Essential point spectra of operator matrices through local spectral theory

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Abstract

For \( A \in B(X) \), \( B \in B(Y) \) and \( C \in B(Y, X) \), let \( M_C \) be the operator defined on \( X \oplus Y \) by \( \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \). In this paper, we study defect set \((\Sigma(A) \cup \Sigma(B)) \setminus \Sigma(M_C)\), where \( \Sigma \) is the Browder spectrum, the essential approximate point spectrum and Browder essential approximate point spectrum. We then give application for Weyl’s and Browder’s theorems.

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1. Introduction

Throughout this paper, \( X \) and \( Y \) are Banach spaces and \( B(X, Y) \) denotes the space of all bounded linear operators from \( X \) to \( Y \). We set \( B(X) \) to denote \( B(X, X) \). For \( T \in B(X) \), let \( T^* \), \( N(T) \), \( R(T) \), \( \sigma(T) \), \( \sigma_p(T) \) and \( \sigma_a(T) \) denote respectively the adjoint, the null space, the range, the spectrum, the point spectrum and the approximate point spectrum of \( T \). Let \( n(T) \) and \( d(T) \) be the nullity and the deficiency of \( T \) defined by

\[
n(T) = \dim N(T) \quad \text{and} \quad d(T) = \text{codim} R(T).
\]

If the range \( R(T) \) of \( T \) is closed and \( n(T) < \infty \) (respectively \( d(T) < \infty \)), then \( T \) is called an upper semi-Fredholm (respectively a lower semi-Fredholm) operator. If \( T \in \mathcal{L}(X) \) is either upper or lower semi-Fredholm, then \( T \) is called a semi-Fredholm operator, and the index of \( T \) is defined by \( \text{ind}(T) = n(T) - d(T) \). If both \( n(T) \) and \( d(T) \) are finite, then \( T \) is a Fredholm operator. An operator \( T \) is called Weyl if it is Fredholm of index zero. The ascent, notated by \( \text{asc}(T) \), and the descent, notated by \( \text{dsc}(T) \), of \( T \) are given by

\[
\text{asc}(T) = \inf \{ n : N(T^n) = N(T^{n+1}) \}, \quad \text{dsc}(T) = \inf \{ n : R(T^n) = R(T^{n+1}) \};
\]

if no such \( n \) exists, then \( \text{asc}(T) = \infty \), respectively \( \text{dsc}(T) = \infty \). For \( T \in B(X) \) we say that \( \lambda \notin \mathcal{F}(T) \) if \( \text{asc}(T - \lambda) = \text{dsc}(T - \lambda) < \infty \).

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A bounded linear operator $T$ is called \textit{Browder} if it is Fredholm of finite ascent and descent. The essential spectrum $\sigma_e(T)$, Weyl spectrum $\sigma_w(T)$, and Browder spectrum $\sigma_b(T)$ of $T$ are defined by

\[
\sigma_e(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm} \},
\]
\[
\sigma_w(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl} \},
\]
\[
\sigma_b(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not Browder} \}.
\]

Evidently
\[
\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup \text{acc } \sigma(T),
\]

where for a subset $K \subseteq \mathbb{C}$, we write acc $K$ (respectively iso $K$) for accumulation (respectively isolated) points of $K$.

We say that \textit{Weyl’s theorem} holds for $T$ if
\[
\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T);
\]
where $\pi_{00}(T)$ is the set of isolated point of $\sigma(T)$ which are eigenvalues of finite multiplicity, and that \textit{Browder’s theorem} holds for $T \in \mathcal{L}(X)$ if
\[
\sigma_w(T) = \sigma_b(T).
\]

Denote
\[
\Phi_l(X) = \{ T \in B(X) : R(T) \text{ is closed and complemented subspace of } X \text{ and } n(T) < \infty \}
\]
and
\[
\Phi_r(X) = \{ T \in B(X) : N(T) \text{ is complemented subspace of } X \text{ and } d(T) < \infty \}
\]
the set of left and right Fredholm operators, respectively. It is well known that
\[
\Phi(X) = \Phi_l \cap \Phi_r(X).
\]

Also, let
\[
SF_+(X) = \{ T \in B(X) : R(T) \text{ is closed and } n(T) < \infty \},
\]
and let $SF_+(X)$ be the class of all $T \in SF_+(X)$ with ind $T \leq 0$. The \textit{essential approximate point spectrum} $\sigma_{ea}(T)$ and the \textit{Browder essential approximate point spectrum} $\sigma_{ab}(T)$ are defined by

\[
\sigma_{ea}(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not in } SF_+(X) \},
\]
\[
\sigma_{ab}(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not in } SF_+(X) \text{ or has infinite ascent} \}.
\]

It is known that
\[
\sigma_{ea}(T) = \bigcap \{ \sigma_a(T + K) : K \in \mathcal{K}(X) \},
\]
\[
\sigma_{ab}(T) = \bigcap \{ \sigma_a(T + K) : TK = KT \text{ and } K \in \mathcal{K}(X) \};
\]  

where $\mathcal{K}(X)$ is the ideal of compact operators on $X$ (see [11]). We say that \textit{a-Weyl’s theorem} holds for $T$ if
\[
\sigma_a(T) \setminus \sigma_{ea}(T) = \pi_{00}^a(T);
\]
where $\pi_{00}^a(T)$ is the set of isolated points of $\sigma_a(T)$ which are eigenvalues of finite multiplicity, and that \textit{a-Browder’s theorem} holds for $T$ if
\[
\sigma_{ea}(T) = \sigma_{ab}(T).
\]

In [6,12], it is shown that for any $T \in \mathcal{B}(X)$ we have the implications:

- Weyl’s theorem
- a-Weyl’s theorem
- Browder’s theorem
- a-Browder’s theorem
We say that $T$ has the single valued extension property (SVEP) at $\lambda \in \mathbb{C}$ if for every open neighborhood $U$ of $\lambda$, the only solution of the equation $(T - \mu)f(\mu) = 0$ that is analytic on $U$ is the constant function $f \equiv 0$. Let $S(T)$ be the set of all $\lambda$ on which $T$ does not satisfy the SVEP.

For $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y, X)$ we denote by $M_C$ the operator defined on $X \oplus Y$ by

$$M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}.$$

Numerous mathematicians were interested by the following equality

$$\Sigma(M_C) = \Sigma(A) \cup \Sigma(B), \quad \text{for every } C \in \mathcal{B}(Y, X)$$

where $\Sigma \in \{\sigma, \sigma_e, \sigma_w, \ldots\}$. See for instance [3,7,9,13] and the references therein. In this paper we describe the defect set $(\Sigma(A) \cup \Sigma(B)) \setminus \Sigma(M_C)$ for $\Sigma \in \{\sigma_b, \sigma_{ea}, \sigma_{ab}\}$ and we give an application for Weyl’s theorem of $M_C$.

2. Browder spectrum

Lemma 2.1. Let $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y, X)$ an arbitrary operators. If $A$ has infinite ascent, then $M_C$ has infinite ascent, and $B$ has infinite descent, then $M_C$ has infinite descent too.

Proof. Let $A$ has infinite ascent. Then for every $n \in \mathbb{N}$ exists $x_n \in X$ such that $x_n \in N(A^{n+1}) \setminus N(A^n)$. Then $x_n \oplus 0 \in N(M_C^{n+1}) \setminus N(M_C^n)$.

Suppose the contrary, that $B$ has infinite descent and $\text{dsc}(M_C) = n < \infty$. Since $\text{dsc}(B)$ is infinite, there exists a $y \in R(B^n) \setminus R(B^{n+1})$, i.e. there exists $z \in Y$ such that $y = B^n(z)$ and $y \neq B^{n+1}(t)$ for all $t \in Y$. Let, for example, $v = 0 \oplus z$ and

$$M_C^n(v) = \tilde{x} \oplus B^n(z) \in R(M_C^n) = R(M_C^{n+1}).$$

There exists a $v_0 = x_0 \oplus z_0 \in X \oplus Y$ such that

$$\tilde{x} \oplus B^n(z) \left( = M_C^n(v) \right) = M_C^{n+1}(v_0) = \tilde{x}_0 \oplus B^{n+1}(z_0).$$

Hence,

$$B^{n+1}(z_0) = B^n(z) = y \in R(B^{n+1})$$

and the contradiction proofs the lemma. \(\Box\)

Lemma 2.2. Let $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y, X)$ an arbitrary operators. If $\lambda \in (\sigma_b(A) \cup \sigma_b(B)) \setminus \sigma_b(M_C)$, then $\lambda \in \sigma_b(A) \cap \sigma_b(B)$.

Proof. Let $\lambda \notin \sigma_b(M_C)$, then $M_C - \lambda$ is a Fredholm operator of index zero and $\text{asc}(M_C - \lambda) = \text{dsc}(M_C - \lambda) < \infty$.

Suppose that $\lambda \notin \sigma_b(A)$. Since $A - \lambda$ is a Fredholm operator of index zero, together with $M_C - \lambda$ is a Fredholm operator of index zero, follows that $\text{ind}(B - \lambda) = 0$. From $\text{dsc}(M_C - \lambda) < \infty$, by Lemma 2.1, follows that $\text{dsc}(B - \lambda) < \infty$ and by [1, Theorem 3.4(iv)] we have that $\text{asc}(B - \lambda) = \text{dsc}(B - \lambda) < \infty$. Hence, $\lambda \notin \sigma_b(B)$.

Similarly, if we suppose that $\lambda \notin \sigma_b(B)$ we consequently have that $\text{ind}(A - \lambda) = 0$. Now by [1, Theorem 3.4(iv)], how it is $\text{asc}(A - \lambda) < \infty$, we have that $\text{asc}(A - \lambda) = \text{dsc}(A - \lambda) < \infty$ and $\lambda \notin \sigma_b(A)$. \(\Box\)

Theorem 2.3. Let $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y, X)$ an arbitrary operators, then

$$(\sigma_b(A) \cup \sigma_b(B)) \setminus \sigma_b(M_C) \subset (\sigma_b(A) \cap \sigma_b(B)) \cup (\sigma_w(A) \cap \sigma_w(B)).$$

Proof. Let $\lambda \in (\sigma_b(A) \cup \sigma_b(B)) \setminus \sigma_b(M_C)$. Then we have $\lambda \in (\sigma_b(A) \cap \sigma_b(B))$ (by Lemma 2.2), $\text{ind}(M_C - \lambda) = 0$ and $\text{asc}(M_C - \lambda) = \text{dsc}(M_C - \lambda) < \infty$. By Lemma 2.1 we have that $\text{asc}(A - \lambda) < \infty$ and $\text{dsc}(B - \lambda) < \infty$, and from [1, Theorem 3.8] follows that $\lambda \notin (\sigma_{w}(A) \cap \sigma_{w}(B))$. Also, $A - \lambda$ is upper semi-Fredholm and $B - \lambda$ is lower semi-Fredholm (see [8, Theorem 3.2]).

Suppose that $\lambda \notin (\sigma_{w}(A) \cap \sigma_{w}(B))$. Then $\lambda \notin (\sigma_{w}(A) \cap \sigma_{w}(B)) \cup (\sigma_{w}(A) \cap \sigma_{w}(B)) \cup (\sigma_{w}(A) \cap \sigma_{w}(B))$ and by [13, Theorem 3.3] follows that $\lambda \notin (\sigma_{w}(A) \cup \sigma_{w}(B))$. Hence, $\lambda \notin \sigma_{w}(A) \cup \sigma_{w}(B)$, i.e. $\text{asc}(A - \lambda) \neq \text{dsc}(A - \lambda)$.
or \( \text{asc}(A - \lambda) = \text{dsc}(A - \lambda) = \infty \), and, \( \text{asc}(B - \lambda) \neq \text{dsc}(B - \lambda) \) or \( \text{asc}(B - \lambda) = \text{dsc}(B - \lambda) = \infty \). Hence, in any case, \( \lambda \in (\mathcal{F}(A^* \cap \mathcal{F}(B)) \).

Now suppose that \( \lambda \notin (\mathcal{F}(A^*) \cap \mathcal{F}(B)) \). If \( \lambda \notin \mathcal{F}(A^*) \), then \( \text{asc}(A - \lambda) = \text{dsc}(A - \lambda) < \infty \). The operator \( A - \lambda \) is upper semi-Fredholm with finite ascend and descent, i.e. \( \lambda \notin \sigma_b(A) \). Contradiction.

Similarly, if we suppose that \( \lambda \notin \mathcal{F}(B) \), follows that \( \lambda \notin \sigma_b(B) \). \( \square \)

**Remark 2.4.** In general, condition \((\sigma_a(A) \cup \sigma_b(B)) \setminus \sigma_b(MC) \subset (\mathcal{S}(A^*) \cap \mathcal{S}(B)) \cup (\mathcal{F}(A^*) \cap \mathcal{F}(B))\) in Theorem 2.3 can be weaker, but if we give this version by connection with works in [13]. Matter in fact, for Fredholm operator \( T \) with zero index (Weyl operator) we have that \( 0 \in \mathcal{S}(T) \) implies \( 0 \notin \mathcal{F}(T) \). Really, if \( T \) has no SVEP at 0 and index zero, by [1, Theorems 3.4 and 3.8] follows that \( \text{asc}(T) \neq \text{dsc}(T) \) or \( \text{asc}(T) = \text{dsc}(T) = \infty \), i.e. \( 0 \notin \mathcal{F}(T) \). Hence, we have
\[
(\sigma_a(A) \cup \sigma_b(B)) \setminus \sigma_b(MC) \subset (\mathcal{F}(A^*) \cap \mathcal{F}(B)).
\]

**Corollary 2.5.** Let \( A \in \mathcal{B}(X) \), \( B \in \mathcal{B}(Y) \), then for every \( C \in \mathcal{B}(Y, X) \) holds \( \sigma_b(MC) \subset (\mathcal{F}(A^*) \cap \mathcal{F}(B)) = \sigma_b(A) \cup \sigma_b(B) \). \( \square \)

**Corollary 2.6.** Let \( A \in \mathcal{B}(X) \), \( B \in \mathcal{B}(Y) \) and \( C \in \mathcal{B}(Y, X) \) an arbitrary operators. If \( (\mathcal{F}(A^*) \cap \mathcal{F}(B)) = \emptyset \), then for every \( C \in \mathcal{B}(Y, X) \) holds \( \sigma_b(MC) = \sigma_b(A) \cup \sigma_b(B) \).

**3. Essential approximate point spectra**

**Lemma 3.1.** Let \( A \in \mathcal{B}(X) \), \( B \in \mathcal{B}(Y) \) and \( C \in \mathcal{B}(Y, X) \) an arbitrary operators. Then
\[
\sigma_{ea}(MC) \subseteq \sigma_{ea}(A) \cup \sigma_{ea}(B).
\]

**Proof.** Let \( \lambda \notin (\sigma_{ea}(A) \cup \sigma_{ea}(B)) \), then it follows from equality (1) that there exists \( K_1 \in \mathcal{K}(X) \) and \( K_2 \in \mathcal{K}(Y) \) such that \( \lambda \notin (\sigma_a(A + K_1) \cup \sigma_a(B + K_2)) \). Let \( K = K_1 \oplus K_2 \), then \( K \in \mathcal{K}(X \oplus Y) \). Since \( \sigma_a(MC) \subseteq \sigma_a(A) \cup \sigma_a(B) \) (see [8, Proposition 5.1 and proof of Proposition 3.1]) then \( \lambda \notin \sigma_a(MC + K) \). Thus \( \lambda \notin \sigma_{ea}(MC) \). \( \square \)

For the sake of completes we will give next lemma that is version of Theorem 2.1 of [4] for the case of Banach spaces \( X \) and \( Y \).

**Lemma 3.2.** Let \( A \in \mathcal{B}(X) \), \( B \in \mathcal{B}(Y) \) and \( C \in \mathcal{B}(Y, X) \) an arbitrary operators. If \( MC \) is upper semi-Fredholm with \( \text{ind}(MC) \leq 0 \), then \( A \) is upper semi-Fredholm and
\[
\begin{align*}
n(B) < \infty \text{ and } \text{ind}(A) + \text{ind}(B) & \leq 0, \text{ or } \\
n(B) = d(A) & = \infty.
\end{align*}
\]

**Proof.** Let \( MC \in SF_+(X \oplus Y) \). Then
\[
MC = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I & C \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}
\]
and by [2, Corollary 1.3.4] follows that \( \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \in SF_+(X \oplus Y) \). Now by [5, Lemma 2.1] we have that \( A \) is a semi-Fredholm operator and, since, for every \( x \in N(A) \), \( x \oplus 0 \in N(MC) \), follows that \( n(A) \leq n(MC) < \infty \). Hence, \( A \) is upper semi-Fredholm and [5, Lemma 2.1] implies that \( B \) is a semi-Fredholm operator.

If we suppose that \( n(B) < \infty \), then we have
\[
\text{ind}(A) + \text{ind}(B) = \text{ind}(MC) \leq 0.
\]
Now, let \( n(B) = \infty \) and let \( \{y_n\} \) be a sequence of linearly independent vectors of \( N(B) \). Since \( n(MC) < \infty \), we can suppose, without lost of generality, that \( C(y_n) \neq 0 \) and, also, \( C(y_n) \notin R(A) \), for all positive integer \( n \). Really, if \( C(y_n) = 0 \), then \( 0 \oplus y_n \in N(MC) \). Also, if \( C(y_n) (= Ax_n) \in R(A) \), then \( -x_n \oplus y_n \in N(MC) \).
Suppose that \( d(A) = n < \infty \) and let \( C_y_1, \ldots, C_y_n \) are linearly independent vectors modulo \( R(A) \). Then, for every \( m > n \), exists scalars \( \alpha^m_1, \ldots, \alpha^m_m \) and \( x_m \in X \) such that
\[
\alpha^m_1 C_y_1 + \cdots + \alpha^m_m C_y_m + C_y_m = Ax_m.
\]

Now the vectors \(-x_m \oplus (\alpha^m_1 C_y_1 + \cdots + \alpha^m_m C_y_n + y_m)\) are linearly independent vectors in \( N(M_C) \) that implies \( n(M_C) = \infty \). The contradiction shows \( d(A) = \infty \). \( \square \)

**Theorem 3.3.** Let \( A \in B(X), B \in B(Y) \) and \( C \in B(Y, X) \) an arbitrary operators. Then
\[
(\sigma_{ea}(A) \cup \sigma_{ea}(B)) \setminus \sigma_{ea}(M_C) \subseteq (S(A) \cap S(B^*)) \cup S(A^*).
\]

**Proof.** Let \( \lambda \in (\sigma_{ea}(A) \cup \sigma_{ea}(B)) \setminus \sigma_{ea}(M_C) \), then \( M_C - \lambda \) is an upper semi-Fredholm operator with \( \text{ind}(M_C - \lambda) \leq 0 \). If \( \text{ind}(M_C - \lambda) = 0 \), then it follows from [13] that \( \lambda \in (S(A) \cap S(B^*)) \cup (S(A^*) \cap S(B)) \). Now assume that \( \text{ind}(M_C - \lambda) < 0 \).

Case 1. \( \lambda \in \sigma_{ea}(A) \). Since \( A \) is upper semi-Fredholm (Lemma 3.2), then \( \text{ind}(A - \lambda) > 0 \) hence it follows from [1, Corollary 3.19(i)] that \( \lambda \in S(A) \). If \( R(B - \lambda) \) is closed then we deduce from Lemma 3.2 that \( B - \lambda \) is upper semi-Fredholm and \( \text{ind}(B - \lambda) < 0 \), so [1, Corollary 3.19(ii)] \( \lambda \in S(B^*) \), or \( n(B) = d(A) = \infty \) which is impossible. Therefore \( \lambda \in S(A) \cap S(B^*) \).

Case 2. \( \lambda \in \sigma_{ea}(B) \). Then \( B - \lambda \) is not upper semi-Fredholm or \( \text{ind}(B - \lambda) > 0 \). Assume that \( B - \lambda \) is not upper semi-Fredholm. If \( R(B - \lambda) \) is closed, then \( n(B - \lambda) = \infty \) hence (Lemma 3.2) \( n(B - \lambda) = d(A - \lambda) = \infty \). Since \( A - \lambda \) is upper semi-Fredholm then \( \text{ind}(A - \lambda) < 0 \). Thus \( \lambda \in S(A^*) \) (see [1]). Hence \( \lambda \in S(A^*) \).

Now if \( B - \lambda \) is upper semi-Fredholm with \( \text{ind}(B - \lambda) > 0 \), then by Lemma 3.2, \( \text{ind}(A - \lambda) < 0 \). Thus \( \lambda \in S(A^*) \). \( \square \)

**Corollary 3.4.** If \( A \) and \( A^* \), or \( A \) and \( B^* \) have the SVEP, then
\[
\sigma_{ea}(M_C) = \sigma_{ea}(A) \cup \sigma_{ea}(B),
\]
for every \( C \in B(Y, X) \).

**Theorem 3.5.** Let \( A \in B(X), B \in B(Y) \) and \( C \in B(Y, X) \) an arbitrary operators. Then
\[
(\sigma_{ab}(A) \cup \sigma_{ab}(B)) \setminus \sigma_{ab}(M_C) \subseteq S(A^*) \cap F(B).
\]

**Proof.** Let \( \lambda \in (\sigma_{ab}(A) \cup \sigma_{ab}(B)) \setminus \sigma_{ab}(M_C) \), then \( M_C - \lambda \) is an upper semi-Fredholm of finite ascent. Hence by Lemmas 2.1 and 3.2, \( A - \lambda \) is an upper semi-Fredholm operator of finite ascent. Then \( \lambda \notin \sigma_{ab}(A) \) and, consequently, by [1, Corollary 3.19] we have \( \lambda \notin S(A^*) \).

Hence, we have \( \lambda \in \sigma_{ab}(B) \). If \( B - \lambda \) is not upper semi-Fredholm, then from Lemma 3.2 we have \( d(A - \lambda) = \infty \). Then \( \text{ind}(A - \lambda) < 0 \) thus \( \lambda \in S(A^*) \).

Now if \( B - \lambda \) is upper semi-Fredholm, then \( B - \lambda \) is not of finite ascent when \( \lambda \in F(B) \). Therefore \( \lambda \in S(A^*) \cap F(B) \). \( \square \)

**Corollary 3.6.** If \( S(A^*) \cap F(B) = \emptyset \), then
\[
\sigma_{ab}(M_C) = \sigma_{ab}(A) \cup \sigma_{ab}(B),
\]
for every \( C \in B(Y, X) \).

4. Applications

**Proposition 4.1.** If \((S(A) \cap S(B^*)) \cup S(A^*) = \emptyset \), then

(a) Browder’s theorem holds for \( \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \) \( \Rightarrow \) Browder’s theorem holds for \( \begin{bmatrix} A \\ 0 & C \end{bmatrix} \).

(b) a-Browder’s theorem holds for \( \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \) \( \Rightarrow \) a-Browder’s theorem holds for \( \begin{bmatrix} A \\ 0 & C \end{bmatrix} \).
**Theorem 4.2.** Assume that \((S(A) \cap S(B^*)) \cup S(A^*) = \emptyset\). If \(A\) is isoloid and obeys Weyl’s theorem, then Weyl’s theorem holds for \([\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}]\) for every \(C \in \mathcal{B}(Y,X)\).

**Proof.** It follows from Proposition 4.1 that \(\sigma(MC) \setminus \sigma_w(MC) = \sigma(MC) \setminus \sigma_b(MC) \subseteq \pi_{00}(MC)\). Let \(\lambda \in \pi_{00}(MC)\), since \(\sigma(MC) = \sigma(A) \cup \sigma(B)\). Then \(\lambda \in \text{iso}(\sigma(A) \cup \sigma(B))\). Now the remaining part of the proof is same as the proof of [10, Theorem 2.4]. □

The condition \(A\) is isoloid is crucial in Theorem 4.2 as showing by the following example:

**Example 4.3.** Let \(A, B\) and \(C\) on \(l_2\) defined by:

\[
A(x_1, x_2, x_3, \ldots) = \left(0, x_1, 0, \frac{1}{2}x_2, 0, \frac{1}{3}x_3, \ldots, \right),
\]

\[
B(x_1, x_2, x_3, \ldots) = (0, x_2, 0, x_4, 0, x_6, \ldots),
\]

\[
C(x_1, x_2, x_3, \ldots) = (0, 0, x_2, 0, x_3, 0, x_4, \ldots).
\]

Then \(\sigma(A) = \sigma_w(A) = \{0\}, \pi_{00}(A) = \emptyset\) and \(\sigma(B) = \sigma_w(B) = \{0, 1\}, \pi_{00}(B) = \emptyset\). Since \(A, A^*, B\) and \(B^*\) satisfy the SVEP then it follows from [7,9,13] that

\[
\sigma\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \sigma\left(\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\right) = \sigma_w\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \sigma_w\left(\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\right) = \{0, 1\},
\]

and \(\pi_{00}\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \emptyset \neq \{0\} = \pi_{00}\left(\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\right)\). Thus \(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\) satisfies Weyl’s theorem but fails for \(\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\). Here \(A\) is not isoloid.

The condition that \(A\) satisfies Weyl’s theorem is also essential. To see this let

\[
A(x_1, x_2, x_3, \ldots) = \left(0, 0, 0, \frac{1}{2}x_2, 0, \frac{1}{3}x_3, 0, \ldots, \right),
\]

\[
B(x_1, x_2, x_3, \ldots) = (0, x_2, 0, x_4, 0, x_6, \ldots),
\]

\[
C(x_1, x_2, x_3, \ldots) = (x_1, 0, x_2, 0, x_3, 0, x_4, \ldots).
\]

Then \((S(A) \cap S(B^*)) \cup S(A^*) = \emptyset\) and \(\sigma(A) = \sigma_w(A) = \{0\} = \pi_{00}(A)\) and \(\sigma(B) = \sigma_w(B) = \{0, 1\}, \pi_{00}(B) = \emptyset\). So \(A\) does not satisfy Weyl’s theorem but is isoloid. Since

\[
\sigma\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \sigma\left(\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\right) = \sigma_w\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \sigma_w\left(\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\right) = \{0, 1\},
\]

and \(\pi_{00}\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \emptyset \neq \{0\} = \pi_{00}\left(\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\right)\). Thus \(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\) satisfies Weyl’s theorem but fails for \(\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\).

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