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Note

The transformation graph G^{xyz} when $xyz = -++$ ☆

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Abstract

The transformation graph G^{-++} of G is the graph with vertex set $V(G) \cup E(G)$ in which the vertex x and y are joined by an edge if one of the following conditions holds: (i) $x, y \in V(G)$, and x and y are not adjacent in G, (ii) $x, y \in E(G)$, and x and y are adjacent in G, (iii) one of x and y is in $V(G)$ and the other is in $E(G)$, and they are incident in G. In this paper, it is shown that for two graphs G and G', $G^{-++} \cong G'^{-++}$ if and only if $G \cong G'$. Simple necessary and sufficient conditions are given for G^{-++} to be planar and hamiltonian, respectively. It is also shown that for a graph G, the edge-connectivity of G^{-++} is equal to its minimum degree. Two related conjectures and some research problems are presented.

Keywords: Transformation; Total graph; Isomorphism

1. Introduction

All graphs considered here are finite and simple. Undefined terminology and notations can be found in [3]. Let $G = (V(G), E(G))$ be a graph. The *connectivity* (*edge-connectivity*)

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of G, denoted by $\kappa(G)(\lambda(G))$, is defined to be the largest integer k for which G is k-connected (k-edge connected). We use $\omega(G)$ to denote the number of components of G. For a vertex v of G, the *eccentricity* $\mathrm{ecc}_G(v)$ of v is the largest distance between v and all the other vertices of G, i.e., $\mathrm{ecc}_G(v) = \max\{d_G(u, v)|u \in V(G)\}$. The *diameter* $\mathrm{diam}(G)$ of G is $\max\{ecc_G(v)|v \in V(G)\}$, equivalently, the maximum distance between two vertices of G. $I_G(v)$ denotes the set of edges incident with v in G, and $|I_G(v)|$ is called the *degree* $d_G(v)$ of v in G. The *neighborhood* $N_G(v)$ of v is the set of all vertices of G adjacent to v. Since G is simple, $|N_G(v)| = d_G(v)$.

Suppose that V' is a nonempty subset of $V(G)$. We call V' an *independent set* if no two vertices of V' are adjacent in G whereas a *clique* if every pair of vertices of V' are adjacent in G. The subgraph $G[V']$ of G *induced* by V' is a graph with $V(G[V']) = V'$ and $uv \in E(G[V'])$ if and only if $uv \in E(G)$. For two disjoint nonempty subsets S and S' of V, we denote by $[S, S']$ the set of edges with one end in S and the other in S'.

Let $G = (V(G), E(G))$ and $H = (V(H), E(G))$ be two graphs. The *union* $G \cup H$ of G and H is the graph whose vertex set is $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H)$. Particularly, we denote their union by $G + H$ if they are disjoint, i.e., $V(G) \cap V(H) = \phi$. The *join* $G \vee H$ of G and H is the graph obtained from $G + H$ by joining each vertex of G to each vertex of H. We call G and H isomorphic, and write $G \cong H$, if there exists a bijection $\theta : V(G) \mapsto V(H)$ with $xy \in E(G)$ if and only if $\theta(x)\theta(y) \in E(H)$ for all $x, y \in V(G)$.

The *line graph* $L(G)$ of G is the graph whose vertex set is $E(G)$, and in which two vertices are adjacent if and only if they are adjacent in G. The *total graph* $T(G)$ of G is the graph whose vertex set is $V(G) \cup E(G)$, and in which two vertices are adjacent if and only if they are adjacent or incident in G. Wu and Meng [7] generalized the concept of total graph, and introduced some new graphical transformations.

Let $G = (V(G), E(G))$ be a graph, and α, β be two elements of $V(G) \cup E(G)$. We say that the associativity of α and β is $+$ if they are adjacent or incident in G, otherwise $-$. Let *xyz* be a 3-permutation of the set $\{+, -\}$. We say that α and β correspond to the first term x (resp. the second term y or the third term z) of *xyz* if both α and β are in $V(G)$ (resp. both α and β are in $E(G)$, or one of α and β is in $V(G)$ and the other is in $E(G)$). The transformation graph G^{xyz} of G is defined on the vertex set $V(G) \cup E(G)$. Two vertices α and β of G^{xyz} are joined by an edge if and only if their associativity in G is consistent with the corresponding term of *xyz*.

Since there are eight distinct 3-permutations of $\{+, -\}$, we obtain eight graphical transformations of G. It is interesting to see that G^{+++} is exactly the total graph $T(G)$ of G, and G^{---} is the complement of $T(G)$. Also, for a given graph G, G^{++-} and G^{--+}, G^{+-+} and G^{-+-}, G^{-++} and G^{+--} are other three pairs of complementary graphs.

One of the classical theorems on line graphs is due to Whitney [6]. That is, for any two connected graphs G and G', $L(G) \cong L(G')$ if and only if either $G \cong G'$, or $\{G, G'\} = \{K_3, K_{1,3}\}$. Behzad and Radjavi [2] also showed that for any two graphs G and G', $G^{+++} \cong G'^{+++}$ if and only if $G \cong G'$. Motivated from the above, we prove that for two graphs G and G', $G^{-++} \cong G'^{-++}$ if and only if $G \cong G'$. In Section 2, it is shown that G^{-++} is planar if and only if $\nu(G) \leqslant 4$, and is hamiltonian if and only if $|\nu(G)| \geqslant 3$. Also, we prove that for any graph G, the edge-connectivity of G^{-++} is equal to its minimum degree.

2. Eccentricity, connectivity, planarity, and hamiltonity

Let G be a graph with n vertices, and $H = G^{-++}$. For every vertex $v \in V(G)$, $d_H(v) = n-1$, and for an edge $e = uw \in E(G)$, $d_H(e) = d_G(u) + d_G(w)$.

Theorem 2.1. *Let G be a graph, and $H = G^{-++}$. Then* $\mathrm{ecc}_H(v) \leqslant 2$ *if* $v \in V(G)$, *and* $\mathrm{ecc}_H(e) \leqslant 3$ *if* $e \in E(G)$, *and* $\mathrm{ecc}_H(e) = 3$ *if and only if* $\mathrm{diam}(L(G)) \geqslant 3$.

Proof. Let $u, w \in V(G)$. If u and w are not adjacent in H, then they are adjacent in G. Let $e = uw$. Since uew is a path joining u and w in H, we have $d_H(u, w) = 2$. Assume $u \in V(G)$ and $e \in E(G)$. If u and e are not adjacent in H, then u is not incident with e in G. Let w be an end vertex of e in G. If u and w are not adjacent in G, then uwe is a path joining u and e in H, so $d_H(u, e) \leqslant 2$; if u and w are adjacent in G, and let $e' = uw$, then $ue'e$ is a path joining u and e in H, so $d_H(u, e) \leqslant 2$ as well. Now assume $e, e' \in E(G)$. If e and e' are not adjacent in H, then they have no common end vertex. Let u be an end vertex of e in G. Then $d_H(e, e') \leqslant d_H(e, u) + d_H(u, e) = 1 + 2 = 3$. Note that if $dima\ (L(G)) \geqslant 3$, and we take $e_1, e_2 \in E(G)$ with $d_{L(G)}(e_1, e_2) \geqslant 3$, then $d_H(e_1, e_2) = 3$. Summing up the argument above, the result follows. □

So, G^{-++} is connected and $\mathrm{diam}(G^{-++}) \leqslant 3$ for any graph G (see [7]). It is well-known that $\kappa(G) \leqslant \lambda(G) \leqslant \delta(G)$ for any graph G. In [1], Bauer and Tindell showed that $\lambda(G^{+++}) = \delta(G^{+++})$ for any connected graph G. We shall obtain the similar result for G^{-++}. Before proving it, recall that the subdivision graph $S_1(G)$ of G is the graph with the vertex set $V(G) \cup E(G)$. Two element x and y are joined by an edge if and only if one of x, y is in $V(G)$ and the other is in $E(G)$, and they are incident in G. So, $G^{-++} = (\overline{G} + L(G)) \cup S_1(G)$.

Theorem 2.2. *Let G be a graph and $H = G^{-++}$. Then $\lambda(H) = \delta(H)$.*

Proof. It suffices to show that $\lambda(H) \geqslant \delta(H)$. Let S be a minimum edge-cut of H. Then $H - S$ has exactly two components, which are denoted by H_1 and H_2. First assume that $V(H_i) \cap V(G) \neq \phi$, and let $u_i \in V(G) \cap V(H_i)$ for $i = 1, 2$. Since $\overline{G} \cup S_1(G)$ is a subdivision of the complete graph on $V(G)$, u_1 and u_2 are connected by at least $|V(G)| - 1$ vertex-disjoint paths in $\overline{G} \cup S_1(G)$. Hence $|S| \geqslant |V(G)| - 1 \geqslant \delta(H)$.

Now we assume that all the vertices of G are contained in the same component of $H - S$. Without loss of generality, suppose $V(G) \subseteq V(H_2)$. Take a vertex e from $V(H_1)$, then $e \in E(G)$. Let x, y be the end vertices of e in G. Then $x, y \in V(H_2)$. As we know, $N_H(e)$ can be partitioned into two vertex-disjoint cliques, which contain x and y, respectively. We denote the two cliques by C_x and C_y. Then $|C_x| = d_G(x)$ and $|C_y| = d_G(y)$. It follows that e and x are connected by at least $d_G(x)$ vertex-disjoint paths in $H[C_x \cup \{e\}]$, and e and y are connected by at least $d_G(y)$ vertex-disjoint paths in $H[C_y \cup \{e\}]$. Thus, we have $|S| \geqslant d_G(x) + d_G(y) \geqslant \delta(H)$. This completes the proof. □

In general does $\kappa(G^{-++})$ equal to $\delta(G^{-++})$? The answer is negative. To see this, let $G = K_n + \overline{K}_m$, where $n \geqslant 3$, $m \geqslant 2$, and $H = G^{-++}$. Suppose $V(G) = \{u_1, \ldots, u_n, v_1, \ldots, v_m\}$ such that the subgraph K_n of G is induced by $V' = \{u_1, \ldots, u_n\}$. Then $H - V'$ is not connected since as vertices of H, the edges of K_n are not adjacent to any v_i $(1 \leqslant i \leqslant m)$ in H. This implies that $\kappa(H) \leqslant |V'| = n$. Note that $d_H(v) = n + m - 1$ for any vertex $v \in V(G)$ and $d_H(u_i u_j) = d_G(u_i) + d_G(u_j) = 2n - 2$ where $i, j = 1, 2, \ldots, n$. Thus $\kappa(H) \leqslant n < \delta(H) = \min\{n + m - 1, 2n - 2\}$ since $n \geqslant 3$ and $m \geqslant 2$.

Theorem 2.3. *For a graph G, G^{-++} is planar if and only if $|V(G)| \leqslant 4$.*

Proof. It is easy to check that G^{-++} is planar for every graph G with $|V(G)| \leqslant 4$.

Note that $\overline{G} \cup S_1(G)$ is a subgraph of G^{-++} and is a subdivision of the complete graph on $V(G)$. If G^{-++} is planar, then it does not contain the subdivision of a complete subgraph with 5 vertices by the well-known Kuratowski's theorem. So $|V(G)| \leqslant 4$. □

Theorem 2.4. *For a graph G, G^{-++} is hamiltonian if and only if $|V(G)| \geqslant 3$.*

Proof. It is obvious that if $|V(G)| < 3$, then G^{-++} is not hamiltonian. For the sufficiency, let G be a graph with $|V(G)| \geqslant 3$. If G is empty, then G^{-++} is a complete graph, and is hamiltonian. Now suppose G is not empty, and let M be a maximum matching of G. Let G' be the complete graph on $V(G)$. Then G is a spanning subgraph of G', and there exists a Hamilton cycle C' of G' containing all the edges of M. Suppose that $e_1, e_2, \ldots, e_m$ are all the edges of G on C', then $M \subseteq \{e_1, e_2, \ldots, e_m\}$. Let $e_i = u_i v_i$ for $i = 1, 2, \ldots, m$, and $u_1, v_1, \ldots, u_2, v_2, \ldots, u_m, v_m, \ldots, u_1$ are the vertices of C' assigned clockwise. Note that if $\{e_1, \ldots, e_m\} = E(G)$, then we may obtain a Hamilton cycle of G^{-++} by replacing each edge $u_i v_i$ of C' by the path $u_i e_i v_i$ of length 2 for $i = 1, \ldots, m$.

Otherwise, $E(G) \backslash \{e_1, \ldots, e_m\} \neq \phi$. Since M is a maximum matching of G, each edge of $E(G) \backslash \{e_1, \ldots, e_m\}$ (denoted by F) is incident with some u_i or v_j for $i, j = 1, 2, \ldots, m$ in G. Next we should insert all the edges of G (as the vertices of G^{-++}) into C' to obtain a Hamilton cycle of G^{-++}. We do this by replacing each edge $u_i v_i$ of C' by a path P_i, $i = 1, 2, \ldots, m$, where P_i is walked along the following vertices: u_i, the edges of F incident with u_i but not incident with v_i (any order), e_i, the edges of F incident with v_i but not incident with u_i (any order), v_i. Note that if there is some edge e of F whose two end-vertices both belonging to $\{u_1, \ldots, u_m, v_1, \ldots, v_m\}$, we just insert e at the end-vertex first appeared on C' and do not insert e at the second one again. Thus we get a Hamilton cycle of G^{-++} from C'. □

3. Isomorphism

For a graph G and $v \in V(G)$, we denote the subgraph $G[N_G(v)]$ of G by G_v for short. Recall that $I_G(v)$ is the set of edges incident with v in G. Next, we start with a few useful remarks. Let G be a graph, $H = G^{-++}$.

Remark 1. For a vertex $v \in V(G)$, we have

(1) $N_H(v) \cap V(G) \neq \phi$ if and only if $d_G(v) \neq |V(G)| - 1$,
(2) for a component F of H_v, $V(F) \subseteq V(G)$ if and only if $V(F) \cap V(G) \neq \phi$, and
(3) if $d_G(v) > 0$, then $H[I_G(v)]$ $(\cong K_d)$ is a component of H_v, where $d = d_G(v)$.

It follows from the fact that $N_H(v) = (V(G)\backslash N_G(v)) \cup I_G(v)$. In particular, if $d_G(v) = |V(G)| - 1$, then $N_H(v)$ is a clique, and if $d_G(v) = 0$, $N_H(v) = V(G)\backslash\{v\}$.

Remark 2. If $e = uw \in E(G)$, then $N_H(e)$ is not a clique of H, but can be partitioned into two cliques of H each containing a vertex of G. So, $\omega(H_e) \leqslant 2$, and H_e is connected if and only if u and w have a common neighbor.

Remark 3. Let G and G' be two graphs with $G^{-++} \cong G'^{-++}$, and θ be an isomorphism from G^{-++} to G'^{-++}. For $v \in V(G)$, if $d_G(v) > 0$ and $G - v - N_G(v)$ has an edge, then $\theta(v) \in V(G')$.

Since $N_{G^{-++}}(v)$ cannot be partitioned into two cliques, and neither is $N_{G'^{-++}}(\theta(v))$. By Remark 2, $\theta(v) \in V(G')$.

Lemma 3.1. *Let G be a graph and $H = G^{-++}$. For a pair of adjacent vertices u and v of H, $N_H(u)\backslash\{v\} = N_H(v)\backslash\{u\}$ if and only if they are isolated vertices of G.*

Proof. If u and v are two isolated vertices of G, then $N_H(u)\backslash\{v\} = V(G)\backslash\{u, v\} = N_H(v)\backslash\{u\}$. Next we prove the necessity. Since u and v are adjacent in H, if $\{u, v\} \subseteq E(G)$, then $u = xy$ and $v = xz$ in G for some vertices $x, y, z \in V(G)$. However, $N_H(u)\backslash\{v\} \neq N_H(v)\backslash\{u\}$ since $y \in N_H(u)\backslash N_H(v)$. So at least one of u and v is in $V(G)$. Assume that $u \in V(G)$, $v \in E(G)$, and let $v = uw \in E(G)$ where $w \in V(G)$. Obviously, $w \in N_H(v)\backslash N_H(u)$. So both u and v are vertices of $V(G)$, and they are not adjacent in G. If $d_G(u) \neq 0$, let e be an edge incident with u in G, then $e \in N_H(u)\backslash N_H(v)$ since v is not incident with e in G. So $d_G(u) = 0$. Similarly, we have $d_G(v) = 0$. This completes the proof. $\square$

Lemma 3.2. *For two graphs G and G', if $G^{-++} \cong G'^{-++}$, then $|V(G)| = |V(G')|$ and $|E(G)| = |E(G')|$.*

Proof. Let θ be an isomorphism from G^{-++} to G'^{-++}. Since $|V(G)| + |E(G)| = |V(G^{-++})| = |V(G'^{-++})| = |V(G')| + |E(G')|$, it suffices to show that $|V(G)| = |V(G')|$. Let $W = \theta(V(G))$. First assume that $W \cap V(G') \neq \phi$. Take a vertex v' from $W \cap V(G')$, then $v = \theta^{-1}(v') \in V(G)$ by the definition of W. Therefore, $|V(G)| - 1 = d_{G^{-++}}(v) = d_{G'^{-++}}(v') = |V(G')| - 1$, and $|V(G)| = |V(G')|$. Now let $W \cap V(G') = \phi$. By the definition of G'^{-++}, each element of W is an edge of G', which is adjacent to exactly two elements of $V(G')$ in G'^{-++}.Hence, we have $|[W, V(G')]| = 2|W| = 2|V(G)|$. On the other hand, $|[W, V(G')]| = |[V(G), \theta^{-1}(V(G'))]| = 2|\theta^{-1}(V(G'))| = 2|V(G')|$. Thus, $|V(G)| = |V(G')|$. $\square$

Theorem 3.3. *For two graphs G and G', $G^{-++} \cong G'^{-++}$ if and only if $G \cong G'$.*

Proof. The sufficiency is obvious.

For the necessity, let θ be an isomorphism from G^{-++} to G'^{-++}, and $W = \theta(V(G))$. Then $|W| = |V(G)| = |V(G')|$ by Lemma 3.2. Since $G'^{-++}[V(G')] = \overline{G'}$ and $G'^{-++}[W] \cong \overline{G}$, if $W = V(G')$, we have $G \cong G'$. So we assume that $W\backslash V(G') \neq \phi$.

Next we see that G has at most one isolated vertex. Suppose it is not, and let u and v be two isolated vertices. Let $u' = \theta(u)$ and $v' = \theta(v)$. Then u' and v' are adjacent in G'^{-++} and $N_{G'^{-++}}(u')\backslash\{v'\} = N_{G'^{-++}}(v')\backslash\{u'\}$ by the isomorphism. By Lemma 3.1, u' and $v'$are isolated vertices of G', too. Therefore $V(G') = \{u'\} \cup N_{G'^{-++}}(u')$, and since $\theta(\{u\} \cup N_{G^{-++}}(u)) = \{u'\} \cup N_{G'^{-++}}(u')$, we have $W = V(G')$, a contradiction.

Now choose $e' \in V(G')\backslash W$, and let $e = \theta^{-1}(e')$. Then $e \in E(G)$, and $\mathrm{ecc}_{G^{-++}}(e) = \mathrm{ecc}_{G'^{-++}}(e') \leqslant 2$ by Theorem 2.1. This implies that G has at most one nontrivial component. Let x_1, y_1 be the two end vertices of e in G. Then x_1 and y_1 are not adjacent in G^{-++}, and so $\theta(x_1)$ and $\theta(y_1)$ are not adjacent in G'^{-++}.

Claim 1. $N_G(x_1) \cap N_G(y_1) = \phi$.

If $N_G(x_1) \cap N_G(y_1) \neq \phi$, then $G'^{-++}{}_{e'}$ is connected. Since $\theta(x_1)$ and $\theta(y_1)$ are two neighbors of e', and are not adjacent in G'^{-++}, $N_{G'^{-++}}(e')$ is not a clique, and thus contains a vertex of G'. Since $G'^{-++}{}_{e'}$ is connected, $N_{G'^{-++}}(e') \subseteq V(G')$ by (2) of Remark 1. Moreover, since $d_{G'^{-++}}(e') = |V(G')| - 1$, $V(G') = \{e'\} \cup N_{G'^{-++}}(e')$. On the other hand, since $d_{G^{-++}}(e) = d_G(x_1) + d_G(y_1) = |V(G)| - 1$, we have $|N_G(x_1) \cup N_G(y_1)| = d_G(x_1) + d_G(y_1) - |N_G(x_1) \cap N_G(y_1)| \leqslant |V(G)| - 2$. Combining with the fact that G has at most one isolated vertex, there exists a vertex $z \in V(G)\backslash(N_G(x_1) \cup N_G(y_1))$ with $d_G(z) > 0$. Since $e \in G - z - N_G(z)$, we have $\theta(z) \in V(G')$ by Remark 3. However, $\theta(z) \notin \{e'\} \cup N_{G'^{-++}}(e') = V(G')$, a contradiction. The claim is true.

Now let $N_G(x_1)\backslash\{y_1\} = \{y_2, \ldots, y_t\}$ and $N_G(y_1)\backslash\{x_1\} = \{x_2, \ldots, x_s\}$. Since $d_{G^{-++}}(e) = d_{G'^{-++}}(e') = |V(G')| - 1$ and $|V(G)| = |V(G')|$, there is the only element z that is neither adjacent to x_1 nor y_1 in G. So $V(G) = \{x_1, \ldots, x_s, y_1, \ldots, y_t, z\}$, and $N_{G^{-++}}(e) = \{x_1, x_1y_2, \ldots, x_1y_t\} \cup \{y_1, x_2y_1, \ldots, x_sy_1\}$ and $\omega(G^{-++}{}_e) = 2$.

Note that $\{\theta(x_1), \theta(y_1)\} \cap V(G') \neq \phi$. If $\theta(x_1), \theta(y_1) \in E(G')$, $\theta(x_1)$ and $\theta(y_1)$ must be adjacent in G'^{-++} by (3) of Remark 1, a contradiction. So, we consider two cases.

Case 1. $\{\theta(x_1), \theta(y_1)\} \subseteq V(G')$.

Since $\{x_1, x_1y_2, \ldots, x_1y_t\}$ and $\{y_1, x_2y_1, \ldots, x_sy_1\}$ are two cliques of G^{-++}, $\{\theta(x_1), \theta(x_1y_2), \ldots, \theta(x_1y_t)\}$ and $\{\theta(y_1), \theta(x_2y_1), \ldots, \theta(x_sy_1)\}$ are also two cliques of G'^{-++}. It follows from $e' \in V(G')$, $\{\theta(x_1), \theta(y_1)\} \subseteq V(G')$, and (2) of Remark 1 that

$$V(G') = \{e'\} \cup \{\theta(x_1), \theta(x_1y_2), \ldots, \theta(x_1y_t)\} \cup \{\theta(y_1), \theta(x_2y_1), \ldots, \theta(x_sy_1)\}.$$

So $\overline{G'} \cong G^{-++}[\{e\} \cup N_{G^{-++}}(e)] \cong (K_s + K_t) \vee K_1$, i.e., $G' \cong K_{s,t} + K_1$. Next we show $G \cong K_{s,t} + K_1$. Since $\theta(z) \in E(G')$ and $G' \cong K_{s,t} + K_1$, we have $G^{-++}{}_z \cong G'^{-++}{}_{\theta(z)} \cong K_s + K_t$. Since $G - z - N_G(z)$ has an edge e, and is not empty, by Remark 3, z must be an isolated vertex of G. Therefore $V(G) = \{z\} \cup N_{G^{-++}}(z)$ and $\overline{G} \cong (K_s + K_t) \vee K_1$. So $G \cong K_{s,t} + K_1 \cong G'$.

Case 2. One of $\theta(x_1)$ and $\theta(y_1)$ is in $V(G')$ and the other is in $E(G')$.

Without loss of generality, assume $\theta(y_1) \in V(G')$ and $\theta(x_1) \in E(G')$. By (2) of Remark 1, $\theta(x_2y_1), \ldots, \theta(x_sy_1) \in V(G')$ and $\theta(x_1y_2), \ldots, \theta(x_1y_t) \in E(G')$. By the similar argument as in the proof of Claim 1, we have $N_G(x_i) \cap N_G(y_1) = \phi$ for $i = 2, \ldots, s$. Combining with $N_G(y_1) = \{x_1, x_2, \ldots, x_s\}$, it follows that

$$\{x_1, x_2, \ldots, x_s\} \text{ is an independent set of } G. \tag{$*$}$$

Since $\theta(x_iy_1) \in V(G')$, we have $d_G(x_i) = t$ for $i = 2, \ldots, s$. Indeed, we can see that

$$N_G(x_i) = \{y_1, y_2, \ldots, y_t\} \quad \text{for each } i = 1, 2, \ldots, s. \tag{$**$}$$

For otherwise, there exists an x_i with $i\geqslant 2$ adjacent to z in G. Then $G - x_1 - N_G(x_1)$ is not empty (since $x_i z \in E(G)$). By Remark 3, $\theta(x_1) \in V(G')$, which contradicts to the assumption $\theta(x_1) \in E(G')$. We consider two subcases.

Subcase 2.1. $d_G(z) > 0$.

By the discussions above, $N_G(z) \subseteq \{y_2, \ldots, y_t\}$. Let $d_G(z) = t - k$, where $k \geqslant 1$. Without loss of generality, assume that $N_G(z) = \{y_{k+1}, \ldots, y_t\}$. Then $N_{G^{-++}}(z) = \{x_1, \ldots, x_s, y_1, \ldots, y_k, zy_{k+1}, \ldots, zy_t\}$, and by Remark 3, $\theta(z) \in V(G')$. By $\theta(x_1) \in E(G')$ and Remark 1, we have $V(G') = \{e', \theta(x_2y_1), \ldots, \theta(x_sy_1)\} \cup \{\theta(y_1), \ldots, \theta(y_k)\} \cup \{\theta(zy_{k+1}), \ldots, \theta(zy_t), \theta(z)\}$. One can see that $\{y_{k+1}, \ldots, y_t\}$ is also an independent set of G. Otherwise, let $y_l y_m \in E(G)$ for some $l, m \in \{k+1, \ldots, t\}$. Since $\theta(zy_l), \theta(zy_m) \in V(G')$, by Remark 1, $\theta(y_l y_m) \in V(G')$, a contradiction. Now we give a bijection $\sigma : V(G) \mapsto V(G')$, defined by $\sigma(x_i) = \theta(x_i y_1)$ for $i = 1, \ldots, s$, $\sigma(z) = \theta(y_1)$, $\sigma(y_1) = \theta(z)$, $\sigma(y_i) = \theta(y_i)$ for $i = 2, \ldots, k$, and $\sigma(y_j) = \theta(zy_j)$ for $j = k+1, \ldots, t$. Then it is easy to check that σ is an isomorphism from $\overline{G}$ to $\overline{G'}$, and so $G \cong G'$.

Subcase 2.2. $d_G(z) = 0$.

Then $N_{G^{-++}}(z)=\{x_1, x_2, \ldots, x_s\}\cup\{y_1, y_2, \ldots, y_t\}$, and so $N_{G'^{-++}}(\theta(z))=\{\theta(x_1), \theta(x_2), \ldots, \theta(x_s)\}\cup\{\theta(y_1), \theta(y_2), \ldots, \theta(y_t)\}$. By $(*)$ and $(**)$, $G'^{-++}[\{\theta(x_1), \theta(x_2), \ldots, \theta(x_2)\}]$ is a component of $G'^{-++}{}_{\theta(z)}$. Hence $\theta(x_i) \in E(G')$ for each i, by $\theta(x_1) \in E(G')$ and Remark 1. Since $e' \in V(G')$, if $\theta(z) \in V(G')$, then by (2) of Remark 1, $\theta(y_2), \ldots, \theta(y_t) \in V(G')$. Thus $V(G')=\{\theta(z), \theta(y_1), \theta(y_2), \ldots, \theta(y_t)\}\cup \{e', \theta(x_2y_1), \ldots, \theta(x_sy_1)\}$. We give a bijection $\sigma : V(G) \longrightarrow V(G')$, defined by $\sigma(x_i) = \theta(x_i y_1)$ for $i = 1, \ldots, s$. $\sigma(z) = \theta(y_1)$, $\sigma(y_1) = \theta(z)$, and $\sigma(y_j) = \theta(y_j)$ for $j = 2, \ldots, t$. One can check that σ is an isomorphism from G to G'.

Now let $\theta(z) \in E(G')$. Then by Remark 2, $\omega(G'^{-++}{}_{\theta(z)}) \leqslant 2$. On the other hand, $(**)$ implies that $\omega(G^{-++}{}_z) \geqslant 2$, and thus $\omega(G'^{-++}{}_{\theta(z)}) \geqslant 2$. Hence $\omega(G'^{-++}{}_{\theta(z)}) = 2$, and $\{\theta(x_1), \theta(x_2), \ldots, \theta(x_s)\}$ and $\{\theta(y_1), \theta(y_2), \ldots, \theta(y_t)\}$ are two cliques of G'^{-++}. Since $W = \theta(V(G)) = \{\theta(z)\} \cup N_{G'^{-++}}(\theta(z))$, we have $\overline{G} \cong G'^{-++}[W] \cong (K_s + K_t) \vee K_1$, i.e., $G \cong K_{s,t} + K_1$. Next we shall see that $G' \cong K_{s,t} + K_1$. Since $\theta(z) \in E(G')$, each of $\{\theta(x_1), \theta(x_2), \ldots, \theta(x_s)\}$ and $\{\theta(y_1), \theta(y_2), \ldots, \theta(y_t)\}$ contains a vertex of G', respectively. By our assumption $\theta(y_1) \in V(G')$, and so assume $\theta(x_s) \in V(G')$ without loss of generality. Since $N_{G'^{-++}}(\theta(x_s)) = \{\theta(x_1), \theta(x_2), \ldots, \theta(x_{s-1}), \theta(z)\} \cup \{\theta(x_s y_1), \theta(x_s y_2), \ldots, \theta(x_s y_t)\}$ and $\theta(x_1) \in E(G')$, we have $\theta(x_s y_1), \theta(x_s y_2), \ldots, \theta(x_s y_t) \in V(G')$, and so $V(G') = \{\theta(x_s y_1)\} \cup \{\theta(y_1), e', \theta(x_2 y_1), \ldots, \theta(x_{s-1} y_1)\} \cup \{\theta(x_s), \theta(x_s y_2), \theta(x_s y_3), \ldots, \theta(x_s y_t)\}$. Hence, $\overline{G'} \cong (K_s + K_t) \vee K_1$, and $G \cong G' \cong K_{s,t} + K_1$.

Thus, $G \cong G'$ in any case. This completes the proof. □

Theorem 3.4. *Let G and G' be two graphs. Then $G^{+--} \cong G'^{+--}$ if and only if $G \cong G'$.*

Proof. Since $\overline{G^{-++}} = G^{+--}$ for any graph G, $G^{+--} \cong G'^{+--}$ if and only if $G^{-++} \cong G'^{-++}$. So the result is immediate from Theorem 3.3. □

Since G^{+++} and G^{---} are also complementary for a graph G, and by the theorem of Behzad and Radjavi in [2], we have $G^{---} \cong H^{---}$ if and only if $G \cong H$. In view of these results, we believe that

Conjecture A. For two graphs G and G', $G^{++-} \cong G'^{++-}$ if and only if $G \cong G'$.

Conjecture B. For two graphs G and G', $G^{+-+} \cong G'^{+-+}$ if and only if $G \cong G'$.

4. For further research

Note that for a graph G, its total graph G^{+++} is connected if and only if G is connected. Wu and Meng [7] proved that if $xyz \neq +++$, G^{xyz} is always connected for every graph G (even G may not be connected) except for a few cases, and the diameter of G^{xyz} does not exceed 3 or 4. It is also interesting to investigate various kinds of properties of G^{xyz}. Vizing, independently Behzad, conjectured that $\chi(G^{+++}) \leqslant \Delta(G) + 2$ for any simple graph G. It is known as the total graph conjecture, and is still open, see [5] for its history and development. Fleischner and Hobbs [4] showed that G^{+++} is hamiltonian if and only if G contains an EPS-subgraph. So, the investigation of the chromatic number and the existence for a Hamilton cycle of G^{xyz} is of special interest.

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