Note

The transformation graph $G^{xyz}$ when $xyz = --+$

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Abstract

The transformation graph $G^{xyz}$ of $G$ is the graph with vertex set $V(G) \cup E(G)$ in which the vertex $x$ and $y$ are joined by an edge if one of the following conditions holds: (i) $x$, $y \in V(G)$, and $x$ and $y$ are not adjacent in $G$, (ii) $x$, $y \in E(G)$, and $x$ and $y$ are adjacent in $G$, (iii) one of $x$ and $y$ is in $V(G)$ and the other is in $E(G)$, and they are incident in $G$. In this paper, it is shown that for two graphs $G$ and $G'$, $G^{xyz} \cong G'^{xyz}$ if and only if $G \cong G'$. Simple necessary and sufficient conditions are given for $G^{xyz}$ to be planar and hamiltonian, respectively. It is also shown that for a graph $G$, the edge-connectivity of $G^{xyz}$ is equal to its minimum degree. Two related conjectures and some research problems are presented.

Keywords: Transformation; Total graph; Isomorphism

1. Introduction

All graphs considered here are finite and simple. Undefined terminology and notations can be found in [3]. Let $G = (V(G), E(G))$ be a graph. The connectivity (edge-connectivity)
of $G$, denoted by $\kappa(G)(\lambda(G))$, is defined to be the largest integer $k$ for which $G$ is $k$-connected ($k$-edge connected). We use $\omega(G)$ to denote the number of components of $G$. For a vertex $v$ of $G$, the eccentricity $\text{ecc}_G(v)$ of $v$ is the largest distance between $v$ and all the other vertices of $G$, i.e., $\text{ecc}_G(v) = \max\{d_G(u, v) | u \in V(G)\}$. The diameter $\text{diam}(G)$ of $G$ is $\max\{\text{ecc}_G(v) | v \in V(G)\}$, equivalently, the maximum distance between two vertices of $G$. $I_G(v)$ denotes the set of edges incident with $v$ in $G$, and $|I_G(v)|$ is called the degree $d_G(v)$ of $v$ in $G$. The neighborhood $N_G(v)$ of $v$ is the set of all vertices of $G$ adjacent to $v$. Since $G$ is simple, $|N_G(v)| = d_G(v)$.

Suppose that $V'$ is a nonempty subset of $V(G)$. We call $V'$ an independent set if no two vertices of $V'$ are adjacent in $G$ whereas a clique if every pair of vertices of $V'$ are adjacent in $G$. The subgraph $G[V']$ of $G$ induced by $V'$ is a graph with $V(G[V']) = V'$ and $uv \in E(G[V'])$ if and only if $uv \in E(G)$. For two disjoint nonempty subsets $S$ and $S'$ of $V$, we denote by $[S, S']$ the set of edges with one end in $S$ and the other in $S'$. Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. The union $G \cup H$ of $G$ and $H$ is the graph whose vertex set is $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H)$. Particularly, we denote their union by $G + H$ if they are disjoint, i.e., $V(G) \cap V(H) = \emptyset$. The join $G \vee H$ of $G$ and $H$ is the graph obtained from $G + H$ by joining each vertex of $G$ to each vertex of $H$. We call $G$ and $H$ isomorphic, and write $G \cong H$, if there exists a bijection $\theta : V(G) \leftrightarrow V(H)$ with $xy \in E(G)$ if and only if $\theta(x)\theta(y) \in E(H)$ for all $x, y \in V(G)$.

The line graph $L(G)$ of $G$ is the graph whose vertex set is $E(G)$, and in which two vertices are adjacent if and only if they are adjacent in $G$. The total graph $T(G)$ of $G$ is the graph whose vertex set is $V(G) \cup E(G)$, and in which two vertices are adjacent if and only if they are adjacent or incident in $G$. Wu and Meng [7] generalized the concept of total graph, and introduced some new graphical transformations.

Let $G = (V(G), E(G))$ be a graph, and $\alpha, \beta$ be two elements of $V(G) \cup E(G)$. We say that the associativity of $\alpha$ and $\beta$ is + if they are adjacent or incident in $G$, otherwise −. Let $xyz$ be a 3-permutation of the set $\{+, −\}$. We say that $\alpha$ and $\beta$ correspond to the first term $x$ (resp. the second term $y$ or the third term $z$) of $xyz$ if both $\alpha$ and $\beta$ are in $V(G)$ (resp. both $\alpha$ and $\beta$ are in $E(G)$, or one of $\alpha$ and $\beta$ is in $V(G)$ and the other is in $E(G)$). The transformation graph $G^{xyz}$ of $G$ is defined on the vertex set $V(G) \cup E(G)$. Two vertices $\alpha$ and $\beta$ of $G^{xyz}$ are joined by an edge if and only if their associativity in $G$ is consistent with the corresponding term of $xyz$.

Since there are eight distinct 3-permutations of $\{+, −\}$, we obtain eight graphical transformations of $G$. It is interesting to see that $G^{+++}$ is exactly the total graph $T(G)$ of $G$, and $G^{−−−}$ is the complement of $T(G)$. Also, for a given graph $G$, $G^{+++}$ and $G^{−−−}$, $G^{++−}$ and $G^{−−+}$, $G^{−++}$ and $G^{+++}$ are other three pairs of complementary graphs.

One of the classical theorems on line graphs is due to Whitney [6]. That is, for any two connected graphs $G$ and $G'$, $L(G) \cong L(G')$ if and only if either $G \cong G'$, or $\{G, G'\} = \{K_3, K_{1,3}\}$. Behzad and Radjavi [2] also showed that for any two graphs $G$ and $G'$, $G^{+++} \cong G'^{+++}$ if and only if $G \cong G'$. Motivated from the above, we prove that for two graphs $G$ and $G'$, $G^{+++} \cong G'^{+++}$ if and only if $G \cong G'$. In Section 2, it is shown that $G^{+++}$ is planar if and only if $\nu(G) \leq 4$, and is hamiltonian if and only if $|\nu(G)| \geq 3$. Also, we prove that for any graph $G$, the edge-connectivity of $G^{+++}$ is equal to its minimum degree.
2. Eccentricity, connectivity, planarity, and hamiltonity

Let $G$ be a graph with $n$ vertices, and $H = G^{+++}$. For every vertex $v \in V(G)$, $d_H(v) = n - 1$, and for an edge $e = uv \in E(G)$, $d_H(e) = d_G(u) + d_G(w)$.

**Theorem 2.1.** Let $G$ be a graph, and $H = G^{+++}$. Then $ecc_H(v) \leq 2$ if $v \in V(G)$, and $ecc_H(e) \leq 3$ if $e \in E(G)$, and $ecc_H(e) = 3$ if and only if $\text{diam}(L(G)) \geq 3$.

**Proof.** Let $u, w \in V(G)$. If $u$ and $w$ are not adjacent in $H$, then they are adjacent in $G$. Let $e = uw$. Since $uw$ is a path joining $u$ and $w$ in $H$, we have $d_H(u, w) = 2$. Assume $u \in V(G)$ and $e \in E(G)$. If $u$ and $e$ are not adjacent in $H$, then $u$ is not incident with $e$ in $G$. Let $w$ be an end vertex of $e$ in $G$. If $u$ and $w$ are not adjacent in $G$, then $uw$ is a path joining $u$ and $w$ in $H$, so $d_H(u, w) \leq 2$; if $u$ and $w$ are adjacent in $G$, and let $e' = uw$, then $ue'w$ is a path joining $u$ and $e$ in $H$, so $d_H(u, w) \leq 2$. Now assume $e, e' \in E(G)$. If $e$ and $e'$ are not adjacent in $H$, then they have no common end vertex. Let $u$ be an end vertex of $e$ in $G$. Then $d_H(e, e') = \min(d_H(u, e), d_H(u, e')) = 1 + 2 = 3$. Note that if $\text{dima}(L(G)) \geq 3$, and we take $e_1, e_2 \in E(G)$ with $d_L(G)(e_1, e_2) \geq 3$, then $d_H(e_1, e_2) = 3$. Summing up the argument above, the result follows. □

So, $G^{+++}$ is connected and $\text{diam}(G^{+++}) \leq 3$ for any graph $G$ (see [7]). It is well-known that $\kappa(G) \leq \lambda(G) \leq \delta(G)$ for any graph $G$. In [1], Bauer and Tindell showed that $\lambda(G^{+++}) = \delta(G^{+++})$ for any connected graph $G$. We shall obtain the similar result for $G^{+++}$. Before proving it, recall that the subdivision graph $S_1(G)$ of $G$ is the graph with the vertex set $V(G) \cup E(G)$. Two element $x$ and $y$ are joined by an edge if and only if one of $x, y$ is in $V(G)$ and the other is in $E(G)$, and they are incident in $G$. So, $G^{+++} = (\overline{G} + L(G)) \cup S_1(G)$.

**Theorem 2.2.** Let $G$ be a graph and $H = G^{+++}$. Then $\lambda(H) = \delta(H)$.

**Proof.** It suffices to show that $\lambda(H) \geq \delta(H)$. Let $S$ be a minimum edge-cut of $H$. Then $H - S$ has exactly two components, which are denoted by $H_1$ and $H_2$. First assume that $V(H_i) \cap V(G) \neq \emptyset$, and let $u_i \in V(G) \cap V(H_i)$ for $i = 1, 2$. Since $\overline{G} \cup S_1(G)$ is a subdivision of the complete graph on $V(G)$, $u_1$ and $u_2$ are connected by at least $|V(G)| - 1$ vertex-disjoint paths in $\overline{G} \cup S_1(G)$. Hence $|S| \geq |V(G)| - 1 \geq \delta(H)$.

Now we assume that all the vertices of $G$ are contained in the same component of $H - S$. Without loss of generality, suppose $V(G) \subseteq V(H_2)$. Take a vertex $e$ from $V(H_1)$, then $e \in E(G)$. Let $x, y$ be the end vertices of $e$ in $G$. Then $x, y \in V(H_2)$. As we know, $N_H(e)$ can be partitioned into two vertex-disjoint cliques, which contain $x$ and $y$, respectively. We denote the two cliques by $C_x$ and $C_y$. Then $|C_x| = d_G(x)$ and $|C_y| = d_G(y)$. It follows that $e$ and $x$ are connected by at least $d_G(x)$ vertex-disjoint paths in $H[C_x \cup \{e\}]$, and $e$ and $y$ are connected by at least $d_G(y)$ vertex-disjoint paths in $H[C_y \cup \{e\}]$. Thus, we have $|S| \geq d_G(x) + d_G(y) \geq \delta(H)$. This completes the proof. □

In general does $\kappa(G^{+++})$ equal to $\delta(G^{+++})$? The answer is negative. To see this, let $G = K_n + K_m$, where $n \geq 3$, $m \geq 2$, and $H = G^{+++}$. Suppose $V(G) = \{u_1, \ldots, u_n, v_1, \ldots, v_m\}$ such that the subgraph $K_n$ of $G$ is induced by $V' = \{u_1, \ldots, u_n\}$. Then $H - V'$ is not
connected since as vertices of $H$, the edges of $K_n$ are not adjacent to any $v_i$ ($1 \leq i \leq m$) in $H$. This implies that $\kappa(H) \leq |V'| = n$. Note that $d_H(v) = n + m - 1$ for any vertex $v \in V(G)$ and $d_H(u_i u_j) = d_G(u_i) + d_G(u_j) = 2n - 2$ where $i, j = 1, 2, \ldots, n$. Thus $\kappa(H) \leq n - \delta(H) = \min\{n + m - 1, 2n - 2\}$ since $n \geq 3$ and $m \geq 2$.

**Theorem 2.3.** For a graph $G$, $G^{++}$ is planar if and only if $|V(G)| \leq 4$.

**Proof.** It is easy to check that $G^{++}$ is planar for every graph $G$ with $|V(G)| \leq 4$.

Note that $\overline{G} \cup S_1(G)$ is a subgraph of $G^{++}$ and is a subvision of the complete graph on $V(G)$. If $G^{++}$ is planar, then it does not contain the subdivision of a complete subgraph with 5 vertices by the well-known Kuratowski’s theorem. So $|V(G)| \leq 4$. \qed

**Theorem 2.4.** For a graph $G$, $G^{++}$ is hamiltonian if and only if $|V(G)| \geq 3$.

**Proof.** It is obvious that if $|V(G)| < 3$, then $G^{++}$ is not hamiltonian. For the sufficiency, let $G$ be a graph with $|V(G)| \geq 3$. If $G$ is empty, then $G^{++}$ is a complete graph, and is hamiltonian. Now suppose $G$ is not empty, and let $M$ be a maximum matching of $G$. Let $G'$ be the complete graph on $V(G)$. Then $G$ is a spanning subgraph of $G'$, and there exists a Hamilton cycle $C'$ of $G'$ containing all the edges of $M$. Suppose that $e_1, e_2, \ldots, e_m$ are all the edges of $G$ on $C'$, then $M \subseteq \{e_1, e_2, \ldots, e_m\}$. Let $e_i = u_i v_i$ for $i = 1, 2, \ldots, m$, and $u_1, v_1, u_2, v_2, \ldots, u_m, v_m, \ldots, u_1$ are the vertices of $C'$ assigned clockwise. Note that if $\{e_1, \ldots, e_m\} = E(G)$, then we may obtain a Hamilton cycle of $G^{++}$ by replacing each edge $u_i v_i$ of $C'$ by the path $u_i e_i v_i$ of length 2 for $i = 1, \ldots, m$.

Otherwise, $E(G) \setminus \{e_1, \ldots, e_m\} \neq \emptyset$. Since $M$ is a maximum matching of $G$, each edge of $E(G) \setminus \{e_1, \ldots, e_m\}$ (denoted by $F$) is incident with some $u_i$ or $v_j$ for $i, j = 1, 2, \ldots, m$ in $G$. Next we should insert all the edges of $G$ (as the vertices of $G^{++}$) into $C'$ to obtain a Hamilton cycle of $G^{++}$. We do this by replacing each edge $u_i v_i$ of $C'$ by a path $P_i$, $i = 1, 2, \ldots, m$, where $P_i$ is walked along the following vertices: $u_i$, the edges of $F$ incident with $u_i$ but not incident with $u_i$ (any order), $e_i$, the edges of $F$ incident with $v_i$ but not incident with $u_i$ (any order), $v_i$. Note that if there is some edge $e$ of $F$ whose two end-vertices both belonging to $\{u_1, \ldots, u_m, v_1, \ldots, v_m\}$, we just insert $e$ at the end-vertex first appeared on $C'$ and do not insert $e$ at the second one again. Thus we get a Hamilton cycle of $G^{++}$ from $C'$. \qed

3. **Isomorphism**

For a graph $G$ and $v \in V(G)$, we denote the subgraph $G[N_G(v)]$ of $G$ by $G_v$ for short. Recall that $I_G(v)$ is the set of edges incident with $v$ in $G$. Next, we start with a few useful remarks. Let $G$ be a graph, $H = G^{++}$.

**Remark 1.** For a vertex $v \in V(G)$, we have

1. $N_H(v) \cap V(G) \neq \emptyset$ if and only if $d_G(v) \neq |V(G)| - 1$,
2. for a component $F$ of $H_v$, $V(F) \subseteq V(G)$ if and only if $V(F) \cap V(G) \neq \emptyset$, and
3. if $d_G(v) > 0$, then $H[I_G(v)] (\cong K_d)$ is a component of $H_v$, where $d = d_G(v)$. 

It follows from the fact that \( N_H(v) = (V(G) \setminus N_G(v)) \cup I_G(v) \). In particular, if \( d_G(v) = |V(G)| - 1 \), then \( N_H(v) \) is a clique, and if \( d_G(v) = 0 \), \( N_H(v) = V(G) \setminus \{v\} \).

**Remark 2.** If \( e = uw \in E(G) \), then \( N_H(e) \) is not a clique of \( H \), but can be partitioned into two cliques of \( H \) each containing a vertex of \( G \). So, \( \omega(H_e) \leq 2 \), and \( H_e \) is connected if and only if \( u \) and \( w \) have a common neighbor.

**Remark 3.** Let \( G \) and \( G' \) be two graphs with \( G^{++} \equiv G'^{++} \), and \( \theta \) be an isomorphism from \( G^{++} \) to \( G'^{++} \). For \( v \in V(G) \), if \( d_G(v) > 0 \) and \( G - v - N_G(v) \) has an edge, then \( \theta(v) \in V(G') \).

Since \( N_{G^{++}}(v) \) cannot be partitioned into two cliques, and neither is \( N_{G'^{++}}(\theta(v)) \). By Remark 2, \( \theta(v) \in V(G') \).

**Lemma 3.1.** Let \( G \) be a graph and \( H = G^{++} \). For a pair of adjacent vertices \( u \) and \( v \) of \( H \), \( N_H(u) \setminus \{v\} = N_H(u) \setminus \{u\} \) if and only if they are isolated vertices of \( G \).

**Proof.** If \( u \) and \( v \) are two isolated vertices of \( G \), then \( N_H(u) \setminus \{v\} = V(G) \setminus \{u, v\} = N_H(v) \setminus \{u\} \). Next we prove the necessity. Since \( u \) and \( v \) are adjacent in \( H \), if \( \{u, v\} \subseteq E(G) \), then \( u = xy \) and \( v = xz \) in \( G \) for some vertices \( x, y, z \in V(G) \). However, \( N_H(u) \setminus \{v\} \neq N_H(v) \setminus \{u\} \) since \( y \in N_H(u) \setminus N_H(v) \). So at least one of \( u \) and \( v \) is in \( V(G) \). Assume that \( u \in V(G) \), \( v \in E(G) \), and let \( v = uw \in E(G) \) where \( w \in V(G) \). Obviously, \( w \in N_H(u) \setminus N_H(u) \). So both \( u \) and \( v \) are vertices of \( V(G) \), and they are not adjacent in \( G \). If \( d_G(u) \neq 0 \), let \( e \) be an edge incident with \( u \) in \( G \), then \( e \in N_H(u) \setminus N_H(v) \) since \( v \) is not incident with \( e \) in \( G \). So \( d_G(u) = 0 \). Similarly, we have \( d_G(v) = 0 \). This completes the proof. \( \square \)

**Lemma 3.2.** For two graphs \( G \) and \( G' \), if \( G^{++} \equiv G'^{++} \), then \( |V(G)| = |V(G')| \) and \( |E(G)| = |E(G')| \).

**Proof.** Let \( \theta \) be an isomorphism from \( G^{++} \) to \( G'^{++} \). Since \( |V(G)| + |E(G)| = |V(G^{++})| = |V(G')| + |E(G')| \), it suffices to show that \( |V(G)| = |V(G')| \). Let \( W = \theta(V(G)) \). First assume that \( W \cap V(G') \neq \emptyset \). Take a vertex \( v' \) from \( W \cap V(G') \), then \( v = \theta^{-1}(v') \in V(G) \) by the definition of \( W \). Therefore, \( |V(G)| - 1 = d_G(v') = d_{G^{++}}(v') = |V(G')| - 1 \), and \( |V(G)| = |V(G')| \). Now let \( W \cap V(G') = \emptyset \). By the definition of \( G'^{++} \), each element of \( W \) is an edge of \( G' \), which is adjacent to exactly two elements of \( V(G') \) in \( G'^{++} \). Hence, we have \( |W| = |V(G')| = 2|W| = 2|V(G)| \). On the other hand, \( |W| = |V(G')| = |V(G), \theta^{-1}(V(G'))| = 2|\theta^{-1}(V(G'))| = 2|V(G')| \). Thus, \( |V(G)| = |V(G')| \). \( \square \)

**Theorem 3.3.** For two graphs \( G \) and \( G' \), \( G^{++} \equiv G'^{++} \) if and only if \( G \cong G' \).

**Proof.** The sufficiency is obvious.

For the necessity, let \( \theta \) be an isomorphism from \( G^{++} \) to \( G'^{++} \), and \( W = \theta(V(G)) \). Then \( |W| = |V(G)| = |V(G')| \) by Lemma 3.2. Since \( G'^{++} = V(G') \setminus W \), \( G'^{++} \) is \( G' \) and \( G^{++} \) is \( G' \). If \( W = V(G') \), we have \( G \cong G' \). So we assume that \( W \setminus V(G') \neq \emptyset \).
Next we see that $G$ has at most one isolated vertex. Suppose it is not, and let $u$ and $v$ be two isolated vertices. Let $u' = \theta(u)$ and $v' = \theta(v)$. Then $u'$ and $v'$ are adjacent in $G'_{r,s}$ and $N_{G'_{r,s}}(u') \cup v' = N_{G'_{r,s}}(v') \cup u'$ by the isomorphism. By Lemma 3.1, $u'$ and $v'$ are isolated vertices of $G'$, too. Therefore $V(G') = \{u' \cup N_{G'_{r,s}}(u')\}$, and since $\theta((u) \cup N_{G'_{r,s}}(u)) = \{u' \cup N_{G'_{r,s}}(u')\}$, we have $W = V(G')$, a contradiction.

Now choose $e' \in V(G') \setminus W$, and let $e = \theta^{-1}(e')$. Then $e \in E(G)$, and $ecc_{G'_{r,s}}(e) = ecc_{G'_{r,s}}(e') \leq 2$ by Theorem 2.1. This implies that $G$ has at most one nontrivial component. Let $x_1, y_1$ be the two end vertices of $e$ in $G$. Then $x_1$ and $y_1$ are not adjacent in $G_{r,s}$, and so $\theta(x_1)$ and $\theta(y_1)$ are not adjacent in $G'_{r,s}$.

Claim 1. $N_G(x_1) \cap N_G(y_1) = \emptyset$. If $N_G(x_1) \cap N_G(y_1) \neq \emptyset$, then $G'_{r,s}$ is connected. Since $\theta(x_1)$ and $\theta(y_1)$ are two neighbors of $e'$, and are not adjacent in $G'_{r,s}$, $N_{G'_{r,s}}(e')$ is not a clique, and thus contains a vertex of $G'$. Since $G'_{r,s}$ is connected, $N_{G'_{r,s}}(e') \subseteq V(G')$ by (2) of Remark 1. Moreover, since $d_{G'_{r,s}}(e') = |V(G')| - 1$, $V(G') = \{e' \cup N_{G'_{r,s}}(e')\}$. On the other hand, since $d_{G'_{r,s}}(e') = d_{G'_{r,s}}(x_1) + d_{G'_{r,s}}(y_1) = |V(G)| - 1$, we have $|N_G(x_1) \cup N_G(y_1)| = d_{G'_{r,s}}(x_1) + d_{G'_{r,s}}(y_1) = |N_G(x_1) \cap N_G(y_1)| \leq |V(G)| - 2$. Combining with the fact that $G$ has at most one isolated vertex, there exists a vertex $z \in V(G) \setminus (N_G(x_1) \cup N_G(y_1))$ with $d_G(z) > 0$.

Since $e \in G - z - N_G(z)$, we have $\theta(z) \in V(G')$ by Remark 3. However, $\theta(z) \notin \{e' \cup N_{G'_{r,s}}(e')\} = V(G')$, a contradiction. The claim is true.

Now let $N_G(x_1) \setminus \{y_1, y_2, \ldots, y_t\}$ and $N_G(y_1) \setminus \{x_1, x_2, \ldots, x_s\}$ have two cliques of $G_{r,s}$, $(\{x_1, x_2, \ldots, x_s\})$ and $(\{y_1, y_2, \ldots, y_t\})$. Since $d_{G'_{r,s}}(e') = d_{G'_{r,s}}(e') = |V(G')| - 1$ and $|V(G)| = |V(G')|$, there is the only element $z$ that is neither adjacent to $x_1$ nor $y_1$ in $G$. So $V(G) = \{x_1, x_2, \ldots, x_s, y_1, \ldots, y_t, z\}$, and $N_{G_{r,s}}(e') = \{x_1, x_2, \ldots, x_s, y_1, \ldots, y_t\}$ and $\omega(G_{r,s}) = 2$.

Note that $\{\theta(x_1), \theta(y_1)\} \cap V(G') \neq \emptyset$. If $\theta(x_1), \theta(y_1) \in E(G')$, $\theta(x_1)$ and $\theta(y_1)$ must be adjacent in $G'_{r,s}$ by (3) of Remark 1, a contradiction. So, we consider two cases.

Case 1. $\theta(x_1), \theta(y_1) \subseteq V(G')$. Since $\{x_1, x_2, \ldots, x_s\}$ and $\{y_1, y_2, \ldots, y_t\}$ are two cliques of $G_{r,s}$, $(\theta(x_1), \theta(x_2), \ldots, \theta(x_s), \theta(y_1), \theta(y_2), \ldots, \theta(y_t))$ are also two cliques of $G'_{r,s}$. If follows from $e' \in V(G')$, $\{\theta(x_1), \theta(y_1)\} \subseteq V(G')$, and (2) of Remark 1 that $V(G') = \{e' \cup \{\theta(x_1), \theta(x_2), \ldots, \theta(x_s), \theta(y_1), \theta(y_2), \ldots, \theta(y_t)\}\}$. So $\overline{G'} \cong G_{r,s} \bigcup N_{G_{r,s}}(e') \cong (K_s + K_t) \cup K_1$, i.e., $G' \cong K_{s,t} + K_1$. Next we show $G \cong K_{s,t} + K_1$. Since $\theta(z) \in E(G')$ and $G' \cong K_{s,t} + K_1$, we have $G_{r,s} \cong G_{r,s} \bigcup N_{G_{r,s}}(z) \cong (K_s + K_t) \cup K_1$. Since $G = G_{r,s}$, $G' = G_{r,s}$, and $G_{r,s}$ has an edge $e$, and is not empty, by Remark 3, $z$ must be an isolated vertex of $G$. Therefore $V(G) = \{z\} \cup N_{G_{r,s}}(z)$ and $\overline{G} \cong (K_s + K_t) \cup K_1$. So $G \cong K_{s,t} + K_1 \cong G'$.

Case 2. One of $\theta(x_1)$ and $\theta(y_1)$ is in $V(G')$ and the other is in $E(G')$. Without loss of generality, assume $\theta(y_1) \in V(G')$ and $\theta(x_1) \in E(G')$. By (2) of Remark 1, $\theta(x_2y_1), \ldots, \theta(x_sv_1) \in V(G')$ and $\theta(x_1y_2), \ldots, \theta(x_1y_t) \in E(G')$. By the similar argument as in the proof of Claim 1, we have $N_G(x_1) \cap N_G(y_1) = \emptyset$ for $i = 2, \ldots, s$. Combining with $N_G(y_1) = \{x_1, x_2, \ldots, x_s\}$, it follows that

\[
\{x_1, x_2, \ldots, x_s\} \text{ is an independent set of } G.
\]

Since $\theta(x_iy_1) \in V(G')$, we have $d_G(x_i) = t$ for $i = 2, \ldots, s$. Indeed, we can see that $N_G(x_i) = \{y_1, y_2, \ldots, y_s\}$ for each $i = 1, 2, \ldots, s$. (***)
For otherwise, there exists an \( x_i \) with \( i \geq 2 \) adjacent to \( z \) in \( G \). Then \( G - x_1 - N_G(x_1) \) is not empty (since \( x_1z \in E(G) \)). By Remark 3, \( \theta(x_1) \in V(G') \), which contradicts to the assumption \( \theta(x_1) \in E(G') \). We consider two subcases.

**Subcase 2.1.** \( d_G(z) > 0 \).

By the discussions above, \( N_G(z) \subseteq \{ y_2, \ldots, y_l \} \). Let \( d_G(z) = t - k \), where \( k \geq 1 \). Without loss of generality, assume that \( N_G(z) = \{ y_k+1, \ldots, y_l \} \). Then \( N_{G'}(z) = \{ x_1, \ldots, x_s, y_1, \ldots, y_k, z, y_{k+1}, \ldots, y_l \} \), and by Remark 3, \( \theta(z) \in V(G') \). By \( \theta(x_1) \in E(G') \) and Remark 1, we have \( V(G') = \{ e', \theta(x_2y_1), \ldots, \theta(x_sy_1) \} \cup \{ \theta(y_1), \ldots, \theta(y_k) \} \cup \{ \theta(zy_k+1), \ldots, \theta(zy_l), \theta(z) \} \). One can see that \( \{ y_k+1, \ldots, y_l \} \) is also an independent set of \( G \). Otherwise, let \( y_l \in E(G) \) for some \( l, m \in \{ k+1, \ldots, t \} \). Since \( \theta(zy_l), \theta(zy_m) \in V(G') \), by Remark 1, \( \theta(y_l) \in V(G') \), a contradiction. Now we give a bijection \( \sigma : V(G) \leftrightarrow V(G') \), defined by \( \sigma(x_i) = \theta(x_iy_1) \) for \( i = 1, \ldots, s \), \( \sigma(z) = \theta(y_1), \sigma(y_1) = \theta(z), \sigma(y_i) = \theta(y_i) \) for \( i = 2, \ldots, k \), and \( \sigma(y_j) = \theta(zy_j) \) for \( j = k+1, \ldots, t \). Then it is easy to check that \( \sigma \) is an isomorphism from \( 
abla G \) to \( 
abla G' \), and so \( G \cong G' \).

**Subcase 2.2.** \( d_G(z) = 0 \).

Then \( N_{G'}(z) = \{ x_1, x_2, \ldots, x_s \} \cup \{ y_1, y_2, \ldots, y_k \} \), and so \( N_{G'}(z) = \{ \theta(x_1), \theta(x_2), \ldots, \theta(x_s) \} \cup \{ \theta(y_1), \theta(y_2), \ldots, \theta(y_k) \} \). By \( (\ast) \) and \( (\ast\ast) \), \( G'^{++} = \{ \theta(x_1), \theta(x_2), \ldots, \theta(x_s) \} \) is a component of \( G'^{++} \). Hence \( \theta(x_i) \in E(G') \) for each \( i \), by \( \theta(x_1) \in E(G') \) and Remark 1. Since \( e' \in V(G) \), if \( \theta(z) \in V(G') \), then by \( (2) \) of Remark 1, \( \theta(y_1), \ldots, \theta(y_k) \in V(G') \). Thus \( V(G') = \{ \theta(z), \theta(y_1), \theta(y_2), \ldots, \theta(y_k) \} \cup \{ e', \theta(x_2y_1), \ldots, \theta(x_sy_1) \} \). We give a bijection \( \sigma : V(G) \leftrightarrow V(G') \), defined by \( \sigma(x_i) = \theta(x_iy_1) \) for \( i = 1, \ldots, s \), \( \sigma(z) = \theta(y_1), \sigma(y_1) = \theta(z) \), and \( \sigma(y_j) = \theta(zy_j) \) for \( j = 2, \ldots, t \). One can check that \( \sigma \) is an isomorphism from \( G \) to \( G' \).

Now let \( \theta(z) \in E(G') \). Then by Remark 2, \( \omega(G'^{++}) = 1 \). On the other hand, \( (\ast\ast) \) implies that \( \omega(G^{++}) = 2 \), and \( \omega(G^{++}) = 2 \). Hence \( \omega(G^{++}) = 2 \), and \( \{ \theta(x_1), \theta(x_2), \ldots, \theta(x_s) \} \) and \( \{ \theta(y_1), \theta(y_2), \ldots, \theta(y_k) \} \) are two cliques of \( G^{++} \). Since \( W = \{ \theta(z) \} \cup \{ \theta(y_1), \theta(y_2), \ldots, \theta(y_k) \} \), we have \( G \cong G'^{++}[W] \cong (K_s + K_t) \cong K_{s+t} \), i.e., \( G \cong K_{s+t} + K_1 \). Next we shall see that \( G' \cong K_{s+t} + K_1 \). Since \( \theta(z) \in E(G') \), each of \( \{ \theta(x_1), \theta(x_2), \ldots, \theta(x_s) \} \) and \( \{ \theta(y_1), \theta(y_2), \ldots, \theta(y_k) \} \) contains a vertex of \( G' \), respectively. By our assumption \( \theta(y_1) \in V(G') \), and so assume \( \theta(x_1) \in V(G') \) without loss of generality. Since \( N_{G^{++}}(\theta(x_1)) = \{ \theta(x_1), \theta(x_2), \ldots, \theta(x_s), \theta(z) \} \cup \{ \theta(y_1), \theta(y_2), \ldots, \theta(x_1y_1) \} \) and \( \theta(x_1) \in E(G') \), we have \( \theta(x_1y_1), \theta(x_2y_2), \ldots, \theta(x_sy_s) \in V(G') \), and so \( V(G') = \{ \theta(x_1y_1), \ldots, \theta(x_sy_s) \} \cup \{ \theta(y_1), e', \theta(x_2y_1), \ldots, \theta(x_{s-1}y_{s-1}) \} \cup \{ \theta(x_s), \theta(x_sy_s) \} \). Hence, \( G' \cong K_{s+t} + K_1 \), and \( G \cong G' \).

Thus, \( G \cong G' \) in any case. This completes the proof. \( \square \)

**Theorem 3.4.** Let \( G \) and \( G' \) be two graphs. Then \( G^{++} \cong G'^{++} \) if and only if \( G \cong G' \).

**Proof.** Since \( G^{++} = G^{++} \) for any graph \( G \), \( G^{++} \cong G^{++} \) if and only if \( G^{++} \cong G^{++} \). So the result is immediate from Theorem 3.3. \( \square \)

Since \( G^{++} \) and \( G^{++} \) are also complementary for a graph \( G \), and by the theorem of Behzad and Radjavi in [2], we have \( G^{++} \cong H^{++} \) if and only if \( G \cong H \). In view of these results, we believe that
Conjecture A. For two graphs $G$ and $G'$, $G^{++-} \cong G'^{++-}$ if and only if $G \cong G'$.

Conjecture B. For two graphs $G$ and $G'$, $G^{+-+} \cong G'^{-+-}$ if and only if $G \cong G'$.

4. For further research

Note that for a graph $G$, its total graph $G^{+++}$ is connected if and only if $G$ is connected. Wu and Meng [7] proved that if $xyz \neq +++$, $G^{xyz}$ is always connected for every graph $G$ (even $G$ may not be connected) except for a few cases, and the diameter of $G^{xyz}$ does not exceed 3 or 4. It is also interesting to investigate various kinds of properties of $G^{xyz}$. Vizing, independently Behzad, conjectured that $\chi(G^{+++}) \leq \Delta(G) + 2$ for any simple graph $G$. It is known as the total graph conjecture, and is still open, see [5] for its history and development. Fleischner and Hobbs [4] showed that $G^{+++}$ is hamiltonian if and only if $G$ contains an EPS-subgraph. So, the investigation of the chromatic number and the existence for a Hamilton cycle of $G^{xyz}$ is of special interest.

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