JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **197,** 227–248 (1996)
ARTICLE NO. 0017

Solutions of the Strong Hamburger Moment Problem

Olav Njåstad*

Department of Mathematical Sciences, University of Trondheim—NTH,
N-7034 Trondheim, Norway

The strong Hamburger moment problem for a bi-infinite sequence $\{c_n: n = 0, \pm 1, \pm 2, \dots\}$ can be described as follows: (1) Find conditions for the existence of a (positive) measure μ on $(-\infty, \infty)$ such that $c_n = \int_{-\infty}^{\infty} t^n \, d\mu(t)$ for all n. (2) When there is a solution, find conditions for uniqueness of the solution. (3) When there is more than one solution, describe the family of all solutions. In this paper a theory concerning question (3) is developed. In particular, an analog to the Nevanlinna parametrization describing the solutions of the classical Hamburger moment problem is given. © 1996 Academic Press, Inc.

1. INTRODUCTION

The classical Hamburger moment problem can be defined as follows: A sequence $\{c_n: n = 0, 1, 2, \dots\}$ of real numbers is given. (1) Find conditions for there to exist a (positive) measure μ on $(-\infty, \infty)$ such that $c_n = \int_{-\infty}^{\infty} t^n \, d\mu(t)$ for $n = 0, 1, 2, \dots$. (2) When there is a solution, i.e., when at least one such measure μ exists, find conditions for uniqueness of the solution. (3) When there is more than one solution, describe the family of all solutions. The problem is called determinate when there exists exactly one solution, indeterminate when there exists more than one solution. The problem was first discussed by Stieltjes [28] for the case that μ has support in $[0, \infty)$ (the classical Stieltjes moment problem), and then by Hamburger [9] for the general case that the support of μ is only required to be contained in $(-\infty, \infty)$. This initial work was followed by an extensive development of a theory of moment problems, where the connection with the theory of orthogonal polynomials plays a central role. We refer the reader to the papers and monographs [1, 6, 9, 13, 17, 19, 20, 26–30].

* Research supported by the Norwegian Research Council.

0022-247X/96 $12.00

The *strong* Hamburger moment problem (and strong Stieltjes moment problem) can be formulated in the same way as the classical problem, except that here bi-infinite sequences $\{c_n: n = 0, \pm 1, \pm 2, \dots\}$ are involved. This problem was introduced by Jones and Thron about 1980 [14–16] (and for the Stieltjes case by Jones *et al.* [17]) A theory of these problems and their connection with orthogonal Laurent polynomials was developed in the ensuing years as far as questions (1) and (2) are concerned. See [7, 10–12, 23–25]. Also in [2] the strong moment problem was briefly discussed. In this paper we continue these investigations to develop a theory concerning question (3), and present an analog of the Nevanlinna parametrization in the classical case.

The classical and strong moment problems are special cases of a more general theory, where moments corresponding to an arbitrary countable sequence $\{\alpha_n\}$ of points are involved (in the classical and strong cases the points are $\{\infty\}$ repeated and $\{\infty, 0\}$ cyclically repeated, respectively), and where orthogonal rational functions play the role of orthogonal polynomials and Laurent polynomials. For an introduction to orthogonal rational functions and discussions of question (1) (and question (2) in the case of a finite number of points cyclically repeated), see [3–5, 8, 21, 22].

2. PRELIMINARIES

The basic theory of orthogonal Laurent polynomials and strong moment problems including the results sketched in this section can be found in [7, 10–12, 14–17, 23–25].

For any pair (p, q) of integers with $p \leq q$, let $\Lambda_{p,q}$ denote the complex linear space spanned by the functions z^j, $j = p, \dots, q$. We shall write $\Lambda_{2m} = \Lambda_{-m,m}$ and $\Lambda_{2m+1} = \Lambda_{-(m+1),m}$ for $m = 0, 1, 2, \dots$, and $\Lambda = \bigcup_{n=0}^{\infty} \Lambda_n$. An element of Λ is called a *Laurent polynomial.*

Let M be a linear functional on Λ, and assume that M is positive on $(-\infty, \infty)$ (i.e., $M[L] > 0$ for all $L \in \Lambda$ where $L(z) \not\equiv 0$, $L(z) \geq 0$ when $z \in (-\infty, \infty)$). The moments c_n are defined for all $n \in \mathbb{Z}$ by $c_n = M[z^n]$. The *strong Hamburger moment problem* (SHMP) consists of finding all positive Borel measures μ on $(-\infty, \infty)$ such that $M[L] = \int_{-\infty}^{\infty} L(\theta)\, d\mu(\theta)$ for all $L \in \Lambda$, or equivalently such that $M[z^n] = \int_{-\infty}^{\infty} \theta^n\, d\mu(\theta)$ for all $n \in \mathbb{Z}$. A necessary and sufficient condition for there to exist at least one such measure is that M is positive on $(-\infty, \infty)$ (as defined above.) The problem is called *determinate* if there is only one solution, *indeterminate* otherwise.

An inner product $\langle\,,\rangle$ is defined on $\Lambda_{\mathbb{R}} \times \Lambda_{\mathbb{R}}$ by

$$\langle P, Q \rangle = M[P(z) \cdot Q(z)]. \tag{2.1}$$

(Here $\Lambda_{\mathbb{R}}$ denotes the real space spanned by z^j, $j = 0, \pm 1, \pm 2, \dots$. By orthonormalization of the base $\{1, z^{-1}, z, z^{-2}, z^2, \dots, z^{-n}, z^n, \dots\}$, orthonormal Laurent polynomials $\{\varphi_n\}$ are obtained. They have the form

$$\varphi_{2m}(z) = \frac{q_{2m,-m}}{z^m} + \cdots + q_{2m,m} z^m, \qquad q_{2m,m} > 0 \tag{2.2}$$

$$\varphi_{2m+1}(z) = \frac{q_{2m+1,-(m+1)}}{z^{m+1}} + \cdots + q_{2m+1,m} z^m,$$

$$q_{2m+1,-(m+1)} > 0 \tag{2.3}$$

for $m = 0, 1, 2, \dots$. *We shall in the following assume that M gives rise to a regular system*, *which means that $q_{2m,-m} \neq 0$, $q_{2m+1,m} \neq 0$ for all m.*

The associated orthogonal Laurent polynomials $\{\psi_n\}$ are defined by

$$\psi_n(z) = M\left[\frac{\varphi_n(\theta) - \varphi_n(z)}{\theta - z}\right]. \tag{2.4}$$

(The functional is applied to its argument as a function of θ.) We note that $\psi_0 \equiv 0$, $\psi_{2m} \in \Lambda_{-m,m-1}$, $\psi_{2m+1} \in \Lambda_{-(m+1),m-1}$. We further define quasi-orthogonal Laurent polynomials $\varphi_n(z,\tau)$ and their associated Laurent polynomials $\psi_n(z,\tau)$ of order n ($n = 2m$ and $n = 2m+1$) by

$$\varphi_{2m}(z,\tau) = \varphi_{2m}(z) - \tau z \varphi_{2m-1}(z) \tag{2.5}$$

$$\varphi_{2m+1}(z,\tau) = \varphi_{2m+1}(z) - \frac{\tau}{z} \varphi_{2m}(z) \tag{2.6}$$

$$\psi_{2m}(z,\tau) = \psi_{2m}(z) - \tau z \psi_{2m-1}(z) \tag{2.7}$$

$$\psi_{2m+1}(z,\tau) = \psi_{2m+1}(z) - \frac{\tau}{z} \psi_{2m}(z) \tag{2.8}$$

Here $\tau \in \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. For $n = 1, 2, \dots$ we may also write

$$\psi_n(z,\tau) = M\left[\frac{\varphi_n(\theta,\tau) - \varphi_n(z,\tau)}{\theta - z}\right]. \tag{2.9}$$

The *quasi-approximants* $R_n(z,\tau)$ are defined by

$$R_n(z,\tau) = \frac{\psi_n(z,\tau)}{\varphi_n(z,\tau)}. \tag{2.10}$$

For $\tau = 0$ they are simply called approximants.

All zeros of $\varphi_n(z,\tau)$ are real and simple, and for $\tau \in \mathbb{R}$ there are n of them (while for $\tau = \infty$ there are $n-1$). Let $\zeta_1^{(n)}(\tau), \dots, \zeta_n^{(n)}(\tau)$ denote these zeros. Then the following quadrature formulas are valid, where $\lambda_1^{(n)}(\tau), \dots, \lambda_n^{(n)}(\tau)$ are positive weights:

$$M[F] = \sum_{k=1}^{n} \lambda_k^{(n)}(\tau) F\big(\zeta_k^{(n)}(\tau)\big) \tag{2.11}$$

valid for $F \in \Lambda_{-2m,2m-2}$ when $n = 2m$, for $F \in \Lambda_{-2m,2m}$ when $n = 2m+1$.

In particular,

$$M[z^p] = \sum_{k=1}^{2m} \lambda_k^{(2m)}(\tau)\big[\zeta_k^{(2m)}(\tau)\big]^p \qquad \text{for } p = -2m, \dots, 2m-2 \tag{2.12}$$

$$M[z^p] = \sum_{k=1}^{2m+1} \lambda_k^{(2m+1)}(\tau)\big[\zeta_k^{(2m+1)}(\tau)\big]^p \qquad \text{for } p = -2m, \dots, 2m. \tag{2.13}$$

The function $f(\theta) = (\varphi_n(\theta,\tau) - \varphi_n(z,\tau))/(\theta - z)$ belongs to $\Lambda_{-2m,2m-2}$ for $n = 2m$, and to $\Lambda_{-2m,2m}$ for $n = 2m+1$. Therefore by the quadrature formulas (taking into account that $\varphi_n(\zeta_k^{(n)}(\tau),\tau) = 0$) we may write

$$R_n(z,\tau) = -\sum_{k=1}^{n} \frac{\lambda_k^{(n)}(\tau)}{\zeta_k^{(n)}(\tau) - z}. \tag{2.14}$$

It follows from (2.12)–(2.14) that

$$R_{2m}(z,\tau) = -\sum_{k=0}^{2m-1} c_{-(k+1)} z^k + O(z^{2m}) \qquad \text{at } 0, \tag{2.15}$$

$$R_{2m}(z,\tau) = \sum_{k=1}^{2m-1} c_{k-1} z^{-k} + O\left(\frac{1}{z^{2m}}\right) \qquad \text{at } \infty, \tag{2.16}$$

$$R_{2m+1}(z,\tau) = -\sum_{k=0}^{2m-1} c_{-(k+1)} z^k + O(z^{2m}) \qquad \text{at } 0, \tag{2.17}$$

$$R_{2m+1}(z,\tau) = \sum_{k=1}^{2m+1} c_{k-1} z^{-k} + O\left(\frac{1}{z^{2m+2}}\right) \qquad \text{at } \infty. \tag{2.18}$$

The following determinant formulas are valid:

$$z\varphi_{2m}(z)\psi_{2m-1}(z) - z\varphi_{2m-1}(z)\psi_{2m}(z) = \frac{q_{2m,-m}}{q_{m-1,-m}} \tag{2.19}$$

$$z\varphi_{2m}(z)\psi_{2m+1}(z) - z\varphi_{2m+1}(z)\psi_{2m}(z) = \frac{q_{2m+1,m}}{q_{2m,m}}. \tag{2.20}$$

The following general Christoffel–Darboux formulas are valid for arbitrary complex coefficients a, b, c, d, and for $z, \zeta \in \mathbb{C} - \{0\}$:

$$\begin{aligned} z&\left[a\psi_{2m-1}(z) + b\varphi_{2m-1}(z)\right] \cdot \left[c\psi_{2m}(\zeta) + d\varphi_{2m}(\zeta)\right] \\ &\quad - \zeta\left[c\psi_{2m-1}(\zeta) + d\varphi_{2m-1}(\zeta)\right] \cdot \left[a\psi_{2m}(z) + b\varphi_{2m}(z)\right] \\ &\quad = \frac{q_{2m,-m}}{q_{2m-1,-m}}\Bigg[(ac - bd) + (z - \zeta)\sum_{j=0}^{2m-1}\big(a\psi_j(z) \\ &\qquad + b\varphi_j(z)\big)\big(c\psi_j(\zeta) + d\varphi_j(\zeta)\big)\Bigg] \end{aligned} \tag{2.21}$$

$$\begin{aligned} z&\left[a\psi_{2m+1}(z) + b\varphi_{2m+1}(z)\right] \cdot \left[c\psi_{2m}(\zeta) + d\varphi_{2m}(\zeta)\right] \\ &\quad - \zeta\left[c\psi_{2m+1}(\zeta) + d\varphi_{2m+1}(\zeta)\right] \cdot \left[a\psi_{2m}(z) + b\varphi_{2m}(z)\right] \\ &\quad = \frac{q_{2m+1,m}}{q_{2m,m}}\Bigg[(ac - bd) + (z - \zeta)\sum_{j=0}^{2m}\big(a\psi_j(z) \\ &\qquad + b\varphi_j(z)\big)\big(c\psi_j(\zeta) + d\varphi_j(\zeta)\big)\Bigg]. \end{aligned} \tag{2.22}$$

Let U denote the open upper half-plane: $U = \{\zeta \in \mathbb{C}: \operatorname{Im} \zeta > 0\}$. For each $z \in U$ and each n the mapping $\tau \to w$ is defined by

$$w = w_n = -R_n(z, \tau). \tag{2.23}$$

This linear fractional transformation maps $\hat{\mathbb{R}}$ onto a circle bounding a disk contained in U. We use the notation $\Delta_n(z)$ for the open disk, $\partial\Delta_n(z)$ for the boundary circle, and $\overline{\Delta}_n(z)$ for the closed disk $\Delta_n(z) \cup \partial\Delta_n(z)$. By solving (2.23) with respect to τ we get

$$\tau = \frac{\varphi_{2m}(z)w_{2m} + \psi_{2m}(z)}{z\left[\varphi_{2m-1}(z)w_{2m} + \psi_{2m-1}(z)\right]}, \qquad n = 2m, \tag{2.24}$$

$$\tau = \frac{z\left[\varphi_{2m+1}(z)w_{2m+1} + \psi_{2m+1}(z)\right]}{\varphi_{2m}(z)w_{2m+1} + \psi_{2m}(z)}, \qquad n = 2m + 1. \tag{2.25}$$

The circle $\partial\Delta_n(z)$ is given by $\operatorname{Im} \tau = 0$, from which follows (by use of (2.15)–(2.22)) that

$$w \in \overline{\Delta}_n(z) \Leftrightarrow \sum_{j=0}^{n-1} |\psi_j(z) + w\varphi_j(z)|^2 \leq \frac{w - \overline{w}}{z - \overline{z}}. \tag{2.26}$$

This shows that

$$\overline{\Delta}_{n+1}(z) \subset \overline{\Delta}_n(z). \tag{2.27}$$

It follows that the intersection $\overline{\Delta}_\infty(z) = \bigcap_{n=1}^{\infty} \overline{\Delta}_n(z)$ is either a single point or a closed disk. We write $\Delta_\infty(z)$ for the interior and $\partial\Delta_\infty(z)$ for the boundary of $\overline{\Delta}_\infty(z)$. The radius $\rho_n(z)$ of $\overline{\Delta}_n(z)$ is given by

$$\rho_n(z) = \left[|z - \bar{z}| \sum_{j=0}^{n-1} |\varphi_j(z)|^2 \right]^{-1} \tag{2.28}$$

and the radius $\rho_\infty(z)$ of $\overline{\Delta}_\infty(z)$ is given by

$$\rho_\infty(z) = \left[|z - \bar{z}| \sum_{j=0}^{\infty} |\varphi_j(z)|^2 \right]^{-1}. \tag{2.29}$$

If $\overline{\Delta}_\infty(z)$ reduces to a single point for some $z \in U$, it reduces to a single point for every $z \in U$ (Theorem of Invariability.) If $\overline{\Delta}_\infty(z)$ is a proper disk, the series $\sum_{j=0}^{\infty} |\varphi_j(z)|^2$ and $\sum_{j=1}^{\infty} |\psi_j(z)|^2$ converge locally uniformly in $\mathbb{C} - \{0\}$. (The uniform convergence is implicitly contained in the Proof of Theorem 3.5 of [24].)

The Stieltjes transform F_μ of a finite measure μ is defined by

$$F_\mu(z) = \int_{-\infty}^{\infty} \frac{d\mu(\theta)}{\theta - z}. \tag{2.30}$$

The quadrature formulas described earlier give rise to discrete measures $\nu^{(n)}(\theta, \tau)$ having masses of magnitude $\lambda_k^{(n)}(\tau)$ at the points $\zeta_k^{(n)}(\tau)$. It follows from (2.12), (2.13) that $\nu^{(2m)}(\theta, \tau)$ solves the truncated moment problem

$$\int_{-\infty}^{\infty} \theta^p \, d\mu(\theta) = c_p \qquad \text{for } p = -2m, \ldots, 2m-2, \tag{2.31}$$

and $\nu^{(2m+1)}(\theta, \tau)$ solves the truncated moment problem

$$\int_{-\infty}^{\infty} \theta^p \, d\mu(\theta) = c_p \qquad \text{for } p = -2m, \ldots, 2m. \tag{2.32}$$

Formula (2.14) shows that

$$F_\mu(z) = -R_n(z, \tau) \qquad \text{when } \mu(\theta) = \nu^{(n)}(\theta, \tau). \tag{2.33}$$

This means that $F_\mu(z) \in \partial\Delta_n(z)$ when $\mu(\theta) = \mu^{(n)}(\theta, \tau)$, and every point in $\partial\Delta_n(z)$ is the value $F_\mu(z)$ of the Stieltjes transform of a measure $\mu(\theta) = \nu^{(n)}(\theta, \tau)$.

We use the following notations for sets of values of Stieltjes transforms for solutions of moment problems:

$$\sum_{q,r}(z) = \Big\{ F_\mu(z) \colon \mu \text{ solution of the truncated moment problem}$$

$$\int_{-\infty}^{\infty} \theta^p \, d\mu(\theta) = c_p \text{ for } p = q, \ldots, r \Big\} \quad (2.34)$$

$$\sum_{\infty}(z) = \Big\{ F_\mu(z) \colon \mu \text{ solution of the moment problem}$$

$$\int_{-\infty}^{\infty} \theta^p \, d\mu(\theta) = c_p \text{ for } p = 0, \pm 1, \pm 2, \ldots \Big\}. \quad (2.35)$$

It follows by compactness arguments (Helly's theorems) that

$$\sum_{\infty}(z) = \bigcap_{-q,r=1}^{\infty} \sum_{q,r}(z). \quad (2.36)$$

From the remarks above it follows that

$$\partial\Delta_{2m}(z) \subset \sum_{-2m,2m-2}(z) \quad \text{and} \quad \partial\Delta_{2m+1}(z) \subset \sum_{-2m,2m}(z).$$

Then also $\overline{\Delta}_{2m}(z) \subset \Sigma_{-2m,2m-2}(z)$, $\overline{\Delta}_{2m+1}(z) \subset \Sigma_{-2m,2m}(z)$ since $\Sigma_{p,q}(z)$ is a convex set. Bessel's inequality for the function $f(\theta) = 1/(\theta - z)$ and the use of (2.26) show that if $w \in \Sigma_{-2m,2m-2}(z)$, respectively, $w \in \Sigma_{-2m,2m}(z)$, then also $w \in \overline{\Delta}_{2m}(z)$, respectively, $w \in \overline{\Delta}_{2m+1}(z)$. Thus

$$\overline{\Delta}_{2m}(z) = \sum_{-2m,2m-2}(z), \qquad \overline{\Delta}_{2m+1}(z) = \sum_{-2m,2m}(z). \quad (2.37)$$

It follows that

$$\overline{\Delta}_{\infty}(z) = \sum_{\infty}(z). \quad (2.38)$$

Thus $\overline{\Delta}_\infty(z)$ consists of exactly the values of Stieltjes transforms of solutions of the moment problem. Since a measure is uniquely determined by its Stieltjes transform, it follows that the moment problem is determinate if and only if $\overline{\Delta}_\infty(z)$ reduces to a single point (for one z, or for all z).

It is seen by the compactness arguments (Helly's theorems) applied above that every subsequence $\{\nu^{(n(k))}(\theta, \tau_{n(k)})\}$ contains a subsequence converging to a measure $\nu(\theta)$ which is a solution of the moment problem.

Solutions that can be obtained in this way shall be called *quasi-natural solutions*. Solutions obtained from the orthogonal Laurent polynomials ($\tau_{n(k)} = 0$ for all k) are the natural solutions.

3. SEPARATE CONVERGENCE AND N-EXTREMAL SOLUTIONS

Let x_0 be an arbitrary, fixed point on $\mathbb{R} - \{0\}$. We define functions $\alpha_n(z), \beta_n(z), \gamma_n(z), \delta_n(z)$ (depending on x_0) by

$$\alpha_n(z) = (z - x_0) \sum_{j=1}^{n-1} \psi_j(x_0) \psi_j(z) \tag{3.1}$$

$$\beta_n(z) = -1 + (z - x_0) \sum_{j=1}^{n-1} \psi_j(x_0) \varphi_j(z) \tag{3.2}$$

$$\gamma_n(z) = 1 + (z - x_0) \sum_{j=1}^{n-1} \varphi_j(x_0) \psi_j(z) \tag{3.3}$$

$$\delta_n(z) = (z - x_0) \sum_{j=0}^{n-1} \varphi_j(x_0) \varphi_j(z). \tag{3.4}$$

Since the coefficients in $\varphi_j(z)$ and $\psi_j(z)$ are real, it follows that $\alpha_n(z), \beta_n(z), \gamma_n(z), \delta_n(z)$ are real for real z.

The definition of $\alpha_n(z)$, $\beta_n(z)$, $\gamma_n(z)$, $\delta_n(z)$ together with the Christoffel–Darboux-type formulas (2.21), (2.22) gives

$$\alpha_{2m}(z) = K_{2m}[z\psi_{2m-1}(z)\psi_{2m}(x_0) - x_0\psi_{2m}(z)\psi_{2m-1}(x_0)] \tag{3.5}$$

$$\alpha_{2m+1}(z) = K_{2m+1}[z\psi_{2m+1}(z)\psi_{2m}(x_0) - x_0\psi_{2m}(z)\psi_{2m+1}(x_0)] \tag{3.6}$$

$$\beta_{2m}(z) = K_{2m}[z\psi_{2m}(x_0)\varphi_{2m-1}(z) - x_0\psi_{2m-1}(x_0)\varphi_{2m}(z)] \tag{3.7}$$

$$\beta_{2m+1}(z) = K_{2m+1}[z\psi_{2m}(x_0)\varphi_{2m+1}(z) - x_0\psi_{2m+1}(x_0)\varphi_{2m}(z)] \tag{3.8}$$

$$\gamma_{2m}(z) = K_{2m}[z\psi_{2m-1}(z)\varphi_{2m}(x_0) - x_0\varphi_{2m-1}(x_0)\psi_{2m}(z)] \tag{3.9}$$

$$\gamma_{2m+1}(z) = K_{2m+1}[z\psi_{2m+1}(z)\varphi_{2m}(x_0) - x_0\varphi_{2m+1}(x_0)\varphi_{2m}(z)] \tag{3.10}$$

$$\delta_{2m}(z) = K_{2m}[z\varphi_{2m-1}(z)\varphi_{2m}(x_0) - x_0\varphi_{2m}(z)\varphi_{2m-1}(x_0)] \tag{3.11}$$

$$\delta_{2m+1}(z) = K_{2m+1}[z\varphi_{2m+1}(z)\varphi_{2m}(x_0) - x_0\varphi_{2m}(z)\varphi_{2m+1}(x_0)], \tag{3.12}$$

where

$$K_{2m} = \frac{q_{2m-1,\,-m}}{q_{2m,\,-m}}, \qquad K_{2m+1} = \frac{q_{2m,\,m}}{q_{2m+1,\,m}}. \tag{3.13}$$

We note that $\beta_{2m}(z), \delta_{2m}(z)$ are quasi-orthogonal Laurent polynomials of order $2m$ and $\alpha_{2m}(z), \gamma_{2m}(z)$ are associated quasi-orthogonal Laurent polynomials of order $2m$, while $z^{-1}\beta_{2m+1}(z), z^{-1}\delta_{2m+1}(z)$ are quasi-orthogonal Laurent polynomials of order $2m+1$ and

$$z^{-1}\alpha_{2m+1}(z), z^{-1}\gamma_{2m+1}(z)$$

are associated quasi-orthogonal Laurent polynomials of order $2m+1$.

By elimination in the formulas (3.5)–(3.12) we can express $\varphi_n(z)$ and $\psi_n(z)$ as follows:

$$\varphi_{2m}(z) = [\psi_{2m}(x_0)\delta_{2m}(z) - \varphi_{2m}(x_0)\beta_{2m}(z)] \tag{3.14}$$

$$\varphi_{2m}(z) = [\psi_{2m}(x_0)\delta_{2m+1}(z) - \varphi_{2m}(x_0)\beta_{2m+1}(z)] \tag{3.15}$$

$$\varphi_{2m+1}(z) = \frac{x_0}{z}[\psi_{2m+1}(x_0)\delta_{2m+1}(z) - \varphi_{2m+1}(x_0)\beta_{2m+1}(z)] \tag{3.16}$$

$$\varphi_{2m-1}(z) = \frac{x_0}{z}[\psi_{2m-1}(x_0)\delta_{2m}(z) - \varphi_{2m-1}(x_0)\beta_{2m}(z)] \tag{3.17}$$

$$\psi_{2m}(z) = [\psi_{2m}(x_0)\gamma_{2m}(z) - \varphi_{2m}(x_0)\alpha_{2m}(z)] \tag{3.18}$$

$$\psi_{2m}(z) = [\psi_{2m}(x_0)\gamma_{2m+1}(z) - \varphi_{2m}(x_0)\alpha_{2m+1}(z)] \tag{3.19}$$

$$\psi_{2m+1}(z) = \frac{x_0}{z}[\psi_{2m+1}(x_0)\gamma_{2m+1}(z) - \varphi_{2m+1}(x_0)\alpha_{2m+1}(z)] \tag{3.20}$$

$$\psi_{2m-1}(z) = \frac{x_0}{z}[\psi_{2m-1}(x_0)\gamma_{2m}(z) - \varphi_{2m-1}(x_0)\alpha_{2m}(z)]. \tag{3.21}$$

It follows that all quasi-orthogonal Laurent polynomials of order $2m$, respectively $2m+1$, are linear combinations of $\beta_{2m}(z), \delta_{2m}(z)$, resp. $z^{-1}\beta_{2m+1}(z), z^{-1}\delta_{2m+1}(z)$.

By substituting from the expressions (3.14)–(3.21) in the determinant formulas (2.19), (2.20) at the point x_0 we get

$$\alpha_{2m}(z)\delta_{2m}(z) - \beta_{2m}(z)\gamma_{2m}(z) = 1 \tag{3.22}$$

$$\alpha_{2m+1}(z)\delta_{2m+1}(z) - \beta_{2m+1}(z)\gamma_{2m+1}(z) = 1. \tag{3.23}$$

It follows from (3.22), (3.23) that for an arbitrary complex constant t, $\alpha_n(z)t - \gamma_n(z)$ and $\beta_n(z)t - \delta_n(z)$ have no common zeros.

Substituting from the formulas (3.14)–(3.21) into the expressions (2.5)–(2.8) for $\varphi_n(z,\tau)$ and $\psi_n(z,\tau)$ we obtain

$$R_{2m}(z,\tau) = \frac{\alpha_{2m}(z)t_{2m}(\tau) - \gamma_{2m}(z)}{\beta_{2m}(z)t_{2m}(\tau) - \delta_{2m}(z)} \tag{3.24}$$

$$R_{2m+1}(z,\tau) = \frac{\alpha_{2m+1}(z)t_{2m+1}(\tau) - \gamma_{2m+1}(z)}{\beta_{2m+1}(z)t_{2m+1}(\tau) - \delta_{2m+1}(z)} \tag{3.25}$$

where

$$t_{2m}(\tau) = \frac{\varphi_{2m}(x_0) - \tau x_0 \varphi_{2m-1}(x_0)}{\psi_{2m}(x_0) - \tau x_0 \psi_{2m-1}(x_0)} \tag{3.26}$$

$$t_{2m+1}(\tau) = \frac{x_0 \varphi_{2m+1}(x_0) - \tau \varphi_{2m}(x_0)}{x_0 \psi_{2m+1}(x_0) - \tau \psi_{2m}(x_0)}. \tag{3.27}$$

We note that the linear fractional transformation $\tau \to t = t_n(\tau)$ maps $\hat{\mathbb{R}}$ bi-uniquely onto $\hat{\mathbb{R}}$.

We set

$$T_n(z,t) = \frac{\alpha_n(z)t - \gamma_n(z)}{\beta_n(z)t - \delta_n(z)}. \tag{3.28}$$

We may then write

$$R_n(z,\tau) = T_n(z,t) \tag{3.29}$$

where t is obtained from τ by the transformations (3.26), (3.27).

We shall denote by $\mu_t^{(n)}(\theta)$ the discrete measure determined by the quadrature formula associated with $\beta_n(z)t - \delta_n(z)$. Then $\mu_t^{(n)}(\theta) = \nu^{(n)}(\theta, \tau)$, where $t = t_n(\tau)$. It follows by (2.14), (2.33), (3.28), and (3.29) that

$$F_\mu(z) = -\frac{\alpha_n(z)t - \gamma_n(z)}{\beta_n(z)t - \delta_n(z)} \qquad \text{when } \mu = \mu_t^{(n)}. \tag{3.30}$$

THEOREM 3.1. *Assume that the moment problem is indeterminate. Then the functions $\alpha_n(z)$, $\beta_n(z)$, $\gamma_n(z)$, $\delta_n(z)$ converge locally uniformly in $\mathbb{C} - \{0\}$ to analytic functions $\alpha(z)$, $\beta(z)$, $\gamma(z)$, $\delta(z)$ given by*

$$\alpha(z) = (z - x_0) \sum_{j=1}^{\infty} \psi_j(x_0)\psi_j(z) \tag{3.31}$$

$$\beta(z) = -1 + (z - x_0) \sum_{j=1}^{\infty} \psi_j(x_0)\varphi_j(z) \tag{3.32}$$

$$\gamma(z) = 1 + (z - x_0) \sum_{j=1}^{\infty} \varphi_j(x_0)\psi_j(z) \tag{3.33}$$

$$\delta(z) = (z - x_0) \sum_{j=0}^{\infty} \varphi_j(x_0)\varphi_j(z). \tag{3.34}$$

The functions $\alpha(z)$, $\beta(z)$, $\gamma(z)$, $\delta(z)$ satisfy the equation

$$\alpha(z)\delta(z) - \beta(z)\gamma(z) = 1. \tag{3.35}$$

For each $t \in \hat{\mathbb{R}}$ *the discrete measures* $\mu_t^{(n)}(\theta)$ *converge to a solution* $\mu_t(\theta)$ *of the SHMP, and*

$$\int_{-\infty}^{\infty} \frac{d\mu_t(\theta)}{\theta - z} = -\frac{\alpha(z)t - \gamma(z)}{\beta(z)t - \delta(z)}. \tag{3.36}$$

Proof. The locally uniform convergence of $\alpha_n(z), \beta_n(z), \gamma_n(z), \delta_n(z)$ and the form of the limit functions $\alpha(z), \beta(z), \gamma(z), \delta(z)$ follow from the locally uniform convergence of the series $\sum_{j=0}^{\infty} |\varphi_j(z)|^2$ and $\sum_{j=1}^{\infty} |\psi_j(z)|^2$ (see Section 2, after formula (2.29)) together with Schwarz' inequality. It follows from (3.30) that $\int_{-\infty}^{\infty} (d\mu_t^{(n)}(\theta)/(\theta - z))$ converge locally uniformly to an analytic function in $\mathbb{C} - R$ (since all the zeros of $\beta_n(z)t - \delta_n(z)$ are real). A standard type of argument (involving Helly's theorems) then shows that $\mu_t^{(n)}(\theta)$ converges to a measure $\mu_t(\theta)$, that $F_{\mu_t}(z) = -(\alpha(z)t - \delta(z))/(\beta(z)t - \delta(z))$, and that $\mu_t(\theta)$ is a solution of the moment problem. Formula (3.35) follows from (3.22), (3.23). ∎

Set $w = -(\alpha(z)t - \gamma(z))/(\beta(z)t - \delta(z))$. It follows by the use of formula (3.35) that

$$\frac{dw}{dt} = \frac{1}{[\beta(z)t - \delta(z)]^2}. \tag{3.37}$$

Hence for real z, $dw/dt > 0$. Thus in this situation w increases along $\mathbb{R}$ as t increases along $\mathbb{R}$, and consequently the mapping $t \to -(\alpha(z)t - \gamma(z))/(\beta(z)t - \delta(t))$ maps U onto U. A continuity argument then shows that for $z \in U$ the mapping $t \to -(\alpha(z)t - \gamma(z))/(\beta(z)t - \delta(z))$ maps $\mathbb{R}$ onto $\partial\Delta_\infty(z)$ and U onto $\Delta_\infty(z)$. See the argument given in the classical situation in [1, p. 98].

THEOREM 3.2. *The mapping* $t \to \mu_t$ *establishes a one-to-one correspondence between* $\hat{\mathbb{R}}$ *and the set of all quasi-natural solutions of the SHMP.*

Proof. Different values of t give different functions $-(\alpha(z)t - \gamma(z))/(\beta(z)t - \delta(z))$, hence the mapping $t \to \mu_t$ is one-to-one from $\hat{\mathbb{R}}$ onto all solutions of the form μ_t. Let ν be a quasi-natural solution. By definition there exists a sequence $\{\nu^{(n(k))}(\theta, \tau_{n(k)})\}$ (cf. Section 2, after formula (2.30)) converging to ν. For every k there is a t_k such that $\nu^{(n(k))}(\theta, \tau_{n(k)}) = \mu_{t_k}^{(n(k))}(\theta)$. Since

$$\int_{-\infty}^{\infty} \frac{d\mu_{t_k}^{(n(k))}(\theta)}{\theta - z} = -\frac{\alpha_{n(k)}(z)t_k - \gamma_{n(k)}(z)}{\beta_{n(k)}(z)t_k - \delta_{n(k)}(z)},$$

since $\alpha_{n(k)}(z), \beta_{n(k)}(z), \gamma_{n(k)}(z), \delta_{n(k)}(z)$ converge to $\alpha(z), \beta(z), \gamma(z), \delta(z)$, and since $\int_{-\infty}^{\infty} (d\mu_{t_k}^{(n(k))}(\theta)/(\theta - z))$ converges, it follows that $\{t_k\}$ converges to a value $t \in \hat{\mathbb{R}}$. Thus $F_\nu(z) = -(\alpha(z)t - \gamma(z))/(\beta(z)t - \delta(z))$, hence $\nu = \mu_t$. ∎

It follows from the way the quasi-natural solutions μ_t are obtained that $F_{\mu_t}(z) \in \delta\Delta_\infty(z)$ for all $z \in U$. A solution μ which has this property $F_\mu(z) \in \partial\Delta_\infty(z)$ for all $z \in U$ shall be called a *Nevanlinna extremal*, or *N-extremal* solution, as in the classical case. Thus all quasi-natural solutions are N-extremal solutions. We shall later show that they are the only N-extremal solutions. (In fact, all other solutions μ have Stieltjes transforms F_μ with values $F_\mu(z)$ in the open disk $\Delta_\infty(z)$ for *all* $z \in U$.)

We close this section with a result concerning separate convergence of subsequences of the orthogonal Laurent polynomials $\varphi_n(z)$ and their associated Laurent polynomials $\psi_n(z)$, and of the discrete measures determined by $\varphi_n(z)$.

THEOREM 3.3. *Assume that for some $x_0 \in \mathbb{R} - \{0\}$ a subsequence $\varphi_{n(k)}(x_0)/\psi_{n(k)}(x_0)$ converges to a value $t_0 \in \hat{\mathbb{R}} - \{0\}$. Then the Laurent polynomials $\{\varphi_{n(k)}(z)/\psi_{n(k)}(x_0)\}$ and $\{\psi_{n(k)}(z)/\psi_{n(k)}(x_0)\}$ converge locally uniformly to analytic functions $\varphi(z), \psi(z)$ respectively in $\mathbb{C} - \{0\}$, and the measures $\nu^{(n(k))}(\theta, 0)$ converge to the solution μ_{t_0}.*

Proof. Let $\alpha_n(z), \beta_n(z), \gamma_n(z), \delta_n(z)$ be defined in terms of the point x_0 in the assumption. It follows from (3.26), (3.27) that $t_{n(k)}(0) = \varphi_{n(k)}(x_0)/\psi_{n(k)}(x_0)$, hence by assumption $t_{n(k)}(0)$ converges to t_0. It follows that $\alpha_{n(k)}(z)t_{n(k)}(0) - \gamma_{n(k)}(z)$ converges to $\alpha(z)t_0 - \gamma(z)$ and $\beta_{n(k)}(z)t_{n(k)}(0) - \delta_{n(k)}(z)$ converges to $\beta(z)t_0 - \delta(z)$. From the formulas (3.14), (3.16), (3.18), and (3.20) we then conclude that $\{\varphi_{n(k)}(z)/\psi_{n(k)}(x_0)\}$ and $\{\psi_{n(k)}(z)/\psi_{n(k)}(x_0)\}$ converge (locally uniformly) in $\mathbb{C} - \{0\}$. Furthermore, $\nu^{(n(k))}(\theta, 0) = \mu^{(n(k))}_{t_{n(k)}(0)}$, hence $\{\nu^{(n(k))}(\theta, 0)\}$ converge to μ_{t_0}.

4. NEVANLINNA PARAMETRIZATION

In the following let $\mathscr{N}$ denote the class of Nevanlinna functions, i.e., analytic functions in U mapping U into $\overline{U} - \{\infty\}$ (the closed finite upper half-plane), and let $\mathscr{N}^*$ denote the extended class of Nevanlinna functions: $\mathscr{N}^* = \mathscr{N} \cup \{\infty\}$. We recall that the Stieltjes transform $F_\mu(z) = \int_{-\infty}^{\infty} (d\mu(\theta)/(\theta - z))$ of a finite positive measure belongs to $\mathscr{N}$. Note that a function φ in $\mathscr{N}$ either maps U into U, or $\varphi(z) \equiv r$, where r is a real constant. Thus $\mathscr{N}^*$ consists of the analytic functions mapping U into U and the constants in $\hat{\mathbb{R}}$.

In the whole of this section we assume that the SHMP is indeterminate.

PROPOSITION 4.1. *Let μ be an arbitrary solution of the moment problem. Then there exists a $\varphi \in \mathscr{N}^*$ such that*

$$\int_{-\infty}^{\infty} \frac{d\mu(\theta)}{\theta - z} = -\frac{\alpha(z)\varphi(z) - \gamma(z)}{\beta(z)\varphi(z) - \delta(z)}. \tag{4.1}$$

Proof. Define the function φ by the formula

$$\varphi(z) = \frac{F_\mu(z)\delta(z) + \gamma(z)}{F_\mu(z)\beta(z) + \alpha(z)}. \tag{4.2}$$

This function φ satisfies (4.1). We shall show that $\varphi \in \mathscr{N}^*$.

Assume first that $F_\mu(z) = -(\alpha(z)/\beta(z))$. Then by (4.2), $\varphi(z) \equiv \infty$, which is consistent with (4.1).

Next assume that $F_\mu(z) \not\equiv -(\alpha(z)/\beta(z))$. Then φ as defined by (4.2) is analytic in U, except possibly for poles. We know that $F_\mu(z) \in \overline{\Delta}_\infty(z)$. We also know that $t \in \overline{U}$ if and only if $-(\alpha(z)t - \gamma(z))/(\beta(z)t - \delta(z)) \in \overline{\Delta}_\infty(z)$. Let $z \in U$. For the value $t = \varphi(z)$ we have $-(\alpha(z)t - \gamma(z))/(\beta(z)t - \delta(z)) = F_\mu(z) \in \overline{\Delta}_\infty(z)$, hence $t = \varphi(z) \in \overline{U}$. It follows that $\varphi(z) \in \overline{U}$ for $z \in U$. This mapping property excludes the possibility of poles in U. Thus φ is an analytic function mapping U into $\overline{U}$. ∎

We recall the formulas

$$\beta_{2m}(z) = -1 + (z - x_0) \sum_{k=1}^{2m-1} \psi_k(x_0)\varphi_k(z) \tag{4.3}$$

$$\beta_{2m+1}(z) = -1 + (z - x_0) \sum_{k=1}^{2m} \psi_k(x_0)\varphi_k(z). \tag{4.4}$$

We may write

$$\beta_{2m}(z) = \frac{b_{2m,-m}}{z^m} + \cdots + b_{2m,m} z^m \tag{4.5}$$

$$\beta_{2m+1}(z) = \frac{b_{2m+1,-(m+1)}}{z^{m+1}} + \cdots + b_{2m+1,m} z^m. \tag{4.6}$$

We shall call $\beta_n(z)$ *regular* if $b_{2m,-m} \neq 0$ and $b_{2m,m} \neq 0$ for $n = 2m$, if $b_{2m+1,-(m+1)} \neq 0$ and $b_{2m+1,m} \neq 0$ for $n = 2m + 1$.

PROPOSITION 4.2. *For each m, either $\beta_{2m}(z)$ or $\beta_{2m+1}(z)$ is regular.*

Proof. It follows from (4.3) and (4.5) that if $\psi_{2m-1}(x_0) \neq 0$, then $b_{2m,-m} \neq 0$ and $b_{2m,m} \neq 0$. Similarly, it follows from (4.4) and (4.6) that if $\psi_{2m}(x_0) \neq 0$, then $b_{2m+1,-(m+1)} \neq 0$ and $b_{2m+1,m} \neq 0$. (Recall that we have assumed that the orthogonal system $\{\varphi_n\}$ is regular.) Thus the coefficients $q_{2m,m}$, $q_{2m+1,-(m+1)}$, $q_{2m,-m}$, $q_{2m+1,m}$ are all different from zero. See (2.2), (2.3) and the following remark. Since $\psi_n(x_0)$ and $\psi_{n+1}(x_0)$ cannot both be zero (cf., e.g., (2.19), (2.20)) we conclude that $b_{2m,-m} \neq 0$, $b_{2m,m} \neq 0$, or $b_{2m+1,-(m+1)} \neq 0$, $b_{2m+1,m} \neq 0$. ∎

Let φ be an arbitrary function in $\mathscr{N}^*$. For every natural number n we define

$$w_n(z) = -\frac{\alpha_n(z)\varphi(z) - \gamma_n(z)}{\beta_n(z)\varphi(z) - \delta_n(z)}. \tag{4.7}$$

First let $\varphi(z) \not\equiv \infty$. Then for $z \in U$ we have $t = \varphi(z) \in \overline{U}$, hence $w_n(z) \in \overline{\Delta}_n(z) \subset \overline{U}$. It follows that w_n is analytic and maps U into $\overline{U}$. Next let $\varphi(z) \equiv \infty$. Then $w_n(z) = -(\alpha_n(z)/\beta_n(z)) \in U$. Thus in all cases, $w_n \in \mathscr{N}$.

An arbitrary function φ in $\mathscr{N}$ can be written in the form

$$\varphi(z) = Az + B + \int_{-\infty}^{\infty} \frac{1 + uz}{u - z}\, d\nu(u), \tag{4.8}$$

where $A \geq 0$, $B \in R$, ν is a finite positive measure. (See, e.g., [1, p. 92; 27, p. 23].) It follows from this and Julia–Carathéodory's lemma that $\varphi(z)$ can be written in the form

$$\varphi(z) = Cz + \frac{D}{z} + \Phi(z) \tag{4.9}$$

where $C \geq 0$, $D \leq 0$, and Φ is a bounded function in $\mathscr{N}$.

By taking into account (2.22), (2.23) we find that

$$w_n(z) + \frac{\alpha_n(z)}{\beta_n(z)} = -\frac{1}{\beta_n(z)[\beta_n(z)\varphi(z) - \delta_n(z)]}. \tag{4.10}$$

LEMMA 4.3. *Assume that* $C \neq 0$, $D \neq 0$ *in* (4.9).

A. *If* $\beta_{2m}(z)$ *is regular, then*

$$w_{2m}(z) + \frac{\alpha_{2m}(z)}{\beta_{2m}(z)} = O\left(\frac{1}{z^{2m+1}}\right) \qquad \text{for } z \to \infty \tag{4.11}$$

$$w_{2m}(z) + \frac{\alpha_{2m}(z)}{\beta_{2m}(z)} = O(z^{2m+1}) \qquad \text{for } z \to 0. \tag{4.12}$$

B. *If $\beta_{2m+1}(z)$ is regular, then*

$$w_{2m+1}(z) + \frac{\alpha_{2m+1}(z)}{\beta_{2m+1}(z)} = O(1/z^{2m+3}) \qquad \text{for } z \to \infty \qquad (4.13)$$

$$w_{2m+1}(z) + \frac{\alpha_{2m+1}(z)}{\beta_{2m+1}(z)} = O(z^{2m+1}) \qquad \text{for } z \to 0. \qquad (4.14)$$

Proof. Under the stated assumptions,

$$\left(\beta_{2m}(z)\left[\beta_{2m}(z)\varphi(z) + \delta_{2m}(z)\right]\right)^{-1} = \frac{1}{z^{2m}z^{2m}zO(1)} \qquad (4.15)$$

$$\left(\beta_{2m}(z)\left[\beta_{2m}(z)\varphi(z) + \delta_{2m}(z)\right]\right)^{-1} = \frac{z^m}{z^{-m}z^{-1}O(1)} \qquad (4.16)$$

$$\left(\beta_{2m+1}(z)\left[\beta_{2m+1}(z)\varphi(z) + \delta_{2m+1}(z)\right]\right)^{-1} = \frac{1}{z^{m+1}z^{m+1}zO(1)} \qquad (4.17)$$

$$\left(\beta_{2m+1}(z)\left[\beta_{2m+1}(z)\varphi(z) + \delta_{2m+1}(z)\right]\right)^{-1} = \frac{z^m}{z^{-(m+1)}O(1)}. \qquad (4.18)$$

The results then follow by (4.10). ▮

LEMMA 4.4. *The following formulas hold*:

$$-\frac{\alpha_{2m}(z)}{\beta_{2m}(z)} = -\sum_{k=1}^{2m-1} c_{k-1}z^{-k} + O(z^{-2m}) \qquad \text{for} \quad z \to \infty \quad (4.19)$$

$$-\frac{\alpha_{2m}(z)}{\beta_{2m}(z)} = \sum_{k=0}^{2m-1} c_{-(k+1)}z^{k} + O(z^{2m}) \qquad \text{for} \quad z \to 0 \quad (4.20)$$

$$-\frac{\alpha_{2m+1}(z)}{\beta_{2m+1}(z)} = -\sum_{k=1}^{2m+1} c_{k-1}z^{-k} + O(z^{-[2m+2]}) \qquad \text{for} \quad z \to \infty \quad (4.21)$$

$$-\frac{\alpha_{2m+1}(z)}{\beta_{2m+1}(z)} = \sum_{k=1}^{2m-1} c_{-(k+1)}z^{k} + O(z^{2m}) \qquad \text{for} \quad z \to 0. \quad (4.22)$$

Proof. The functions $\alpha_n(z)/\beta_n(z)$ are quasi-approximants, so the results follow from (2.15)–(2.18). ▮

PROPOSITION 4.5. *Assume that* $C \neq 0$, $D \neq 0$ *in* (4.9).

A. *If* $\beta_{2m}(z)$ *is regular, then*

$$w_{2m}(z) + \sum_{k=1}^{2m-1} c_{k-1} z^{-k} = O(z^{-2m}) \qquad \text{for} \quad z \to \infty \tag{4.23}$$

$$w_{2m}(z) - \sum_{k=0}^{2m-1} c_{-(k+1)} z^{k} = O(z^{2m}) \qquad \text{for} \quad z \to 0 \tag{4.24}$$

B. *If* $\beta_{2m+1}(z)$ *is regular, then*

$$w_{2m+1}(z) + \sum_{k=1}^{2m+1} c_{k-1} z^{-k} = O(z^{-[2m+2]}) \qquad \text{for} \quad z \to \infty \tag{4.25}$$

$$w_{2m+1}(z) - \sum_{k=0}^{2m-1} c_{-(k+1)} z^{k} = O(z^{2m}) \qquad \text{for} \quad z \to 0. \tag{4.26}$$

Proof. The results follow by combining Lemmas 4.3 and 4.4. ∎

PROPOSITION 4.6. *Assume that* $C \neq 0$, $D \neq 0$ *in* (4.9).

A. *If* $\beta_{2m}(z)$ *is regular, then there exists a positive measure* σ_{2m} *such that*

$$w_{2m}(z) = \int_{-\infty}^{\infty} \frac{d\sigma_{2m}(\theta)}{\theta - z} \tag{4.27}$$

and

$$c_k = \int_{-\infty}^{\infty} \theta^k \, d\sigma_{2m}(\theta), \qquad k = -2m, \dots, 2m-2. \tag{4.28}$$

B. *If* $\beta_{2m+1}(z)$ *is regular, then there exists a positive measure* σ_{2m+1} *such that*

$$w_{2m+1}(z) = \int_{-\infty}^{\infty} \frac{d\sigma_{2m+1}(\theta)}{\theta - z} \tag{4.29}$$

and

$$c_k = \int_{-\infty}^{\infty} \theta^k \, d\sigma_{2m+1}(\theta), \qquad k = -2m, \dots, 2m. \tag{4.30}$$

Proof. It follows from (4.23), (4.25) that

$$\sup_{y \geq 1} |y w_n(iy)| < \infty \tag{4.31}$$

($z = x + iy$) when $\beta_n(z)$ is regular. Since $w_n(z)$ belongs to $\mathscr{N}$ there exists a measure σ_n such that

$$w_n(z) = \int_{-\infty}^{\infty} \frac{d\sigma_n(\theta)}{\theta - z}. \tag{4.32}$$

(see, e.g., [1, p. 93; 27, pp. 24–25].) From the asymptotic expansions

$$\int_{-\infty}^{\infty} \frac{d\sigma_n(\theta)}{\theta - z} \sim \sum_{k=0}^{\infty} z^k \int_{-\infty}^{\infty} \frac{d\sigma_n(\theta)}{\theta^{k+1}} \tag{4.33}$$

$$\int_{-\infty}^{\infty} \frac{d\sigma_n(\theta)}{\theta - z} \sim -\sum_{k=1}^{\infty} z^{-k} \int_{-\infty}^{\infty} \theta^{k-1}\, d\sigma_n(\theta) \tag{4.34}$$

together with (4.23)–(4.24), respectively (4.25)–(4.26), we conclude that (4.28), respectively (4.29), holds under the conditions stated. ∎

PROPOSITION 4.7. *Assume that* $C \neq 0$, $D \neq 0$ *in* (4.9). *Then there exists a measure* μ *which solves the SHMP*

$$c_k = \int_{-\infty}^{\infty} \theta^k\, d\mu(\theta), \qquad k = 0, \pm 1, \pm 2, \dots, \tag{4.35}$$

and such that

$$\int_{-\infty}^{\infty} \frac{d\mu(\theta)}{\theta - z} = -\frac{\alpha(z)\varphi(z) - \delta(z)}{\beta(z)\varphi(z) - \delta(z)}. \tag{4.36}$$

Proof. For each n such that $\beta_n(z)$ is regular there exists by Proposition 4.6 a measure σ_n which solves the truncated moment problem (4.28), respectively (4.30), and such that

$$\int_{-\infty}^{\infty} \frac{d\sigma_n(\theta)}{\theta - z} = -\frac{\alpha_n(z)\varphi(z) - \gamma_n(z)}{\beta_n(z)\varphi(z) - \delta_n(z)}. \tag{4.37}$$

(cf. (4.7), (4.27), and (4.19).) It follows from Proposition 4.2 that there are infinitely many indices such that $\beta_n(z)$ is regular. By using Helly's theorems and the convergence of $\alpha_n(z)$, $\beta_n(z)$, $\gamma_n(z)$, $\delta_n(z)$ we then establish the existence of a measure μ satisfying (4.35) and (4.36). ∎

THEOREM 4.8. *Assume that the functional* M *gives rise to a regular system. Then there exists a one-to-one correspondence between the functions* φ *in the extended Nevanlinna class* $\mathscr{N}^*$ *and the measures* μ *in the class* $\mathscr{M}$ *of solutions of the SHMP. The correspondence is given by*

$$\int_{-\infty}^{\infty} \frac{d\mu(\theta)}{\theta - z} = -\frac{\alpha(z)\varphi(z) - \gamma(z)}{\beta(z)\varphi(z) - \delta(z)}. \tag{4.38}$$

Proof. According to Proposition 4.1 there exists for every μ in $\mathscr{M}$ a function φ in $\mathscr{N}^*$ satisfying (4.38). Now let φ be a function in $\mathscr{N}$. It follows from Proposition 4.7 that there exists a solution μ of the moment problem such that (4.38) is satisfied if $C \neq 0$, $D \neq 0$ in (4.9). If this is not the case, we consider a Nevanlinna function

$$\varphi_{A,B}(z) = Az + \frac{B}{z} + \varphi(z) \tag{4.39}$$

of the desired form. Then according to Proposition 4.7 a solution $\mu_{A,B}$ of the moment problem exists such that

$$\int_{-\infty}^{\infty} \frac{d\mu_{A,B}(z)}{\theta - z} = -\frac{\alpha(z)\varphi_{A,B}(z) - \gamma(z)}{\beta(z)\varphi_{A,B}(z) - \delta(z)}. \tag{4.40}$$

Letting $A, B \to 0$, hence $\varphi_{A,B}(z) \to \varphi(z)$, and again using Helly's theorems, we obtain a solution μ of the SHMP such that (4.38) holds. Finally, if $\varphi(z) \equiv \infty$, then $w_n(z) = -(\alpha_n(z)/\beta_n(z))$. Then the existence of measures σ_n as in Proposition 4.6 follows directly from Lemma 4.4. Hence the existence of a solution μ with the desired properties follows as in the proof of Proposition 4.7.

That the correspondence is one-to-one follows directly from formulas (4.1), (4.2) together with the fact that a measure μ is determined by its Stieltjes transform $F_\mu(z) = \int_{-\infty}^{\infty}(d\mu(\theta)/(\theta - z))$. ∎

5. CANONICAL SOLUTIONS

A solution μ of the SHMP is called a *canonical solution* if the Nevanlinna function φ in (4.38) is of the form $\varphi(z) = P(z)/Q(z)$, where P and Q are polynomials with real coefficients. (This is analogous to the definition of canonical solution in the classical situation.) Note that all real constants and the constant ∞ are among these functions. The canonical solution is said to be of order r if max(degree P, degree Q) $= r$, where P and Q have no common factors.

A canonical solution of order 0 is then a solution μ where

$$\int_{-\infty}^{\infty} \frac{d\mu(\theta)}{\theta - z} = -\frac{\alpha(z)t - \gamma(z)}{\beta(z)t - \delta(z)}, \qquad t \in \hat{\mathbb{R}}. \tag{5.1}$$

According to Theorem 3.2 these solutions are exactly the quasi-natural solutions. It follows from Theorem 4.8 that these solutions are also exactly the N-extremal solutions.

We note that if P and Q are polynomials, then the functions

$$P(z)\alpha(z) - Q(z)\gamma(z) \qquad \text{and} \qquad P(z)\beta(z) - Q(z)\delta(z)$$

are analytic in $\mathbb{C} - \{0\}$. The zeros of these functions thus form discrete sets in $\mathbb{C} - \{0\}$, with 0 and ∞ as the only possible limit points. We also note that if P and Q have no common zeros, then $\alpha(z)P(z) - \gamma(z)Q(z)$ and $\beta(z)P(z) - \delta(z)Q(z)$ have no common zeros. This follows from the fact that z_0 being a common zero would imply $\alpha(z_0)\delta(z_0) - \beta(z_0)\gamma(z_0) = 0$, which is not the case (cf. (3.35)). Thus the function $-(\alpha(z)P(z) - \gamma(z)Q(z))/(\beta(z)P(z) - \delta(z)Q(z))$ has singularities exactly at the zeros of $\beta(z)P(z) - \delta(z)Q(z)$, and possibly at the origin.

The zeros of $\beta(z)P(z) - \delta(z)Q(z)$ are real and simple since the function $F_\mu(z) = (\alpha(z)P(z) - \gamma(z)Q(z))/(\beta(z)P(z) - \delta(z)Q(z))$ is analytic in U and maps U into $\overline{U}$. Thus all singularities of $F_\mu(z)$ are simple poles on the real axis, except for 0 which is a limit point for the poles if it is a singularity. We note that μ cannot have a mass point at the origin, since the negative moments c_{-n} of μ are assumed to exist.

To obtain a description of the measure μ we shall use the Stieltjes–Perron inversion formula (see, e.g., [1, pp. 124–125]). It follows from this that

$$\frac{\mu(b+) + \mu(b-)}{2} - \frac{\mu(a+) + \mu(a-)}{2} = \lim_{\eta \to 0} \frac{1}{\pi} \int_a^b \operatorname{Im} F_\mu(\xi + i\eta)\, d\xi \tag{5.2}$$

for arbitrary points a, b on $\mathbb{R}$. We note that if $(\mu(x+) + \mu(x-))/2$ is constant on an open interval, then $\mu(x)$ is also constant on that interval. (We have here used $\mu(x)$ for the distribution function which determines the measure μ as well as for the measure itself.)

THEOREM 5.1. *Let μ be a canonical solution of the SHMP, with $\varphi(z) = P(z)/Q(z)$, where P and Q are polynomials with no common factors. Then the spectrum of μ consists of a discrete subset of $\mathbb{R} - \{0\}$, namely the set $\{z_k : k = 1, 2, \ldots\}$ of zeros of $\beta(z)P(z) - \delta(z)Q(z)$, and in addition the origin. The measure μ has a mass of magnitude $\lambda_k = -\rho_k$ at z_k, where ρ_k is the residue of $F_\mu(z)$ at z_k. The origin is a point of continuity.*

Proof. Formula (4.38) may be written as

$$\int_{-\infty}^{\infty} \frac{d\mu(\theta)}{\theta - z} = -\frac{\alpha(z)P(z) - \gamma(z)Q(z)}{\beta(z)P(z) - \delta(z)Q(z)} \tag{5.3}$$

in this case. It is known that the integral to the left is analytic for z not in the spectrum of μ. It then follows from the foregoing discussion that all the zeros of $\beta(z)P(z) - \delta(z)Q(z)$ belong to the spectrum, and the origin if (and only if) it is a limit point for the zeros. We shall show that the spectrum contains no other points.

Let ζ_1 and ζ_2 be two zeros of $\beta(z)P(z) - \delta(z)Q(z)$ such that there are no zeros between, and such that both are positive or both are negative. Assume $\zeta_1 < \zeta_2$, and let $\zeta_1 < a < b < \zeta_2$. Let Γ denote the contour in the complex plane consisting of the line segments Γ_0 from $(b, 0)$ to $(a, 0)$, Γ_a from $(a, 0)$ to $(a, i\zeta)$, Γ_ζ from $(a, i\zeta)$ to $(b, i\zeta)$, and Γ_b from $(b, i\zeta)$ to $(b, 0)$. By Cauchy's integral theorem $\int_\Gamma F_\mu(z)\,dz = 0$. Clearly, $\int_{\Gamma_a} F_\mu\,dz$ and $\int_{\Gamma_b} F\mu(z)\,dz$ tend to 0 when η tends to 0. It follows that

$$\lim_{\eta\to 0}\left[\int_{\Gamma_0} F_\mu(z)\,dz + \int_{\Gamma_\eta} F_\mu(z)\,dz\right] = 0, \tag{5.4}$$

i.e.,

$$\lim_{\eta\to 0}\left[\int_b^a F_\mu(\xi)\,d\xi + \int_a^b F_\mu(\xi + i\eta)\,d\xi\right] = 0, \tag{5.5}$$

hence also

$$\lim_{\eta\to 0}\left[\int_b^a \operatorname{Im} F_\mu(\xi)\,d\xi + \int_a^b \operatorname{Im} F_\mu(\xi + i\eta)\,d\xi\right] = 0. \tag{5.6}$$

Since $F_\mu(z)$ is real for real z, this implies

$$\lim_{\eta\to 0}\int_a^b \operatorname{Im} F_\mu(\xi + i\eta)\,d\xi = 0. \tag{5.7}$$

So from (5.2) we conclude that $\mu(x)$ is constant on (ζ_1, ζ_2). This shows that the spectrum of μ contains no other points than the zeros of $\beta(z)P(z) - \delta(z)Q(z)$, and possibly the origin, and since the zeros are isolated, μ has point masses there. It remains to determine the magnitude of the masses.

Let z_k be one of the zeros of $\beta(z)P(z) - \delta(z)Q(z)$, and let ρ_k denote the residue of $F\mu(z) = -(\alpha(z)P(z) - \gamma(z)Q(z))/(\beta(z)P(z) - \delta(z)Q(z))$ at z_k. Let $a < z_k < b$, let there be no other zeros in (a, b), and let Δ denote the contour consisting of the half circle S_η: $z - z_k = \frac{1}{2}\eta e^{i\varphi}$, $\varphi \in [0, \pi]$, and the line segments Δ_a from $z_k - \frac{1}{2}\eta$ to a; Γ_a from a to $a + i\eta$; Γ_η from $a + i\eta$ to $b + i\eta$; Γ_b from $b + i\eta$ to b^0, and Δ_b from b to $z_k + \frac{1}{2}\eta$. By Cauchy's integral theorem $\int_\Delta F_\mu(z)\,dz = 0$. Clearly $\int_{\Gamma_a} F_\mu(z)\,dz$, $\int_{\Gamma_b} F_\mu(z)\,dz$ tend to zero when η tends to zero. Hence

$$\lim_{\eta\to \infty}\left[\int_{\Delta_a} F_\mu(z)\,dz + \int_{\Delta_b} F_\mu(z)\,dz + \int_{S_\eta} F_\mu(z)\,dz + \int_{\Gamma_\eta} F_\mu(z)\,dz\right] = 0. \tag{5.8}$$

Since $F_\mu(z)$ is real for real z, this implies

$$\lim_{\eta\to 0}\left[\operatorname{Im}\int_{S_\eta} F_\mu(z)\,dz + \int_a^b \operatorname{Im} F_\mu(\xi + i\eta)\,d\xi\right] = 0. \tag{5.9}$$

Since z_k is a simple pole of $F_\mu(z)$, we have

$$\lim_{\eta \to 0} \int_{S_\eta} F_\mu(z)\, dz = \pi i \rho_k, \tag{5.10}$$

hence

$$\lim_{\eta \to 0} \operatorname{Im} \int_{S_\eta} F_\mu(z)\, dz = \pi \rho_k. \tag{5.11}$$

It follows from (5.2), (5.9), (5.10), and the fact that $\mu(x)$ is constant on (a, z_k) and (z_k, b) that

$$\mu(b) - \mu(a) = -\rho_k. \tag{5.12}$$

This shows that $\mu(x)$ has a jump of magnitude $\lambda_k = -\rho_k$ at z_k. Just as in the classical case a Hamburger moment problem is determinate if a solution has bounded support; it can be shown that a SHMP is determinate if a solution has support which is bounded or bounded away from zero. Thus for an indeterminate problem the origin belongs to the support. ∎

REFERENCES

1. N. I. Akhiezer, "The Classical Moment Problem and Some Related Questions in Analysis," Hafner, New York, 1965.
2. C. Berg, J. P. R. Christensen, and P. Ressel, "Harmonic Analysis on Semigroups," Springer-Verlag, Berlin, 1984.
3. A. Bultheel, P. Gonzalez-Vera, E. Hendriksen, and O. Njåstad, The computation of orthogonal rational functions and their interpolating properties, *Numer. Algorithms* **2** (1992), 85–114.
4. A. Bultheel, P. Gonzalez-Vera, E. Hendriksen, and O. Njåstad, Moment problems and orthogonal functions. *J. Comput. Appl. Math.* **48** (1993), 217–228.
5. A. Bultheel, P. Gonzalez-Vera, E. Hendriksen, and O. Njåstad, Orthogonal rational functions with poles on the unit circle, *J. Math. Anal. Appl.* **182** (1994), 221–243.
6. T. S. Chihara, "An Introduction to Orthogonal Polynomials," Mathematics and Its Applications Series, Gordon and Breach, New York, 1978.
7. L. Cochran and S. Clement Cooper, Orthogonal Laurent polynomials on the real line, *in* "Continued Fractions and Orthogonal Functions" (S. Clement Cooper and W. J. Thron, Eds.), 47–101, Dekker, New York, 1994.
8. M. M. Djrbashian, A survey on the theory of orthogonal systems and some open problems, *in* "Orthogonal Polynomials: Theory and Practice" (P. Nevai, Ed.), pp. 135–146, Kluwer, Dordrecht/Boston/London, 1990.
9. H. Hamburger, Über eine Erweiterung des Stieltjesschen Momenten problems, I, II, III, *Math. Annalen* **81** (1920), 235–319; **82** (1921), 120–164, 168–187.
10. E. Hendriksen and H. van Rossum, Orthogonal Laurent polynomials, *Indag. Math.* **89** (1986), 17–36.

11. W. B. Jones, O. Njåstad, and W. J. Thron, Orthogonal Laurent polynomials and the strong Hamburger moment problem, *J. Math. Anal. Appl.* **98** (1984), 528–554.
12. W. B. Jones, O. Njåstad, and W. J. Thron, Continued fractions and strong Hamburger moment problems, *Proc. Lond. Math. Soc. (3)* **47** (1983), 363–384.
13. W. B. Jones, O. Njåstad, and W. J. Thron, Moment theory, orthogonal polynomials, quadrature, and continued fractions associated with the unit circle, *Bull. London Math. Soc.* **21** (1989), 133–152.
14. W. B. Jones and W. J. Thron, Orthogonal Laurent polynomials and Gaussian quadrature, "Quantum Mechanics in Mathematics, Chemistry, and Physics" (K. Gustafson and W. P. Reinhardt, Eds.), pp. 449–455, Plenum, New York, 1981.
15. W. B. Jones and W. J. Thron, Survey of continued fractions methods of solving moment problems and related topics, *in* "Analytic Theory of Continued Fractions" (W. B. Jones, W. J. Thron, H. Waadeland, Eds.), Lecture Notes in Mathematics, Vol. 932, pp. 4–37, Springer-Verlag, Berlin, 1982.
16. W. B. Jones and W. J. Thron, Continued fractions: Analytic theory and applications, *in* "Encyclopedia of Mathematics and Its Applications," Vol. 11, Addison–Wesley, Reading, MA, 1980.
17. W. B. Jones, W. J. Thron and H. Waadeland, A strong Stieltjes moment problem, *Trans. Amer. Math. Soc.* **206** (1980), 503–528.
18. H. J. Landau, The classical moment problem: Hilbertian proofs, *J. Funct. Anal.* **38** (1980), 255–272.
19. H. J. Landau (Ed.), "Moments in Mathematics," Proceedings of Symposia in Applied Mathematics, Vol. 37, Amer. Math. Soc., Providence, RI, 1987.
20. R. Nevanlinna, Über beschränkte analytische Funktionen, *Ann. Acad. Sci. Fenn. A* **18**, No. 5, 1–52.
21. O. Njåstad, An extended Hamburger moment problem, *Proc. Edinburgh Math. Soc.* **28** (1985), 167–183.
22. O. Njåstad, Unique solvability of an extended Hamburger moment problem, *J. Math. Anal. Appl.* **124** (1987), 502–519.
23. O. Njåstad and W. J. Thron, The theory of sequences of orthogonal L-polynomials, *in* "Padé Approximants and Continued Fractions" (H. Waadeland and H. Wallin, Eds.), Det. Kongelige Norske Videnskabers Selskab, Vol. 1, pp. 54–91, 1983.
24. O. Njåstad and W. J. Thron, Unique solvability of the strong Hamburger moment problem, *J. Austral. Math. Soc. Ser. A* **40** (1986), 5–19.
25. A. Sri Ranga, J-fractions and strong moment problems, *in* "Analytic Theory of Continued Fractions" (W. J. Thron, Ed.), Lecture Notes in Mathematics, Vol. 1199, pp. 269–284, Springer-Verlag, Berlin, 1986.
26. M. Riesz, Sur le problème des moments, I, II, III, *Arkiv. Matematik, Astron. Fys.* **16**, No. 12 (1921), 23; **16**, No. 119 (1922), 21; **17**, No. 16 (1923), 52.
27. J. A. Shohat and J. D. Tamarkin, "The Problem of Moments," Mathematical Surveys, Vol. 1, Amer. Math. Soc., Providence, RI, 1943.
28. T. J. Stieltjes, Recherches sur les fractions continues, *Ann. Fac. Sci. Toulouse* **8** (1894), J1–J122; **9** (1894), A1–A47; Oeuveres **2**, 402–566.
29. M. H. Stone, "Linear Transformations in Hilbert Space and Their Applications to Analysis," Amer. Math. Soc. Colloquium Publ., Vol. 15, Amer. Math. Soc., New York, 1932.
30. G. Szegö, "Orthogonal Polynomials," Amer. Math. Soc. Coll. Publ. Vol. 23, 4th ed., Amer. Math. Soc., Providence, RI, 1925.