A Jacobson Radical for Hopf Module Algebras*

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INTRODUCTION

Let $H$ be a bialgebra over the commutative associative ring $K$ with unit. This paper examines the concept of an $H$-radical for (associative) $H$-module algebras (also called algebras over $H$), based on the Amitsur–Kurosh general radical theory (Definitions 2, 3, Propositions 1–5, below). In particular, a Jacobson-type $H$-radical $\mathcal{J}$ is constructed as the upper $H$-radical generated by the left $H$-primitive $H$-module algebras (Definition 3, Theorem 1). Another $H$-radical of interest is $J_H$, which consists of all associative $H$-module algebras whose underlying algebra is in $J$, the ordinary Jacobson radical for associative $K$-algebras (Propositions 2, 3).

The main theorems on $\mathcal{J}$ are in Section 2, where we show that if $H$ is irreducible (also called filtered), and if $H$ is a flat $K$-module, then (Theorem 2) for any $H$-module algebra $A$, $\mathcal{J}(A)$ is equal to the intersection of all left $H$-primitive ideals of $A$; (Theorem 3 and Corollary) $\mathcal{J}$ is a strongly hereditary $H$-radical; (Theorem 4) $\mathcal{J}(A) = J(A \# H) \cap A$, where $A \# H$ is the smash product of $A$ with $H$; (Corollary 1 to Theorem 4) $\mathcal{J}(A)$ is the intersection of all right $H$-primitive ideals of $A$; (Theorem 5) $\mathcal{J}(A)$ contains all the left or right $H$-ideals of $A$ which are in $\mathcal{J}$; (Theorem 6) $\mathcal{J} \subseteq J_H$. An example is then provided showing that it is possible to have $\mathcal{J}(A) \neq J_H(A)$ for a non-Artinian $H$-module algebra $A$, whereas Theorem 7 shows that $\mathcal{J}(A) = J_H(A)$ if $A$ is (left or right) Artinian.

An example motivating this study is the case in which $A$ is an (associative) $K$-algebra and $H$ is the universal enveloping algebra of the Lie algebra of derivations of $A$.

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Throughout this paper $K$ will denote a commutative associative ring with unit. Algebras, bialgebras, and tensor products over base ring $K$ are considered. The reader is referred to [6, p. 53] for the definition of a bialgebra $H$ over $K$, and to [6, p. 153] for the definition of an $H$-module algebra, except that we do not assume that $H$-module algebras are necessarily unital. Reference [6] defines these concepts in the case that $K$ is a field, but the same definitions (as well as that of a coalgebra over $K$) make sense in the general case considered here. For this general approach we will use results from [3, Section 1]. To be explicit, $A$ is an $H$-module algebra if $A$ is a $K$-algebra which is an $H$-module with the measuring condition written out as follows. If $\mu: H \otimes A \to A$ is the measuring of $A$ by $H$ (or action of $H$ on $A$), we will also write $\mu(h \otimes a) = h \cdot a$ so that the measuring condition reads $h \cdot (ab) = \sum (h) (h_{(1)} \cdot a)(h_{(2)} \cdot b)$, for all $a, b \in A$, $h \in H$. For more description of the summation notation in the last statement see [6, p. 10]. It is assumed that $1_H \cdot a = a$ for all $a \in A$, where $1_H$ is the unit of $H$. The measuring is called unital if $A$ has a unit element $1$ and if $h \cdot 1 = \epsilon(h) 1$ for all $h \in H$, where $\epsilon$ is the counit of $H$. Note that $H$-module algebras are the multiplicative objects in the monoidal category of $H$-modules.

Let $\mathcal{H}$ be the category of all associative $H$-module algebras, where $H$ is a given bialgebra over $K$. The objects of $\mathcal{H}$ are all associative $H$-module algebras. The morphisms of $\mathcal{H}$ are those algebra homomorphisms $\varphi: A \to B$, $A, B \in \mathcal{H}$, which are also $H$-module maps. Such a $\varphi$ will be called an $H$-homomorphism. An ideal $I$ of an $H$-module algebra $A$ is called an $H$-ideal if the action of $H$ on $A$ leaves $I$ invariant. An $H$-ideal is the same thing as the kernel of an $H$-homomorphism. In particular, if $I$ is an $H$-ideal of $A$, then $I$ is the kernel of the natural $H$-homomorphism $A \to A/I$, where $A/I$ is an $H$-module algebra via $h \cdot (a + I) = (h \cdot a) + I$ for all $h \in H$, $a \in A$. The sum and intersection of $H$-ideals are $H$ ideals. The image $\varphi(A)$ of an $H$-module algebra $A$ by an $H$-homomorphism $\varphi$ is naturally $H$-isomorphic to $A/I$ for the $H$-ideal $I = \ker \varphi$.

The concept of a module for an $A$ in the category $\mathcal{H}$ is made explicit by means of the following definition.

**Definition 1.** Suppose $A$ is an $H$-module algebra and $M$ is a left $A$-module. Then $M$ is a left $A$, $H$-module provided $M$ is also a unital left $H$-module (where $H$ is thought of as an algebra), and

$$h(am) = \sum (h) (h_{(1)} \cdot a) h_{(2)}(m)$$

for all $h \in H$, $a \in A$, $m \in M$, where $\Delta h = \sum (h) h_{(1)} \otimes h_{(2)}$. If $A$ has a unit,
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then the $A, H$-module $M$ is called unital if $M$ is unital as a left $A$-module.

Note that $A, H$-modules are the multiplicative actions in the monoidal category of $H$-modules. The $A, H$-module $M$ is irreducible if $AM \neq 0$ and $M$ has no proper nonzero $A, H$-submodule (i.e., no $K$-subspace closed under action by $A$ and $H$); in addition, if $A$ has a unit then it is further required that $M$ be unital. An $H$-module algebra $A$ is left $H$-primitive provided $A$ has a left, $A, H$-module $M$ which is faithful as an $A$-module, and irreducible as an $A, H$-module.

Suppose $A$ is an $H$-module algebra. The smash product (or semidirect product) $A \# H$ of $A$ by $H$ is the associative algebra consisting of the elements of $A \otimes H$ ($a \otimes h$ written $a \# h$) with products defined by

$$(a \# g)(b \# h) = \sum_{(o)} a(g_{(1)} \cdot b) \# g_{(2)} h.$$ 

If $A$ has a unit $1_A$ and the measuring of $H$ on $A$ is unital, then $1_A \# 1_H$ is a unit for $A \# H$.

**Lemma 1.** (i) If $A$ is an $H$-module algebra such that either $A$ does not have a unit, or $A$ does have a unit but the measuring is not unital, then one can adjoin a unit to $A$ to obtain an $H$-module algebra $A_* = A + K$ (direct as $K$-spaces) for which the measuring is unital, where the action of $H$ on $A$ is defined by

$$h \cdot (a + k) = h \cdot a + \epsilon(h)k,$$

for all $h \in H$, $a \in A$, $k \in K$. $A$ is then embedded as an $H$-ideal in $A_*$ in the natural fashion. If $M$ is an $A, H$-module, then $M$ is a unital $A_*, H$-module under the action $(a + k)m = am + km$ for all $a \in A$, $k \in K$, $m \in M$.

(ii) If $M$ is an $A, H$-module, then $M$ is an $A \# H$-module under the action $(a \# h)m = ah(m)$ for all $a \in A$, $h \in H$, $m \in M$. If $M$ is an irreducible left $A, H$-module, then $M$ is an irreducible left $A \# H$-module.

(iii) If the measuring of $H$ on $A$ is unital, and if $M$ is an (irreducible) left $A \# H$-module, then $M$ is an (irreducible) left $A, H$-module under the action $am = (a \# 1_H)m$, $h(m) = (1_A \# h)m$ for all $a \in A$, $h \in H$, $m \in M$.

**Proof.** The details of the proof are mostly straight-forward, being based on definitions. However, the last statement in (ii) needs comment. As stated, $A$ need not have a unit. If $A$ does have a unit, then the proof is easy. Assume then that $A$ does not have a unit. As in (i) adjoin a unit to $A$ to get $A_* = A + K$. $M$ is then an irreducible $A_*$, $H$-module and an irreducible $A_* \# H$-module. Since $A$ is a direct summand (as a $K$-space) of $A_*$, $A \# H$ is embedded in $A_* \# H$ in the natural fashion. Let $S = \{m \in M: (A \# H)m = 0\}$. $S$ is an $A_* \# H$-submodule of $M$, hence $S = M$ or $S = 0$. If $S = M$,
then $AM = 0$, contrary to hypothesis. Thus $S = 0$. So for any nonzero $m \in M$, $(A \# H)m = M$. Now suppose $N$ is a nonzero $A \# H$-submodule of $M$. Then $(A \# H)N \subseteq N$ and $(A \# H)N \supseteq (A \# H)n = M$ for any nonzero $n \in N$, thus $N = M$. It has been shown that $M$ is an irreducible $A \# H$-module.

**DEFINITION 2.** A nonempty subset $\mathcal{R}$ of $\mathcal{H}$ is an $H$-radical provided

(a) If $A \in \mathcal{R}$, then $\varphi(A) \in \mathcal{R}$ for every $H$-homomorphism $\varphi$ of $A$.

(b) If $A \in \mathcal{H}$, $A \notin \mathcal{R}$, then there exists a nonzero $H$-homomorphism $\varphi$ of $A$ such that $\varphi(A)$ has no nonzero $H$-ideals in $\mathcal{R}$.

The following notation will be useful. For $\mathcal{X} \subseteq \mathcal{H}$,

$\mathcal{G}(\mathcal{X}) = \{A \in \mathcal{H} : A$ has no nonzero $H$-ideal in $\mathcal{X}\}$,

$\mathcal{R}(\mathcal{X}) = \{A \in \mathcal{H} : A$ has no nonzero $H$-homomorphic image in $\mathcal{X}\}$.

Given an $H$-radical $\mathcal{R}$ and $A \in \mathcal{H}$, $A$ is said to be $\mathcal{R}$-radical provided $A \in \mathcal{R}$ and $A$ is said to be $\mathcal{R}$-semisimple provided $A \in \mathcal{G}(\mathcal{R})$. The $H$-ideal

$\mathcal{R}(A) = \sum \{I : I$ is an $H$-ideal of $A$, and $I \in \mathcal{R}\}$

is called the $\mathcal{R}$-radical of $A$. For each $A \in \mathcal{H}$, $\mathcal{R}(A) \in \mathcal{R}$, $A/\mathcal{R}(A) \in \mathcal{G}(\mathcal{R})$, and $\mathcal{R}(A) = \bigcap \{I : I$ is an $H$-ideal of $A$, and $A/I \in \mathcal{G}(\mathcal{R})\}$.

**PROPOSITION 1.** Suppose $\mathcal{S} \subseteq \mathcal{H}$ satisfies the following condition: $A \in \mathcal{S}$ implies every nonzero $H$-ideal of $A$ has a nonzero $H$-homomorphic image in $\mathcal{S}$. Then (i) $\mathcal{R}(\mathcal{S})$ is an $H$-radical; (ii) $\mathcal{G}(\mathcal{R}(\mathcal{S}))$ is the minimal semisimple class in $\mathcal{H}$ containing $\mathcal{S}$; (iii) if $\mathcal{R}$ is an $H$-radical for which $\mathcal{G}(\mathcal{R}) \supseteq \mathcal{S}$, then $\mathcal{R} \subseteq \mathcal{R}(\mathcal{S})$.

Because of (iii) $\mathcal{R}(\mathcal{S})$ is called the upper $H$-radical generated by $\mathcal{S}$. Generally, the proofs of the propositions in this section are similar to known proofs in general radical theory, or are otherwise straightforward. In particular, the proof of Proposition 1 resembles [4, Lemma 3, p. 6].

We wish to apply Proposition 1 to the class $\mathcal{S}$ of all left $H$-primitive $H$-module algebras. The condition in the hypothesis of Proposition 2 is verified in Theorem 1 below.

**THEOREM 1.** If $A$ is left $H$-primitive and $I$ is a nonzero $H$-ideal of $A$, then $I$ is left $H$-primitive.

**Proof.** Suppose $M$ is an irreducible left $A$, $H$-module which is faithful as an $A$-module. Then $M$ is an $I$, $H$-module which is faithful as an $I$-module. Suppose $N$ is an $I$, $H$-submodule of $M$. Let $C$ denote the $K$-subspace of $A \# H$ generated by $\{x \# h : x \in I, h \in H\}$. Then $C$ is an ideal of $A \# H$. 
Thus \( CN \) is an \( A \# H \)-submodule of \( M \), whereas \( M \) is an irreducible \( A \# H \)-module, using Lemma 1(ii). Hence \( CN = 0 \) or \( CN = M \). If \( CN = M \), then \( N = M \), and the proof is completed. On the other hand, if \( CN = 0 \), let \( S = \{ m \in M, Cm = 0 \} \). Now \( S \supseteq N \) and \( S \) is an \( A \# H \)-submodule of \( M \), so \( S = 0 \) or \( S = M \). If \( S = 0 \), then \( N = 0 \), and the proof is again completed. If \( S = M \), then \( CM = 0 \), so \( IM = 0 \), which would imply \( I = 0 \) since \( M \) is faithful as an \( A \)-module. This case is therefore not possible and so \( I \) is left \( H \)-primitive.

Theorem 1 justifies the use of Proposition 1 to form the upper \( H \)-radical \( \mathcal{J} = \mathcal{R}(\mathcal{P}) \), where \( \mathcal{P} \) is the class of all left \( H \)-primitive \( H \)-module algebras. More can be proved about \( \mathcal{J} \), and this is done in Section 2, if one assumes further conditions on \( H \). The conditions of interest in this paper are stated explicitly, and explained at the end of this section and the beginning of Section 2.

Another \( H \)-radical of interest is obtained from the (ordinary) Jacobson radical \( J \) for associative rings (or \( K \)-algebras). The general procedure is spelled out in the following proposition.

**Proposition 2.** Assume that \( \rho \) is an ordinary radical for associative \( K \)-algebras. Then \( \rho_H \), the class of all \( H \)-module algebras whose underlying algebra is in \( \rho \), is an \( H \)-radical.

Hence \( J_H \) is an \( H \)-radical. Section 2 gives the relationships, under appropriate conditions, among \( \mathcal{J}(A) \), \( J_H(A) \), and \( J(A \# H) \), where \( A \) is an \( H \)-module algebra. Structure theorems for \( A/J_H(A) \) where \( H = \mathcal{R}(\text{der } A) \), with certain finiteness conditions, can be found in [1, p. 452].

**Definition 3.** The \( H \)-radical \( \mathcal{R} \) is a hereditary \( H \)-radical provided \( A \in \mathcal{R} \) implies \( I \in \mathcal{R} \) for every \( H \)-ideal \( I \) of \( A \). \( \mathcal{R} \) is strongly hereditary provided \( \mathcal{R}(I) = \mathcal{R}(A) \cap I \) for every \( H \)-ideal \( I \) of \( A \).

As usual, every strongly hereditary \( H \)-radical is hereditary. If \( \mathcal{R} \) is a hereditary \( H \)-radical then \( \mathcal{R}(A) \cap I \subseteq \mathcal{R}(I) \) for all \( H \)-ideals \( I \) of \( A \). If \( \mathcal{R} \) is a strongly hereditary \( H \)-radical and \( A \in \mathcal{E}(\mathcal{R}) \), then \( I \in \mathcal{E}(\mathcal{R}) \) for every \( H \)-ideal \( I \) of \( A \).

**Proposition 3.** Suppose \( \rho \) is an ordinary radical for associative \( K \)-algebras. If \( \rho \) is hereditary, then \( \rho_H \) is a hereditary \( H \)-radical and

\[
\rho_H(A) = \sum \{ I : I \text{ is an } H \text{-ideal of } A, \text{ and } I \subseteq \rho(A) \}.
\]

If \( \rho \) is strongly hereditary, then \( \rho_H \) is a strongly hereditary \( H \)-radical.
As an immediate application of Proposition 3, one gets that $J_{H}$ is a strongly hereditary $H$-radical.

**Proposition 4.** Suppose $\mathcal{R}$ is a hereditary $H$-radical such that all $A \in \mathcal{H}$ with $A^2 = 0$ are in $\mathcal{R}$. Then $\mathcal{R}$ is a strongly hereditary $H$-radical.

*Proof.* Let $I$ be an $H$-ideal of $A \in \mathcal{H}$. It suffices to show that $\mathcal{R}(I)$ is an $H$-ideal of $A$. Set $R = \mathcal{R}(I)$. Then $(AR + R)^2 \subseteq R$ and hence $(AR + R)/R \in \mathcal{R}$ and $(AR + R)/R$ is an $H$-ideal of $I/R \in \mathcal{Z}(\mathcal{R})$. Hence $(AR + R)/R = 0$, i.e., $AR \subseteq R$. Similarly $RA \subseteq R$ and thus $R = \mathcal{R}(I)$ is an $H$-ideal of $A$.

For ordinary radical theory, Proposition 4 can be proved without the assumption that $\mathcal{R}$ contains all $A$ such that $A^2 = 0$. Whether or not this assumption can be deleted for hereditary $H$-radicals is left open in this paper. In Section 2, Proposition 4 will be applied to show that $\mathcal{J}$ is a strongly hereditary $H$-radical.

**Proposition 5.** Suppose $\mathcal{J} \subseteq \mathcal{H}$ satisfies the condition in the hypothesis of Proposition 1, and let $\mathcal{R} = \mathcal{R}(\mathcal{J})$. Suppose further that for all $A \in \mathcal{H}$:

(i) If $I$ is a nonzero $H$-ideal of $A$ and $I \subseteq \mathcal{J}$, then there exists an $H$-ideal $B$ of $A$ such that $A/B \in \mathcal{J}$ and $I \nsubseteq B$. (ii) $A^2 = 0$ implies $A \nsubseteq \mathcal{J}$. Then, for every $A \in \mathcal{H}$,

$$\mathcal{R}(A) = \bigcap \{I : I \text{ is an } H\text{-ideal of } A, \text{ and } A/I \in \mathcal{J}\}.$$ 

A similar result is that if $\rho$ is a hereditary ordinary upper radical generated by a class $\sigma$, and if $\rho(A) = \bigcap \{I : I \text{ is an ideal of } A, \text{ and } A/I \in \sigma\}$, then $\rho_H(A) = \bigcap \{I_H : A/I \in \sigma\}$, where $I_H$ is the sum of all the $H$-ideals of $A$ contained in $I$. This was used for $J_{H}$ in [1, p. 452], using $H = \mathcal{R}(\text{der } A)$, $\sigma$ equal to the set of primitive associative rings.

Proposition 5 resembles [4, Lemma 80, p. 139], which is concerned with the topic of special radicals for associative rings. Under appropriate conditions, Proposition 5 will be applied to $\mathcal{J} = \mathcal{R}(\mathcal{J})$, $\mathcal{J}$ the left $H$-primitive $H$-module algebras, in Theorem 2 in Section 2.

**Definition 4.** $H$ is an irreducible bialgebra over $K$ provided there exists a denumerable sequence of $K$-subspaces $H_i$ of $H$, $H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots \subseteq H$ where $H_0 = K1_h$, $H = H_i$ for $i \geq 0$ and $H_iH_j \subseteq H_{i+j}$, and $\Delta H_n \subseteq \sum_{i=0}^{n} \text{Im}(H_i \otimes H_{n-i})$. Here $\text{Im}(H_i \otimes H_{n-i})$ denotes the image of the canonical map of $H_i \otimes H_{n-i}$ into $H \otimes H$.

This definition of irreducible bialgebra is the same as that of filtered bialgebra in [3, p. 10], and is equivalent to $H$ being irreducible as a coalgebra in the sense of [7] where $K$ is a field.
The following are important examples of irreducible bialgebras over $K$.

1. $K$ a field and $H$ irreducible as a coalgebra.

2. $K$ not necessarily a field, but $H$ generated as an algebra by $P(H) = \{h \in H : \Delta h = h \otimes 1 + 1 \otimes h\}$, the "primitive" elements of $H$. One easily checks in this case that a filtration is provided by setting $H_n = \sum_{i=0}^{\infty} P(H)^i$, $n = 0, 1, 2, \ldots$, where, by convention $P(H)^0 = K1_H$.

3. $H = U(L)$, the universal enveloping algebra of the Lie algebra $L$. This is a special case of (2).

If $\{H_i\}$ is a filtration of the bialgebra $H$ over $K$, and if we set $H_n^+ = H_n \cap (\ker \epsilon)$, $H^+ = H \cap (\ker \epsilon)$, then one has the following decompositions:

$$H = H^+ \oplus K1_H, \quad H^+ = \bigcup_{i=0}^{\infty} H_i^+,$$

where the sum is direct as $K$-spaces. As is shown in [3, p. 10], if $H$ is irreducible and $h \in H_n^+$, then $\Delta h = h \otimes 1 + 1 \otimes h + y$ for some $y \in \sum_{i=1}^{n-1} \text{Im}(H_i \otimes H_{n-i})$.

Lemma 2 below proves one fact about irreducible bialgebras that will be useful in Section 2.

**Lemma 2.** Assume that $H$ is irreducible and that $A$ is an $H$-module algebra. Then the annihilator in $A$ of a left $A, H$-module $M$ is an $H$-ideal of $A$.

**Proof.** Let $I = \{a \in A : aM = 0\}$, an ideal of $A$. It needs to be shown that $a \in I$ implies $h \cdot a \in I$ for all $h \in H$. Writing, as above $H = H^+ + K1_H$, one can assume $h \in H^+ = \bigcup_{i=0}^{\infty} H_i^+$. This makes $h$ an element of some $H_n^+$. If $n = 0$, then $h = 0$ and $h \cdot a = 0 \cdot a = 0$ is in $I$. The induction assumption is that $g \cdot a$ is in $I$ for all $g$ in $H_i^+$ and for all $i$ less than $n$. Since $H$ is irreducible one can write

$$\Delta h = h \otimes 1 + 1 \otimes h + \sum g_i \otimes f_i,$$

where $g_i, f_i$ belong to subspaces of index less than $n$. Then for any $m \in M$,

$$(h \cdot a)m = h(am) - ah(m) - \sum (g_i \cdot a)f_i(m)$$

$$= 0 - 0 - 0 = 0,$$

since $aM = 0$ and $g_i \cdot a \in I$ for all $i$ by the induction assumption. Therefore, $h \cdot a \in I$, as claimed.

As a slight generalization note that essentially the same argument shows that if $N$ is an $A, H$-submodule of $M$ then $\{a \in M : aM \subseteq N\}$ is also an $H$-ideal of $A$. Also, analogous right-handed versions for the above are true.
The following two basic assumptions on the bialgebra $H$ over $K$ occur frequently in this section:

1. $H$ is irreducible,
2. $H$ is a flat $K$-module.

For example, if $K$ is a field then $H$ is flat; if $K = Z$ then $H$ is flat if and only if $H$ is torsion-free. As a consequence, if $A$ is an associative $H$-module algebra, and if $S$ is an $H$-invariant subalgebra of $A$, then $S \# H$ is embedded injectively in $A \# H$. Therefore, if $I$ is an $H$-ideal of $A$, one can naturally consider $I \# H$ as an ideal of $A \# H$, and it is for the sake of this type of application that we assume that $H$ is flat.

Theorem 2 states that $\mathcal{J}(A)$ is the intersection of the left $H$-primitive ideals of $A$. As expected, an $H$-ideal $P$ of $A$ is defined to be a left $H$-primitive ideal provided $A/P$ is left $H$-primitive; i.e., $P$ is the annihilator in $A$ of an irreducible left $A$, $H$-module. (Lemma 2 shows immediately that every such annihilator is an $H$-ideal of $A$.)

**Theorem 2.** Assume that $H$ is an irreducible bialgebra over $K$, and that $H$ is a flat $K$-module. Then for an $H$-module algebra $A$,

$$\mathcal{J}(A) = \bigcap \{P: P \text{ is an } H\text{-ideal of } A \text{ and } A/P \text{ is left } H\text{-primitive}\}.$$ 

The following lemma establishes one of the sufficient conditions (see Proposition 5).

**Lemma 3.** Assume the hypotheses on $H$ in the statement of Theorem 2. Suppose $I$ is a nonzero $H$-ideal of the $H$-module algebra $A$, and that $I$ is itself a left $H$-primitive $H$-module algebra. Then there exists an $H$-ideal $B$ of $A$ such that $A/B$ is left $H$-primitive and $I \subseteq B$.

**Proof.** Let $M$ be an irreducible left $I$, $H$-module, faithful as an $I$-module. Then $M$ is an irreducible left $I \# H$-module by Lemma 1(ii). As in the proof of Lemma 1(iii), one has that for any nonzero $n \in M$, $(I \# H)n = M$. Work with some such fixed generator $n$. Since $H$ is a flat $K$-module, consider $I \# H$ as an ideal of $A \# H$ and make $M$ into an $A \# H$-module by defining $u(vn) = (uv)n$ for all $u \in A \# H$, $v \in I \# H$. To show that this action is well-defined it must be shown that if $vn = 0$, then $(uv)n = 0$. One has the conventional calculation, assuming $vn = 0$:

$$(I \# H)((uv)n) = ((I \# H)(uv))n = ((I \# H)u)(vn) = 0.$$
As before, anything annihilated by \( I \# H \) is zero, so \((uv)n = 0\). Now make \( M \) into an \( A \)-module by setting \( am = (a \# 1)m \) for any \( m \in M, a \in A \). If \( a \) is in \( I \), then ("new action") \( am = (a \# 1)m = am \) ("old action"), giving the correct module action of \( I \) on \( M \). Thus any \( A \)-submodule of \( M \) is an \( I \)-submodule of \( M \). Once it is shown that \( M \) is an \( A, H \)-module, it follows that \( M \) is an irreducible \( A, H \)-module. So we claim that

\[
h(am) = \sum_{(h)} (h_{(1)} \cdot a) h_{(2)}(m) \quad (\dagger)
\]

for all \( h \in H, a \in A, m \in M \). But

\[
am = (a \# 1)m = (a \# 1) \left[ \left( \sum_{i=1}^{k} x_i \# g_i \right) n \right] = \sum_{i=1}^{k} (a \# 1)[(x_i \# g_i)n]
\]

for some \( x_i \in I, g_i \in H \) and

\[
h(am) = \sum_{i=1}^{k} h((a \# 1)[x_i \# g_i]n)).
\]

So one needs show (\( \dagger \)) when \( m \) has the form \( m = (x \# g)n, x \in I \). We have

\[
h(am) = h((a \# 1)((x \# g)n)) = h((ax \# g)n) = h((ax)g(n)) \quad \text{since} \quad ax \in I
\]

\[
= \sum_{(h)} (h_{(1)} \cdot (ax)) h_{(2)}g(n) = \sum_{(h)} (h_{(1)} \cdot a)(h_{(2)} \cdot x) h_{(3)}g(n),
\]

where here and below the fact that \( \sum_{(h)} \Delta h_{(1)} \otimes h_{(2)} = \sum_{(h)} h_{(1)} \otimes \Delta h_{(2)} \) (i.e., coassociativity) is used, which justifies the use of three subscripts as displayed. On the other hand,

\[
\sum_{(h)} (h_{(1)} \cdot a) h_{(2)}(m) = \sum_{(h)} (h_{(1)} \cdot a) h_{(2)}((x \# g)n)
\]

\[
= \sum_{(h)} (h_{(1)} \cdot a) h_{(2)}(xg(n)) = \sum_{(h)} (h_{(1)} \cdot a)(h_{(2)} \cdot x) h_{(3)}g(n)
\]

and the two end results are equal. So \( M \) is indeed an \( A, H \)-module.
Now let $B = \{ a \in A : aM = 0 \}$. $B$ is an $H$-ideal of $A$ since $H$ is irreducible, by Lemma 2. It must be the case that $B \cap I = 0$ since $BM = 0$ and $M$ is a faithful $I$-module (in fact $B = \{ a \in A : aI = 0 \}$). Also, $M$ is a faithful $A/B$-module, and an irreducible $A/B$, $H$-module via $(a + B)m = am$. Thus $A/B$ is left $H$-primitive, and $I \subseteq B$. This finishes the lemma.

Proof of Theorem 2. In addition to Lemma 3 all one needs to observe is that if $A$ is an $H$-module algebra with $A^2 = 0$, then $A$ cannot be left $H$-primitive, since $AM$ would be zero whenever $M$ was an irreducible $A$, $H$-module, a contradiction. Proposition 5 is now applied to finish the proof.

An $H$-module algebra $A$ is $H$-simple provided the only $H$-ideals of $A$ are 0 and $A$, and $A^2 \neq 0$.

Corollary. An $H$-simple algebra $A$ is $\mathfrak{J}$-semisimple if and only if $A$ is left $H$-primitive.

Proof. If $A$ is $H$-primitive then $A$ is $\mathfrak{J}$-semisimple. If $A$ is $H$-simple and $H$-semisimple, then $A$ has an irreducible left $A$, $H$-module $M$ such that $AM \neq 0$. But the annihilator of $M$ is an $H$-ideal of $A$, not equal to $A$, and hence is zero, so $A$ is left $H$-primitive.

Theorem 3. Assume that $H$ is an irreducible bialgebra over $K$, and that $H$ is flat as a $K$-module. Then $\mathfrak{J}$ is a hereditary $H$-radical.

Proof. It must be shown that if $A \in \mathfrak{J}$ and if $I$ is an $H$-ideal of $A$, then $I \in \mathfrak{J}$. By Theorem 2, $A \in \mathfrak{J}$, i.e., $A = \mathfrak{J}(A)$, means that $A$ has no irreducible left $A$, $H$-modules. We show that $I$ also has no irreducible left $I$, $H$-modules, in which case $I = \mathfrak{J}(I)$, so $I \in \mathfrak{J}$. Suppose that $M$ is an irreducible left $I$, $H$-module, hence $IM \neq 0$. Lemma 3 shows how to make $M$ into an irreducible left $A$, $H$-module such that the action of $A$ on $M$ when restricted to $I$ gives the original action of $I$ on $M$. (Here it is noted that to make $M$ into an $A$, $H$-module requires only $IM \neq 0$, and not necessarily that $M$ be a faithful $I$-module.) But by the assumption about $A$ (that it has no irreducible left $A$, $H$-modules) this gives a contradiction. Therefore $I = \mathfrak{J}(I)$ and $I \in \mathfrak{J}$, so that $\mathfrak{J}$ is hereditary.

Corollary. $\mathfrak{J}$ is a strongly hereditary $H$-radical. That is, for an $H$-module algebra $A$ and an $H$-ideal $I$ of $A$,

$$\mathfrak{J}(I) = \mathfrak{J}(A) \cap I.$$ 

Proof. This follows from Theorem 3 and Proposition 4 since if $A$ is an $H$-module algebra such that $A^2 = 0$. then $A \in \mathfrak{J}$. 

In what follows, regard $A$ as a subalgebra of $A \# H$ via the canonical embedding $a \rightarrow a \# 1$.

**Theorem 4.** Assume that $H$ is an irreducible bialgebra over $K$ and that $H$ is a flat $K$-module. Then $\mathcal{J}(A) = J(A \# H) \cap A$.

*Proof.* The theorem can be proved if one first assumes that the measuring on $A$ is unital (hence that $A$ has a unit), and then remove this restriction. Assuming then that the measuring is unital, first observe that the irreducible left $A, H$-modules are exactly the irreducible left $A \# H$-modules by (ii) and (iii) of Lemma 1. Applying the constructions in (ii) and (iii), first one, then the other, preserves the module action with which one starts. Now $J(A \# H)$ is the intersection of the annihilators of irreducible left $A \# H$-modules, and $\mathcal{J}(A)$ is the intersection of the annihilators of irreducible left $A, H$-modules. These annihilators correspond as follows. Consider $P = \{a \in A : aM = 0\}$, where $M$ is an irreducible left $A, H$-module. Consider $M$ as an irreducible left $A \# H$-module as in (ii) of Lemma 1. Then $(P \# 1)M = P1(m) = PM = 0$. On the other hand, if $(u \# 1)M = 0$, then $a1(M) = aM = 0$, and so $a \in P$. So if $Q = \{u \in A \# H : uM = 0\}$, then $Q \cap A = P$. These arguments are reversible, using (iii) of Lemma 1 this time. Hence $\bigcap (Q \cap A) = \bigcap P$, where $Q$ ranges over the left primitive ideals of $A \# H$, $P$ ranges over the left $H$-primitive ideals of $A$. Hence $J(A \# H) \cap A = \mathcal{J}(A)$.

If the measuring is not unital, then use $A_1$, constructed in (i) of Lemma 1. Since $\mathcal{J}$ is a strongly hereditary $H$-radical and $A$ is an $H$-ideal of $A_1$, we have

$$\mathcal{J}(A) = \mathcal{J}(A_1) \cap A$$

$$= (J(A_1 \# H) \cap A_1) \cap A \text{ by the above,}$$

$$= J(A_1 \# H) \cap A$$

$$= (J(A_1 \# H) \cap A \# H) \cap A$$

$$= J(A \# H) \cap A$$

since $J$ is a strongly hereditary ordinary radical. This establishes the theorem in general.

**Corollary 1.** $\mathcal{J}(A)$ is the intersection of all right $H$-primitive ideals of $A$.

The definition of right $H$-primitive is the obvious one. The corollary follows easily from the theorem, since $J$ is left or right definable, and makes $\mathcal{J}$ symmetrically definable as either the upper $H$-radical generated by the left $H$-primitive algebras or as the upper $H$-radical generated by the right $H$-primitive algebras.
COROLLARY 2. If $A \in \mathcal{J}$, then $A \# H \in \mathcal{J}$.

Proof. If $A \in \mathcal{J}$, then in fact $A$ does not have a unit, since otherwise $1_A \# 1_H \in J(A \# H)$, which is impossible. However, adjoin a unit to $A$ as in (i) of Lemma 1, obtaining $A_1$, so that the following argument can be given. Since $A \in \mathcal{J}$, $A = \mathcal{J}(A) = J(A \# H) \cap A$, so that $A \# 1 \subseteq J(A \# H) = J(A_1 \# H) \cap (A \# H)$. Hence $A \# H = (A \# 1)(1 \# H) \subseteq J(A_1 \# H) \cap (A \# H)$ and so $A \# H = J(A \# H)$ or $A \# H \in \mathcal{J}$.

THEOREM 5. Assume that $H$ is an irreducible bialgebra over $K$, and that $H$ is a flat $K$-module. Then $\mathcal{J}(A)$ contains all the left (or right) $H$-ideals of $A$ which are in $\mathcal{J}$.

Proof. Assume $L \in \mathcal{J}$ is a left $H$-ideal of $A$. Then since $L \in \mathcal{J}, L \# H \in \mathcal{J}$ by Corollary 2 above, and $L \# H$ is a left ideal of $A \# H$, hence $L \# H \subseteq J(A \# H)$. Thus, $L \# 1 \subseteq J(A \# H) \cap (A \# 1) = \mathcal{J}(A) \# 1$ and so $L \subseteq \mathcal{J}(A)$, as required. Similar argument applies to the right $H$-ideals.

The next theorem gives the general relationship between $\mathcal{J}$ and $J_H$.

THEOREM 6. Assume that $H$ is an irreducible bialgebra over $K$, and that $H$ is a flat $K$-module. Then $J(A \# H) \cap A \subseteq J_H(A)$, hence $\mathcal{J} \subseteq J_H$ and $\mathcal{J}(A) \# H \subseteq J(A \# H)$.

LEMMA 4. Assume the hypotheses of Theorem 6. If $S$ is an ideal of $A \# H$, then $S \cap A$ is an $H$-ideal of $A$.

Proof. We show that if $a \# 1$ is in $S \cap (A \# 1)$, then $(h \cdot a) \# 1$ is in $S \cap (A \# 1)$ for all $h \in H$. Write $H = H^+ + K1_H$, $H^+ = \bigcup_{i=0}^{\infty} H_i^+$. It suffices to show $(h \cdot a) \# 1 \in S \cap (A \# 1)$ for all $h \in H^+$. If $h \in H_{0+} = K^+ = 0$, then $h = 0$ and $0 \# 1 = 0$ is in $S \cap (A \# 1)$. So assume the conclusion is true for all $g \in H_j^+$, for all $j$ less than $n$, and let $h$ be in $H_n^+$. Then

$$\Delta h = h \otimes 1 + 1 \otimes h + \sum g_i \otimes f_i,$$

where $g_i, f_i$ belong to subspaces of index less than $n$. Then $(1 \# h)(a \# 1) = (h \cdot a) \# 1 + a \# h + \sum (g_i \cdot a) \# f_i$, which is in $S$ since $a \# 1 \in S$ and $S$ is an ideal of $A \# H$. Also, each $(g_i \cdot a) \# 1 \in S$ by the induction assumption, so $((g_i \cdot a) \# 1)(1 \# f_i) = (g_i \cdot a) \# f_i \in S$. Also $(a \# 1)(1 \# h) = a \# h \in S$. Going back to the original expression for $(1 \# h)(a \# 1)$, we get $(h \cdot a) \# 1 \in S \cap A$, finishing the lemma.

LEMMA 5. Assume the hypotheses of Theorem 6. Suppose the measuring of $H$ on $A$ is unital. If $b \# 1$ is right invertible in $A \# H$, then $b$ is right invertible in $A$. 
Proof. Suppose \( b \neq 1 \) has right inverse \( \sum_{i=1}^{n} c_i \# h_i \). Then \( (b \neq 1) (\sum c_i \# h_i) = 1 \# 1 \) or \( \sum bc_i \# h_i = 1 \# 1 \). Now \( A \) is a unital \( A \# H \)-module via the basic action \( (a \# h)x = a(h \cdot x) \), for all \( a, x \in A, h \in H \). Apply both sides of \( \sum bc_i \# h_i = 1 \# 1 \) to \( 1_A \). Then \( \sum bc_i e(h_i) = 1_A \), i.e., \( b(\sum c_i e(h_i)) = 1_A \), which shows \( b \) is right invertible in \( A \).

Proof of Theorem 6. First assume that the measuring is unital. Since \( J(A \# H) \) is an ideal of \( A \# H \), Lemma 4 shows that \( J(A \# H) \cap A \) is an \( H \)-ideal of \( A \). Every element \( a \neq 1 \) in \( J(A \# H) \cap A \) is right-quasi-regular, that is \( 1 \# 1 + a \# 1 = (1 + a) \# 1 \) has a right inverse in \( A \# H \). Hence, by Lemma 5, \( 1 + a \) is right invertible in \( A \). Thus \( J(A \# H) \cap A \) is a right-quasi-regular \( H \)-ideal of \( A \), and so is contained in \( J_H(A) \). This proves the theorem when the measuring is unital. If the measuring is not unital, consider \( A_1 \). By the corollary to Theorem 3, \( \mathcal{J}(A) = \mathcal{J}(A_1) \cap A \subseteq J_H(A_1) \cap A = J_H(A) \) since \( J_H \) is a strongly hereditary \( H \)-radical. This gives the theorem for general \( A \). The other conclusions in Theorem 6 now follow easily.

The question naturally arises as to whether \( J_H \subseteq \mathcal{J} \). It is shown in Theorem 7 below that this is the case for (left or right) Artinian algebras, but first a non-Artinian counterexample is given.

Example with \( \mathcal{J}(A) \neq J_H(A) \). Let \( R \) denote the real field and let \( A = R(x, y) \) be the algebra of all formal power series in commuting indeterminants \( x \) and \( y \). Let \( d = d/dx \) and let \( H \) be the bialgebra over \( R \) generated by \( d \); a typical element of \( H \) is a finite polynomial in powers of \( d \) with coefficients in \( R \). The Jacobson radical of \( A \), \( J(A) \), consists of all power series with zero constant term, and \( J(A) \) contains the \( H \)-ideal \( B \) of \( A \) consisting of all power series of the form \( p_0 + p_1x + \cdots + p_nx + \cdots \), where each \( p_i \) is a power series in \( y \) with zero constant term. Now \( B \subseteq J_H(A) \) and \( y \in B \), so \( y \in J_H(A) \).

We propose to show \( y \neq d \) is not in \( J(A \# H) \), verifying the example. The reason this will suffice is the following. If \( J_H(A) = J_H(A) \# 1 \subseteq \mathcal{J}(A) = J(A \# H) \cap A \), then \( J_H(A) \# H \) must also be contained in \( J(A \# H) \), since the latter is an ideal of \( A \# H \). But \( y \# d \in J_H(A) \# H \) and \( y \# d \notin J(A \# H) \), a contradiction.

Now every element \( u \) of \( A \# H \) can be expressed in the form \( u = \sum a_i \# d_i \), \( a_i \in A \), for some nonnegative integer \( n \). Suppose \( y \# d \) were in \( J(A \# H) \). Then \( 1 \# 1 + y \# d \) must have a left inverse \( u \):

\[
u(1 \# 1 + y \# d) = \left( \sum_{i=0}^{n} a_i \# d_i \right) (1 \# 1 + y \# d) = 1 \# 1.
\]

This equation can be solved for the \( a_i \) by applying both sides to elements \( 1, x, x^2, \ldots \), of \( A \), since \( A \) is an \( A \# H \)-module. For example, applying both
sides to 1, one gets $a_0 = 1$; applying both sides to $x$, one gets $a_0 x + a_0 y + a_1 = x$ or $a_1 = -y$, etc. Summing up,

$$u = 1 \# 1 - y \# d + y^2 \# d^2 - y^3 \# d^3 + \cdots \pm y^n \# d^n.$$  

However,

$$u(1 \# 1 + y \# d) = u + y \# d - y^2 \# d^2 + y^3 \# d^3 - \cdots \mp y^n \# d^n \\
= 1 \# 1 \pm y^{n+1} \# d^{n+1}$$

since $y^{n+1} \# d^{n+1} \neq 0$. Hence one concludes that $A \# H$ does not contain a left-quasi-inverse for $y \# d$, and so $J_H(A) \subseteq \mathcal{J}(A)$.

**Theorem 7.** Assume that $H$ is an irreducible bialgebra over $K$, and that $H$ is a flat $K$-module. If $J_H(A)$ is nilpotent, then $J_H(A) = \mathcal{J}(A)$. Hence if $A$ is (left or right) Artinian, then $J_H(A) = \mathcal{J}(A)$.

**Proof.** This follows from the fact that if $T$ is an $H$-ideal of $A$, then $(T \# H)^n \subseteq T^n \# H$. Applying this to $T = J_H(A)$, where $T^m = 0$ some $m$, one gets $(J_H(A) \# H)^m = 0$. Now $J_H(A) \# H$ is an ideal of $A \# H$, so $J_H(A) \# H \subseteq J(A \# H)$, or $J_H(A) \subseteq J(A \# H) \cap A = \mathcal{J}(A)$.

It is easy to show that for left Artinian $H$-module algebras, $A$ is left $H$-primitive if and only if $A$ has an irreducible left module, the annihilator of which contains no nonzero $H$-ideal of $A$. Similarly, for left Artinian $H$-module algebras, a left $H$-primitive ideal $I$ is the largest $H$-ideal contained in some primitive ideal. (Statements in this paragraph do not require any of the restrictions on $H$.)

One would like answers to the following questions, which have been left open here:

1. When is $\mathcal{J}(A) \# H = J(A \# H)$? Theorem 6 says only that $\mathcal{J}(A) \# H \subseteq J(A \# H)$.

2. Is $J(A \# H) \subseteq J_H(A) \# H$? Again, Theorem 6 says only that $J(A \# H) \cap A \subseteq J_H(A)$.

Whereas this paper concerns itself with the abstract theory of the $H$-radical $\mathcal{J}$, a paper by R. E. Block [2] will give structure theorems for certain $H$-primitive algebras with finiteness conditions (either on the algebra or the module), carrying further the work in [1].

**References**