to use the algebraic (respectively analytic) nature of the monodromy action on $M_B(X_o, n)$ (respectively $M_{DR}(X_o, n)$) and to describe the size of the monodromy orbits on those spaces in geometric, rather than measure-theoretic terms.

Our first result is of essentially topological nature. Before we state it we will need to introduce some notation. Let as before $f : X \rightarrow B$ be a smooth holomorphic family of genus g curves and let $o \in B$ be a base point. Let X_o be the fiber of f over o and let

$$\text{map} : \pi_1(B, o) \rightarrow \text{Map}(X_o) \subset \text{Out}(\pi_1(X_o))$$

be the corresponding geometric monodromy representation. Here $\text{Map}(X_o)$ denotes the mapping class group of X_o . By definition $\text{Map}(X_o) := \pi_0(\text{Diff}^+(X_o))$ is the group of connected components of the group of orientation preserving diffeomorphisms of X_o . Alternatively, $\text{Map}(X_o)$ can be identified with the subgroup of index two in $\text{Out}(\pi_1(X_o))$ consisting of all outer automorphisms which act trivially on $H^2(X_o, \mathbb{Z})$. Fix some topological double cover $v : X_o \rightarrow \mathbb{P}^1$ and let $\iota \in \text{Map}(X_o)$ be the mapping class of the covering involution. The *hyperelliptic mapping class group* of X_o is defined to be the centralizer $\Delta(X_o)$ of ι in $\text{Map}(X_o)$:

$$\Delta(X_o) := \{\phi \in \text{Map}(X_o) \mid \phi \iota \phi^{-1} = \iota\}.$$

Note that the definition of $\Delta(X_o)$ depends on the choice of the double cover v and so the hyperelliptic mapping class group is defined as a subgroup in $\text{Map}(X_o)$ only up to conjugation. We will say that the geometric monodromy of f *dominates the hyperelliptic monodromy* if up to conjugation in $\text{Map}(X_o)$ the monodromy group $\text{mon}(\pi_1(B, o)) \subset \text{Map}(X_o)$ contains a subgroup of finite index in $\Delta(X_o)$.

Theorem A. *Assume that the monodromy of f dominates the hyperelliptic monodromy, e.g. assume that the image $\text{mon}(\pi_1(B, o)) \subset \text{Out}(\pi_1(X_o))$ is of finite index in $\text{Out}(\pi_1(X_o))$. Then there exists a positive integer g_0 so that if $g \geq g_0$ and n is any fixed odd integer, we have*

- (i) *There is no meromorphic function on $M_B(X_o, n)^{\text{an}}$ which is invariant under the action of $\text{mon}_B^n(\pi_1(B, o))$ (equivalently there is no meromorphic function on $M_{DR}(X/B, n)^{\text{an}}$ which is ∇_{DR}^n -invariant);*
- (ii) *In the case of $M_B(X_o, n)$, considered with its natural structure of an affine algebraic variety, there exists a point $x_B \in M_B(X_o, n)$ so that the orbit*

$$\text{mon}_B^n(\pi_1(B, o)) \cdot x_B \subset M_B(X_o, n)$$

is Zariski-dense in $M_B(X_o, n)$; or in the case of $M_{DR}(X/B, n) \rightarrow B$ there is a leaf of the foliation defined by ∇_{DR}^n which is Zariski-dense in the algebraic Zariski topology.

This theorem suggests that for families $f : X \rightarrow B$ with a “large enough” geometric monodromy one should expect Zariski-dense monodromy actions on non-Abelian

cohomology. Geometrically, families with large monodromy naturally arise from hyperplane sections and Lefschetz fibrations. In this context we prove the following

Theorem B. *Let Z be a smooth projective surface with $b_1(Z) = 0$. Let $\mathcal{O}_Z(1)$ be an ample line bundle on Z and let $n > 1$ be a fixed odd integer. Then there exists a positive integer ℓ (depending only on Z and $\mathcal{O}_Z(1)$), such that for every $k \geq \ell$ and for every Lefschetz fibration $f : \widehat{Z} \rightarrow \mathbb{P}^1$ in the linear system $|\mathcal{O}_Z(k)|$ we have*

- (i) *there is no meromorphic function on $M_B(Z_o, n)^{\text{an}}$ which is invariant under the action of $\text{mon}_B^n(\pi_1(\mathbb{P}^1 \setminus \{p_1, \dots, p_\mu\}, o))$ (equivalently there is no meromorphic function on $M_{\text{DR}}(\widehat{Z}/\mathbb{P}^1 \setminus \{p_1, \dots, p_\mu\}, n)^{\text{an}}$ which is ∇_{DR}^n -invariant);*
- (ii) *in the case of $M_B(Z_o, n)$, there exist a point $x_B \in M_B(Z_o, n)$ so that the orbit*

$$\text{mon}_B^n(\pi_1(\mathbb{P}^1 \setminus \{p_1, \dots, p_\mu\}, o)) \cdot x_B \subset M_B(Z_o, n)$$

is Zariski-dense in $M_B(Z_o, n)$; or in the case of the space $M_{\text{DR}}(\widehat{Z}/\mathbb{P}^1 \setminus \{p_1, \dots, p_\mu\}, n)$ the foliation defined by ∇_{DR}^n has a Zariski-dense leaf.

Here as usual \widehat{Z} is the blow-up of Z at the base points of the pencil and $p_1, \dots, p_\mu \in \mathbb{P}^1$ are the points where the map $\widehat{Z} \rightarrow \mathbb{P}^1$ is not submersive.

These statements can be viewed as non-Abelian analogues of Deligne's irreducibility theorem [3, Section 4.4; 7], which asserts that the monodromy group on the first cohomology of a Lefschetz pencil of curves is a subgroup of finite index in the full symplectic group of the lattice of vanishing cycles.

The paper is organized as follows. In Section 2.1 we examine the action of a finitely generated group on an affine algebraic variety. We show how the existence of a Zariski-dense orbit can be deduced from the existence of an open orbit for the linearized action on the tangent space at a fixed point. Section 2.2 describes a particular point in the moduli space of representations of the fundamental group of a curve X_o , which corresponds to the Schrödinger representation of a suitably chosen finite dihedral Heisenberg group. This point is smooth and fixed by a subgroup of finite index in the monodromy. Moreover, the tangent space of the moduli of representations at the 'Schrödinger point' is naturally identified with the first cohomology of an étale cover Y_o of X_o . Finally in Section 4.1, we discuss the necessity of the hypotheses of Theorems A and B for the existence of a dense monodromy orbit. We conjecture that the density holds under very mild assumptions and give some additional evidence supporting the conjecture.

2. Preliminary reductions

We start with some general results about linear group actions on algebraic varieties, which will allow us to localize at a point the Zariski density property of an action.

2.1. Open orbits and dense actions

Suppose M is a reduced irreducible affine scheme of finite type over \mathbb{C} . Write $M = \text{Spec}(A)$ and let Γ be a finitely presented discrete group acting on M by algebraic automorphisms. Thus Γ acts on A by \mathbb{C} -algebra automorphisms.

Lemma 2.1. *If there is one point in M whose orbit under Γ is Zariski-dense, then there is a countable union of proper closed subvarieties of M such that for any x not on this countable union, the orbit of x is Zariski-dense.*

Proof. The fact that A is of finite type over \mathbb{C} means that we have a surjection from a polynomial ring to A . In particular (doing an enumeration of the monomial basis for this polynomial ring) we obtain a filtration of A :

$$A = \bigcup_{i=0}^{\infty} A_i$$

by finite-dimensional sub- \mathbb{C} -vector spaces $A_i \subset A$.

For each pair of integers $i, k \geq 0$ let $G'_{i,k}$ denote the space of k -tuples of elements of A_i . It is a finite-dimensional affine space. For $V \in G'_{i,k}$ let $I_A(V)$ denote the ideal in A generated by $V = (v_1, \dots, v_k)$ and let $Z_V \subset M$ denote the reduced closed subvariety of M defined by V .

There is a closed algebraic subset

$$\mathcal{Z}'_{i,k} \subset G'_{i,k} \times M$$

such that for each $V \in G'_{i,k}$ the fiber over V is equal to Z_V . To see this, note that the coordinate ring of $G'_{i,k}$ is the symmetric algebra on the dual of $A_i^{\oplus k}$. The coordinate ring of $G'_{i,k} \times M$ is thus the tensor product and the projections–inclusions $A_i^{\oplus k} \rightarrow A$ can be viewed as elements of this coordinate ring; they generate the ideal of the closed subset $\mathcal{Z}'_{i,k}$. Let $G_{i,k} \subset G'_{i,k}$ be the complement of the origin and let $\mathcal{Z}_{i,k}$ be the inverse image of $G_{i,k}$. The family

$$\mathcal{Z}_{i,k} \subset G_{i,k} \times M$$

parameterizes precisely the closed proper subvarieties of M which are cut out by ideals generated by k -tuples of elements in A_i . (Note that since $\text{Spec}(A)$ is reduced and irreducible, any non-zero k -tuple generates the ideal of a proper subvariety.)

For each $\gamma \in \Gamma$ we obtain the translate

$$\gamma \mathcal{Z}_{i,k} \subset G_{i,k} \times M.$$

Let $G_{i,k}(\gamma) \subset G_{i,k}$ denote the subset of points V such that $\gamma Z_V = Z_V$. We claim that this is a constructible subset. It may be described as the set of points $V \in G_{i,k}$ such that the intersection $\gamma \mathcal{Z}_{i,k} \cap \mathcal{Z}_{i,k} \cap (\{V\} \times M)$ contains $\mathcal{Z}_{i,k} \cap (\{V\} \times M)$. This set is

the complement in $G_{i,k}$ of the image of the map

$$(\mathcal{L}_{i,k} - (\gamma \mathcal{L}_{i,k} \cap \mathcal{L}_{i,k})) \rightarrow G_{i,k}$$

so it is constructible.

Now as γ_j runs through a finite set of generators we obtain the intersection of this finite collection of subsets, which is again a constructible subset

$$G_{i,k}(\Gamma) := \bigcap_j G_{i,k}(\gamma_j) \subset G_{i,k}.$$

Each V in $G_{i,k}(\Gamma)$ defines a Γ -invariant closed proper subvariety $Z_V \subset M$, and conversely it is clear that any Γ -invariant closed proper subvariety of M appears as a Z_V for some i, k and some $V \in G_{i,k}(\Gamma)$.

Let $N_{i,k}$ be the union of the points contained in all of the subvarieties Z_V for all $V \in G_{i,k}(\Gamma)$. This is again a constructible set since it is the image of the projection

$$\mathcal{L}_{i,k} \times_{G_{i,k}} G_{i,k}(\Gamma) \rightarrow M.$$

If a point $x \in M$ is contained in any proper closed Γ -invariant subvariety then it is contained in some $N_{i,k}$. Note also that $N_{i,k}$ is Γ -invariant; indeed it is a union of the Γ -invariant subsets Z_V for all the $V \in G_{i,k}(\Gamma)$.

We now have two possibilities: either

- (a) one of the constructible subsets $N_{i,k}$ is dense in M ; or
- (b) all of the constructible subsets $N_{i,k}$ are contained in proper closed subvarieties $\overline{N}_{i,k}$.

In case (b), we obtain a countable union of closed subvarieties $\bigcup_{i,k} \overline{N}_{i,k}$ such that if $x \in M$ is a point which is not in this countable union, then x is never contained in a proper closed Γ -invariant subvariety.

In case (a) we claim that no point has a Zariski-dense orbit. Indeed the complement of the constructible set which is dense, has a closure which is itself a proper Zariski closed and Γ -invariant subvariety (note that all of our constructible sets were Γ -invariant). Thus any point here has non-Zariski-dense orbit. On the other hand, any point in the complement of this closed set is in the open interior of the constructible set in question, so it is by definition in the image of one of the Z_V , i.e. it is in a proper Γ -invariant closed subvariety. This proves that in case (a) no point has a Zariski-dense orbit.

Assume now that there is some point in M which has a Zariski-dense orbit, then we are in case (b), so there is a countable union of closed subvarieties such that if x is not in here then x has Zariski-dense orbit. The lemma is proven. \square

We are now ready to prove the main result of this section.

Theorem 2.2. *Suppose that Γ acts on an irreducible complex affine algebraic variety M . There are two possibilities: either*

- (1) *there exists a non-constant Γ -invariant meromorphic function; or*
- (2) *there exists a point $x \in M$ with Γx Zariski-dense in M .*

Proof. Assume that no point in M has a Zariski-dense orbit. This means that we are in situation (a) discussed in the proof of Lemma 2.1. In other words, there exist integers $i, k \geq 0$ so that the Γ -invariant constructible set $N_{i,k} \subset M$ is dense in M . To simplify notation put $S := G_{i,k}(\Gamma)$ and $\mathcal{Z} := \mathcal{Z}_{i,k} \times_{G_{i,k}} G_{i,k}(\Gamma)$. By construction S and \mathcal{Z} are both schemes of finite type over \mathbb{C} and the natural maps

$$\begin{array}{ccc} \mathcal{Z} & \longrightarrow & M \\ \downarrow & & \\ S & & \end{array}$$

constitute a Γ equivariant family of closed Γ -invariant proper subvarieties of M . Moreover, the total space \mathcal{Z} of this family maps onto the Γ -invariant constructible subset $N_{i,k} \subset M$.

Let x be a point in the open interior of $N_{i,k} \subset M$ and let $Z \subset M$ be the Zariski-closure of Γx . Passing to a subgroup of finite index in Γ we can assume that Z is geometrically irreducible. Let $V \in S$ be such that $Z = Z_V$. Then $x \in Z_V$ and by further localizing S we may assume that all the fibers of $\mathcal{Z} \rightarrow S$ are irreducible and of the same dimension.

Let $W \subset \mathcal{Z}$ denote the set of all points which are contained in two or more distinct fibers Z_{V_1} and Z_{V_2} (i.e. two fibers with $Z_{V_1} \neq Z_{V_2}$). We claim that this is a constructible subset. To see this, look at the closed subvariety $\mathcal{Z} \times_M \mathcal{Z} \subset \mathcal{Z} \times \mathcal{Z}$ and the subset I of $S \times S$ given by the condition

$$I := \{(V_1, V_2) \in S \times S \mid \text{such that } Z_{V_1} = Z_{V_2} \text{ as subvarieties in } M\}.$$

Note that $I \subset S \times S$ is a constructible subset (see this by taking a compactification of M , compactifying the family $\mathcal{Z} \rightarrow S$ and then using Chow schemes) and so the preimage \mathcal{I} of I in $\mathcal{Z} \times \mathcal{Z}$ is also constructible. The subset W is the projection of $\mathcal{Z} \times_M \mathcal{Z} - \mathcal{I}$ on one of the factors \mathcal{Z} , so W is constructible.

Next, we claim that W does not contain our original point x thought of as a point in the fiber $V \in S$. For if it did, this would mean that there was a distinct $Z_{V'} \neq Z_V$ containing x , and then x would be contained in the Γ -invariant set $Z_{V'} \cap Z_V$; but this latter set is a proper subset of Z_V , contradicting the fact (by definition of our family) that Γx is Zariski-dense in Z_V .

It follows that there is an open set of the total space \mathcal{Z} which does not meet W . Note that it is clear from the definition that W is the inverse image of a constructible subset in M ; thus this subset does not contain the generic point of M so there is a closed set C such that W is contained in the preimage of C in \mathcal{Z} . Since W is a Γ -invariant set, we may replace C here by the intersection of all of its translates so we

can assume that C is Γ -invariant as well. Finally then we can throw C out of M (i.e. replace M by $M - C$ in the whole discussion) so we may assume that W is empty. Note that the new M will no longer be affine but only quasi-affine. This does not affect the rest of the argument though since the existence of meromorphic functions can be detected on opens.

By taking a compactification of M and looking at the Chow scheme of subvarieties of this compactification, we can replace our family by a family indexed by a new base scheme S where each fiber (considered as a subset of M) occurs exactly once.

Now by the above reduction the morphism $\mathcal{Z} \rightarrow M$ is injective on points; also its image hits a generic point of M . Thus there is a largest open subset of M over which this is an isomorphism and we can replace M by this open subset (which is Γ -invariant). Hence we obtain a Γ -invariant fibration $M \rightarrow S$. Now any meromorphic function on S pulls back to give a Γ -invariant meromorphic function on M . This essentially proves the theorem. The only problem we need to address is that in the construction of the family $\mathcal{Z} \rightarrow S$ we had to pass to a finite index subgroup of Γ and so the function just constructed may be invariant only under a subgroup of finite index of Γ . This however is easily remedied—by taking the different invariant polynomials in the Galois translates of our meromorphic function we obtain a meromorphic function invariant by the full Γ . \square

In view of Theorem 2.2 we need to find effective criteria for the non-existence of invariant rational functions on an affine variety. One such criterion is given by the following lemma:

Lemma 2.3. *Let Γ be a finitely presented discrete group.*

- (alg) *Suppose M is a reduced irreducible scheme of finite type over \mathbb{C} . Suppose that Γ acts on M by algebraic automorphisms and let $y \in M$ be a point in the smooth locus of M , fixed by the action, so that Γ acts linearly on the tangent space $T_y M$. Let $G \subset \mathrm{GL}(T_y M)$ be the Zariski-closure of $\mathrm{im}[\Gamma \rightarrow \mathrm{GL}(T_y M)]$. Assume that G acts on $T_y M$ with an open orbit, and that the connected component G^o of G has no non-trivial characters. Then there is no Γ -invariant rational function on M .*
- (an) *Suppose that N is an irreducible analytic space on which Γ acts by analytic automorphisms. Suppose that $y \in N$ is a point in the smooth locus of N , fixed by the action, so that Γ acts linearly on the tangent space $T_y N$. Let $G \subset \mathrm{GL}(T_y N)$ be the Zariski-closure of $\mathrm{im}[\Gamma \rightarrow \mathrm{GL}(T_y N)]$. Assume that G acts on $T_y N$ with an open orbit, and that the connected component G^o of G has no non-trivial characters. Then there is no Γ -invariant analytic-meromorphic function on N .*

Proof. Clearly, the statement of the lemma is insensitive to passing to a finite index subgroup of Γ and so we may assume that $G = G^o$. We will only give the proof in the algebraic case. The analytic case is completely analogous.

Suppose that h is such a function, and write the germ of h at y as f/g with $f, g \in \mathcal{O}_{M,y}$ relatively prime. Then for any $\gamma \in \Gamma$,

$$\frac{f}{g} = h = \gamma^* h = \frac{\gamma^* f}{\gamma^* g},$$

which implies that there is a unit $u(\gamma) \in \mathcal{O}_{M,y}^\times$ with

$$\gamma^* f = u(\gamma) f,$$

$$\gamma^* g = u(\gamma) g.$$

Note that $\gamma \mapsto u(\gamma)$ is a cocycle for Γ acting on the multiplicative group of units $\mathcal{O}_{M,y}^\times$. In particular we get that the value $u(\gamma)(y)$ is a character of Γ .

Without loss of generality, we may assume that h is not an invertible function at o (otherwise subtract the constant function with the same value at o) hence we may assume that one of f or g has a non-trivial leading term of some degree. We may suppose that f has such (otherwise replace h by h^{-1}). Let f_m be the leading term of f . Note that f_m (of degree m)—modulo higher order terms—is an element of $\text{Sym}^m T^\vee(M)_y$. The action of Γ on this leading term factors through the group G . Our previous formula gives

$$\gamma^*(f_m) = u(\gamma)(y) f_m.$$

This shows that $\gamma \mapsto u(\gamma)(y)$ comes from a character of G ; but by assumption there are no such characters, Therefore we get $\gamma^*(f_m) = f_m$, so f_m is a G -invariant homogeneous form of degree m .

In particular, we can think of f_m as a G -invariant polynomial function on the tangent space $T_y M$. This contradicts the supposed existence of an open orbit in the action of G on $T_y M$. The lemma is proven. \square

The essential consequence of Theorem 2.2 that we need is the following localization statement.

Corollary 2.4. *Suppose that $y \in M$ is a point in the smooth locus of M , fixed by the action, so that Γ acts linearly on the tangent space $T_y M$. Let $G \subset \text{GL}(T_y M)$ be the Zariski-closure of $\text{im}[\Gamma \rightarrow \text{GL}(T_y M)]$. Assume that G acts on $T_y M$ with an open orbit, and that the connected component G^o of G has no non-trivial characters. Then there exists a point $x \in M$ with Γx Zariski-dense in M .*

Proof. By Theorem 2.2 we only have to rule out the possibility that M admits a non-constant Γ -invariant meromorphic function. This however is precisely the content of Lemma 2.3(alg). The corollary is proven. \square

Theorem 2.2, Lemma 2.3 and Corollary 2.4 not only provide a convenient localization criterion for the density of an action but also suggest another geometric

notion of ‘largeness’ of the Γ -orbits. Motivated by Theorem 2.2, we define various degrees of analytic generic Zariski denseness (AGZD for short) as follows:

Definition 2.5. Suppose that a finitely generated group Γ acts by analytic automorphisms on an irreducible analytic space N . Let $m : \Gamma \times N \rightarrow N$ be the action map.

- We say that the action m is AGZD1 if there is no Γ -invariant analytic meromorphic function f on N .
- We say that the action m is AGZD2 if there is no pair (U, f) , where $U \subset N$ is a Γ -invariant analytically Zariski-dense open subset of N and $f : U \rightarrow Z$ is a Γ -equivariant holomorphic map from U to a complex analytic space Z with $\dim Z < \dim N$.
- We say that the action m is AGZD3 if there is a point $x \in N$ such that $m(\Gamma \times \{x\})$ is analytically Zariski-dense in N .
- We say that the action m is AGZD4 if there is an analytically Zariski-dense open subset $U \subset N$ such that for every $x \in U$ the Γ orbit of x is analytically Zariski-dense in N .

Clearly for an analytic action m one has the implications:

$$\text{AGZD4} \Rightarrow \text{AGZD3} \Rightarrow \text{AGZD2} \Rightarrow \text{AGZD1},$$

but we do not think that the converse implications are true. Similarly, it is clear that if m is actually an algebraic action, then AGZD1 implies that m is Zariski-dense in the algebraic sense of Theorem 2.2.

Suppose now that B is a base scheme and that $p : M \rightarrow B$ is a morphism equipped with a connection ∇ (by which we mean a stratification over the crystalline site of S [6,17]). For the following definition it is not necessary to assume that ∇ is integrable.

Definition 2.6. Suppose that B and the generic geometric fiber of M/B are irreducible. We say that $(p : M \rightarrow B, \nabla)$ is *generically Zariski-dense* (or GZD) if there is no algebraic meromorphic function f on the total space M which is invariant under ∇ .

If the connection ∇ is integrable, then the corresponding analytic family is associated to a local system of complex analytic spaces over B , which in turn corresponds to the monodromy action m of $\Gamma := \pi_1(B, o)$ on a fiber M_o^{an} . It is clear that if m is AGZD1, then $(p : M \rightarrow B, \nabla)$ is GZD.

In particular, the AGZD1 property for $M_B(X_o, n)$ or equivalently $M_{\text{DR}}(X_o, n)$ implies the algebraic generic Zariski-denseness property GZD for the Gauss–Manin connection ∇_{DR}^n .

Consider now a family of smooth projective connected curves $f : X \rightarrow B$ and let $o \in B$ be a base point. We will show that when the geometric monodromy of $f : X \rightarrow B$ is of finite index in the mapping class group or when f comes from a Lefschetz pencil as in Theorem B, then the monodromy action of $\pi_1(B, o)$ on $M_B(X_o, n)$ is AGZD1.

In combination with Theorem 2.2 this fact yields statement (i) of Theorems A and B. As explained above this automatically gives the analytic statement (ii) in both theorems. In fact, it follows from the above considerations that $M_{\text{DR}}(X/B, n) \rightarrow B$ together with the non-Abelian Gauss–Manin connection is GZD in the sense of Definition 2.6.

In view of all this it only remains to show that under the hypothesis of Theorems A or B the algebraic action

$$\text{mon}_{\mathbb{B}}^n : \pi_1(B, o) \rightarrow \text{Aut}(M_{\mathbb{B}}(X_o, n)).$$

on the affine variety $M_{\mathbb{B}}(X_o, n)$ is AGZD1. (Note that since X_o is a smooth curve the variety $M_{\mathbb{B}}(X_o, n)$ is irreducible by Simpson [16, Section 11].) In view of Corollary 2.4, to achieve this we only need to find a smooth point $\rho \in M_{\mathbb{B}}(X_o, n)$ which is fixed by the monodromy group $\text{mon}_{\mathbb{B}}^n(\pi_1(B, o))$, and for which the Zariski-closure of $\text{mon}_{\mathbb{B}}^n(\pi_1(B, o)) \subset \text{GL}(T_{[\rho]}M_{\mathbb{B}}(X_o, n))$ acts on $T(M_{\mathbb{B}}(X_o, n))_{\rho}$ with an open orbit and has a connected component of the identity which admits no non-trivial characters.

In the next section we describe a proposal for such a point ρ which utilizes the Schrödinger representation of a finite dihedral Heisenberg group. Later on, we will show in Sections 3.1 and 3.2 that the open orbit property for the monodromy action on the tangent space at ρ holds, provided that the geometric monodromy of $f : X \rightarrow B$ is large enough.

We have stated the additional properties AGZD2-4 in order to pose the question: which of these properties hold for families whose monodromy has finite index in the mapping class group? For (sufficiently ample) Lefschetz pencils?

2.2. The Schrödinger representation

Since ρ is supposed to be fixed by the monodromy a natural choice would be to take ρ to be the trivial representation of $\pi_1(X_o)$ in $\text{GL}(n, \mathbb{C})$. However, the trivial representation is a singular point of $M_{\mathbb{B}}(X_o, n)$ and so is unsuitable for our purposes. On the other hand, any representation

$$\rho : \pi_1(X_o) \rightarrow \text{GL}(n, \mathbb{C})$$

which has finite image will be fixed under some finite index subgroup of $\text{mon}_{\mathbb{B}}^n(\pi_1(B, o))$. Furthermore, the properties AGZD1–AGZD4 and GZD are obviously stable under passage to a finite index subgroup of Γ . Hence we are free to replace B by any finite étale cover of B , and so it is enough to find a finite representation ρ which satisfies the open orbit condition.

In order to apply Corollary 2.4 we also need to choose $\rho \in M_{\mathbb{B}}(X_o, n)$ to be a smooth point. This is equivalent to choosing ρ to be an irreducible representation.

To construct such a representation we proceed as follows. Let $\mu_n \subset \mathbb{C}^{\times}$ be the group of all n th roots of unity. Let $\widehat{\mathbb{Z}/n} := \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{C}^{\times})$ denote the group of

characters of the cyclic group \mathbb{Z}/n . Consider the finite Heisenberg group H_n . By definition H_n is the central extension

$$0 \rightarrow \mu_n \rightarrow H_n \rightarrow \mathbb{Z}/n \times \widehat{\mathbb{Z}/n} \rightarrow 0$$

corresponding to the cocycle $e : (\mathbb{Z}/n \times \widehat{\mathbb{Z}/n})^2 \rightarrow \mu_n \subset \mathbb{C}^\times$, $e((a, \alpha), (a', \alpha')) = \alpha'(a)$. Explicitly, H_n can be identified with the set $\mu_n \times \mathbb{Z}/n \times \widehat{\mathbb{Z}/n}$ with a group law given by

$$(\lambda; a, \alpha)(\lambda'; a', \alpha') = (\lambda\lambda'\alpha'(a); a + a', \alpha\alpha'). \tag{2.1}$$

Let $\phi_n : H_n \rightarrow \text{GL}(V_n)$ be the Schrödinger representation of H_n [14]. By definition ϕ_n is the unique n -dimensional irreducible representation of H_n which has a tautological central character. One way to construct ϕ_n is to observe that the natural injective map

$$\mu_n \times \widehat{\mathbb{Z}/n} \hookrightarrow H_n, \quad (\lambda, \alpha) \mapsto (\lambda; 0, \alpha)$$

is a group monomorphism, i.e. H_n contains $\mu_n \times \widehat{\mathbb{Z}/n}$ as an abelian subgroup. Let \mathbb{T} be the one-dimensional complex representation of $\mu_n \times \widehat{\mathbb{Z}/n}$ which corresponds to the pullback of the tautological character of μ_n under the projection $\mu_n \times \widehat{\mathbb{Z}/n} \rightarrow \mu_n$. In other words, $\mathbb{T} = (\mathbb{C}, \tau)$ where $\tau : \mu_n \times \widehat{\mathbb{Z}/n} \rightarrow \mathbb{C}^\times$ is given by $\tau(\lambda, \alpha) = \lambda$. In terms of \mathbb{T} then we have

$$(V_n, \phi_n) = \text{Ind}_{\mu_n \times \widehat{\mathbb{Z}/n}}^{H_n}(\mathbb{T}).$$

Explicitly, we can identify V_n with the vector space of all complex valued functions on the finite set \mathbb{Z}/n and the action ϕ_n by the formula

$$[\phi_n(\lambda; a, \alpha)f](x) = \lambda\alpha(x)f(x + a),$$

for all $x \in \mathbb{Z}/n, f \in V_n$ and $(\lambda; a, \alpha) \in H_n$.

The irreducibility of the representation ϕ_n follows from Frobenius reciprocity or directly by noticing that V_n has a basis consisting of the characteristic functions of the elements in \mathbb{Z}/n and that the subgroup $\widehat{\mathbb{Z}/n} \subset H_n$ acts transitively on the elements of this basis. In particular if we compose ϕ_n with some surjective homomorphism $\pi_1(X_o) \twoheadrightarrow H_n$ we will get a representation of $\pi_1(X_o)$ in $\text{GL}(V_n)$ which is irreducible and has finite image. Unfortunately, it turns out (see Remark 2.10) that this representation cannot be used directly to obtain an open orbit action on the tangent space to $M_B(X_o, n)$. However a slight modification of this representation does the job. The modification involves an extension of H_n of dihedral type which we proceed to describe.

Let $\mu_2 = \{-1, +1\} \subset \mathbb{C}^\times$ be the group of square roots of one. The group μ_2 acts naturally on $\mathbb{Z}/n \times \widehat{\mathbb{Z}/n}$ as the inversion on both factors. This action clearly

preserves the cocycle defining the Heisenberg central extension $0 \rightarrow \mu_n \rightarrow H_n \rightarrow \mathbb{Z}/n \times \widehat{\mathbb{Z}/n} \rightarrow 0$ and so we get a natural action of μ_2 on H_n . Explicitly, if we think of H_n as the set $\mu_n \times \mathbb{Z}/n \times \widehat{\mathbb{Z}/n}$ equipped with the group law (2.1), then an element $\varepsilon \in \mu_2$ acts on H_n via $(\lambda; a, \alpha) \mapsto (\lambda, \varepsilon a, \alpha^\varepsilon)$. We define the *dihedral Heisenberg group* as the semidirect product

$$\mathfrak{D}H_n := \mu_2 \ltimes H_n$$

for the above action. Thus $\mathfrak{D}H_n$ can be identified with the set $\mu_2 \times \mu_n \times \mathbb{Z}/n \times \widehat{\mathbb{Z}/n}$ with a group law given by

$$(\varepsilon, \lambda, a, \alpha) \cdot (\varepsilon', \lambda', a', \alpha') = (\varepsilon\varepsilon', \lambda\lambda' \alpha'^{\varepsilon}(a), a + \varepsilon a', \alpha\alpha'^{\varepsilon}). \tag{2.2}$$

In particular, for each $(\varepsilon, \lambda, a, \alpha) \in \mathfrak{D}H_n$ we have

$$\begin{aligned} (\varepsilon, \lambda, a, \alpha) &= (1, \lambda, a, \alpha) \cdot (\varepsilon, 1, 0, \mathbf{1}) \\ &= (\varepsilon, 1, 0, \mathbf{1}) \cdot (1, \lambda, \varepsilon a, \alpha^\varepsilon), \end{aligned}$$

where $\mathbf{1}: \mathbb{Z}/n \rightarrow \mathbb{C}^\times$ stands for the trivial character, i.e. $\mathbf{1}(a) = 1$ for all a .

Observe next that the Schrödinger representation $\phi_n: H_n \rightarrow \text{GL}(V_n)$ extends naturally to a *dihedral Schrödinger representation*

$$\mathfrak{d}\phi_n: \mathfrak{D}H_n \rightarrow \text{GL}(V_n)$$

defined by

$$(\mathfrak{d}\phi_n(\varepsilon, \lambda, a, \alpha)f)(x) = \lambda\alpha(x)f(\varepsilon(x+a))$$

for all $f \in V_n$ and all $x \in \mathbb{Z}/n$.

Recall that if C is any smooth curve of genus g , then there is a surjective homomorphism $\pi_1(C) \twoheadrightarrow \mathbf{F}_g$ onto a free group of g generators. This homomorphism is obtained by moding $\pi_1(C)$ out by the normal subgroup generated by the a -cycles for a standard basis in the first homology of C . Note furthermore that $\mathfrak{D}H_n$ is generated by the three elements $(-1, 1, 0, \mathbf{1})$, $(1, 1, 1, \mathbf{1})$ and $(1, 1, 0, \alpha)$, where $\alpha \in \widehat{\mathbb{Z}/n}$ is any generator. Hence we can find a surjective homomorphism $\pi_1(C) \twoheadrightarrow \mathbf{F}_g \twoheadrightarrow \mathfrak{D}H_n$ as long as $g \geq 3$.

By hypothesis the genus of X_o is big enough and so we can find a surjective homomorphism $\psi_n: \pi_1(X_o) \twoheadrightarrow \mathfrak{D}H_n$. Let $\rho: \pi_1(X_o) \rightarrow \text{GL}(V_n)$ denote the composition $\rho := \psi_n \circ \mathfrak{d}\phi_n$. By construction ρ is irreducible and so represents a smooth point of the moduli of representations. Moreover by a standard deformation theory argument (see e.g. [11]), we can identify the Zariski tangent space $T_{[\rho]} \mathcal{M}_B(X_o, n)$ with the group cohomology $H^1(\pi_1(X_o), \text{ad}(\rho))$, where

$$\text{ad}(\rho): \pi_1(X_o) \rightarrow \text{GL}(\text{End}(V_n))$$

is the natural representation induced from ρ . Explicitly,

$$\text{ad}(\rho) = (\mathfrak{d}\phi_n^\vee \otimes \mathfrak{d}\phi_n) \circ \psi_n$$

and since $\text{ad}(\phi_n) = \phi_n^\vee \otimes \phi_n$ has a trivial central character we see that $\text{ad}(\rho)$ factors through the quotient group $\mathfrak{D}H_n \rightarrow \mu_2 \ltimes (\mathbb{Z}/n \times \widehat{\mathbb{Z}/n})$.

It is not hard to calculate $H^1(\pi_1(X_o), \text{ad}(\rho))$ in terms of geometric data on the curve X_o . The action of μ_2 on $\mathbb{Z}/n \times \widehat{\mathbb{Z}/n}$ induces an obvious action (inversion on both factors) of μ_2 on the group characters

$$\text{Hom}(\mathbb{Z}/n \times \widehat{\mathbb{Z}/n}, \mathbb{C}^\times) = \widehat{\mathbb{Z}/n} \times \mathbb{Z}/n.$$

Now for each orbit $u \in (\widehat{\mathbb{Z}/n} \times \mathbb{Z}/n) / \mu_2$ we get an irreducible representation W_u of $\mu_2 \ltimes (\mathbb{Z}/n \times \widehat{\mathbb{Z}/n})$. The representation $W_{(1,0)}$ corresponding to the trivial character is just the trivial one dimensional representation of $\mu_2 \ltimes (\mathbb{Z}/n \times \widehat{\mathbb{Z}/n})$. For any other orbit u we have that $u = \{\chi, \chi^{-1}\}$ for some non-trivial character $\chi \in \widehat{\mathbb{Z}/n} \times \mathbb{Z}/n$ and so W_u is the representation of $\mu_2 \ltimes (\mathbb{Z}/n \times \widehat{\mathbb{Z}/n})$ induced from the one-dimensional representation (\mathbb{C}, χ) of $\mathbb{Z}/n \times \widehat{\mathbb{Z}/n}$. Thus $W_u = (\mathbb{C}, \chi) \oplus (\mathbb{C}, \chi^{-1})$ as a representation of $\mathbb{Z}/n \times \widehat{\mathbb{Z}/n}$ and the generator of μ_2 acts by switching the two summands. In particular, W_u is a two-dimensional irreducible (we are assuming that n is odd here) representation of $\mu_2 \ltimes (\mathbb{Z}/n \times \widehat{\mathbb{Z}/n})$.

With this notation we have

Lemma 2.7. *The tangent space to $M_B(X_o, n)$ at the dihedral Schrödinger representation ρ is given by*

$$T_{[\rho]} M_B(X_o, n) = H^1(X_o, \text{ad}(\rho)) = \bigoplus_{u \in (\widehat{\mathbb{Z}/n} \times \mathbb{Z}/n) / \mu_2} H^1(X_o, \mathbb{W}_u), \tag{2.3}$$

where \mathbb{W}_u is the local system on X_o corresponding to the representation

$$\pi_1(X_o) \xrightarrow{\psi_n} \mathfrak{D}H_n \rightarrow \mu_2 \ltimes (\mathbb{Z}/n \times \widehat{\mathbb{Z}/n}) \rightarrow \text{GL}(W_u).$$

Proof. Note that a representation $\kappa : \mathbb{Z}/n \times \widehat{\mathbb{Z}/n} \rightarrow \text{GL}(V)$ of the abelian group $\mathbb{Z}/n \times \widehat{\mathbb{Z}/n}$ will extend to representation of the ‘dihedral’ group $\mu_2 \ltimes (\mathbb{Z}/n \times \widehat{\mathbb{Z}/n})$ if and only if κ is self-dual. Furthermore, each self-dual representation κ has a canonical dihedral extension:

$$\mathfrak{d}\kappa : \mu_2 \ltimes (\mathbb{Z}/n \times \widehat{\mathbb{Z}/n}) \rightarrow \text{GL}(V),$$

in which μ_2 acts as the self-duality automorphism of V . Concretely if we decompose (V, κ) into a direct sum of characters of $\mathbb{Z}/n \times \widehat{\mathbb{Z}/n}$, then the self-duality of V will identify the multiplicity space of each character χ with the multiplicity space of the character χ^{-1} . In particular, the multiplicity spaces in the character decomposition of (V, κ) depend not on the individual characters but rather on the μ_2 -orbits $u \in (\widehat{\mathbb{Z}/n} \times \mathbb{Z}/n)/\mu_2$. Hence $(V, \mathfrak{d}\kappa)$ decomposes as

$$(V, \mathfrak{d}\kappa) = \bigoplus_{u \in (\widehat{\mathbb{Z}/n} \times \mathbb{Z}/n)/\mu_2} W_u \otimes M_u,$$

where M_u denotes the multiplicity space of a character $\chi \in u$ in (V, κ) .

Consider now the representation $\text{ad}(\phi_n)$ of H_n . Since it has a trivial central character, it factors through a representation

$$\text{ad}(\phi_n)^{\text{ab}} : \mathbb{Z}/n \times \widehat{\mathbb{Z}/n} \rightarrow \text{GL}(\text{End}(V_n)).$$

This is the *abelian part* of $\text{ad}(\phi_n)$. The representation $\text{ad}(\phi_n)^{\text{ab}}$ is self-dual by construction and so admits a canonical dihedral extension

$$\mathfrak{d} \text{ad}(\phi_n)^{\text{ab}} : \mu_2 \ltimes (\mathbb{Z}/n \times \widehat{\mathbb{Z}/n}) \rightarrow \text{GL}(\text{End}(V_n)).$$

This dihedral extension fits in the commutative diagram

$$\begin{CD} \pi_1(X_\rho) @>\text{ad}(\rho)>> \text{GL}(\text{End}(V_n)) \\ @V\psi_nVV @AA\mathfrak{d} \text{ad}(\phi_n)^{\text{ab}}A \\ \mathfrak{D}H_n @>>> \mu_2 \ltimes (\mathbb{Z}/n \times \widehat{\mathbb{Z}/n}) \end{CD}$$

and so understanding $\text{ad}(\rho)$ is equivalent to understanding $\text{ad}(\phi_n)^{\text{ab}}$. But $\text{ad}(\phi_n)^{\text{ab}}$ is just the regular representation of the abelian group $\mathbb{Z}/n \times \widehat{\mathbb{Z}/n}$. To see this note first that $\dim \text{End}(V_n) = n^2 = \dim \mathbb{C}[\mathbb{Z}/n \times \widehat{\mathbb{Z}/n}]$. Now since every irreducible representation of $\mathbb{Z}/n \times \widehat{\mathbb{Z}/n}$ occurs in $\mathbb{C}[\mathbb{Z}/n \times \widehat{\mathbb{Z}/n}]$ with multiplicity one we need only to check that for every character $\chi : \mathbb{Z}/n \times \widehat{\mathbb{Z}/n} \rightarrow \mathbb{C}^\times$ we have

$$\dim \text{Hom}_{(\mathbb{Z}/n \times \widehat{\mathbb{Z}/n})\text{-mod}}(\chi, \text{ad}(\phi_n)^{\text{ab}}) \geq 1.$$

But the group of characters of $\mathbb{Z}/n \times \widehat{\mathbb{Z}/n}$ is naturally isomorphic to $\widehat{\mathbb{Z}/n} \times \mathbb{Z}/n$ and so each character χ as above is given by a pair $(\xi, \alpha) \in \widehat{\mathbb{Z}/n} \times \mathbb{Z}/n$ via the formula $\chi(a, \alpha) = \xi(a) \cdot \alpha(x)$. Therefore, we only need to show that for any pair (ξ, α) there exists a non-zero element $A_{(\xi, \alpha)} \in \text{End}(V_n) = V_n^\vee \otimes V_n$ so that

$$(a, \alpha)A_{(\xi, \alpha)} = \xi(a)\alpha(x)A_{(\xi, \alpha)}$$

for all $(a, \alpha) \in \mathbb{Z}/n \times \widehat{\mathbb{Z}/n}$.

To construct the element $A_{(\zeta,x)}$ recall that the vector space $V_n = \mathbb{C}[\mathbb{Z}/n]$ has a natural basis $\{e_0, e_1, \dots, e_{n-1}\}$ consisting of characteristic functions of elements of \mathbb{Z}/n , i.e. $e_i(j) := \delta_{ij}$. Let $\{e_0^\vee, e_1^\vee, \dots, e_{n-1}^\vee\}$ be the dual basis of V_n^\vee . Then in terms of the basis $\{e_i^\vee \otimes e_j\}_{i,j=0}^{n-1}$ of $V_n^\vee \otimes V_n$ the representation $\text{ad}(\phi_n)^{\text{ab}}$ is given by the formula

$$[\text{ad}(\phi_n)^{\text{ab}}(a, \alpha)](e_i^\vee \otimes e_j) = \alpha(j - i)e_{i-a}^\vee \otimes e_{j-a}.$$

In view of this we may take

$$A_{(\zeta,x)} := \sum_{i=0}^{n-1} \zeta(i)e_i^\vee \otimes e_{i+x}$$

which is obviously a non-zero eigenvector corresponding to the character $\chi = (\zeta, x)$.

This shows that

$$(\text{End}(V_n), \text{ad}(\phi_n)^{\text{ab}}) = \mathbb{C}[\mathbb{Z}/n \times \widehat{\mathbb{Z}/n}] = \bigoplus_{\chi \in \widehat{\mathbb{Z}/n \times \mathbb{Z}/n}} (\mathbb{C}, \chi),$$

and so the lemma is proven. \square

Let us now go back to the problem of checking whether $\text{mon}(\pi_1(B, o))$ has a dense orbit on $M_B(X_o, n)$. As mentioned above, the fact that ρ has a finite image implies that the conjugacy class of ρ will be fixed by some finite-index subgroup of $\text{mon}(\pi_1(B, o))$. In particular, applying Corollary 2.4 to this subgroup, it follows that in order to show the existence of a dense $\text{mon}(\pi_1(B, o))$ -orbit on $M_B(X_o, n)$, it suffices to check the following two items:

- (i) The Zariski-closure G of

$$\text{im}[\text{mon}(\pi_1(B, o)) \rightarrow \text{GL}(H^1(X_o, \text{ad}(\rho)))]$$

in $\text{GL}(H^1(X_o, \text{ad}(\rho)))$ has an open orbit on $H^1(X_o, \text{ad}(\rho))$.

- (ii) The identity component G^o of G does not have any non-trivial characters.

Condition (ii) follows easily from isomorphism (2.3):

Lemma 2.8. *The identity component G^o of the Zariski-closure of*

$$\text{im}[\text{mon}(\pi_1(B, o)) \rightarrow \text{GL}(H^1(X_o, \text{ad}(\rho)))]$$

in $\text{GL}(H^1(X_o, \text{ad}(\rho)))$ has no non-trivial characters.

Proof. Indeed, let $p: Y_o \rightarrow X_o$ be the dihedral Galois cover of X_o corresponding to the surjection

$$\pi_1(X_o) \xrightarrow{\psi_n} \mathfrak{D}H_n \rightarrow \mu_2 \ltimes (\mathbb{Z}/n \times \widehat{\mathbb{Z}/n}). \tag{2.4}$$

Then Y_o is a smooth connected curve and the pushforward of the trivial one-dimensional local system \mathbb{C}_{Y_o} via p is precisely

$$p_*\mathbb{C}_{Y_o} = \bigoplus_{u \in (\widehat{\mathbb{Z}/n \times \mathbb{Z}/n})/\mu_2} \mathbb{W}_u.$$

Thus we can identify $H^1(X_o, \text{ad}(\rho))$ with $H^1(Y_o, \mathbb{C})$.

Next, observe that, without a loss of generality, we may assume that the family $f : X \rightarrow B$ has an algebraic section $\sigma : B \rightarrow X$. Indeed, since X is quasi-projective a generic hyperplane section on X will be a multisection of f . But replacing B by an étale cover of a Zariski open set of B will replace $\text{mon}(\pi_1(B, o))$ by a subgroup of finite index. In particular, such a replacement will not affect the property that $\text{mon}(\pi_1(B, o))$ fixes the conjugacy class of ρ . In fact, by taking another étale cover if necessary we can ensure that not only the conjugacy class of ρ is fixed under $\text{mon}(\pi_1(B, o))$ but that the actual representation

$$\rho : \pi_1(X_o, \sigma(o)) \rightarrow \text{GL}(V_n)$$

remains fixed under $\text{mon}(\pi_1(B, o))$. Indeed, to achieve this we only need to pass to the finite index subgroup of the monodromy which preserves the kernel of ρ and acts trivially on the finite group $\text{im}(\rho) = \mathfrak{D}H_n$.

Assume that we are in this situation. Then ρ lifts to a well-defined representation of $\pi_1(B, o) \rtimes_{\text{mon}} \pi_1(X_o, \sigma(o))$ and so defines a $\mathfrak{D}H_n$ -cover of X . In particular under the identification $H^1(X_o, \text{ad}(\rho)) \cong H^1(Y_o, \mathbb{C})$ the action of $\text{mon}(\pi_1(B, o))$ on $H^1(X_o, \text{ad}(\rho))$ becomes just the monodromy action of $\pi_1(B, o)$ on $H^1(Y_o, \mathbb{C})$ corresponding to the family of curves $Y \rightarrow B$. But by Deligne’s semisimplicity theorem [1, Corollaire 4.2.9] the monodromy action on the middle dimensional cohomology of any smooth projective family over a quasi-projective base has a semisimple Zariski-closure. Thus G^ρ must be a connected semisimple algebraic group and so has no non-trivial characters. The lemma is proven. \square

In order to check that condition (i) is satisfied we need to make sure that the monodromy group of the family $f : X \rightarrow B$ is sufficiently large.

Before we explain how this is achieved we need to introduce some notation. On the way we will also rephrase condition (i) in a slightly more general context.

Let $f : X \rightarrow B$ be a smooth family of connected curves of genus $g \geq 3$. Let as before $o \in B$ be a fixed base point and let

$$\text{mon} : \pi_1(B, o) \rightarrow \text{Map}(X_o) \subset \text{Out}(\pi_1(X_o))$$

be the corresponding geometric monodromy representation.

If $f : X \rightarrow B$ has a holomorphic section $\sigma : B \rightarrow X$ the representation

$$\text{mon} : \pi_1(B, o) \rightarrow \text{Map}(X_o)$$

can be lifted to a geometric monodromy representation respecting the base point:

$$\text{mon}^\sigma : \pi_1(B, o) \rightarrow \text{Map}^1(X_o) \subset \text{Aut}(\pi_1(X_o, \sigma(o))).$$

Here $\text{Map}^1(X_o)$ denotes the mapping class group of the once punctured surface $X_o - \{\sigma(o)\}$.

Fix a finite abelian group A and let $\mathfrak{DA} := \mu_2 \rtimes A$ denote the standard dihedral extension of A in which the generator $(-1) \in \mu_2$ acts as the inversion on A . Fix a surjective homomorphism $\pi_1(X_o, \sigma(o)) \twoheadrightarrow \mathfrak{DA}$ and let $p^{\mathfrak{DA}} : Y_o^{\mathfrak{DA}} \rightarrow X_o$ be the corresponding Galois cover. Let $\text{Map}^1(X_o, \mathfrak{DA})$ be the group of $p^{\mathfrak{DA}}$ -liftable mapping classes, i.e.

$$\text{Map}^1(X_o, \mathfrak{DA}) = \left\{ \varphi \in \text{Map}^1(X_o) \left| \begin{array}{l} \varphi \text{ preserves } \ker[\pi_1(X_o, \sigma(o)) \twoheadrightarrow \mathfrak{DA}] \\ \text{and } \varphi \text{ induces the identity on } \mathfrak{DA} \end{array} \right. \right\}.$$

Clearly $\text{Map}^1(X_o, \mathfrak{DA}) \subset \text{Map}^1(X_o)$ is of finite index and consists precisely of the mapping classes on X_o which lift to mapping classes on $Y_o^{\mathfrak{DA}}$. Furthermore, since by definition each $\varphi \in \text{Map}^1(X_o, \mathfrak{DA})$ induces the identity on \mathfrak{DA} it follows that any lift $\tilde{\varphi} \in \text{Map}(Y_o^{\mathfrak{DA}})$ of φ commutes with the action of \mathfrak{DA} on $Y_o^{\mathfrak{DA}}$, and that \mathfrak{DA} acts transitively on the set of all such lifts. Thus, if we define $\text{LMap}^1(X_o, \mathfrak{DA}) \subset \text{Map}^1(Y_o^{\mathfrak{DA}})$ to be the subgroup consisting of all lifts of elements in $\text{Map}^1(X_o)$ we see that $\text{LMap}^1(X_o, \mathfrak{DA})$ fits in a short exact sequence of groups

$$1 \rightarrow \mathfrak{DA} \rightarrow \text{LMap}^1(X_o, \mathfrak{DA}) \rightarrow \text{Map}^1(X_o, \mathfrak{DA}) \rightarrow 1.$$

Assume now that $\text{mon}^\sigma(\pi_1(B, o)) \subset \text{Map}^1(X_o, \mathfrak{DA})$. In particular, $\text{mon}^\sigma(\pi_1(B, o))$ preserves $\pi_1(Y_o^{\mathfrak{DA}})$ and so we get a short exact sequence of groups

$$1 \rightarrow \pi_1(B, o) \times_{\text{mon}^\sigma} \pi_1(Y_o^{\mathfrak{DA}}) \rightarrow \pi_1(X_o, \sigma(o)) \rightarrow \mathfrak{DA} \rightarrow 1.$$

Let $Y^{\mathfrak{DA}} \rightarrow X$ denote the \mathfrak{DA} -Galois cover of X corresponding to the homomorphism $\pi_1(X, \sigma(o)) \twoheadrightarrow \mathfrak{DA}$. By construction $\pi_1(Y^{\mathfrak{DA}}) \cong \pi_1(B, o) \times_{\text{mon}^\sigma} \pi_1(Y_o^{\mathfrak{DA}})$ and the corresponding monodromy representation

$$\text{mon}_{\mathfrak{DA}} : \pi_1(B, o) \rightarrow \text{Map}(Y_o^{\mathfrak{DA}}) \subset \text{Out}(\pi_1(Y_o^{\mathfrak{DA}}))$$

lands in $\text{LMap}^1(X_o, \mathfrak{DA})$.

Furthermore, note that from the viewpoint of the density properties we are interested in, the conditions that $f : X \rightarrow B$ has a section and that $\text{mon}(\pi_1(B, o)) \subset \text{Map}^1(X_o, \mathfrak{DA})$ are harmless. Indeed, as explained in the proof of Lemma 2.8, if $f : X \rightarrow B$ is an arbitrary smooth projective family of curves with B smooth and connected and X quasi-projective, then we can always find a Zariski open set $U \subset B$ containing the point $o \in B$, and a finite étale cover $(B', o') \rightarrow (U, o)$, so that the pulled-back family $X \times_B B' \rightarrow B'$ has a holomorphic section, and a geometric

monodromy which is contained in $\text{Map}^1(X_o, \mathfrak{D}A)$. Since $\pi_1(U, o) \twoheadrightarrow \pi_1(B, o)$ is surjective and $\pi_1(B', o') \subset \pi_1(U, o)$ is a subgroup of finite index, it follows that the geometric monodromy $\text{mon}(\pi_1(B', o'))$ of the family $X \times_B B' \rightarrow B'$ is a subgroup of finite index in $\text{mon}(\pi_1(B, o))$. In particular, any density statement we can make for the action of $\pi_1(B, o)$ will be equivalent to the corresponding density statement for the action of $\pi_1(B', o')$.

The previous reasoning also shows that for any smooth family of curves $f : X \rightarrow B$, such that B is smooth and X is quasi-projective, and any surjective homomorphism $\pi_1(X_o, \sigma(o)) \twoheadrightarrow \mathfrak{D}A$, there is an appropriate $(B', o') \rightarrow (B, o)$ and an $\mathfrak{D}A$ -Galois cover $Y^{\mathfrak{D}A} \rightarrow X \times_B B'$, so that:

- the image of the monodromy representation

$$\text{mon}_{\mathfrak{D}A} : \pi_1(B', o') \rightarrow \text{Map}(Y_o^{\mathfrak{D}A})$$

is contained in $\text{LMap}^1(X_o, \mathfrak{D}A)$;

- the natural map $\text{mon}_{\mathfrak{D}A}(\pi_1(B', o')) \rightarrow \text{mon}(\pi_1(B, o))$ has finite kernel and cokernel.

Motivated by the discussion in Sections 2.1 and 2.2 we make the following definition:

Definition 2.9. Let A be a finite abelian group. A pair

$$(f : X \rightarrow B, \pi_1(X_o) \twoheadrightarrow \mathfrak{D}A)$$

is called *good* if the Zariski-closure of

$$\text{im}[\pi_1(B', o') \xrightarrow{\text{mon}_{\mathfrak{D}A}} \text{LMap}^1(X_o, \mathfrak{D}A) \subset \text{Map}(Y_o^{\mathfrak{D}A}) \rightarrow \text{Sp}(H_1(Y_o^{\mathfrak{D}A}, \mathbb{Z}))]$$

in $\text{Sp}(H_1(Y_o^{\mathfrak{D}A}, \mathbb{C}))$ acts on $H_1(Y_o^{\mathfrak{D}A}, \mathbb{C})$ with an open orbit.

Clearly now, condition (i) is equivalent to the statement that if $A = \mathbb{Z}/n \times \widehat{\mathbb{Z}/n}$ and if the homomorphism $\pi_1(X_o) \twoheadrightarrow \mathfrak{D}A$ is induced from a surjective homomorphism

$$\pi_1(X_o) \twoheadrightarrow \mathfrak{D}H_n,$$

then the pair $(f : X \rightarrow B, \pi_1(X_o) \twoheadrightarrow \mathfrak{D}A)$ is good.

In other words, we need to find geometric restrictions on a family $f : X \rightarrow B$ and a homomorphism $\pi_1(X_o) \twoheadrightarrow \mathfrak{D}A$, which will guarantee that the pair $(f : X \rightarrow B, \pi_1(X_o) \twoheadrightarrow \mathfrak{D}A)$ is good.

As a first approximation one has to understand the image of $\text{LMap}^1(X_o, \mathfrak{D}A)$ into the symplectic group $\text{Sp}(H_1(Y_o^{\mathfrak{D}A}, \mathbb{Z}))$. In the next section, we will analyze the hyperelliptic part of this image for a suitably chosen surjection $\pi_1(X_o) \twoheadrightarrow \mathfrak{D}A$.

Remark 2.10. Satisfying condition (i) is a somewhat subtle task. In a preliminary version of this paper we attempted to work with a representation $\rho : \pi_1(X_o) \rightarrow \text{GL}(V_n)$ which comes from a choice of a surjective homomorphism $\pi_1(X_o) \twoheadrightarrow H_n$ onto

the Heisenberg group H_n rather than its dihedral extension $\mathfrak{D}H_n$. This representation ρ is also irreducible and so gives a smooth point in $M_B(X_o, n)$ which satisfies condition (ii). Furthermore, the image of $\text{LMap}(X_o, A)$ into the corresponding symplectic group $\text{Sp}(H_1(Y_o^A, \mathbb{Z}))$ was described explicitly by Looijenga [10, Theorem 2.5]. Unfortunately, the self-duality pairing on the representation $\text{ad}(\phi_n)^{\text{ab}}$ gives rise to a quadratic function on $H^1(X_o, \text{ad}(\rho))$ which will be preserved by all elements of the geometric monodromy and so we cannot hope that G will have an open orbit for this choice of ρ . Replacing ρ by a representation coming from a surjection onto the dihedral Heisenberg group repairs this problem as we will see below. However this changes the setup and forces us to work with the two-dimensional dihedral representations W_u instead of the characters of A . In particular, this setup lies beyond the scope of Looijenga’s analysis in [10] and forces us to look for a description of the image of $\text{LMap}^1(X_o, \mathfrak{D}A)$ into the symplectic group $\text{Sp}(H_1(Y_o^{\mathfrak{D}A}, \mathbb{Z}))$ based on first principles only.

Remarkably enough, it turns out that such a concrete description is possible and that it leads to a stronger result which uses only hyperelliptic mapping classes to obtain an open orbit. However note that we need to assume that n is odd in the explicit argument.

3. Proofs of the main theorems

In this section we prove Theorems A and B.

3.1. The case of a hyperelliptic monodromy

In this section $f : X \rightarrow B$ will denote a smooth family of all hyperelliptic curves of genus g . For us the hyperelliptic curves will be represented as branched double covers of \mathbb{P}^1 having $2g + 2$ branch points. From that point of view it is natural to take B to be the configuration space Conf_{2g+2} of $2g + 2$ distinct points in \mathbb{P}^1 . However, it is well known [13] that no universal hyperelliptic family exists on that space. This can be remedied by either passing to an unramified double cover of Conf_{2g+2} (see [13, Remark 4]) or, alternatively, by taking an open subfamily of Conf_{2g+2} . We take the second approach since it is better suited for our purposes.

Concretely, we take B to be the configuration space of $2g + 2$ distinct points in the affine line $\mathbb{C} = \mathbb{P}^1 - \{\infty\}$. To see that the universal hyperelliptic family on B exists, we only need to show that the incidence divisor

$$\Sigma := \{(\{b_1, \dots, b_{2g+2}\}, x) \in B \times \mathbb{P}^1 \mid x \in \{b_1, \dots, b_{2g+2}\}\} \subset B \times \mathbb{P}^1$$

corresponds to a section σ in the line bundle $p_{\mathbb{P}^1}^* \mathcal{O}(2g + 2)$ on $B \times \mathbb{P}^1$. Indeed, if this is the case we have a divisor $\sigma(B)$ in the total space $\text{tot}(p_{\mathbb{P}^1}^* \mathcal{O}(2g + 2))$ of the line

bundle $p_{\mathbb{P}^1}^* \mathcal{O}(2g + 2)$ and so X can be constructed simply as the preimage of $\sigma(B)$ in $\text{tot}(p_{\mathbb{P}^1}^* \mathcal{O}(g + 1))$ under the natural squaring map.

To see that Σ is in the linear system $|p_{\mathbb{P}^1}^* \mathcal{O}(2g + 2)|$ one can argue as follows. Fix an affine coordinate z on $\mathbb{C} = \mathbb{P}^1 - \{\infty\}$. For any integer $k > 0$ let $S_k := H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k))$. On the product $S_k \times \mathbb{P}^1$ we have the line bundle $p_{\mathbb{P}^1}^* \mathcal{O}(k)$. Moreover, the direct image $p_{S_k} p_{\mathbb{P}^1}^* \mathcal{O}(k)$ is a vector bundle of rank $k + 1$ on S_k which is canonically isomorphic to $\mathcal{O}_{S_k} \otimes S_k$ and so has a tautological section corresponding to id_{S_k} . Let σ_k denote the corresponding section of $p_{\mathbb{P}^1}^* \mathcal{O}(k)$. By construction, the divisor of σ_k consists of all pairs $(s, x) \in S_k \times \mathbb{P}^1$ with $s(x) = 0$. Consider now the subspace $E_k \subset S_k$ of all monic polynomials in z of degree k . Since by definition B can be identified with the open subset of the affine subspace $E_{2g+2} \subset S_{2g+2}$ consisting of monic polynomials with simple zeros, we can take $\sigma = \sigma_{2g+2}|_B$.

Therefore, we have constructed a universal double cover

$$\begin{array}{ccc}
 X & \xrightarrow{v} & B \times \mathbb{P}^1 \\
 & \searrow f & \swarrow p_B \\
 & & B
 \end{array}$$

over the configuration space B . Note that the fundamental group of B is the braid group B_{2g+2} on $2g + 2$ strands and that the monodromy homomorphism for $f: X \rightarrow B$ can be interpreted as the standard surjection from B_{2g+2} onto the hyperelliptic mapping class group.

Fix as base point $o \in B$ the double cover $v_o: X_o \rightarrow \mathbb{P}^1$ with branch points $1, 2, \dots, 2g + 2 \in \mathbb{R} \subset \mathbb{P}^1$ on the real axis. Make branch cuts C_i on \mathbb{P}^1 from $2i - 1$ to $2i$ along the real axis. Topologically, the surface X_o is obtained (see Fig. 1) by gluing together two copies of the sliced-up \mathbb{P}^1 along the rims of the branch cuts.

For concreteness we label the two \mathbb{P}^1 -sheets as the *upper* and the *lower* sheet of X_o . The covering map $v_o: X_o \rightarrow \mathbb{P}^1$ projects each sheet onto \mathbb{P}^1 and the corresponding covering involution $\iota_o: X_o \rightarrow X_o$ interchanges the two sheets.

While working with loops on X_o (either as representatives of elements in $\pi_1(X_o)$ or as circles determining Dehn twists on X_o) it will be convenient to describe these loops in terms of their v_o -images in \mathbb{P}^1 . We will only look at loops which do not pass through the branch points and which are transversal to the boundary circles of our sheets. Every such loop L projects via v_o onto a simple closed path in \mathbb{P}^1 which does not pass through any branch point and is transversal to the branch cuts. Therefore specifying a loop L in X_o is the same thing as specifying a simple closed path in \mathbb{P}^1 (not passing through the branch points and transversal to the branch cuts), together with a labeling at each point indicating whether it is on the upper or lower sheet and such that this labeling changes upon crossing a branch cut. To avoid introducing additional notation we will write L both for the loop in X_o and for the corresponding labeled path in \mathbb{P}^1 . This will not create any confusion since it will always be clear from the context which incarnation of L we have in mind.

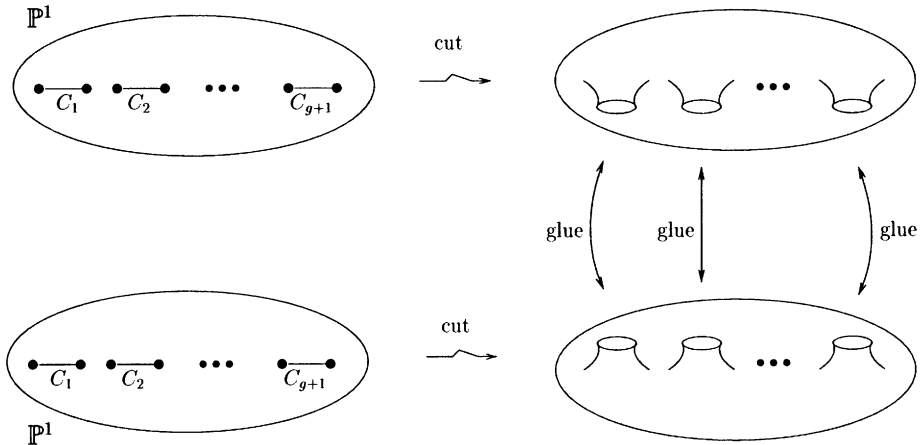


Fig. 1. Gluing two \mathbb{P}^1 sheets into a hyperelliptic curve.

The image of the geometric monodromy representation

$$\text{mon} : \pi_1(B, o) \rightarrow \text{Map}(X_o)$$

for the family $f : X \rightarrow B$, is the full hyperelliptic mapping class group

$$\Delta(X_o) = \{ \phi \in \text{Map}(X_o) \mid \phi \iota_o \phi^{-1} = \iota_o \}.$$

It is generated by the right-handed Dehn twists along the sequence of loops a_1, \dots, a_{2g+1} depicted in Fig. 2. In this figure we use the convention that for paths in \mathbb{P}^1 the solid pieces are on the upper sheet and the dotted pieces are on the lower sheet.

Note that the above Dehn twists define a surjective homomorphism from the braid group B_{2g+2} on $2g + 2$ strands to the hyperelliptic mapping class group $\Delta(X_o)$. Indeed, by definition B_{2g+2} can be presented as

$$B_{2g+2} = \left\langle t_1, t_2, \dots, t_{2g+1} \left| \begin{array}{l} t_i t_j = t_j t_i, \text{ for } |i - j| \geq 2 \\ t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1} \end{array} \right. \right\rangle$$

and so the assignment $t_i \mapsto$ (Dehn twist along a_i) induces a (necessarily surjective) group homomorphism $\kappa_g : B_{2g+2} \rightarrow \Delta(X_o)$.

Fix a positive odd integer n . To fix notation, choose a primitive n th root of unity $\gamma \in \mu_n$ and let $\alpha : \mathbb{Z}/n \rightarrow \mathbb{C}^\times, k \mapsto \gamma^k$ be the corresponding character. For future use we denote the corresponding standard generators of $\mathfrak{D}H_n$ as follows:

$$\sigma := (-1, 1, 0, \mathbf{1}), \quad \mathbf{a} := (1, 1, 1, \mathbf{1}), \quad \alpha := (1, 1, 0, \alpha).$$

We also write $e := (1, 1, 0, \mathbf{1})$ for the identity element in $\mathfrak{D}H_n$.

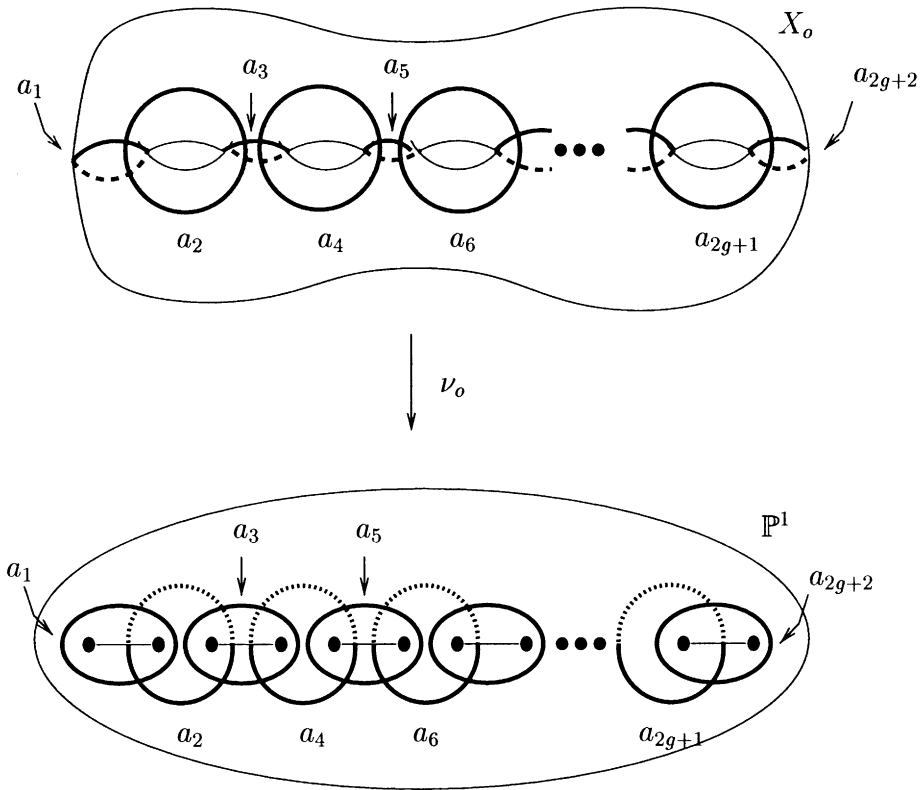


Fig. 2. Dehn twists generating $\Delta(X_0)$.

With this notation we are now ready to define the base representation $\rho \in M_{\mathbb{B}}(X_0, n)$ at which we will be checking the open orbit condition (i) from Section 2.2. For this we only need to exhibit a surjective homomorphism $\psi_n : \pi_1(X_0) \rightarrow \mathfrak{D}H_n$. We define ψ_n to be trivial on the complement of the branch cuts on both sheets and we postulate that the passing transformations $P_i \in \mathfrak{D}H_n$ corresponding to going through the branch cut C_i should be

$$P_1 = \sigma, \quad P_2 = a\sigma, \quad P_3 = \alpha\sigma, \quad P_4 = a, \quad P_5 = \alpha, \quad P_i = e, \quad \text{for all } i \geq 6.$$

Assume that $g \geq 6$, so there are at least two branch cuts with passing transformation equal to the identity.

Consider next any two element orbit $u = \{\chi, \chi^{-1}\} \in (\widehat{\mathbb{Z}/n} \times \mathbb{Z}/n) / \mu_2$ and let W_u be the corresponding two-dimensional irreducible representation of $\mathfrak{D}H_n$. As we saw in Section 2.2, the action of $\mathfrak{D}H_n$ on W_u factors through $\mu_2 \ltimes (\mathbb{Z}/n \times \widehat{\mathbb{Z}/n})$. Choose a basis $\{v_+, v_-\}$ of W_u consisting of eigenvectors for the $\mathbb{Z}/n \times \widehat{\mathbb{Z}/n}$ action. To fix notation assume that v_+ corresponds to the character χ and that v_- corresponds to the character χ^{-1} . If we write the character χ as $\chi = (\alpha^b, \alpha^c)$ for some integers b and c ,

then in the basis $\{v_+, v_-\}$ the representation W_u is given by associating

$$\sigma \mapsto R, \quad \mathbf{a} \mapsto P^b, \quad \boldsymbol{\alpha} \mapsto P^c,$$

where P and R are the 2×2 matrices:

$$P := \begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{pmatrix}, \quad R := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This gives the matrices for the action of passing transformations on the local system \mathbb{W}_u :

$$P_1 = R, \quad P_2 = P^b R, \quad P_3 = P^c R, \quad P_4 = P^b, \quad P_5 = P^c \quad (3.1)$$

and the rest are equal to the identity matrix $I \in \text{GL}_2(\mathbb{C})$.

Note that by our assumption on n it follows that P is of odd order so these matrices are never equal to $-I$.

For any $1 \leq i < j \leq 2g + 2$ let L_{ij} denote the loop which goes around the branch points i and j , passing under any other branch points which are in between on the real axis, and going in the clockwise direction. (Some sample loops L_{ij} are illustrated in Fig. 3.) Assume that the lower part of the curve is on the upper sheet, and let $M_{ij} \in \text{GL}_2(\mathbb{C}) \cong \text{GL}(W_u)$ denote the monodromy transformation around L_{ij} .

Observe next that the representation W_u is self-dual: in our basis $\{v_+, v_-\}$ the invariant pairing $Q: W_u \otimes W_u \rightarrow \mathbb{C}$ is given by

$$Q\left(\begin{pmatrix} r \\ s \end{pmatrix}, \begin{pmatrix} r' \\ s' \end{pmatrix}\right) := rs' + r's.$$

The pairing Q induces an intersection pairing on $H_1(X_o, \mathbb{W}_u)$ and an isomorphism of homology and cohomology $H_1(X_o, \mathbb{W}_u) \cong H^1(X_o, \mathbb{W}_u)$, both compatible with the monodromy action of the hyperelliptic mapping class group $\pi_1(B, o) \cong \mathcal{A}(X_o)$. Thus it suffices to calculate the monodromy action on the homology $H_1(X_o, \mathbb{W}_u)$.

We will represent the elements in $H_1(X_o, \mathbb{W}_u)$ by loop-like chains. Similarly to ordinary loops, a loop-like chain with coefficients in W_u is given by

- an oriented simple closed path in \mathbb{P}^1 (not passing through the branch points and transversal to the branch cut), together with a labeling at each point indicating whether it is on the upper or the lower sheet, and such that the labeling changes upon crossing of a branch cut;
- a specification at each point of the path of a vector in W_u , such that upon crossing a branch cut C_i from the upper to the lower sheet this vector is modified by the corresponding passing matrix P_i .

A loop-like chain will typically be denoted by vL , where L is the loop and v is the corresponding vector in W_u .

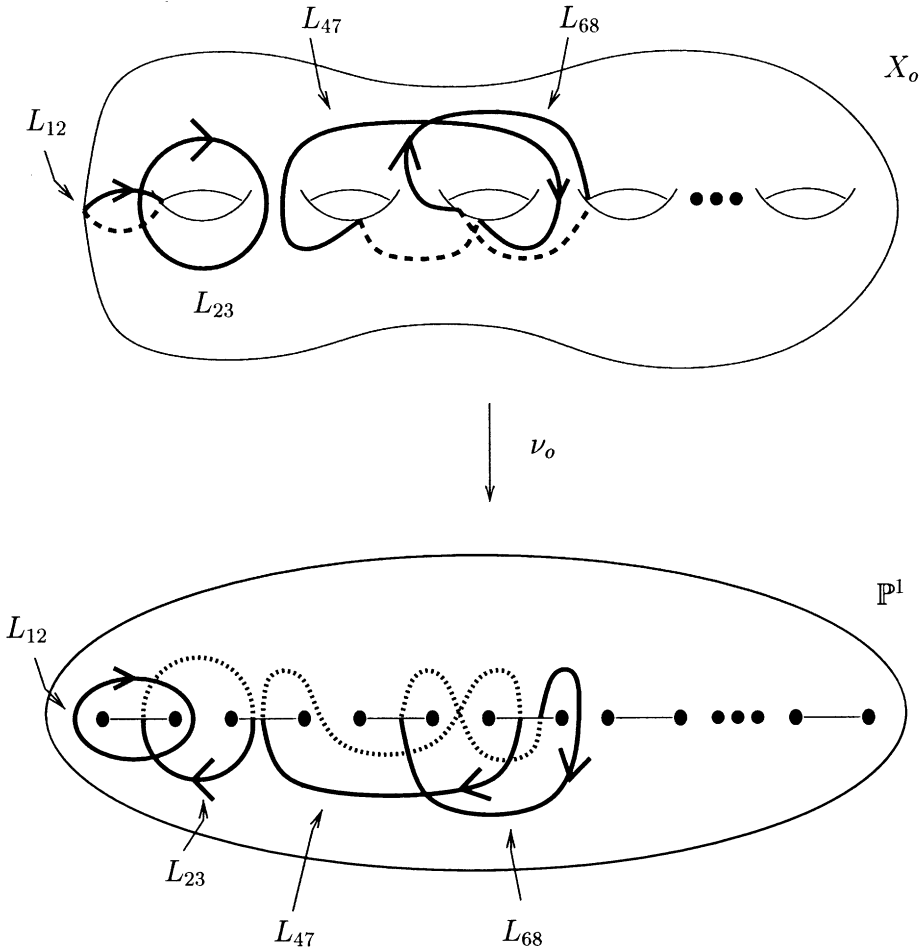


Fig. 3. Some L_{ij} 's.

Our convention for the intersection pairing on $H_1(X_o, \mathbb{W}_u)$ will be that we intersect the loop-like chains by intersecting the underlying loops according to the right-hand rule (a first path intersects positively a second path which, to him is coming from the right) and at each intersection point we pair the corresponding elements in \mathbb{W}_u via \mathcal{Q} .

The homology group $H_1(X_o, \mathbb{W}_u)$ contains elements of the form $v_{ij}L_{ij}$, where $v_{ij} \in \mathbb{W}_u$ is any vector which is invariant under M_{ij} . We can distinguish three cases:

- if M_{ij} is the identity then v_{ij} can be any vector so there is a two-dimensional space of such cycles;
- if M_{ij} is a reflection (of the form $P^k R$) then v_{ij} is of the form $(v + M_{ij}v)$ for any sufficiently general vector in \mathbb{W}_u , and in fact we may take $v = v_+$;

- if M_{ij} is a rotation of the form P^k for k different from 0 modulo the order of P , then there are no non-zero cycles of this form.

Using these elements we can now prove the following:

Lemma 3.1. *The cycles of the form $v_{ij}L_{ij}$ span the homology $H_1(X_o, \mathbb{W}_u)$ of the hyperelliptic curve with coefficients in \mathbb{W}_u .*

Proof. We need to separate into cases depending on b and c . Assume first of all that b and c are different and different from 0 (modulo n). For the purposes of this lemma, we can apply a braid transformation to arrange things so that the passing matrices are (in order starting with P_1):

$$P^b, P^c, P^bR, P^cR, R, I, \dots$$

Now we consider an element of the homology of the hyperelliptic curve with coefficients in \mathbb{W}_u . It can be moved to a cycle supported over the real axis, necessarily on the interval between 1 and $2g + 2$. Look first at the interval $[2g + 1, 2g + 2]$. By subtracting off a cycle of the form $v_{2g+1,2g+2}L_{2g+1,2g+2}$ we obtain a cycle which is zero on the upper sheet along the interval in question (note that the vector $v_{2g+1,2g+2}$ can be arbitrary). Now the cycle condition at the point $2g + 2$ implies that the cycle is also zero on the lower sheet. We get to a cycle supported on the interval $[1, 2g + 1]$. Continuing this way by induction we get to a cycle supported on the interval $[8]$ (the last non-trivial passing matrix is P_5).

The monodromy transformation $M_{9,10}$ is the identity and so again we can take $v_{9,10}$ to be an arbitrary vector. By subtracting a multiple of $v_{9,10}L_{9,10}$ we get to a cycle supported in $[7]$.

Next, look at the monodromy matrices

$$M_{2,9} = RP^b \quad \text{and} \quad M_{4,9} = RP^c.$$

Thus we get that up to scalars

$$v_{2,9} = (1 + RP^b)v_+ = \begin{pmatrix} 1 \\ \gamma^b \end{pmatrix}$$

and

$$v_{4,9} = (1 + RP^c)v_+ = \begin{pmatrix} 1 \\ \gamma^c \end{pmatrix}.$$

The determinant of the matrix with these two vectors in the columns is $\gamma^c - \gamma^b$ which is non-zero under our assumption that b is different from c . Therefore by subtracting off an appropriate combination of the cycles $v_{2,9}L_{2,9}$ and $v_{4,9}L_{4,9}$ we obtain a cycle which is zero in the interval $[6,7]$, in other words it is supported in $[6]$. Again the

monodromy matrix $M_{7,8}$ is the identity so we can subtract off a vector of the form $v_{7,8}$ to get a cycle supported in [5].

We repeat the argument above using

$$v_{2,7} = (1 + P^b R P^b)v_+ = v_+$$

and

$$v_{4,7} = (1 + P^b R P^c)v_+ = v_+ + \gamma^{c-b}v_-$$

which are linearly independent. This, combined with subtracting off a $v_{5,6}L_{5,6}$, gets us to a cycle supported in [3]; and repeating again the same argument with $v_{2,5}$ and $v_{4,5}$ we get to a cycle supported in [1]. On the other hand, the monodromy transformation $M_{1,2}$ is trivial so by subtracting off a cycle of the form $v_{1,2}L_{1,2}$ with $v_{1,2}$ arbitrary, we get to a cycle supported in [1], which must be a multiple of $v_{2,3}L_{2,3}$ so we are done. One can note in passing that this last cycle must automatically be zero since the monodromy $M_{2,3}$ does not have any fixed vectors but this is not really important. This completes the proof in the case $b \neq c$.

Suppose now we are in the case $b = c$ (modulo n). Then b and c are non-zero. Thus, we can repeat the same argument as above but arranging things so that the passing matrices are (in order)

$$P^b, P^c, I, P^b R, P^c R, R, I, \dots$$

In this case the same argument as before (but using vectors such as $v_{4,11}, v_{6,11}$, etc.) allows us to get to a cycle supported on [1, 5]. Now the first monodromy matrices are the identities:

$$M_{1,2} = M_{2,3} = M_{3,4} = I,$$

so we can subtract off cycles of the form $v_{1,2}L_{1,2}, v_{2,3}L_{2,3}$, and $v_{3,4}L_{3,4}$ to get to a cycle supported on [2,3] and again we are done. This completes the proof of the lemma. \square

For the remainder of the argument we return to the labeling (3.1) for the order of the branch cuts.

Now let $t_{ij} \in \text{Map}(X_o)$ denote the right-handed Dehn twist along the loop L_{ij} . For the three types of behavior of M_{ij} we have

- if M_{ij} is the identity, then $t_{ij} \in \Delta(X_o)$ and acts on the local system $R^1 f_* \mathbb{W}_u$. In particular t_{ij} maps to a well-defined element $D_{ij} \in \text{Sp}(H_1(X_o, \mathbb{W}_u))$;
- if M_{ij} is a reflection, then $t_{ij}^2 \in \Delta(X_o)$ and acts on the local system $R^1 f_* \mathbb{W}_u$. In particular t_{ij}^2 maps to a well-defined element $D_{ij}^2 \in \text{Sp}(H_1(X_o, \mathbb{W}_u))$ (Fig. 4 illustrates the typical action of D_{ij}^2);
- if M_{ij} is a rotation then we do not consider the Dehn twist.

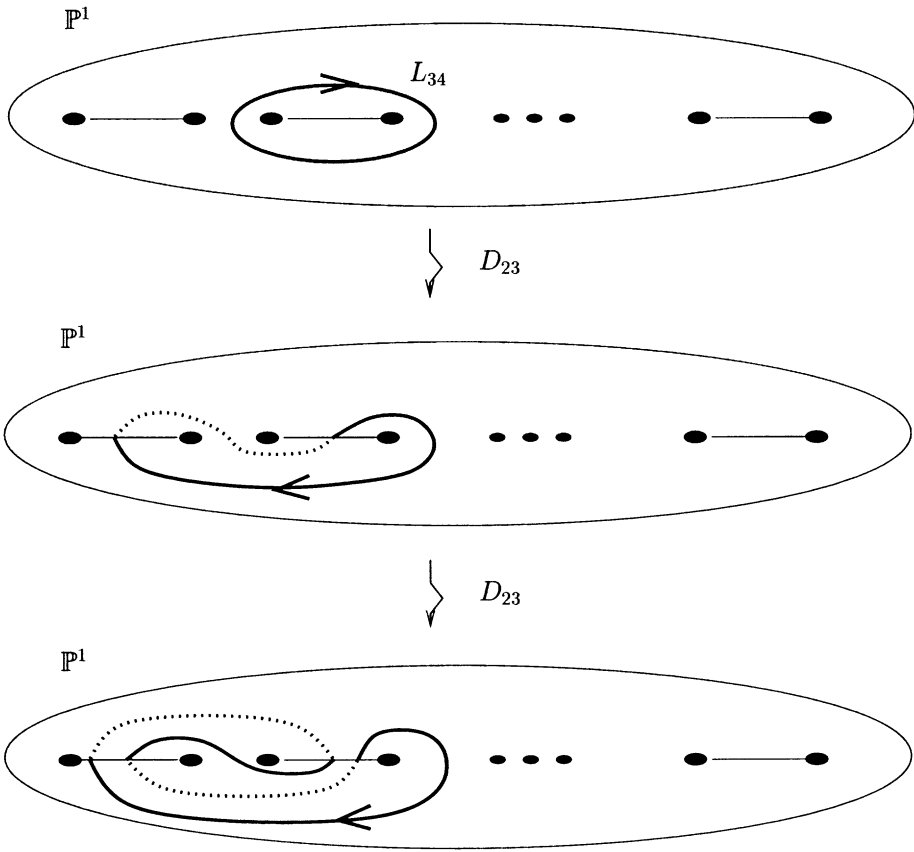


Fig. 4. The action of D_{23}^2 on the loop L_{34} .

For uniformity, we will always consider D_{ij}^2 . We now have the following lemma:

Lemma 3.2. *The subgroup of $\text{Sp}(H_1(X_o, \mathbb{W}_u))$ generated by the elements*

$$\left\{ D_{ij}^2 \mid \begin{array}{l} 1 \leq i < j \leq 2g + 2 \\ M_{ij} \neq \text{rotation} \end{array} \right\}$$

acts irreducibly on the complex vector space $H_1(X_o, \mathbb{W}_u)$.

Proof. Consider the elements of the form $D_{ij}^2 - 1$ in the group algebra of $\text{Sp}(H_1(X_o, \mathbb{W}_u))$. By general principles this algebra is semisimple, so it suffices to find a vector w_0 such that the subspace generated by the action of the $D_{ij}^2 - 1$ starting

with w_0 , spans the whole $H_1(X_o, \mathbb{W}_u)$. We will start with

$$w_0 := v_{2,11}L_{2,11}$$

and show that using the $D_{ij}^2 - 1$ we can get to any vector $v_{ij}L_{ij}$. In view of Lemma 3.1 this will complete the proof of the irreducibility.

Note that P_6 is the identity and P_1 is a reflection, so $M_{2,11} = R$ is a reflection. Thus $v_{2,11}$ is of the form $v_+ + Rv_+ = v_+ + v_-$.

Using $D_{12}^2 - 1$ we get to $(v_+ + Rv_+)L_{12}$ (note that we allow ourselves to multiply by a factor for example $\frac{1}{2}$ or $-\frac{1}{2}$ when we say this).

Now one of b or c is different from zero modulo the order of P . Assume for example that b is different from zero. Then the monodromy transformation $M_{2,7}$ is $P^{-b}R$ so using $D_{27}^2 - 1$ we get to

$$(1 + P^{-b}R)(1 + R)v_+L_{27}.$$

Applying again $D_{12}^2 - 1$ we get back to

$$(1 + P^{-b}R)(1 + R)v_+L_{12}.$$

In the other case where b is zero but c non-zero we could use $D_{29}^2 - 1$ and get to

$$(1 + P^{-c}R)(1 + R)v_+L_{12}.$$

In the first case, note that the image of $(1 + R)$ is not contained in the kernel of $(1 + P^{-b}R)$, and the image of $(1 + P^{-b}R)$ is linearly independent from the image of $(1 + R)$, so with the vector we obtained previously we obtain both vectors v_+L_{12} and v_-L_{12} . The same holds in the second case where we used the matrix P^c .

A similar argument gets us to any of the vectors vL_{34} and vL_{56} .

Next, using again the fact that one of the P^bR or P^cR is different from R , and using the appropriate transformation $D_{4,11}^2 - 1$ or $D_{6,11}^2 - 1$ as well as $D_{2,11}^2 - 1$ and following by $D_{11,12}^2 - 1$, we get to any vector of the form $vL_{11,12}$. Now using the Dehn twists for i, j with $11 \leq i < j \leq 2g + 2$ we obtain all of the vectors of the form vL_{ij} for $11 \leq i < j \leq 2g$.

From these using the Dehn twists $D_{i,j}^2 - 1$ for $i < 11$ and $j \geq 11$ we get to all vectors of the form $v_{ij}L_{ij}$ for $i < 11$ and $j \geq 11$.

Similarly we get to all vectors of the form $v_{ij}L_{ij}$ when the monodromy transformations M_{ij} are reflections, by using the Dehn twist $D_{ij}^2 - 1$ on a vector $v_{kl}L_{kl}$ where one of k or l is either i or j and where $[k, l]$ is a branch cut on which the passing matrix is a reflection (note that such k, l always exist when the monodromy

M_{ij} is a reflection). Here from above we have already gotten to the $v_{kl}L_{kl}$ with v_{kl} arbitrary.

This argument also works to obtain $v_{ij}L_{ij}$ whenever $1 \leq i < j \leq 6$ is an exceptional case where the monodromy is the identity due to a special equality of the form $b = c$ or $b = 0$ or $c = 0$.

The only cycles which remain to be obtained are the $v_{ij}L_{ij}$ for $7 \leq i < j \leq 10$. We first obtain $v_{78}L_{78}$. To do this, note that two among the three matrices R , P^bR and P^cR are different, and for appropriate choices of i and j corresponding to these two, chosen among 2, 4 and 6, we have that the images of the rank one matrices $(1 + M_{i,7})$ and $(1 + M_{j,7})$ are linearly independent and span our two-dimensional space (this is similar to the argument used in Lemma 3.1). Thus applying the Dehn twist $D_{78}^2 - 1$ to the vectors $v_{i,7}$ and $v_{j,7}$ we span a two-dimensional space so we can get to any vector of the form $v_{78}L_{78}$ with v_{78} arbitrary. The same argument yields any vector of the form $v_{9,10}L_{9,10}$ with $v_{9,10}$ arbitrary. Finally, in the exceptional case where M_{89} is the identity (this is when $b = c$), using its Dehn twist we get to the vectors of the form $v_{89}L_{89}$.

This completes the proof of the lemma. \square

Let now \mathfrak{g} denote the Lie algebra of the monodromy group acting on the representation $\mathbb{C}^{4g-4} \cong H^1(X_o, \mathbb{W}_u)$. Note that \mathfrak{g} is semisimple, by general theory. In Lemma 3.2 we proved that $H^1(X_o, \mathbb{W}_u)$ is an irreducible representation of \mathfrak{g} . The above proof works for the Lie algebra since the D_{ij}^2 are unipotent matrices with Jordan blocks of length at most one and thus $A_{ij} := D_{ij}^2 - 1$ are elements of \mathfrak{g} . Hence the above proof shows that the A_{ij} act irreducibly.

For some i, j we have A_{ij} decomposing into two Jordan blocks of length one (this is the case for $A_{1,2}$, $A_{3,4}$ etc.). However there are some i, j where the monodromy $M_{i,j}$ is a reflection (for example $i = 2$, $j = 11$), where the $A_{i,j}$ has a single Jordan block of length one.

Next, note that by isolating the monodromy representation on the part of the curve where the local system is trivial, we obtain a monodromy representation of the direct sum of two copies of the cohomology of a hyperelliptic curve of genus g' with the monodromy acting diagonally. If we take all branch points for $i \geq 11$ then this has genus $g' = g - 5$. Deligne's argument from [3, Section 4.4] or the argument from [7] works for the hyperelliptic monodromy action on the standard cohomology (the monodromy is generated by conjugate Dehn twists) so this monodromy group is $\mathrm{Sp}(2g - 10)$. In particular, we have

$$\mathrm{sp}(2g - 10) \subset \mathfrak{g} \subset \mathrm{sp}(4g - 4),$$

where the composite inclusion is the linear embedding of the diagonal action on the direct sum of two copies of the standard representation of $\mathrm{sp}(2g - 10)$.

This shows that our monodromy action satisfies the hypothesis of the following purely algebraic theorem:

Theorem 3.3. *There exists g_0 with the following property: suppose $g \geq g_0$ and suppose \mathfrak{g} is a semisimple Lie algebra sitting in a pair of inclusions*

$$\mathfrak{sp}(2g - 10) \subset \mathfrak{g} \subset \mathfrak{sp}(4g - 4),$$

where the composite inclusion is the linear embedding of the diagonal action on the direct sum of two copies of the standard representation of $\mathfrak{sp}(2g - 10)$. Suppose that the action of \mathfrak{g} on \mathbb{C}^{4g-4} is irreducible, and suppose \mathfrak{g} contains an element A which acts on \mathbb{C}^{4g-4} with a single Jordan block of length one. Then $\mathfrak{g} = \mathfrak{sp}(4g - 4)$.

Proof. We first claim that \mathfrak{g} is simple. If that were not the case, then we could write $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$ and so the representation $W = \mathbb{C}^{4g-4}$ would decompose as an exterior tensor product $= W_1 \otimes W_2$ of representations of \mathfrak{g}_1 and \mathfrak{g}_2 . This however can be ruled out by looking at the element A . If it is non-trivial in both factors then it would act on W by a Jordan normal form which is the tensor product of two non-trivial Jordan normal forms, in particular it would have a Jordan block of length > 1 ; if it was non-trivial in only one of the factors then it would act by the tensor product of a non-trivial Jordan form, by a trivial vector space (of dimension > 1); thus it would have at least two Jordan blocks. In either case this contradicts the hypothesis that A acts on \mathbb{C}^{4g-4} with a single Jordan block of length one. This proves that \mathfrak{g} is simple.

Now we can choose g_0 big enough so that the dimension of $\mathfrak{sp}(2g - 10)$ is bigger than the dimension of the exceptional simple Lie algebras. By classification, this means that \mathfrak{g} is of one of $\mathfrak{so}(2m), \mathfrak{so}(2m - 1), \mathfrak{sp}(2m)$, or $\mathfrak{sl}(m)$. Looking at dimensions, we get $m \geq g - C$ where C is some constant.

On the other hand, note that all the fundamental representations of the classical groups are essentially (the only exception being the spin representation) the wedge powers of the standard representation. Combined with Weyl’s dimension formula this implies that there is an m_0 such that for $m \geq m_0$ the only irreducible representations of dimension $< 5m$ of one of the classical groups above, are the fundamental representation or (in the last case) the dual of the fundamental representation.

This claim gives that \mathfrak{g} acts by the standard representation, which immediately implies that it is equal to $\mathfrak{sp}(4g - 4)$ (it cannot be orthogonal or special linear because we already know it is contained in the symplectic group). \square

We are now in a position to complete the

Proof of Theorem A. Let g_0 be such that Theorem 3.3 holds. In view of Corollary 2.4 and Lemma 2.8 we only need to show that the Zariski-closure G of the monodromy action of $\pi_1(B, o)$ on $H_1(X_o, \text{ad}(\rho))$ acts with an open orbit on $H_1(X_o, \text{ad}(\rho))$.

By Theorem 3.3, we get that each subspace $H_1(X_o, \mathbb{W}_u) \subset H_1(X_o, \text{ad}(\rho))$ yields a monodromy representation equal to the symplectic group. Note that Theorem 3.3 implies that the Lie algebra of the monodromy group is equal to $\mathfrak{sp}(4g - 4)$. However, in view of the fact that we already have an inclusion of the monodromy

group of $H_1(X_o, \mathbb{W}_u)$ in $\mathrm{Sp}(4g - 4)$, we get that this inclusion is surjective so the monodromy group is equal to $\mathrm{Sp}(4g - 4)$ which we now write as $\mathrm{Sp}(H_1(X_o, \mathbb{W}_u))$.

In particular, we have a natural inclusion

$$G \subset \mathrm{Sp}(H_1(X_o, \mathbb{C})) \times \prod_{u \neq (1,0)} \mathrm{Sp}(H_1(X_o, \mathbb{W}_u)),$$

so that the projection on each factor is surjective. On the other hand, G is semisimple. Going back to the level of Lie algebras, this implies that the simple summands of \mathfrak{g} are $\mathfrak{sp}(2g)$ occurring once, and $\mathfrak{sp}(H_1(X_o, \mathbb{W}_u)) = \mathfrak{sp}(4g - 4)$ occurring a certain number of times. Call these summands $\mathfrak{s}_1, \dots, \mathfrak{s}_k$.

Each irreducible factor $H_1(X_o, \mathbb{W}_u)$ in the representation $H_1(X_o, \mathrm{ad}(\rho))$, is an irreducible representation of the Lie algebra

$$\mathfrak{sp}(2g) \oplus \mathfrak{s}_1 \oplus \dots \oplus \mathfrak{s}_k.$$

As such, it decomposes a priori into an exterior tensor product of representations of the summands; but by dimension considerations, this tensor product must just be an irreducible representation of one of the summands \mathfrak{s}_i . Also this representation is isomorphic to the standard representation of $\mathfrak{s}_i = \mathfrak{sp}(4g - 4)$. Thus each $H_1(X_o, \mathbb{W}_u)$ comes from the standard representation composed with a projection onto one of the factors. Note also that the representation \mathbb{C}^{2g} comes from the standard representation composed with the projection onto the factor $\mathfrak{sp}(2g)$.

The above statements on the level of Lie algebras imply the same things for the connected components of the Lie groups. We obtain that the connected component of the monodromy group G decomposes as a product

$$G^o = \mathrm{Sp}(2g) \times S_1 \times \dots \times S_k,$$

where each S_i is equal to $\mathrm{Sp}(4g - 4)$ and S_i acts on a direct sum of r_i copies of its standard representation.

Our goal now is to prove that all of the r_i are equal to 1. Suppose the contrary, i.e. suppose that there are two distinct components u and u' such that the same S_i acts on $H_1(X_o, \mathbb{W}_u)$ and $H_1(X_o, \mathbb{W}_{u'})$. In particular this means that $H_1(X_o, \mathbb{W}_u)$ and $H_1(X_o, \mathbb{W}_{u'})$ are isomorphic as representations of G^o .

The elements D_{ij}^2 are unipotent so they go into G^o . Let Γ denote the subgroup of the monodromy group which maps into G^o . We obtain a map

$$\mathbb{C}[\Gamma] \rightarrow \mathbb{C}[G^o].$$

In particular, the action of the group algebra $\mathbb{C}[\Gamma]$ on $\bigoplus_u H_1(X_o, \mathbb{W}_u)$ factors through the action of the group algebra of G^o . Thus, with our assumption of the previous paragraph that some r_i is > 1 , we would get two components $H_1(X_o, \mathbb{W}_u)$ and $H_1(X_o, \mathbb{W}_{u'})$ which are isomorphic as representations of the group algebra $\mathbb{C}[\Gamma]$. Therefore for every element $E \in \mathbb{C}[\Gamma]$, the eigenvalues of E acting on $H_1(X_o, \mathbb{W}_u)$ and $H_1(X_o, \mathbb{W}_{u'})$ are the same. We will write down elements E which act with single

non-zero eigenvalues; thus these values are the same for u and u' . We will show that this implies that u and u' are the same component. This will contradict the assumption that $r_i > 1$ so it will prove the statement that all of the r_i are equal to 1.

We will write down our E as products of elements of the form $A_{ij} := D_{ij}^2 - 1$. The component u (respectively u') is determined by the numbers b and c (respectively b' and c') which occur above. These are taken modulo the order n of the root of unity γ , and interchanging $b \leftrightarrow -b$ and $c \leftrightarrow -c$ does not change the dihedral component u . Thus, proving that the two components are the same means that we want to show

$$(b, c) = \pm(b', c') \text{ in } (\mathbb{Z}/n)^2.$$

Let $i, j = 7, 9, 11$. Look for example at

$$A_{2,i}A_{1,2}.$$

This takes the vector $v_{2,j}L_{2,j}$ first to $v_{2,j}L_{1,2}$ and then to

$$(1 + P^{-x_i}R)v_{2,j}L_{2,i}$$

where $x_i = b, c$ or 0 depending on whether $i = 7, 9$ or 11. Look now at

$$E = A_{2,i}A_{1,2}A_{2,j}A_{1,2}.$$

It has an image vector which is a multiple of $v_{2,i}L_{2,i}$ so this can be its only non-zero eigenvector. Its action on this vector is

$$Ev_{2,i}L_{2,i} = (1 + P^{-x_i}R)(1 + P^{-x_j}R)v_{2,i}L_{2,i}.$$

Note also that the matrix $(1 + P^{-x_i}R)$ is itself of rank one so the product of matrices appearing above also has a single non-zero eigenvalue. In particular, the unique non-zero eigenvalue of B is equal to the unique non-zero eigenvalue of the matrix $(1 + P^{-x_i}R)(1 + P^{-x_j}R)$. This matrix may be written as

$$\begin{pmatrix} 1 + \gamma^{x_j - x_i} & \gamma^{-x_i} + \gamma^{-x_j} \\ \gamma^{x_i} + \gamma^{x_j} & 1 + \gamma^{x_i - x_j} \end{pmatrix}.$$

Its eigenvector is

$$v_{2,i} = (1 + P^{-x_i}R)v_+ = \begin{pmatrix} 1 \\ \gamma^{x_i} \end{pmatrix}.$$

Calculating

$$\begin{pmatrix} 1 + \gamma^{x_j - x_i} & \gamma^{-x_i} + \gamma^{-x_j} \\ \gamma^{x_i} + \gamma^{x_j} & 1 + \gamma^{x_i - x_j} \end{pmatrix} \begin{pmatrix} 1 \\ \gamma^{x_i} \end{pmatrix} = \begin{pmatrix} 2 + \gamma^{x_j - x_i} + \gamma^{x_i - x_j} \\ 2\gamma^{x_i} + \gamma^{x_j} + \gamma^{2x_i - x_j} \end{pmatrix}$$

says that the eigenvalue is equal to

$$2 + \gamma^{x_j - x_i} + \gamma^{x_i - x_j}.$$

Now take various values of i and j , and compare the results for the components u and u' . We obtain

from $i = 11, j = 7$,

$$\gamma^b + \gamma^{-b} = \gamma^{b'} + \gamma^{-b'};$$

from $i = 11, j = 9$,

$$\gamma^c + \gamma^{-c} = \gamma^{c'} + \gamma^{-c'};$$

and from $i = 9, j = 7$,

$$\gamma^{b-c} + \gamma^{c-b} = \gamma^{b'-c'} + \gamma^{c'-b'}.$$

In \mathbb{Z}/n these equations give

$$b = \pm b', \quad c = \pm c', \quad (b - c) = \pm (b' - c').$$

The first two equations admit four possibilities: either

$$(b, c) = (b', c') \quad \text{or} \quad (b, c) = -(b', c'),$$

or else

$$(b, c) = \pm (b', -c').$$

The first two possibilities are what we want to show. In the last two possibilities we have

$$b - c = \pm (b' + c').$$

Thus the third equation above says

$$b' + c' = \pm (b' - c').$$

This says that either $c' = -c'$ or else $b' = -b'$. In either case, the equation $(b, c) = \pm (b', -c')$ gets transformed into the equation $(b, c) = \pm (b', c')$ so we are done. In fact, we could have noted here that since n is odd, the equations $c' = -c'$ or $b' = -b'$ do not occur, but the present argument works even when n is even.

We have now shown that if $H_1(X_o, \mathbb{W}_u)$ and $H_1(X_o, \mathbb{W}_{u'})$ are isomorphic as representations of G^o then u and u' represent the same dihedral component of our representation. This implies that all of the r_i are equal to one, which in turn gives that

$$G^o = \text{Sp}(2g) \times \prod_{u \neq (1,0)} \text{Sp}(H_1(X_o, \mathbb{W}_u)).$$

(Actually, the same is true of the full monodromy group because the full group is also contained in this product of symplectic groups: $G = G^o$.) This group acts on its representation $\mathbb{C}^{2g} \oplus \bigoplus_{u \neq (1,0)} H_1(X_o, \mathbb{W}_u)$ with an open orbit. Combined with Lemma 2.8 and Theorem 2.2 this completes the proof of Theorem A. \square

3.2. Lefschetz pencils

We would now like to extend the techniques of the previous section in order to prove Theorem B.

Let Z be a smooth projective surface with $b_1(Z) = 0$. Let $\mathcal{O}_Z(1)$ be a very ample line bundle on Z and let $\mathbb{P}^1 \subset \mathbb{P}(H^0(Z, \mathcal{O}_Z(k)))$ be a generic line. Denote by $\epsilon : \widehat{Z} \rightarrow Z$ the blow-up of Z at the base points of the pencil of curves $\{D_t\}_{t \in \mathbb{P}^1}$, and let $f : \widehat{Z} \rightarrow \mathbb{P}^1$ be the corresponding Lefschetz fibration. Let $p_1, \dots, p_\mu \in \mathbb{P}^1$ be the critical points of f and let $B = \mathbb{P}^1 - \{p_1, \dots, p_\mu\}$ and $X = f^{-1}(B)$. Let $X_t, t \in B$ denote the fiber of f over t , or equivalently, the strict transform of the divisor $D_t \subset Z$.

Fix g_0 so that Theorem 3.3 applies. Then, as we saw at the end of the previous section, it follows that the monodromy group of the hyperelliptic family of genus g_0 acting on the cohomology of the full local system $\text{End}(\mathbb{V}_n)$ corresponding to $\text{ad}(\rho)$ is equal to $\text{Sp}(2g_0) \times \prod_u \text{Sp}(H_1(X_o, \mathbb{W}_u))$ with $\text{Sp}(H_1(X_o, \mathbb{W}_u)) = \text{Sp}(4g_0 - 4)$. We also assume that $g_0 \geq 6$, so there are several branch cuts along which the representation ρ is the identity. Set $m := 2g_0 + 1$.

Let $\mathcal{L} := \mathcal{O}_Z(k)$ denote our line bundle. Our assertions will be made for k big enough. Let $\mathbb{D} \subset \mathbb{P}H^0(Z, \mathcal{L})$ denote the discriminant locus consisting of the sections defining singular curves. We will fix a base point $o \in \mathbb{P}H^0(Z, \mathcal{L}) - \mathbb{D}$ (chosen specially below); and as always X_o will denote the smooth curve defined by the section o . Then $\pi_1(\mathbb{P}(H^0) - \mathbb{D}, o)$ acts by diffeomorphisms on X_o and hence it acts on $M_B(X_o, n)$. Furthermore, by the Lefschetz hyperplane section theorem, the geometric monodromy action $\pi_1(B, o) \rightarrow \text{Map}(X_o)$ for the family $f : X \rightarrow B$ factors through the natural map $\pi_1(B, o) \rightarrow \pi_1(\mathbb{P}(H^0) - \mathbb{D}, o)$ and so it suffices to show that $\pi_1(\mathbb{P}(H^0) - \mathbb{D}, o)$ acts on $M_B(X_o, n)$ with a Zariski-dense orbit.

As before we shall fix a local system ρ on X_o , with finite monodromy factoring through a representation of the dihedral Heisenberg group. Then there is a subgroup of finite index in $\pi_1(\mathbb{P}H^0 - \mathbb{D}, o)$ which preserves ρ , so it acts on the space

$$T_{[\rho]} M_B(X_o, n) = H^1(X_o, \text{ad}(\rho)) = H^1(X_o, \mathbb{C}) \oplus \left(\bigoplus_{u \neq (1,0)} H^1(X_o, \mathbb{W}_u) \right).$$

According to Corollary 2.4 and Lemma 2.8 we only need to show that the Zariski-closure G of the image of this monodromy action acts with an open orbit on $H^1(X_o, \text{ad}(\rho))$. Our technique will be to show that G is as big as possible, given the above decomposition and the fact that it preserves symplectic forms on everything. To achieve this we will use a family of curves in the linear system $|\mathcal{L}|$ which have

hyperelliptic handles and will apply to the curves the results for the hyperelliptic case obtained in Section 3.1.

We will define a particular subspace $\mathbb{E} \subset \mathbb{P}H^0(Z, \mathcal{L})$, and among other things choose $o \in \mathbb{E} - \mathbb{D} \cap \mathbb{E}$. Then the fundamental group $\pi_1(\mathbb{E} - \mathbb{D} \cap \mathbb{E}, o)$ is contained in $\pi_1(H^0 - \mathbb{D}, o)$ and again a finite index subgroup will act on $H^1(X_o, \text{ad}(\rho))$. The subspace \mathbb{E} will be designed so that $\pi_1(\mathbb{E} - \mathbb{D} \cap \mathbb{E}, o)$ preserves the handle decomposition of X_o . Using this we will obtain first a smaller subgroup of G and then apply an argument using loops in the full space $\mathbb{P}H^0(Z, \mathcal{L}) - \mathbb{D}$ (which do not preserve the handle decomposition) to obtain the full group G .

Fix a point $P \in Z$. Choose a collection of sections $a_0, \{f_i\}_{i=1}^m, \{s_j\}_{j=0}^N \in H^0(Z, \mathcal{L})$ so that for a suitable local trivialization of \mathcal{L} and local coordinates (x, y) near P we have

- (a) near P we have $a_0 = x^m - y^2$;
- (b) the section a_0 has no singularities other than P ;
- (c) near P the sections f_i have the form x^i ;
- (d) the sections s_j vanish to order at least $m + 2$ at P and together with a_0 form a basis for a linear system which has no base points outside of P .

It is clear that by taking k big enough we can always find such a_0, f_i and s_j . In addition, we will need to choose the s_j 's so that they satisfy certain connectedness conditions which we shall describe further on (Lemmas 3.4 and 3.8).

With these choices we let \mathbb{E} be the affine space of sections of \mathcal{L} of the form

$$a_0 + \sum_{i=0}^{m-1} t_i f_i + \sum_{j=0}^N z_j s_j,$$

and we take as a base point the point $o \in \mathbb{E}$ corresponding to the values $z_j = 0$ and $t_i = 0$ for $i > 0$, with $t_0 = 1$. Write this as $o = (1, 0, \dots, 0)$. We will work on open polydisks in \mathbb{E} of the form $|t_0 - 1| < \mathbf{B}, |t_i| < \mathbf{B}$ for $i > 0$ and $|z_j| < \mathbf{C}$ for all j (with \mathbf{B} and \mathbf{C} to be determined later), possibly after rescaling the f_i .

To achieve the desired behavior of the monodromy we start by looking at the choice of the s_j and \mathbf{C} . For k big enough we can choose a linearly independent family of sections s_j such that the s_j all vanish to order $> m$ at P , and such that the linear system they generate is without base points away from P . Let $\mathbb{G} \subset \mathbb{P}H^0(X, \mathcal{L})$ denote the subspace generated by the s_j and by a_0 . It is a projective space, with a codimension one projective subspace $\mathbb{G}_\infty \subset \mathbb{G}$ corresponding to the linear system spanned by the s_j without a_0 . The family of sections $a_0 + \sum_j z_j s_j$ provides a system of affine coordinates z_j for the complementary affine space $\mathbb{A}_\mathbb{G} := \mathbb{G} - \mathbb{G}_\infty$. On the other hand, over $\mathbb{A}_\mathbb{G}$ the universal family of curves is a family which is holomorphically locally trivial near the singular point P . Indeed, over any small enough disc in the coordinates z_j , one can choose local coordinates at P , depending on the z_j 's, such that $a_0 + \sum_j z_j s_j$ has the form $x(z_j)^m - y(z_j)^2$. Let $\mathbb{D}_\mathbb{G}$ denote the subset of points in \mathbb{G} parametrizing curves that have singularities outside

of P , union the \mathbb{G}_∞ which corresponds to curves with bigger singularities than usual at P . Over $\mathbb{G} - \mathbb{D}_\mathbb{G}$ we obtain a family of curves which are smooth except for their singularities at P , and the family is holomorphically locally trivial (hence topologically locally trivial) along the section corresponding to the point P .

Let s' be a point in the complement $\mathbb{G} - \mathbb{D}_\mathbb{G}$ and let $X_{s'}$ be the fiber over s' . With m odd, the singularity of each fiber at P is a higher-order cusp, so in fact the fibers such as $X_{s'}$ are singular curves which are topologically (but not differentially) manifolds. The local topological triviality of the family means that it makes sense to speak of the monodromy action of $\pi_1(\mathbb{G} - \mathbb{D}_\mathbb{G}, s')$ on the cohomology of the fiber $X_{s'}$ with coefficients in the trivial local system.

Now we come to the first of the connectedness conditions referred to above. In fact, the subject of the monodromy of pencils having singularities especially at the base locus, has been intensively studied recently notably by Tibar [18,19]. Our situation above is a very special easy case of this phenomenon so we do not need to call upon his general results.

Lemma 3.4. *For k big enough and by choosing a big enough family of sections s_j , we can ensure that the Zariski closure of the monodromy action of $\pi_1(\mathbb{G} - \mathbb{D}_\mathbb{G}, s')$ on $H^1(X_{s'}, \mathbb{C})$ is the full symplectic group.*

Proof. If we choose a general line $\mathbb{A}^1 \subset \mathbb{A}_\mathbb{G}$ then by standard Lefschetz theory (see e.g. [8,9]) the fundamental group of $\mathbb{G} - \mathbb{D}_\mathbb{G}$ is generated by the loops in $\mathbb{A}^1 - \mathbb{A}^1 \cap \mathbb{D}_\mathbb{G}$. Here (and this is the important point of the argument) we can take only the loops which go around the points in $\mathbb{A}^1 \cap \mathbb{D}_\mathbb{G}$; we do not need to look at the loop going around the point at infinity since it is the product of the others. Thus, the fundamental group of $\mathbb{G} - \mathbb{D}_\mathbb{G}$ is generated by loops going around the affine part of the discriminant $\mathbb{D}_\mathbb{G} - \mathbb{G}_\infty = \mathbb{D}_\mathbb{G} \cap \mathbb{A}_\mathbb{G}$.

For k big enough and by choosing a big enough family of sections s_j , the affine part of the discriminant divisor $\mathbb{D}_\mathbb{G} - \mathbb{G}_\infty$ is irreducible, hence connected. Note that we could never assure that $\mathbb{D}_\mathbb{G}$ is connected since it contains \mathbb{G}_∞ as an irreducible component—thus the importance of saying that the monodromy is generated by loops around the affine piece. To get this connectedness we follow the standard argument in the theory of Lefschetz pencils: the discriminant divisor in the affine piece is the image of an affine space bundle over the surface $Z - P$. For big enough values of k this family of affine spaces (which to a point $x \in Z - P$ associates the subspace of sections in $\mathbb{A}_\mathbb{G}$ which are singular at x) is a vector bundle over $Z - P$; thus its image is irreducible. Of course, we also choose k so that the general point in this divisor corresponds to an ordinary double point of the curve.

Now the Kazhdan–Margulis result as reported by Deligne [3, 5.10] works the same way to show that the monodromy of the fundamental group of $\mathbb{G} - \mathbb{D}_\mathbb{G}$ on the cohomology of the family of curves with trivial coefficients, has Zariski-closure equal to the full symplectic group. Indeed, the monodromy around a point of $\mathbb{D}_\mathbb{G} - \mathbb{G}_\infty$ is

a symplectic transvection (because the singular curve has an ordinary node), and the connectedness of the divisor means that all of these elements are conjugate. As we have seen above, they generate the monodromy group, so we have a group generated by a family of conjugate symplectic transvections. Furthermore, the monodromy representation has no fixed vectors (a fixed vector would correspond to a class in $H^1(Z, \mathbb{C})$ which we have assumed is trivial). \square

Choose k and the s_j as per the above lemma. Choose an explicit collection of loops γ_k in $\mathbb{G} - \mathbb{D}_{\mathbb{G}}$ which generate the monodromy, and choose C big enough so that these loops are contained in the region $|z_j| < C$.

Our coordinate patch around P will consist of a nested pair of balls $U \subset U' \subset Z$ together with a pair of coordinate functions $(x, y) : U' \rightarrow \mathbb{C}^2$ sending U (respectively U') to the ball of radius T (respectively T') in \mathbb{C}^2 . We can assume that x and y come from sections of $\mathcal{O}_Z(k_0)$ for some k_0 , via a trivialization of this last line bundle over U' . Furthermore, the trivialization can be assumed to come from a given section u of $\mathcal{O}_Z(1)$ which does not vanish at P . Then provisionally put

$$f_i := x^i u^{k-ik_0} \text{ for } 0 \leq i \leq m-1 \quad \text{and} \quad a_0 := x^m u^{k-mk_0} - y^2 u^{k-2k_0}.$$

These come from global sections of $\mathcal{O}_Z(k)$ (choose $k > mk_0$).

Lemma 3.5. *For any fixed choice of the sections s_j and a_0 , of the constants T and T' , and for any $\delta > 0$, we can make a rescaling of our local coordinates and of the f_i so that the following hold:*

- we can retain properties (a) and (c), while a_0 and the s_j remain fixed in $H^0(Z, \mathcal{L})$;
- the sections f_i become arbitrarily small inside $H^0(Z, \mathcal{L})$;
- the coordinate patches U and U' become arbitrarily small inside Z , and

$$\left| \sum_j z_j s_j \right| < \delta$$

on the coordinate patch U' , for all $|z_j| < C$.

Proof. We can rescale x by a factor of λ^2 and y by a factor of λ^m , and the trivialization by a factor of λ^{-2m} (in other words scale u by a factor of λ^{2m/k_0}). We retain expressions (a) and (c), a_0 remains fixed in $H^0(Z, \mathcal{L})$, and the sections f_i become arbitrarily small inside $H^0(Z, \mathcal{L})$. Also the coordinate patches U and U' become arbitrarily small inside Z . Finally, since the s_j (which are fixed) vanish to order $> m$ at P , for any $\delta > 0$ we can choose the rescaling so that the required estimate holds. \square

The family of curves of the form

$$a_0 + \sum_{i=0}^{m-1} t_i f_i = x^m - y^2 + \sum_{i=0}^{m-1} t_i x^i$$

gives the full family of hyperelliptic curves in our coordinate patch. In particular, the monodromy of this family (i.e. the fundamental group of the complement of the discriminant locus) acts as the braid group of braids on m strands. Choose loops generating this monodromy and choose \mathbf{B} so that the loops are contained in the region $|t_i| < \mathbf{B}$.

Let ∂U be the spherical boundary of the coordinate patch. In terms of the local coordinates the equation for ∂U is $|(x, y)| = \mathbf{T}$. For \mathbf{T} big enough, the intersection of the curve

$$a_0 + \sum_{i=0}^{m-1} t_i f_i$$

with the sphere ∂U will remain approximately equal to a single fixed circle as t_i vary in the region $|t_i| < \mathbf{B}$. As pointed out above, if we made a sufficient rescaling at the start then the ball U will still be small in \mathbf{Z} . Furthermore, by the estimate of Lemma 3.5, addition of the terms $\sum_j z_j s_j$ will not move the intersection circle by very much either.

We have thus found parameters for our family of curves \mathbb{E} (and a rescaling of the f_i) such that for any point (t, z) in the family satisfying the bounds $|t_i| < \mathbf{B}$ and $|z_j| < \mathbf{C}$, the intersection $X_{(t,z)} \cap \partial U$ remains close to a single fixed circle.

Finally, we choose as a base point $o = (1, 0, \dots, 0)$ i.e. $t_0 = 1$. By choosing δ small enough in the above choices, we can insure using the estimate of Lemma 3.5 that when we let the z coordinates go around the loops γ_k , the piece of the curve inside U does not move too far from the base curve and in particular the monodromy action on this piece is trivial. Similarly, note that with a very small scaling factor λ the sections f_i become very small compared to a_0 (or more precisely, they become small compared to the differential of a_0 along the zero-set of a_0). So when t goes around loops generating the braid group action, this does not move very much the curve outside of U' . Furthermore, we choose U so that these loops act trivially in the coordinate patch on $U' - U$. Thus when t moves we obtain a braid group action on the piece of the curve inside U and a trivial action on the piece outside U . We can sum all this up as follows:

Corollary 3.6. *Over \mathbb{E} the family of curves decomposes into a family of hyperelliptic curves of genus g_0 joined onto a family of curves of genus $g - g_0$ along a circle which stays essentially fixed. There is a collection of paths in \mathbb{E} which generate a monodromy action on the genus $g - g_0$ piece whose Zariski-closure is the full symplectic group of the cohomology of the genus $g - g_0$ curves (Lemma 3.4). These*

paths act trivially on the hyperelliptic piece. On the other hand, there are paths in \mathbb{E} generating the braid group action on the hyperelliptic piece, which in turn act trivially on the piece of genus $g - g_0$.

Next, fix our base representation ρ to be trivial on the genus $g - g_0$ piece and equal to the dihedral Heisenberg representation chosen in the hyperelliptic argument of the previous sections for the hyperelliptic genus g_0 piece. Note that since $m = 2g_0 + 1$ rather than $2g_0 + 2$, one of the branch points on the hyperelliptic handle is at infinity; we assume that this branch point is part of a branch cut on which the passing matrix is trivial. Furthermore, there is at least one other branch cut on which the passing matrix is trivial too.

If we consider monodromy elements coming from the family \mathbb{E} , these preserve the cutting-up of our curve into pieces $X_o \cap U$ of genus g_0 and $X_o - X_o \cap U$ of genus $g - g_0$. This monodromy group preserves the decomposition of the cohomology of $\text{ad}(\rho)$ into a sum of two pieces, one of dimension $2n^2(g - g_0)$ and the other of dimension

$$2(n^2 - 1)(g_0 - 1) + 2g_0.$$

The first piece corresponds to the monodromy on the sum of n^2 copies of the trivial representation, since ρ and thus $\text{ad}(\rho)$ are trivial outside $X_o \cap U$. Now the cohomology of $X_o - X_o \cap U$ with coefficients in the trivial local system is isomorphic to the cohomology of the Riemann surface obtained by glueing in a disc along the boundary $X_o \cap \partial U$. In turn, this Riemann surface is homeomorphic to the cusp curve considered in the family parametrized by \mathbb{G} above. Thus the monodromy result of Lemma 3.4 implies that the monodromy action of the loops γ_k on the cohomology of $X_o - X_o \cap U$ with coefficients in the trivial local system, has Zariski-closure equal to the symplectic group $\text{Sp}(2(g - g_0))$. Now taking into account the fact that the restriction of \mathbb{V}_n to $X_o - X_o \cap U$ is the direct sum of n^2 copies of the trivial representation; we get that the monodromy action on this first piece is a diagonal copy of $\text{Sp}(2(g - g_0))$ embedded in the product of n^2 copies of its standard representation.

The second piece corresponds to the monodromy action on the cohomology of our genus g_0 handle with coefficients in $\text{ad}(\rho)$. We have an action of the braid group B_m on this second piece equal to the braid monodromy action for the family of hyperelliptic curves. Of course not all elements of B_m preserve the representation ρ , and as before we look only at elements which preserve ρ . The results of Section 3.1 apply here, giving the Zariski closure of the monodromy of this braid action on the cohomology of the hyperelliptic piece.

We are now in the following situation. Consider the cohomology $H^1(X_o, \text{ad}(\rho))$. Let $H^1(X_o, \text{ad}(\rho))^{\text{hyper}}$ and $H^1(X_o, \text{ad}(\rho))^{g-g_0}$, respectively, denote the cohomologies of $\text{ad}(\rho)$ over the hyperelliptic piece and the complementary piece. Note that in $H^1(X_o, \text{ad}(\rho))^{\text{hyper}}$ we restrict to classes which are zero on the boundary circle joining the two pieces. By Mayer–Vietoris, restriction gives an

isomorphism

$$H^1(X_o, \text{ad}(\rho)) \xrightarrow{\cong} H^1(X_o, \text{ad}(\rho))^{\text{hyper}} \oplus H^1(X_o, \text{ad}(\rho))^{g-g_0}.$$

Furthermore, both pieces have natural symplectic forms and the symplectic form on $H^1(X_o, \text{ad}(\rho))$ corresponds to the direct sum.

Combined with the decomposition

$$H^1(X_o, \text{ad}(\rho)) = H^1(X_o, \mathbb{C}) \oplus \left(\bigoplus_{u \neq (1,0)} H^1(X_o, \mathbb{W}_u) \right),$$

this yields splittings

$$H^1(X_o, \text{ad}(\rho))^{\text{hyper}} = H^1(X_o, \mathbb{C})^{\text{hyper}} \oplus \left(\bigoplus_{u \neq (1,0)} H^1(X_o, \mathbb{W}_u)^{\text{hyper}} \right),$$

and similarly

$$H^1(X_o, \text{ad}(\rho))^{g-g_0} = H^1(X_o, \mathbb{C})^{g-g_0} \oplus \left(\bigoplus_{u \neq (1,0)} H^1(X_o, \mathbb{W}_u)^{g-g_0} \right).$$

Again these decompositions are compatible with the symplectic form. The action of $\pi_1(\mathbb{E} - \mathbb{E} \cap \mathbb{D}, o)$ on $H^1(X_o, \text{ad}(\rho))$ preserves this decomposition.

Let G denote the global monodromy group i.e. the complex Zariski-closure of the monodromy image of $\pi_1(\mathbb{P}H^0(Z, \mathcal{L}) - \mathbb{D}, o)$ acting on $H^1(X_o, \text{ad}(\rho))$. Let G^{hyper} (respectively G^{g-g_0}) denote the Zariski closures of the monodromy of the family \mathbb{E} (i.e. of the image of $\pi_1(\mathbb{E} - \mathbb{E} \cap \mathbb{D}, o)$) acting on each of the pieces in the above decomposition.

Lemma 3.7. *With these notations, the product group is contained in the global monodromy*

$$G^{\text{hyper}} \times G^{g-g_0} \subset G.$$

Furthermore,

$$G^{\text{hyper}} = \text{Sp}(2g_0) \times \prod_{u \neq (1,0)} \text{Sp}(H^1(X_o, \mathbb{W}_u)^{\text{hyper}}),$$

and G^{g-g_0} is the diagonal copy of $\text{Sp}(2(g - g_0))$ acting on $H^1(X_o, \mathbb{C})^{g-g_0} \cong \mathbb{C}^{2n^2(g-g_0)}$.

Proof. The loops discussed in Corollary 3.6 generate a subgroup of G which factors as a product of subgroups of G^{hyper} and G^{g-g_0} , since the loops preserve the decomposition along ∂U and act trivially on one side or the other. The subgroup generated by these loops is also a subgroup of $G^{\text{hyper}} \times G^{g-g_0}$. However, these loops

generate inside G^{g-g_0} the diagonal copy of $\mathrm{Sp}(2(g-g_0))$ acting on $H^1(X_o, \mathbb{C})^{g-g_0} \cong \mathbb{C}^{2n^2(g-g_0)}$ (cf the discussion above), and for the hyperelliptic case by the result of the previous section, the group

$$\mathrm{Sp}(2g_0) \times \prod_{u \neq (1,0)} \mathrm{Sp}(H^1(X_o, \mathbb{W}_u)^{\mathrm{hyper}}) \subset G^{\mathrm{hyper}}.$$

For general reasons, neither G^{g-g_0} nor G^{hyper} can be any bigger than these subgroups generated by our loops. Therefore, our loops generate all of G^{g-g_0} (respectively G^{hyper}), and we obtain that

$$G^{\mathrm{hyper}} \times G^{g-g_0} \subset G.$$

The lemma is proven. \square

To finish the proof of Theorem B we need some elements of G which mix up the factors $H^1(X_o, \mathrm{ad}(\rho))^{\mathrm{hyper}}$ and $H^1(X_o, \mathrm{ad}(\rho))^{g-g_0}$. We get these by the following an “interchange of singularities” argument, which basically comes down to saying that certain singularities in the hyperelliptic family and singularities in the complementary family are conjugate under the global monodromy.

Lemma 3.8. *Let ξ in G^{hyper} denote the monodromy element which acts by the Dehn twist $D_{m-2, m-1}$ on the cohomology of the hyperelliptic piece. Let $\eta \in G^{g-g_0}$ denote a Dehn twist coming from a double point on the complement of U . Then there is an element ψ of the global monodromy group G such that $\psi^{-1} \xi \psi = \eta$.*

Proof. This is because the Dehn twists generating the fundamental group of $\mathbb{P}H^0(Z, \mathcal{L}) - \mathbb{D}$ are all conjugate since the discriminant divisor \mathbb{D} is connected. However, one must be a bit careful since we are looking at cohomology with coefficients in a local system that is not necessarily preserved by the full fundamental group: our representation ρ is preserved by a subgroup of finite index which corresponds to a covering of $\mathbb{P}H^0(Z, \mathcal{L})$ ramified along \mathbb{D} , and in this covering the inverse image of the discriminant locus might no longer be connected. We remedy this by stating somewhat more explicitly how to construct ψ , but without actually writing down the equations for this loop in $\mathbb{P}H^0(Z, \mathcal{L})$ since that would be tedious. Recall that we have assumed that the last two branch points of the hyperelliptic curve were in branch cuts where the passing transformation was the identity. The curve acquires a node when these two branch points come together; this node corresponds to the monodromy element ξ . Then this node can move out of our coordinate patch U and into the complementary region $Z - U$. At this point we are left with a hyperelliptic handle whose equation is a small deformation of $x^{m-2} - y^2$ rather than $x^m - y^2$. The connectedness of the discriminant locus analogous to $\mathbb{D}_{\mathbb{G}} - \mathbb{G}_{\infty}$ but for $m - 2$ rather than m , and also for two nodes at once, allows us to choose a path whereby the node which came out of U gets interchanged with another node

corresponding to η . This connectedness statement, which holds for k large enough, is the second statement referred to in condition (d) at the start of the argument. After interchanging the two singularities, go backwards along the path to send the ξ node back into U . All of this corresponds to a path in the subvariety of \mathbb{D} corresponding to sections with two nodes. (Geometrically, this subvariety is the codimension two nodal locus of the discriminant variety; we choose a path which interchanges the two sheets of the discriminant which come together along the nodal locus.) Choose a path ψ which is near to this path, but in the complement of the discriminant. This has the effect of giving the conjugation above. Finally, notice that all of this took place in a region where the representation ρ is trivial, so we can follow ρ along the path ψ , i.e. the path ψ lifts to a path in the ramified covering on which ρ is defined. \square

Let $\xi \in G^{\text{hyper}}$, $\eta \in G^{g-g_0}$, and $\psi \in G$ be the elements from the above lemma. Note that η is a generating symplectic transvection in $\text{Sp}(2(g - g_0))$. Of course, η no longer acts as a symplectic transvection but as a direct sum of n^2 copies of a transvection on $H^1(X_o, \mathbb{C})^{g-g_0}$.

Now the proof of Theorem B will be a consequence of the following statement.

Lemma 3.9. *The global monodromy group is the full product*

$$G = \text{Sp}(H^1(X_o, \mathbb{C})) \times \prod_{u \neq (1,0)} \text{Sp}(H^1(X_o, \mathbb{W}_u))$$

acting on $H^1(X_o, \mathbb{C}) \oplus (\oplus_{u \neq (1,0)} H^1(X_o, \mathbb{W}_u))$ by the sum of the standard representations.

Proof. Note first that G is contained in the product. Its image in the first factor is the full group $\text{Sp}(2g) = \text{Sp}(H^1(X_o, \mathbb{C}))$, which is just the usual statement (the Deligne–Kazhdan–Margulis theorem again) for cohomology with the trivial coefficient system \mathbb{C} .

Look at one of the pieces $H^1(X_o, \mathbb{W}_u)$, proceeding in the spirit of the hyperelliptic discussion in the previous section (the proof of Lemma 3.2 and Theorem A). First we show that the action of G on $H^1(X_o, \mathbb{W}_u)$ is irreducible. For this, it suffices to consider the action of the group algebra. There is a vector v in $H^1(X_o, \mathbb{W}_u)^{\text{hyper}}$ such that

$$\mathbb{C}[G^{\text{hyper}}] \cdot v = H^1(X_o, \mathbb{W}_u)^{\text{hyper}}.$$

We claim that

$$\mathbb{C}[G] \cdot v = H^1(X_o, \mathbb{W}_u).$$

Let A be the image of the element $\xi - 1$. It is a two-dimensional subspace of $H^1(X_o, \mathbb{W}_u)$, and in particular it is contained in $\mathbb{C}[G] \cdot v$. Thus there are elements v_1

and v_2 of $\mathbb{C}[G] \cdot v$ such that $(\xi - 1)v_1$ and $(\xi - 1)v_2$ span A . On the other hand, the image B of $\eta - 1 = \psi^{-1}(\xi - 1)\psi$ is an isomorphic image of A , a two-dimensional space contained in $H^1(X_o, \mathbb{W}_u)^{g-g_0}$. The isomorphism is given by

$$\psi^{-1} : A \xrightarrow{\cong} B.$$

The vectors

$$\psi^{-1}(\xi - 1)\psi(\psi^{-1}v_1) \quad \text{and} \quad \psi^{-1}(\xi - 1)\psi(\psi^{-1}v_2)$$

span B . In particular B is contained in $\mathbb{C}[G] \cdot v$.

Now we can write $H^1(X_o, \mathbb{W}_u)^{g-g_0} = \mathbb{C}^{2(g-g_0)} \oplus \mathbb{C}^{2(g-g_0)}$ and the image of the action of G contains the diagonal copy of $\text{Sp}(2(g - g_0))$ (this is the image of G^{g-g_0}). The subspace B is transverse to the above decomposition, in other words it contains one basis element in each piece. It follows that the translates of B by elements of G^{g-g_0} span $H^1(X_o, \mathbb{W}_u)^{g-g_0}$. Therefore

$$H^1(X_o, \mathbb{W}_u)^{g-g_0} \subset \mathbb{C}[G] \cdot v,$$

so putting this together with the above we get that $\mathbb{C}[G] \cdot v = H^1(X_o, \mathbb{W}_u)$, so the action of G on $H^1(X_o, \mathbb{W}_u)$ is irreducible as claimed (to get this last deduction we use the standard fact that G is semisimple so its action decomposes as a direct sum of irreducible pieces).

Next, note that the image of G acting on $H^1(X_o, \mathbb{W}_u)$ is a simple group. This is because the group G^{hyper} acting on $H^1(X_o, \mathbb{W}_u)$ contains an element whose Jordan normal form has a single Jordan block of length one (see the argument of the previous section for the hyperelliptic case). As before this implies that the image is simple.

Next, we show that the image of G acting on $H^1(X_o, \mathbb{W}_u)$ is the full symplectic group $\text{Sp}(H^1(X_o, \mathbb{W}_u))$. This again is by the same argument as in the previous section, using the fact that the image of G acting on W_u contains a copy of $\text{Sp}(2(g - g_0))$, and noting that we can insure that $g - g_0$ is large enough (by choosing k big).

Finally, complete the proof of Lemma 3.9 by noting that inside G^{hyper} we can find an element which acts with different eigenvalues on each of the different pieces $H^1(X_o, \mathbb{W}_u)$, the same element as exhibited in the previous section. As then, this implies that

$$G = \text{Sp}(2g) \times \prod_{u \neq (1,0)} \text{Sp}(H^1(X_o, \mathbb{W}_u)).$$

This completes the proof of the lemma. \square

Proof of Theorem B. The dihedral Heisenberg representation ρ which we chose here corresponds to a smooth point in $M_B(X_o, n)$, fixed under a finite index subgroup of the monodromy group. The action of the monodromy group which fixes ρ on the

tangent space $T_{[\rho]}M_{\mathbb{B}}(X_o, n)$ at ρ is exactly the action on $H^1(X_o, \text{ad}(\rho))$ we considered above. The Zariski-closure G as described in Lemma 3.9, acts with an open orbit. Either from the general consideration in Lemma 2.8 or else just by inspection, this monodromy group has no characters. Therefore by Corollary 2.4 we get that the monodromy action on $M_{\mathbb{B}}(X_o, n)$ is Zariski-dense. \square

4. Further remarks

4.1. Topologically irreducible families

The requirement that our families have relatively large monodromy was used in an essential way in the proofs of Theorems A and B. However this requirement seems to be more an artifact of the method of proof rather than a real condition on the family $f : X \rightarrow B$ which is necessary for the density of the monodromy action. In this section we briefly examine some consequences of the density, which will allow us to probe the necessity of the ‘large monodromy’ condition.

Recall that a smooth family of curves $f : X \rightarrow B$ is called *topologically irreducible* if and only if there is no finite collection of disjoint embedded circles in X_o which is preserved by the geometric monodromy. We have the following simple.

Lemma 4.1. *Let $f : X \rightarrow B$ be a family of smooth curves, such that the monodromy action of $\pi_1(B, o)$ has a Zariski-dense orbit on the Betti moduli space $M_{\mathbb{B}}(X_o, n)$ for some $n \geq 1$. Then $f : X \rightarrow B$ is topologically irreducible.*

Proof. Assume that one can find simple disjoint loops $a_1, \dots, a_k \subset X_o$ such that the collection $\{a_1, \dots, a_k\}$ of free homotopy classes on X_o is preserved by $\text{mon}(\pi_1(B, o)) \subset \text{Map}(X_o)$. Then for every $N \in \mathbb{Z}$ we have a well-defined $\text{mon}(\pi_1(B, o))$ -invariant regular function $\psi_N : M_{\mathbb{B}}(X_o, n) \rightarrow \mathbb{C}$ on $M_{\mathbb{B}}(X_o, n)$, given by $\psi_N([\rho]) := \text{Tr}(\prod_{i=1}^k (\rho(a_i))^N)$. But clearly for some N the function ψ_N will be non-constant and so $\text{mon}(\pi_1(B, o))$ cannot have a Zariski-dense orbit on $M_{\mathbb{B}}(X_o, n)$. The lemma is proven. \square

In particular, all families of curves satisfying the hypothesis of Theorems A and B will be topologically irreducible. Recently McMullen has shown that topological irreducibility holds very generally: every non-isotrivial holomorphic family of curves is topologically irreducible [12, Proof of Theorem 3.1]. In particular, this corollary of our main results was already known.

When we started the current project we were hoping that topological irreducibility will allow one to distinguish symplectic Lefschetz pencils (whose topology tends to be much softer) from projective Lefschetz pencils. In the meantime however, Ivan Smith succeeded in showing [16] that all symplectic Lefschetz fibrations over \mathbb{P}^1 are topologically irreducible. We still expect that the stronger property GZD (or the open orbit property from Theorem B) will allow one to distinguish projective from

symplectic Lefschetz pencils. We hope to return to examples of this type in a future paper.

McMullen’s result (with the alternative symplectic proof by Smith when the base is \mathbb{P}^1) is the only evidence we have for the following conjectural generalization of Theorems A and B:

Conjecture 4.2. *For a family $f : X \rightarrow B$ assume that $\text{mon}(\pi_1(B, o))$ is not a finite group. Then*

- (i) *there is no meromorphic function on $M_B(X_o, n)^{\text{an}}$ which is invariant under the action of $\text{mon}_B^n(\pi_1(B, o))$ (equivalently there is no meromorphic function on $M_{\text{DR}}(X_o, n)^{\text{an}}$ which is $\text{mon}_{\text{DR}}^n(\pi_1(B, o))$ -invariant);*
- (ii) *in the case of $M_B(X_o, n)$, considered with its natural structure of an affine algebraic variety, there exist a point $x_B \in M_B(X_o, n)$ so that the orbit*

$$\text{mon}_B^n(\pi_1(B, o)) \cdot x_B \subset M_B(X_o, n)$$

is Zariski-dense in $M_B(X_o, n)$.

Procesi’s theorem [15] implies that the field of rational functions on $M_B(X_o, n)$ is generated by traces of evaluation maps for conjugacy classes of simple loops on X_o . One might hope (although we did not find an argument) that the field of $\pi_1(B, o)$ -invariant rational functions on $M_B(X_o, n)$ is similarly generated by the traces of evaluation maps at finite invariant collections of simple loops. If this were the case then McMullen’s theorem would imply the validity of the variant of Conjecture 4.2 concerning algebraic meromorphic functions. The property AGZD1 (i.e. the conjecture as it is stated using analytic meromorphic functions) would seem to remain more elusive.

4.2. Points Γ -near to a finite representation

We briefly describe here another variation on the basic result. Essentially, we have constructed a finite-image representation, the dihedral Schrödinger representation ρ , which corresponds to a smooth point in $M_B(X_o, n)$ and which turns out to be sufficient in order to get the Zariski-denseness property. In an attempt to better understand what is going on, we can explore a bit further the sense in which ρ is near the rest of $M_B(X_o, n)$.

Suppose a finitely presented group Γ acts on an affine variety M , and suppose $p \in M$ is a closed point in the smooth set of M , fixed by the action. We say that another point $q \in M$ is Γ -near to p if p lies in the closure of the orbit $\overline{\Gamma \cdot q}$. Let $N = \text{Near}(M, p, \Gamma) \subset M$ denote the subset of points q which are Γ -near to p . It is Γ -invariant. Let $T_p N$ denote its tangent cone at p , defined to be the set of limits of secants to M going from p to points $q \in N$ which approach p (the limits of secants may be taken in any real embedding of M). Note that $T_p N \subset T_p M$ is an invariant

subset of the tangent space to M at p . An easy argument similar to that of Corollary 2.4 shows that if $T_p N$ is Zariski-dense in $T_p M$ then N is Zariski-dense in M .

Suppose $\gamma \in \Gamma$. Let $R \subset T_p M$ denote the span of the eigenvectors of γ whose eigenvalues have norm < 1 . We claim that $R \subset T_p N$. Choose a smooth submanifold $V \subset M$ tangent to R . Let D be a ball neighborhood of p in M . Then the collection

$$\{V^k := \gamma^{-k}(W \cap \gamma^k D)\}$$

is a collection of manifolds with boundaries lying in the boundary of D , which are tangent to R at the origin, and whose curvature is bounded (the $V \cap \gamma^k D$ all lie in a sector preserved by γ^{-1} and in which γ^{-1} smooths things out). Thus, these converge to a manifold V^∞ which is preserved by the action of γ and on which γ acts with all eigenvalues < 1 . In particular, $V^\infty \subset N$ which shows that $R \subset T_p N$.

Lemma 4.3. *Suppose $\rho \in M_B(X_o, n)$ is the dihedral Schrödinger representation we have considered above. Suppose that a group $\Gamma = \pi_1(B, o)$ acts, satisfying one of the hypotheses of Theorem A or B. Let $N = \text{Near}(M_B(X_o, n), \rho, \Gamma)$. Then $T_p N$ is Zariski-dense in $T_p M_B(X_o, n)$.*

Proof. Recall that

$$T_{[\rho]} M_B(X_o, n) = H^1(X_o, \text{ad}(\rho)),$$

and that we have a decomposition

$$H^1(X_o, \text{ad}(\rho)) = \left(\bigoplus_u H^1(X_o, \mathbb{W}_u) \right)$$

such that the monodromy group Γ is Zariski-dense in the product $G = \prod_u G_u$ with $G_u = \text{Sp}(H^1(X_o, \mathbb{W}_u))$. Each component of the decomposition corresponds to a weight one variation of Hodge structure over B which is irreducible and not unitary (since both Hodge subspaces are nontrivial). Therefore Γ actually lies in a real form which decomposes

$$\Gamma \subset G_{\mathbb{R}} = \prod_u G_{u, \mathbb{R}}$$

and the $G_{u, \mathbb{R}}$ are noncompact real forms of the symplectic groups. (One cannot have a real component whose complexification splits into two components, because that would be a complex group considered as a real group, which is never of Hodge type.) The real Zariski closure of Γ , i.e. the intersection of all real algebraic subsets of G containing Γ , is $G_{\mathbb{R}}$, since anything smaller would lead to a smaller complex Zariski closure.

Let $pr_u : G_{\mathbb{R}} \rightarrow G_{u, \mathbb{R}}$ denote the projection. Let $E_u \subset G_{\mathbb{R}}$ be the real algebraic subset of elements g such that all of the eigenvalues of $pr_u(g) \in G_{u, \mathbb{R}}$ have norm 1. This is a

proper subset since $G_{u, \mathbb{R}}$ is noncompact. The union $\bigcup_u E_u$ is again a proper real algebraic subset of $G_{\mathbb{R}}$, so it cannot contain Γ .

Thus there is an element $\gamma \in \Gamma$ such that every projection $pr_u(\gamma)$ has at least one eigenvalue of norm different from 1. On the other hand these projections have determinant one, so each $pr_u(\gamma)$ has at least one eigenvalue of norm < 1 .

In particular, there is a vector $v \in H^1(X_o, \text{ad}(\rho))$ such that v is in the span of the eigenvectors of γ corresponding to eigenvalues of absolute value < 1 , and such that v has a nonzero component in all of the irreducible factors of $H^1(X_o, \text{ad}(\rho))$. Using the fact that the complex Zariski closure of Γ contains a product of symplectic groups (corresponding to the decomposition into irreducible pieces of the representation $H^1(X_o, \text{ad}(\rho))$), we find that the orbit of the vector v under the action of Γ is Zariski-dense in $T_{[\rho]}M_B(X_o, n)$. On the other hand, from our discussion prior to the present lemma, v is in the Γ -invariant subset $T_\rho N$. Thus $T_\rho N$ is Zariski-dense. \square

Corollary 4.4. *The moduli space $M_B(X_o, n)$ contains a smooth point, the dihedral Schrödinger representation ρ , such that the set of points $\text{Near}(M_B(X_o, n), \rho, \Gamma) \subset M_B(X_o, n)$ which are Γ -near to ρ , is Zariski-dense in $M_B(X_o, n)$.*

It is clear that any Γ -invariant regular (i.e. holomorphic algebraic) function takes the same value at ρ as at every point of $\text{Near}(M_B(X_o, n), \rho, \Gamma)$. In particular, this corollary implies the result that there are no Γ -invariant regular functions. This result is weaker than our main results about non-existence of invariant meromorphic functions, but does provide a slightly different conceptual route to the topological irreducibility result referred to above.

Our motivation for introducing the notion of Γ -nearness is that the first and second authors asked some time ago whether there was any sense in which the finite-image representations could take up a big place in $M_B(X_o, n)$. The short answer to that question is that, by Jordan's theorem, the finite image representations occupy a rather small place in that there are only finitely many outside of representations which factor through a normalizer of a torus (and those which factor in this way lie in a closed subset of relatively high codimension). However, Corollary 4.4 provides the slightly more subtle answer that, in the presence of a large monodromy action, if you start out very near to a certain finite-image representation (such as one of our dihedral Schrödinger representations) and then let the monodromy act, then you can get out to a significant part of $M_B(X_o, n)$.

4.3. Other groups

Finally, let us explicitly state that we expect that all of the results and conjectures of this paper to hold for Betti and deRham cohomology with coefficients in an arbitrary complex reductive group G . Specifically we make the following

Conjecture 4.5. *Let $f : X \rightarrow B$ be a smooth algebraic family of curves and G a complex reductive group. Then:*

(i) *Assume that f is not isotrivial. Then the families*

$$M_{\mathbb{B}}(X/B, G) \rightarrow B \quad \text{and} \quad M_{\text{DR}}(X/B, G) \rightarrow B$$

of relative Betti and de Rham cohomology with coefficients in G are GZD when equipped with the non-Abelian Gauss–Manin connection.

(ii) *Assume that f comes from a projective Lefschetz pencil of sufficiently high degree. Then there exists a smooth point $\rho \in M_{\mathbb{B}}(X_o, G)$ which is fixed by a finite index subgroup $\Gamma \subset \pi_1(B, o)$ and for which the Zariski-closure of Γ in $\text{GL}(T_{\rho}M_{\mathbb{B}}(X_o, G))$ acts with an open orbit on $T_{\rho}M_{\mathbb{B}}(X_o, G)$.*

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