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Occupation time limits of inhomogeneous Poisson systems of independent particles

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Abstract

We prove functional limits theorems for the occupation time process of a system of particles moving independently in $\mathbb{R}^d$ according to a symmetric α-stable Lévy process, and starting from an inhomogeneous Poisson point measure with intensity measure $\mu(\mathrm{d}x) = (1 + |x|^{\gamma})^{-1}\mathrm{d}x, \gamma > 0$, and other related measures. In contrast to the homogeneous case ($\gamma = 0$), the system is not in equilibrium and ultimately it becomes locally extinct in probability, and there are more different types of occupation time limit processes depending on arrangements of the parameters γ, d and α. The case $\gamma < d < \alpha$ leads to an extension of fractional Brownian motion.

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1. Introduction

Several authors have studied systems of particles moving independently in $\mathbb{R}^d$ according to a Markov process (usually a symmetric α-stable Lévy process, $0 < \alpha \leq 2$), and also

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systems having in addition a branching mechanism (e.g. [5–8,10,12–14,17,18,20–22,25,26] and references therein). A typical assumption in the cited references is that the system starts from a homogeneous Poisson point measure, i.e., with intensity the Lebesgue measure (denoted here by λ). This assumption represents a strong technical simplification because in the special cases usually studied λ is invariant for the semigroup of the motion, and this implies that the particle system without branching is in equilibrium, and for $d > \alpha$ a critical branching system converges towards equilibrium [17]. In this case the systems have been extensively studied. New situations appear if the initial condition is an inhomogeneous Poisson point measure.

In this paper we consider the system without branching, with symmetric α-stable Lévy process for the particle motion, and initial inhomogeneous Poisson point measure with intensity measure μ of the form

$$\mu(\mathrm{d}x) = \frac{\mathrm{d}x}{1 + |x|^{\gamma}}, \quad \gamma > 0,$$

and other more general related measures. In this case the system is not in equilibrium and ultimately it becomes locally extinct in probability (see Proposition 2.1). Therefore one should expect different types of results from those of the homogeneous case. Our purpose is to obtain functional limits for the rescaled occupation time process of the particle system in different cases.

The particle system is described as follows. Given a Poisson point measure on $\mathbb{R}^d$ with intensity measure μ, particles evolve from its atoms, moving independently according to a symmetric α-stable Lévy process (called the standard α-stable process). Let $N = (N_t)_{t \geq 0}$ denote the empirical measure process of the system, i.e.,

$$N_t = \sum_i \delta_{x_i(t)}, \tag{1.1}$$

where $\{x_i(t)\}_i$ are the positions of the particles at time t. Note that N_t converges in probability to the null measure as $t \to \infty$ (Proposition 2.1). From the proof of Proposition 2.1 it follows also that EN_t properly normalized converges towards λ as t tends to infinity, so, on average, the system becomes uniformly spread out in space.

Let $X_T = (X_T(t))_{t \geq 0}$ denote the normalized occupation time fluctuation process of the system, defined by

$$X_T(t) = \frac{1}{F_T} \int_0^{Tt} (N_s - EN_s)\mathrm{d}s, \tag{1.2}$$

where T is the time scaling and F_T is a norming. The problem is to find F_T such that the process X_T converges in distribution as $T \to \infty$ (i.e., the time is accelerated), and to describe the limit process X in the cases where it exists.

In the homogeneous case (corresponding to $\gamma = 0$), the occupation time fluctuation limit process has three different forms, for $d < \alpha$ [5], $d = \alpha$ and $d > \alpha$ [6]. In the inhomogeneous case there are more results depending on the values of γ relative to d and α when $\gamma \leq d$: $\gamma < d < \alpha, \gamma < d = \alpha, \gamma < \alpha < d, \gamma = \alpha < d, \gamma = d < \alpha, \gamma = d = \alpha, \alpha < \gamma \leq d$. For "small" γ, i.e., $\gamma < \alpha$, the results are analogous to those of the homogeneous case, while for "large" γ, i.e., $\gamma \geq \alpha$, and this seems unexpected, they are of a different kind. The case $\gamma < d < \alpha$ leads to a long range dependence, self-similar, centered Gaussian process ξ with covariance

$$E\xi_t\xi_s = \int_0^{s\wedge t} u^a((t-u)^b + (s-u)^b)\mathrm{d}u,$$

where $a = -\gamma/\alpha \in (-1, 0), b = 1 - 1/\alpha \in (0, 1/2]$, which is an extension of fractional Brownian motion with Hurst parameter $H \in (1/2, 3/4]$ (corresponding to $\gamma = 0$; the process ξ with maximal ranges for the values of the parameters a and b is discussed in [9]). Nevertheless, although the process ξ depends on γ, its dependence exponent [7] is independent of γ. The cases $\gamma = \alpha < d$ and $\gamma = d = \alpha$ give a new type of limits (with no counterpart in the homogeneous case), namely, centered, constant (and hence continuous) Gaussian processes on $(0, \infty)$, discontinuous at 0.

For $\gamma > d$ the measure μ is finite, and the results are in sharp contrast to those for $\gamma \le d$. In this case we give the results for a finite measure μ in general, and for $d \le \alpha$ they are akin to the famous limit theorem of Darling and Kac [11] for the occupation time (without centering), and its generalization to path space by Bingham [2].

All the occupation time limit theorems are formulated in the context of $\mathcal{S}'(\mathbb{R}^d)$-valued processes, where $\mathcal{S}'(\mathbb{R}^d)$ is the usual space of tempered distributions (dual of the space $\mathcal{S}(\mathbb{R}^d)$ of smooth rapidly decreasing functions). In some cases the limit process is of the form λ multiplied by a real valued process, but in others the limit is "truly" $\mathcal{S}'(\mathbb{R}^d)$-valued. More precisely, these two different qualitative behaviours depend on whether the particle motion is recurrent ($d \le \alpha$, the first type of limit) or transient ($d > \alpha$, the second type of limit). In the recurrent case, for all γ's for which an occupation time fluctuation limit exists, the spatial covariance kernel of the limit is constant; hence the occupation system fluctuates around the mean jointly the same way in every region of space at each time t, and the dependence of t is governed by a real centered Gaussian process. In the transient case, when an occupation time fluctuation limit exists, the spatial covariance kernel of the limit is given by the Riesz potential kernel $1/|x|^{d-\alpha}$, and the limit process is "truly" $\mathcal{S}'(\mathbb{R}^d)$-valued.

The methods of proof for the fluctuation limit theorems are analogous to those developed in [5,6], with some new technical complexities because the measure μ is not invariant for the semigroup of the motion. On the other hand, there is a significant difference in the tightness proofs, as they require estimates for moments of arbitrarily high order (whereas in [5,6] order 2 or 4 was enough). For the results of Darling–Kac type we proceed similarly to [2]. However, in our setting the uniform convergence condition (A) for that kind of result is not satisfied, and some additional work is needed.

Convergence in distribution in the space of continuous functions $C([0, \tau], \mathcal{S}'(\mathbb{R}^d))$ for any $\tau > 0$ is denoted by $\Rightarrow_C$. In some cases the interval $[0, \tau]$ is replaced by $[\varepsilon, \tau], 0 < \varepsilon < \tau$, because the limit process is discontinuous at $t = 0$.

The duality between the spaces $\mathcal{S}'(\mathbb{R}^k)$ and $\mathcal{S}(\mathbb{R}^k)$ is denoted by $\langle\cdot, \cdot\rangle$.

Generic constants are written as $C, C_1, C_2, \ldots$, with possible dependencies in parentheses.

In Section 2 we present the results. In Section 3 we give an explanation of the general method used for the proofs of the occupation time fluctuation limits, and we prove most of the results. Some proofs that are similar to others are omitted, with some comments.

Other initial conditions for the particle system may be considered, for example, configurations such that N_t converges towards a homogeneous Poisson point measure as $t \to \infty$ [24], but we have not tried to investigate this.

The branching particle systems in the inhomogeneous case produce fewer results, but there are other kinds of difficulties related to extinction of the system. These results will be presented elsewhere.

2. Results

Let N and X_T be the processes defined in (1.1) and (1.2). As stated in the introduction, for simplicity most of our results are formulated for μ of the form

$$\mu(\mathrm{d}x) = \frac{\mathrm{d}x}{1+|x|^\gamma}, \quad \gamma > 0. \tag{2.1}$$

Note that μ is finite for $d < \gamma$. More general measures μ will be also considered in this section. Our main objective is to study fluctuations of the occupation times around the mean, but it is natural to describe first the asymptotics of the mean itself.

Proposition 2.1. (a) *Let*

$$m_t = \begin{cases} t^{-\gamma/\alpha} & \text{if } \gamma < d, \\ t^{-d/\alpha}\log t & \text{if } \gamma = d, \\ t^{-d/\alpha} & \text{if } \gamma > d. \end{cases}$$

Then for each $\varphi \in \mathcal{S}(\mathbb{R}^d)$ there exists a finite limit $\lim_{t\to\infty}(1/m_t)E\langle N_t, \varphi\rangle$*, which is non-zero if* $\int_{\mathbb{R}^d}\varphi(x)\mathrm{d}x \neq 0$.

(b) *Let*

$$M_t = \begin{cases} t^{1-\gamma/\alpha} & \text{if } \gamma < d, \gamma < \alpha, & \text{(a)} \\ \log t & \text{if } \gamma = \alpha < d, & \text{(b)} \\ t^{1-d/\alpha}\log t & \text{if } \gamma = d < \alpha, & \text{(c)} \\ (\log t)^2 & \text{if } \gamma = d = \alpha, & \text{(d)} \\ t^{1-d/\alpha} & \text{if } \gamma > d, d < \alpha, & \text{(e)} \\ \log t & \text{if } \gamma > d, d = \alpha, & \text{(f)} \\ 1 & \text{if } \gamma > \alpha, d > \alpha. & \text{(g)} \end{cases} \tag{2.2}$$

Then for each $\varphi \in \mathcal{S}(\mathbb{R}^d)$ there exists a finite limit

$$\lim_{t\to\infty}\frac{1}{M_t}E\int_0^t \langle N_s, \varphi\rangle \mathrm{d}s, \tag{2.3}$$

which is non-zero if $\int_{\mathbb{R}^d}\varphi(x)\mathrm{d}x \neq 0$.

We now pass to the fluctuation process. In the theorems below, K is a number depending on α, d, μ, which may vary from case to case, and may be computed explicitly in each specific case.

Different arrangements of α, γ, d yield different results, and we order them according to the relationship between γ and d. We start with $\gamma < d$.

Theorem 2.2. *Let $\gamma < d < \alpha$ (and hence $d = 1$) and*

$$F_T = T^{1-(d+\gamma)/2\alpha}. \tag{2.4}$$

Then $X_T \Rightarrow_C K\lambda\xi$ as $T \to \infty$, where ξ is a real centered Gaussian process with covariance

$$E\xi_t\xi_s = \int_0^{t\wedge s} u^{-\gamma/\alpha}\left((t-u)^{1-d/\alpha} + (s-u)^{1-d/\alpha}\right)\mathrm{d}u. \tag{2.5}$$

This theorem is a generalization of Theorem 2.1 in [5], which corresponds to $\gamma = 0$.

Remark 2.3. The following properties of the process ξ are easy to obtain.

(a) For $\gamma = 0$, ξ is a fractional Brownian motion with Hurst parameter $1 - 1/2\alpha$ [5].

(b) ξ is self-similar with index $1 - (1+\gamma)/2\alpha$. This is immediate from (2.5), but more generally, from the form of the fluctuation process given by (1.2) it follows that if F_T has the form $T^\kappa f(T)$, where f is a function slowly varying at infinity and $\kappa \geq 0$ (as in our cases), then the limit process is self-similar with index κ.

(c) ξ is a long range dependence process where

$$E(\xi_{t+T} - \xi_{s+T})(\xi_v - \xi_r) = O(T^{-1/\alpha}) \quad \text{as } T \to \infty,$$

for $0 \leq r < v$, $0 \leq s < t$. Note that the dependence exponent [7] $1/\alpha$ is independent of γ.

(d) ξ is not a Markov process and not a semimartingale. The non-semimartingale property can be proved by Lemma 2.1 in [4].

The next two theorems are generalizations of Theorem 2.1 in [6] (for $\gamma = 0$).

Theorem 2.4. *Let $\gamma < d = \alpha$ (=1 or 2) and*

$$F_T = (T \log T)^{1/2} T^{-\gamma/2\alpha}. \tag{2.6}$$

Then $X_T \Rightarrow_C K\lambda\beta$ as $T \to \infty$, where β is an inhomogeneous real Wiener process with covariance

$$E\beta_t\beta_s = (t \wedge s)^{1-\gamma/\alpha}. \tag{2.7}$$

Theorem 2.5. *Let $\gamma < \alpha < d$ and*

$$F_T = T^{(1-\gamma/\alpha)/2}. \tag{2.8}$$

Then $X_T \Rightarrow_C KW$ as $T \to \infty$, where W is an $\mathcal{S}'(\mathbb{R}^d)$-valued time inhomogeneous Wiener process with covariance functional

$$E\langle W(t), \varphi_1\rangle\langle W(s), \varphi_2\rangle = (t \wedge s)^{1-\gamma/\alpha} \int_{\mathbb{R}^d} \varphi_1(x) G\varphi_2(x)\mathrm{d}x, \quad \varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d), \tag{2.9}$$

where G is the α-potential operator, i.e.

$$G\varphi(x) = C_{\alpha,d} \int_{\mathbb{R}^d} \frac{\varphi(y)}{|x-y|^{d-\alpha}}\mathrm{d}y, \tag{2.10}$$

with $C_{\alpha,d} = \Gamma(\frac{d-\alpha}{2})(2^\alpha \pi^{\frac{d}{2}} \Gamma(\frac{\alpha}{2}))^{-1}$.

The analogy with the case $\gamma = 0$ breaks down for "large" γ, i.e., $\gamma \geq \alpha$.

Theorem 2.6. *Let $\gamma = \alpha < d$ and*

$$F_T = (\log T)^{1/2}. \tag{2.11}$$

Then $X_T \Rightarrow KX$ in $C([\varepsilon, \tau], \mathcal{S}'(\mathbb{R}^d))$ as $T \to \infty$ for any $0 < \varepsilon < \tau$, where X is an $\mathcal{S}'(\mathbb{R}^d)$-valued Gaussian process constant in time on $(0, \infty)$, $X(t) \equiv X(1)$, and $X(1)$ is centered with covariance functional

$$E\langle X(1), \varphi_1\rangle\langle X(1), \varphi_2\rangle = \int_{\mathbb{R}^d} \varphi_1(x) G\varphi_2(x)\mathrm{d}x, \quad \varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d), \tag{2.12}$$

where G is given by (2.10).

Note that the limit process is discontinuous at $t = 0$ since $X_T(0) = 0$.

To complete the case $\gamma < d$ it remains to consider $\alpha < \gamma < d$.

However, by Proposition 2.1 we know that if $\alpha < d$ and $\alpha < \gamma$, then the relationship between γ and d is irrelevant and the total occupation time is bounded (see (2.2)(g) and (2.3)), so it does not make sense to investigate the fluctuation process.

We now proceed to the critical case $\gamma = d$.

Theorem 2.7. *Let* $1 = d = \gamma < \alpha$ *and*

$$F_T = T^{1-d/\alpha}(\log T)^{1/2}. \tag{2.13}$$

Then $X_T \Rightarrow_C K\lambda\xi$ *as* $T \to \infty$*, where* ξ *is as in Theorem 2.2.*

The next case is "doubly critical".

Theorem 2.8. *Let* $\gamma = d = \alpha$ $(=1$ *or* $2)$ *and*

$$F_T = (\log T)^{3/2}. \tag{2.14}$$

Then $X_T \Rightarrow K\lambda\eta$ *in* $C([\varepsilon, \tau], \mathcal{S}'(\mathbb{R}^d))$ *as* $T \to \infty$ *for any* $0 < \varepsilon < \tau$*, where* η *is a real Gaussian process constant in time on* $(0, \infty)$*,* $\eta_t \equiv \eta_1$*, and* η_1 *is standard normal.*

Here, as in Theorem 2.6, the limit process is discontinuous at $t = 0$.

Note that, as should be expected, in each case F_T grows more slowly than the corresponding M_T (see (2.4), (2.6), (2.8), (2.11), (2.13), (2.14) and (2.2)(a–d)).

So far we have assumed that μ is of the form (2.1). It is rather clear that for $\gamma < d$ we can take $\mu(\mathrm{d}x) = |x|^{-\gamma}\mathrm{d}x$. Moreover, a careful analysis of the proofs shows that in this case μ can have a more general form, given in the following proposition. (For the case $\gamma = d$, see the discussion after the proof of Proposition 2.9.)

Proposition 2.9. *All the previous results for the case* $\gamma < d$ *remain true (with possibly different constants* K*) for an intensity measure* μ *of the form*

$$\mu(\mathrm{d}x) = \nu(\mathrm{d}x) + \frac{h(x)}{1 + |x|^\gamma}\mathrm{d}x, \tag{2.15}$$

where ν *is a finite measure, and* h *is a non-negative bounded function such that there exists a strictly positive limit*

$$\lim_{R\to\infty} \frac{1}{R^d}\int_{|x|\le R} h(x)\mathrm{d}x. \tag{2.16}$$

It seems interesting and perhaps unexpected that it is not sufficient to assume that $\mu(\mathrm{d}x) = g(x)\mathrm{d}x$ with

$$\frac{C_1}{1 + |x|^\gamma} \le g(x) \le \frac{C_2}{1 + |x|^\gamma}. \tag{2.17}$$

We have the following counterexample.

Example 2.10. Let $\gamma < d < \alpha$ ($d = 1$) and let μ be of the form (2.15) with $\nu \equiv 0$, and

$$h(x) = \begin{cases} 1 & \text{for } |x| \le 4, \\ 1 & \text{for } (2k)^{2k} < |x| \le (2k+1)^{2k+1}, \\ 2 & \text{for } (2k+1)^{2k+1} < |x| \le (2(k+1))^{2(k+1)}, \end{cases}$$

$k = 1, 2, \ldots$. The limit (2.16) does not exist for this measure, whereas (2.17) obviously holds. The only non-trivial normalization (cf. Theorem 2.2) is that given by (2.4), but we will explain later that the corresponding X_T does not converge as $T \to \infty$.

There remains the case $\gamma > d$. Here the situation changes dramatically and the results are of an entirely different nature. In particular, they do not depend on γ, but only on the fact that the measure μ is finite. Therefore we will formulate our results for a general finite measure μ. It turns out that the appropriate normalization is

$$F_T = T^{1-1/\alpha} \tag{2.18}$$

if $1 = d < \alpha$, and

$$F_T = \log T \tag{2.19}$$

if $d = \alpha$.

By Proposition 2.1 we know that in both cases $\frac{1}{F_T} E \int_0^T \langle N_s, \varphi\rangle \mathrm{d}s$ converges to a finite non-zero limit as $T \to \infty$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\int \varphi \neq 0$ (see (2.2)(e) and (f)); hence there is no reason to consider fluctuation processes and it suffices to investigate the occupation process

$$Y_T(t) = \frac{1}{F_T} \int_0^{Tt} N_s \mathrm{d}s. \tag{2.20}$$

For a given $\alpha > 1$, let L denote the local time process (at 0) of a standard real α-stable process. See, e.g., [1] for properties of L. In particular, L is a continuous increasing process, $L(0) = 0$.

The relation between the processes Y_T and L is given in the following theorem.

Theorem 2.11. *Let $1 = d < \alpha$ and μ be a finite measure on $\mathbb{R}$. Let $L_1, L_2, \ldots$ be independent copies of L and let ν be a Poisson random variable with parameter $\mu(\mathbb{R})$ independent of $L_1, L_2, \ldots$. Then for F_T defined by (2.18),*

$$Y_T \Rightarrow_C K \sum_{j \le \nu} L_j \lambda$$

as $T \to \infty$.

This theorem is based on the following lemma, which is of interest in itself.

Lemma 2.12. *Let α, d and F_T be as in Theorem 2.11, let ζ be a real standard α-stable process, and denote by Z_T its normalized occupation process, i.e.,*

$$\langle Z_T(t), \varphi\rangle = \frac{1}{F_T} \int_0^{tT} \varphi(\zeta_s) \mathrm{d}s, \quad \varphi \in \mathcal{S}(\mathbb{R}^d), t \ge 0. \tag{2.21}$$

Then

$$Z_T \Rightarrow_C K L \lambda \tag{2.22}$$

as $T \to \infty$.

This lemma is closely related to the famous Darling–Kac result [11]. Their theorem was generalized by Bingham [2], who obtained the limit in path space for more general Markov processes and for fixed positive φ with compact support. For such φ this result can also (and more easily) be obtained using the self-similarity of ζ (see (3.2) below) and the fact that the local time of ζ at x, $L(t, x)$, is a continuous occupation density. Fitzsimmons and Getoor [16] mention the limit of (2.21) for fixed general φ. We will present an outline of a proof of the lemma in the next section.

It remains to consider the case $d = \alpha$.

Theorem 2.13. *Let $d = \alpha$ (=1 or 2) and μ be a finite measure. Let $\rho_1, \rho_2, \ldots$ be i.i.d. standard exponential random variables and ν a Poisson random variable with parameter $\mu(\mathbb{R})$, independent of $\rho_1, \rho_2, \ldots$. Then for F_T defined by (2.19),*

$$Y_T \Rightarrow K \sum_{j \le \nu} \rho_j \lambda$$

in $C([\varepsilon, \tau], \mathcal{S}'(\mathbb{R}^d))$ as $T \to \infty$ for any $0 < \varepsilon < \tau$.

So the limit process is constant in time on $(0, \infty)$.

Remark 2.14. (a) As we have noticed, for $\gamma < d$, the case of the measure μ of the form (2.1) is essentially the same as the case of $\mu(\mathrm{d}x) = |x|^{-\gamma}\mathrm{d}x$. For the latter measure one can also consider $\gamma < 0$, provided that $|\gamma| < \alpha$ if $\alpha < 2$ (this condition ensures finiteness of the mean). It turns out that Theorems 2.2, 2.4 and 2.5 hold for such γ, too. Also the proofs are essentially the same, and therefore we omit them.

(b) Observe that in all the cases where the norming is of the form $F_T = (\log T)^\kappa$, the limit process is constant in time on $(0, \infty)$. It is clear that no other form of the limit could be expected in this case.

3. Proofs

3.1. *Proof of Proposition* 2.1

(a) Given $0 < \alpha \le 2$, let $\mathcal{T}_t$ denote the transition semigroup of the standard α-stable process ζ in $\mathbb{R}^d$, i.e., $\mathcal{T}_t\varphi = p_t * \varphi$, where p_t is the transition density of ζ.

It is well known that, by the Poisson property,

$$E\langle N_t, \varphi\rangle = \int_{\mathbb{R}^d} \mathcal{T}_t\varphi(x)\mu(\mathrm{d}x), \quad \varphi \in \mathcal{S}(\mathbb{R}^d). \tag{3.1}$$

Hence, for μ of the form (2.1), by the self-similarity of the α-stable density, i.e.,

$$p_{at}(x) = a^{-d/\alpha} p_t(xa^{-1/\alpha}), \tag{3.2}$$

we have

$$E\langle N_t, \varphi\rangle = t^{-d/\alpha} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_1((x-y)t^{-1/\alpha})\varphi(y)\mathrm{d}y \frac{\mathrm{d}x}{1+|x|^\gamma}. \tag{3.3}$$

This proves the assertion in the case $\gamma > d$ since p_1 is bounded.

Next, assume $\gamma < d$. Substituting $xt^{-1/\alpha} = x'$ in (3.3) we obtain

$$t^{\gamma/\alpha} E\langle N_t, \varphi\rangle = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_1(x - yt^{-1/\alpha})\varphi(y)\frac{t^{\gamma/\alpha}}{1 + t^{\gamma/\alpha}|x|^\gamma}\mathrm{d}y\mathrm{d}x. \tag{3.4}$$

We use, consecutively, a well-known estimate,

$$p_1(x) \le \frac{C}{1 + |x|^{d+\alpha}}, \tag{3.5}$$

and an obvious inequality,

$$\frac{1}{1 + |x - y|^{d+\alpha}} \le C_1 \frac{1 + |y|^{d+\alpha}}{1 + |x|^{d+\alpha}}. \tag{3.6}$$

Hence, the integrand in (3.4) is estimated by

$$C_2 \frac{1}{(1 + |x|^{d+\alpha})|x|^\gamma}(1 + |y|^{d+\alpha})|\varphi(y)|,$$

which is integrable on $\mathbb{R}^{2d}$. So, by the dominated convergence theorem we obtain

$$\lim_{t\to\infty} t^{\gamma/\alpha} E\langle N_t, \varphi\rangle = \int_{\mathbb{R}^d} p_1(x)|x|^{-\gamma}\mathrm{d}x \int_{\mathbb{R}^d}\varphi(y)\mathrm{d}y.$$

It remains to consider the case $\gamma = d$. Without loss of generality we may assume $\varphi \ge 0$. We use the following simple fact checked with the L'Hôpital rule:

If f is a real continuous function on $\{x \in \mathbb{R}^d : |x| \le 1\}$, then

$$\lim_{t\to\infty}\frac{1}{\log t}\int_{|x|\le 1} f(x)\frac{t^{d/\alpha}}{1 + t^{d/\alpha}|x|^d}\mathrm{d}x = \frac{d}{\alpha} f(0)\int_{\mathbb{R}^d}(1 + |x|^d)^{-2}\mathrm{d}x.$$

Now, an elementary argument (splitting the integrals appropriately) permits us to show that

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{m_t}E\langle N_t, \varphi\rangle &= \lim_{t\to\infty}\frac{1}{\log t}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_1(x - yt^{-1/\alpha})\varphi(y)\frac{t^{d/\alpha}}{1 + t^{d/\alpha}|x|^d}\mathrm{d}y\mathrm{d}x\\ &= \frac{d}{\alpha}p_1(0)\int_{\mathbb{R}^d}(1 + |x|^d)^{-2}\mathrm{d}x\int_{\mathbb{R}^d}\varphi(y)\mathrm{d}y.\end{aligned}$$

This completes the proof of part (a) of the proposition.

(b) The existence of the limit (2.3) in the cases (2.2)(a)–(f) follows from part (a) by the L'Hôpital rule. If $\gamma > \alpha$ and $d > \alpha$ (case (2.2)(g)), for $\varphi \ge 0$ we have, by (3.1),

$$E\int_0^\infty \langle N_s, \varphi\rangle \mathrm{d}s = \int_0^\infty\int_{\mathbb{R}^d}\mathcal{T}_s\varphi(x)\frac{1}{1 + |x|^\gamma}\mathrm{d}x\mathrm{d}s = \int_{\mathbb{R}^d} G\varphi(x)\frac{1}{1 + |x|^\gamma}\mathrm{d}x, \tag{3.7}$$

since

$$\int_0^\infty \mathcal{T}_s\varphi\mathrm{d}s = G\varphi \tag{3.8}$$

(see (2.10)). The right-hand side of (3.7) is finite because

$$G\varphi(x) \le \frac{C}{1 + |x|^{d-\alpha}}$$

for $d > \alpha$ ([19], Lemma 5.3). □

3.2. General scheme

We describe a general method used in the proofs of Theorems 2.2 and 2.4–2.8 and Proposition 2.9.

For a continuous $\mathcal{S}'(\mathbb{R}^d)$-valued process X we define an $\mathcal{S}'(\mathbb{R}^{d+1})$ random variable $\widetilde{X}$ by

$$\langle \widetilde{X}, \Phi \rangle = \int_0^\tau \langle X(t), \Phi(\cdot, t)\rangle \mathrm{d}t, \qquad \Phi \in \mathcal{S}(\mathbb{R}^{d+1}). \tag{3.9}$$

As explained in [3], in order to prove $X_T \Rightarrow_C X$, where X is the limit process occurring in the specific case, it suffices to show that

$$\langle \widetilde{X}_T, \Phi \rangle \Rightarrow \langle \widetilde{X}, \Phi \rangle, \qquad \Phi \in \mathcal{S}(\mathbb{R}^{d+1}), \tag{3.10}$$

and that the family $\{\langle X_T, \varphi \rangle\}_{T \geq 2}$ is tight in $C([0, \tau], \mathbb{R})$ for any $\tau > 0$, for each $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

This scheme should be modified in an obvious way if we consider $C([\varepsilon, \tau], \mathcal{S}'(\mathbb{R}^d))$. Without loss of generality we will always assume $\tau = 1$.

Since the limits are Gaussian, in order to obtain (3.10) it suffices to show that

$$\lim_{T\to\infty} E\mathrm{e}^{-\langle \widetilde{X}_T, \Phi \rangle} = E\mathrm{e}^{-\langle \widetilde{X}, \Phi \rangle} \tag{3.11}$$

for any non-negative $\Phi \in \mathcal{S}(\mathbb{R}^{d+1})$ (see, e.g., [5]).

Given such Φ we define

$$\Psi(x, t) = \int_t^1 \Phi(x, s)\mathrm{d}s, \qquad \Psi_T(x, t) = \frac{1}{F_T} \Psi\left(x, \frac{t}{T}\right). \tag{3.12}$$

By (1.2), (3.1) and (3.9), we have

$$\langle \widetilde{X}_T, \Phi \rangle = \int_0^T \langle N_u, \Psi_T(\cdot, u)\rangle \mathrm{d}u - \int_0^T \int_{\mathbb{R}^d} \mathcal{T}_u \Psi_T(\cdot, u)\mu(\mathrm{d}x)\mathrm{d}u. \tag{3.13}$$

Hence, by the compound Poisson structure of the system,

$$E\mathrm{e}^{-\langle \widetilde{X}_T, \Phi \rangle} = \exp\left\{\int_0^T \int_{\mathbb{R}^d} \mathcal{T}_u \Psi_T(\cdot, u)(x)\mu(\mathrm{d}x)\mathrm{d}u\right\} \exp\left\{-\int_{\mathbb{R}^d} v_T(x, T)\mu(\mathrm{d}x)\right\}, \tag{3.14}$$

where

$$v_T(x, t) = 1 - E\exp\left\{-\int_0^t \Psi_T(x + \zeta_u, T - t + u)\mathrm{d}u\right\}, \qquad 0 \leq t \leq T. \tag{3.15}$$

(Recall that ζ is the standard α-stable process.)

We know that (repeating the argument of [5] for $V = 0$), by the Feynman–Kac formula, v_T satisfies

$$v_T(x, t) = \int_0^t \mathcal{T}_{t-s}(\Psi_T(\cdot, T - s)(1 - v_T(\cdot, s)))(x)\mathrm{d}s. \tag{3.16}$$

We will often use an immediate consequence of (3.15) and (3.16):

$$v_T(x, t) \leq \int_0^t \mathcal{T}_{t-s} \Psi_T(\cdot, T - s)(x)\mathrm{d}s. \tag{3.17}$$

Putting (3.16) into (3.14) and then using (3.16) once more we obtain

$$Ee^{-\langle \widetilde{X}_T, \Phi\rangle} = e^{I(T)-II(T)}, \tag{3.18}$$

where

$$I(T) = \int_{\mathbb{R}^d}\int_0^T \mathcal{T}_{T-s}\left(\Psi_T(\cdot, T-s)\int_0^s \mathcal{T}_{s-u}\Psi_T(\cdot, T-u)\mathrm{d}u\right)(x)\mathrm{d}s\mu(\mathrm{d}x) \tag{3.19}$$

and

$$II(T) = \int_{\mathbb{R}^d}\int_0^T \mathcal{T}_{T-s}\left(\Psi_T(\cdot, T-s)\int_0^s \mathcal{T}_{s-u}(\Psi_T(\cdot, T-u)v_T(\cdot, u))\mathrm{d}u\right)(x)\mathrm{d}s\mu(\mathrm{d}x). \tag{3.20}$$

To prove (3.11) we will show that

$$\lim_{T\to\infty} e^{I(T)} = Ee^{-\langle \widetilde{X}, \Phi\rangle}, \tag{3.21}$$

and

$$\lim_{T\to\infty} II(T) = 0. \tag{3.22}$$

For simplicity we will prove (3.21) and (3.22) for Φ of the form

$$\Phi(x,t) = \varphi(x)\psi(t), \quad \varphi \in \mathcal{S}(\mathbb{R}^d), \psi \in \mathcal{S}(\mathbb{R}), \varphi, \psi \geq 0. \tag{3.23}$$

It will be clear from the proofs that for general Φ the argument is analogous. For Φ of the form (3.23) it will be convenient to define

$$\chi(t) = \int_t^1 \psi(s)\mathrm{d}s, \qquad \chi_T(t) = \chi\left(\frac{t}{T}\right), \tag{3.24}$$

and then

$$\Psi_T(x,t) = \frac{1}{F_T}\varphi(x)\chi_T(t). \tag{3.25}$$

Note that expressions $I(T)$ and $II(T)$ have more complicated forms than those corresponding to $\gamma = 0$ [5,6], since the measure μ is not invariant under $\mathcal{T}_t$ and it is infinite, so in particular the Fourier transform technique that we have used before is not applicable.

In order to prove tightness of $\{\langle X_T, \varphi\rangle\}_{T\geq 2}$ for a given $\varphi \in \mathcal{S}(\mathbb{R}^d), \varphi \geq 0$ (it suffices to take φ non-negative), we need a formula for the Laplace transform of $\langle X_T(t_2) - X_T(t_1), \varphi\rangle$ for $0 \leq t_1 < t_2 \leq 1$. We take $\Psi_{T,n}$ of the form (3.25) with $\theta\varphi$ instead of φ ($\theta > 0$), and with smooth χ_n approximating $\chi = \mathbb{1}_{[t_1,t_2]}$. Using (3.16), (3.14) and (3.13) and letting $n \to \infty$ we obtain

$$Ee^{-\theta\langle X_T(t_2)-X_T(t_1),\varphi\rangle} = e^{H_T(\theta)}, \tag{3.26}$$

where

$$H_T(\theta) = \frac{\theta}{F_T}\int_{\mathbb{R}^d}\int_0^T \mathcal{T}_{T-s}(\varphi\chi_T(T-s)v_{\theta,T}(\cdot, s))(x)\mathrm{d}s\mu(\mathrm{d}x), \tag{3.27}$$

and $v_{\theta,T}$ is defined by (3.15) for $\Psi_T(x,t) = \theta\varphi(x)\chi_T(t)$. This $v_{\theta,T}$ also satisfies (3.16).

Unlike [6], where fourth moments were employed, we need moments of $\langle X_T(t_2)-X_T(t_1),\varphi\rangle$ of arbitrarily high order. By (3.26) we have

$$E\langle X_T(t_2)-X_T(t_1),\varphi\rangle^k = (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}\theta^k}\mathrm{e}^{H_T(\theta)}\bigg|_{\theta=0}, \quad k=1,2,\ldots. \tag{3.28}$$

Using (3.16) and (3.27) we have

$$\begin{aligned}
&H_T(0)=0,\\
&H_T'(0)=0,\\
&H_T^{(k)}(0) = (-1)^k\frac{k!}{F_T^k}\int_{\mathbb{R}^d}\int_0^T\int_0^{s_k}\cdots\int_0^{s_2}\mathcal{T}_{T-s_k}(\varphi\mathcal{T}_{s_k-s_{k-1}}(\varphi\ldots\mathcal{T}_{s_2-s_1})\ldots)(x)\\
&\qquad\times\chi_T(T-s_k)\ldots\chi_T(T-s_1)\mathrm{d}s_1\ldots\mathrm{d}s_k\mu(\mathrm{d}x), \quad k\geq 2.
\end{aligned} \tag{3.29}$$

By (3.28) and (3.29), tightness will be proved if we show that there exists $\delta>0$ such that

$$|H_T^{(k)}(0)|\leq C(k,\varphi)(t_2-t_1)^{k\delta} \quad \text{for } k=2,3,\ldots. \tag{3.30}$$

The scheme described above is employed in the proofs of all results for $\gamma\leq d$. The proof of each specific case, however, requires slightly different and non-trivial calculations; nevertheless, for brevity we will omit some proofs, concentrating on arguments which are either the most typical or the most involved.

3.3. *Proof of Theorem* 2.2

We will prove the theorem for $\mu(\mathrm{d}x)=|x|^{-\gamma}\mathrm{d}x$ since in this case the formulas are slightly simpler. It will be obvious that the same type of argument applies for μ of the form (2.1). It is easy to see by straightforward calculation that in this case the right-hand side of (3.21) with Φ given by (3.23) is of the form

$$\exp\left\{K_1\left(\int\varphi(x)\mathrm{d}x\right)^2\int_0^1\int_0^u(u-s)^{-d/\alpha}s^{-\gamma/\alpha}\chi(s)\chi(u)\mathrm{d}s\mathrm{d}u\right\}. \tag{3.31}$$

Using (3.19) and (3.25) and substituting $u'=1-u/T$, $s'=1-s/T$ we obtain

$$\begin{aligned}
I(T) = {}&\frac{T^2}{F_T^2}\int_{\mathbb{R}^d}\int_0^1\int_0^u\int_{\mathbb{R}^{2d}}p_{Ts}(x-y)\varphi(y)p_{T(u-s)}(y-z)\\
&\times\varphi(z)\chi(s)\chi(u)|x|^{-\gamma}\mathrm{d}y\mathrm{d}z\mathrm{d}s\mathrm{d}u\mathrm{d}x.
\end{aligned}$$

We apply the self-similarity of the α-stable density (3.2), substitute $x'=xT^{-1/\alpha}$ and use (2.4); then

$$I(T)=\int_{\mathbb{R}^d}g_T(x)|x|^{-\gamma}\mathrm{d}x, \tag{3.32}$$

where

$$\begin{aligned}
g_T(x) = {}&\int_0^1\int_0^u\int_{\mathbb{R}^{2d}}\chi(s)\chi(u)p_s(x-yT^{-1/\alpha})p_{u-s}((y-z)T^{-1/\alpha})\\
&\times\varphi(y)\varphi(z)\mathrm{d}y\mathrm{d}z\mathrm{d}s\mathrm{d}u.
\end{aligned} \tag{3.33}$$

By self-similarity again the integrand in (3.33) is bounded by $Cs^{-d/\alpha}(u-s)^{-d/\alpha}\varphi(y)\varphi(z)$, which is integrable since $d < \alpha$. Hence, by the dominated convergence theorem we obtain

$$\begin{aligned}\lim_{T\to\infty} g_T(x) = g_\infty(x) &= \int_0^1 \int_0^u \int_{\mathbb{R}^{2d}} \chi(s)\chi(u) p_s(x) p_{u-s}(0)\varphi(y)\varphi(z)\mathrm{d}y\mathrm{d}z\mathrm{d}s\mathrm{d}u \\ &= p_1(0) \int_0^1 \int_0^u \chi(s)\chi(u) s^{-d/\alpha}(u-s)^{-d/\alpha} p_1(xs^{-1/\alpha})\mathrm{d}s\mathrm{d}u \left(\int_{\mathbb{R}^d} \varphi(y)\mathrm{d}y\right)^2 .\end{aligned} \tag{3.34}$$

From (3.2) and (3.5) we easily deduce that for $d < \alpha$,

$$\int_0^1 p_s(x)\mathrm{d}s \le \frac{C_1}{1+|x|^{d+\alpha}}. \tag{3.35}$$

This and (3.6) (we take $yT^{-1/\alpha}$ instead of y) imply that

$$g_T(x) \le C_3 \frac{1}{1+|x|^{d+\alpha}}. \tag{3.36}$$

By (3.32), (3.34) and (3.36) and taking into account (3.31), we obtain (3.21).

We proceed to the proof of (3.22). We use (3.20), (3.17) and (3.25) and boundedness of χ to get

$$II(T) \le \frac{C}{F_T^3} \int_{\mathbb{R}^d} \int_0^T \int_0^s \int_0^u \mathcal{T}_{T-s}(\varphi \mathcal{T}_{s-u}(\varphi \mathcal{T}_{u-v}\varphi))(x)\mathrm{d}v\mathrm{d}u\mathrm{d}s|x|^{-\gamma}\mathrm{d}x. \tag{3.37}$$

We substitute $v' = \frac{u-v}{T}$, then $u' = \frac{s-u}{T}$, then $s' = \frac{1-s}{T}$, and we increase the time intervals to $[0, 1]$, obtaining

$$\begin{aligned} II(T) \le C\frac{T^3}{F_T^3} &\int_{\mathbb{R}^d} \int_{\mathbb{R}^{3d}} \int_0^1 p_{Ts}(x-y)\mathrm{d}s\varphi(y) \int_0^1 p_{Tu}(y-z)\mathrm{d}u\varphi(z) \\ &\times \int_0^1 p_{Tv}(z-w)\mathrm{d}v\varphi(w)\mathrm{d}y\mathrm{d}z\mathrm{d}w|x|^{-\gamma}\mathrm{d}x. \end{aligned} \tag{3.38}$$

Define

$$f(x) = \int_0^1 p_s(x)\mathrm{d}s \tag{3.39}$$

and

$$\widetilde{\varphi}_T(x) = T^{d/\alpha}\varphi(T^{1/\alpha}x). \tag{3.40}$$

Note that f is integrable, and by (3.35) it is bounded, and

$$\int_{\mathbb{R}^d} \widetilde{\varphi}_T(x)\mathrm{d}x = \int_{\mathbb{R}^d} \varphi(x)\mathrm{d}x. \tag{3.41}$$

Using (3.2), substituting $x' = xT^{-1/\alpha}$, $y' = yT^{-1/\alpha}$, $z' = zT^{-1/\alpha}$, $w' = wT^{-1/\alpha}$, we write (3.38) as

$$II(T) \leq C \frac{T^{3-(2d+\gamma)/\alpha}}{F_T^3} \int_{\mathbb{R}} a_T(x)|x|^{-\gamma} \mathrm{d}x, \tag{3.42}$$

where

$$a_T(x) = f * (\widetilde{\varphi}_T * (f * (\widetilde{\varphi}_T (f * \widetilde{\varphi}_T))))(x).$$

The properties of f and $\widetilde{\varphi}_T$ easily imply that

$$\sup_T \int_{\mathbb{R}^d} a_T(x)\mathrm{d}x < \infty \quad \text{and} \quad \sup_T \sup_{x\in\mathbb{R}^d} a_T(x) < \infty.$$

Hence (3.22) follows from (3.42) since $\gamma < d$ and $T^{3-(2d+\gamma)/\alpha}/F_T^3 \to 0$ (see (2.4)). This completes the proof of (3.10).

According to the general scheme, in order to prove tightness we show (3.30). We substitute $s_j' = 1 - \frac{s_j}{T}$ in (3.29) and we obtain

$$|H_T^{(k)}(0)| = k! \frac{T^k}{F_T^k} \int_{\mathbb{R}^d} \int_0^1 \int_{s_k}^1 \dots \int_{s_2}^1 \mathcal{T}_{Ts_k}(\varphi \mathcal{T}_{T(s_{k-1}-s_k)}(\varphi \dots) \dots)(x)$$
$$\times \chi(s_k) \dots \chi(s_1) \mathrm{d}s_1 \dots \mathrm{d}s_k \frac{1}{|x|^\gamma} \mathrm{d}x. \tag{3.43}$$

We need the following estimate:

$$\frac{T}{F_T} \int_s^1 \mathcal{T}_{T(u-s)} \varphi(y) \chi(u) \mathrm{d}u \leq C T^{(\gamma-d)/2\alpha} (t_2 - t_1)^{1-d/\alpha} \tag{3.44}$$

(recall that $\chi = \mathbb{1}_{[t_1,t_2]}$). By (2.4) and (3.2) and boundedness of p_1 we have

$$\frac{T}{F_T} \int_s^1 \mathcal{T}_{T(u-s)} \varphi(y) \chi(u) \mathrm{d}u \leq C T^{(\gamma-d)/2\alpha} \int_{\mathbb{R}^d} \varphi(z) \mathrm{d}z \int_s^1 (u-s)^{-d/\alpha} \chi(u) \mathrm{d}u. \tag{3.45}$$

Hence (3.44) follows. We iterate (3.44) $k-1$ times in (3.43), estimate $(T^{(\gamma-d)/2\alpha})^{k-2}$ by 1, arriving at

$$|H_T^{(k)}(0)| \leq C(t_2 - t_1)^{(k-1)(1-d/\alpha)} T^{(\gamma-d)/2\alpha} \frac{T}{F_T}$$
$$\times \int_{\mathbb{R}^d} \int_0^1 \int_{\mathbb{R}^d} p_{Ts_k}(x-y) \varphi(y) \mathrm{d}y \mathrm{d}s_k |x|^{-\gamma} \mathrm{d}x$$
$$= C(t_2 - t_1)^{(k-1)(1-d/\alpha)} \int_{\mathbb{R}^d} f * \widetilde{\varphi}_T(x) |x|^{-\gamma} \mathrm{d}x,$$

where we have used (2.4), self-similarity and the usual substitutions $x' = xT^{-1/\alpha}$, $y' = yT^{-1/\alpha}$, where f and $\widetilde{\varphi}_T$ are defined by (3.39) and (3.40). Hence we obtain (3.30) by the properties of f and $\widetilde{\varphi}_T$.

The proof of the theorem is complete. □

3.4. Some properties of the α-stable semigroup in the critical case $d = \alpha$

We will need the following facts, valid for $d = \alpha$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\varphi \geq 0$:

$$\sup_{T>2}\sup_{x\in\mathbb{R}^d}\frac{1}{\log T}\int_0^T \mathcal{T}_u\varphi(x)\mathrm{d}u<\infty, \tag{3.46}$$

$$\lim_{T\to\infty}\frac{1}{\log T}\int_0^T \mathcal{T}_u\varphi(x)\mathrm{d}u=p_1(0)\int_{\mathbb{R}^d}\varphi(y)\mathrm{d}y, \tag{3.47}$$

$$\lim_{T\to\infty}\frac{1}{\log T}\int_{|x|^d>T}\int_0^T \mathcal{T}_u\varphi(x)\mathrm{d}u|x|^{-d}\mathrm{d}x=0. \tag{3.48}$$

These properties are perhaps known but we have not been able to find references for them, so we show briefly how to derive them.

To prove (3.46) and (3.47), it is clear that it suffices to consider $\int_1^T$. We use self-similarity, then make a substitution which turns out to be particularly useful in the critical cases and will be applied several times. Namely, we put

$$u'=\frac{\log u}{\log T}, \tag{3.49}$$

obtaining

$$\frac{1}{\log T}\int_1^T \mathcal{T}_u\varphi(x)\mathrm{d}u=\int_0^1\int_{\mathbb{R}^d}p_1\left((x-y)T^{-u/d}\right)\varphi(y)\mathrm{d}y\mathrm{d}u.$$

Hence (3.46) and (3.47) follow immediately.

To prove (3.48) we again replace $\int_0^T$ by $\int_1^T$ and make the substitution (3.49). We then have

$$\begin{aligned}&\frac{1}{\log T}\int_{|x|^d>T}\int_1^T \mathcal{T}_u\varphi(x)\mathrm{d}u|x|^{-d}\mathrm{d}x\\ &=\int_{|x|^d>T}\int_0^1\int_{\mathbb{R}^d}p_1((x-y)T^{-u/d})\varphi(y)|x|^{-d}\mathrm{d}y\mathrm{d}u\mathrm{d}x,\\ &\le C\int_{|x|^d>T}\int_0^1\int_{\mathbb{R}^d}\frac{1}{1+|x|^{2d}T^{-2u}}(1+|y|^{2d}T^{-2u})\varphi(y)|x|^{-d}\mathrm{d}y\mathrm{d}u\mathrm{d}x,\end{aligned}$$

where the last estimate follows from (3.5) and (3.6) (recall that $d=\alpha$). As $\varphi\in\mathcal{S}(\mathbb{R}^d)$, this expression is estimated by

$$C_1\int_{|x|^d>T}|x|^{-3d}\mathrm{d}x\int_0^1 T^{2u}\mathrm{d}u\le\frac{C_2}{\log T}$$

by calculus. This proves (3.48).

3.5. *Proof of Theorem 2.4*

We will present only an outline of the proof.

Following the general scheme, and again taking for simplicity $\mu(\mathrm{d}x)=|x|^{-\gamma}\mathrm{d}x$, we prove that (see (3.19), (3.24) and (3.25))

$$\lim_{T\to\infty}I(T)=K_1\int_0^1 s^{-\gamma/\alpha}\chi^2(s)\mathrm{d}s\left(\int_{\mathbb{R}^d}\varphi(x)\mathrm{d}x\right)^2. \tag{3.50}$$

In (3.19) we substitute $u' = T - u, s' = 1 - \frac{s}{T}$, use self-similarity and put $x' = xT^{-1/\alpha}s^{-1/\alpha}$. By (2.6) we obtain

$$I(T) = \frac{1}{\log T}\int_{\mathbb{R}^d}\int_0^1\int_{\mathbb{R}^d}\int_0^{T(1-s)} \chi(s)\chi\left(s+\frac{u}{T}\right)s^{-\gamma/\alpha}p_1(x - ys^{-1/\alpha}T^{-1/\alpha}) \times \varphi(y)\mathcal{T}_u\varphi(y)|x|^{-\gamma}\mathrm{d}u\mathrm{d}y\mathrm{d}s\mathrm{d}x. \tag{3.51}$$

Using

$$\sup_{z\in\mathbb{R}^d}\int_{\mathbb{R}^d} p_1(x+z)|x|^{-\gamma}\mathrm{d}x < \infty, \tag{3.52}$$

it is easy to see that

$$\lim_{T\to\infty} I(T) = \lim_{T\to\infty} I'(T),$$

where

$$I'(T) = \frac{1}{\log T}\int_{\mathbb{R}^d}\int_0^{1-\frac{1}{T}}\int_{\mathbb{R}^d}\int_1^{T(1-s)} \chi(s)\chi\left(s+\frac{u}{T}\right)s^{-\gamma/\alpha}p_1(x - ys^{-1/\alpha}T^{-1/\alpha}) \times \varphi(y)\mathcal{T}_u\varphi(y)|x|^{-\gamma}\mathrm{d}u\mathrm{d}y\mathrm{d}s\mathrm{d}x.$$

We use self-similarity again and make substitution (3.49). Then

$$I'(T) = \int_{\mathbb{R}^d}\int_0^{1-\frac{1}{T}}\int_{\mathbb{R}^d}\int_0^{\frac{\log T(1-s)}{\log T}} \chi(s)\chi(s+T^{u-1})s^{-\alpha/\gamma}p_1(x - ys^{-1/\alpha}T^{-1/\alpha}) \times \varphi(y)\int_{\mathbb{R}^d} p_1((y-z)T^{-u/\alpha})\varphi(z)\mathrm{d}z\mathrm{d}u\mathrm{d}y\mathrm{d}s|x|^{-\gamma}\mathrm{d}x.$$

Now it is clear that the limit of $I'(T)$ should have the form (3.50). We omit details.

We proceed to (3.22). We use (3.37), substitute $v' = u - v$, then $u' = s - u$, then $s' = \frac{1-s}{T}$, and increase the time intervals appropriately, obtaining

$$II(T) \le C\frac{T}{F_T^3}\int_{\mathbb{R}^d}\int_0^1 \mathcal{T}_{Ts}\left(\varphi\int_0^T \mathcal{T}_u\left(\varphi\int_0^T \mathcal{T}_v\varphi\mathrm{d}v\right)\mathrm{d}u\right)(x)\mathrm{d}s|x|^{-\gamma}\mathrm{d}x. \tag{3.53}$$

By (3.46) applied twice we have

$$II(T) \le C_1\frac{T(\log T)^2}{F_T^3}\int_{\mathbb{R}^d}\int_0^1 \mathcal{T}_{Ts}\varphi(x)\mathrm{d}s|x|^{-\gamma}\mathrm{d}x.$$

Hence (3.22) follows by self-similarity, substitution $x' = x(Ts)^{-1/\alpha}$, (3.52) and (2.6).

Tightness is proved similarly to in Theorem 2.2. Here are the main steps. Instead of (3.42) we show that

$$\frac{T}{F_T}\int_s^1 \mathcal{T}_{T(u-s)}\varphi(y)\chi(u)\mathrm{d}u \le C(t_2 - t_1)^{\frac{1}{4}(1-\gamma/\alpha)},$$

we iterate this estimate $k-2$ times in (3.45), and we obtain

$$|H_T^{(k)}(0)| \le C_1(t_2 - t_1)^{(1-\gamma/\alpha)(k-2)/4}H''(0).$$

Using (3.46) it is not difficult to prove that

$$H_T''(0) \le C_2(t_2^{1-\gamma/\alpha} - t_1^{1-\gamma/\alpha}) \le C_2(t_2 - t_1)^{1-\gamma/\alpha}.$$

Hence (3.30) follows. □

3.6. *Proof of Theorem 2.5*

Again we give only a sketch of the proof. We make the same substitutions as at the beginning of the previous proof, and by (2.8) we obtain

$$I(T) = \int_{\mathbb{R}^d}\int_0^1\int_{\mathbb{R}^d}\int_0^{T(1-s)} \chi(s)\chi\left(s+\frac{u}{T}\right)s^{-\gamma/\alpha} \times p_1(x - ys^{-1/\alpha}T^{-1/\alpha})\varphi(y)\mathcal{T}_u\varphi(y)|x|^{-\gamma}\mathrm{d}u\mathrm{d}y\mathrm{d}s\mathrm{d}x$$

(cf. (3.51)). Hence it is not hard to see that

$$\lim_{T\to\infty} I(T) = \int_{\mathbb{R}^d} p_1(x)|x|^{-\gamma}\mathrm{d}x \int_0^1 s^{-\gamma/\alpha}\chi^2(s)\mathrm{d}s \int_{\mathbb{R}^d}\varphi(y)G\varphi(y)\mathrm{d}y,$$

by (3.8). This implies (3.21). Next, by (3.53) (which is always valid),

$$II(T) \le C\frac{T}{F_T^3}\int_{\mathbb{R}^d}\int_0^1 \mathcal{T}_{Ts}(\varphi G(\varphi G\varphi))(x)\mathrm{d}s|x|^{-\gamma}\mathrm{d}x.$$

Hence (3.22) follows by the usual argument since $G\varphi$ is bounded.

Tightness can be proved in the same manner, even more easily, as in Theorem 2.4. □

3.7. *Comments on the proofs of Theorems 2.6 and 2.7*

The proof of Theorem 2.6, though by no means straightforward, is slightly simpler than the proof for the doubly critical case (Theorem 2.8), which will be given in detail. Therefore we omit it.

The proof of Theorem 2.7 is similar (but not identical) to the argument carried out for Theorem 2.2. We omit it for brevity.

3.8. *Proof of Theorem 2.8*

We apply the general scheme. By (3.19) and (3.23)–(3.25) and the substitutions $u' = s - u$, then $s' = T - s$, we have

$$I(T) = I_1(T) + I_2(T) + I_3(T) + I_4(T), \tag{3.54}$$

where

$$I_1(T) = \frac{1}{F_T^2}\int_{1\le|x|^d\le T}\int_1^{T-1}\int_1^{T-s}\mathcal{T}_s(\varphi\mathcal{T}_u\varphi)(x) \times\chi\left(\frac{s}{T}\right)\chi\left(\frac{s}{T}+\frac{u}{T}\right)\frac{1}{1+|x|^d}\mathrm{d}u\mathrm{d}s\mathrm{d}x, \tag{3.55}$$

$$I_2(T) = \frac{1}{F_T^2}\int_{1\le|x|^d\le T}\left(\int_0^T\int_0^{T-s} - \int_1^{T-1}\int_1^{T-s}\right)\dots, \tag{3.56}$$

$$I_3(T) = \frac{1}{F_T^2} \int_{|x|^d > T} \int_0^T \int_0^{T-s} \dots, \tag{3.57}$$

$$I_4(T) = \frac{1}{F_T^2} \int_{|x|<1} \int_0^T \int_0^{T-s} \dots, \tag{3.58}$$

where ... denotes the same integrand as in $I_1(T)$.

We will show that

$$\lim_{T\to\infty} I_1(T) = K_1 \chi^2(0) \left(\int_{\mathbb{R}^d} \varphi(x) \mathrm{d}x \right)^2, \tag{3.59}$$

and the remaining integrals converge to 0.

By (2.14)

$$I_2(T) \le \frac{1}{(\log T)^3} \int_{|x|^d \le T} \left(\int_0^T \int_0^1 \dots \mathrm{d}u \mathrm{d}s + \int_0^1 \int_0^T \dots \mathrm{d}u \mathrm{d}s \right) \mathrm{d}x.$$

Using (3.46) we obtain

$$I_2(T) \le \frac{C_1}{(\log T)^2} \int_{|x|^d \le T} \frac{1}{1+|x|^d} \mathrm{d}x \le \frac{C_2}{\log T} \to 0 \quad \text{as } T \to \infty. \tag{3.60}$$

The fact that

$$\lim_{T\to\infty} I_3(T) = 0 \tag{3.61}$$

follows immediately from (3.46) and (3.48), and

$$\lim_{T\to\infty} I_4(T) = 0 \tag{3.62}$$

is also a consequence of (3.46).

By (3.55), (2.14) and (3.2) we have

$$I_1(T) = \frac{1}{(\log T)^3} \int_{1 \le |x|^d \le T} \int_{\mathbb{R}^{2d}} \int_1^{T-1} \int_1^{T-s} s^{-1}$$
$$\times p_1((x-y)s^{-1/d}) \varphi(y) u^{-1} p_1((y-z)u^{-1/d})$$
$$\times \varphi(z) \chi\left(\frac{s}{T}\right) \chi\left(\frac{s}{T} + \frac{u}{T}\right) \frac{1}{1+|x|^d} \mathrm{d}u \mathrm{d}s \mathrm{d}y \mathrm{d}z \mathrm{d}x.$$

We make the substitution (3.49) for both u and s, obtaining

$$I_1(T) = \frac{1}{\log T} \int_{1 \le |x|^d \le T} \int_{\mathbb{R}^{2d}} \int_0^{\frac{\log(T-1)}{\log T}} \int_0^{\frac{\log(T-T^s)}{\log T}} p_1((x-y)T^{-s/d}) p_1((y-z)T^{-u/d})$$
$$\times \varphi(y) \varphi(z) \chi(T^{s-1}) \chi(T^{s-1} + T^{u-1}) \frac{1}{1+|x|^d} \mathrm{d}u \mathrm{d}s \mathrm{d}y \mathrm{d}z \mathrm{d}x.$$

In the integral $\int \mathrm{d}x$ we pass to polar coordinates (r, w) ($r = |x|$) and then substitute $r' = r^d$.

We have

$$I_1(T) = \frac{1}{\log T} \frac{1}{d} \int_1^T \int_{S_{d-1}} \int_{\mathbb{R}^{2d}} \int_0^1 \int_0^1 \mathbb{1}_{[0, \frac{\log(T-1)}{\log T}]}(s) \mathbb{1}_{[0, \frac{\log(T-T^s)}{\log T}]}(u)$$

$$\times p_1((wr^{1/d} - y)T^{-s/d}) p_1((y - z)T^{-u/d})$$
$$\times \varphi(y)\varphi(z)\chi(T^{s-1})\chi(T^{s-1} + T^{u-1})\frac{1}{1+r} \mathrm{d}u\mathrm{d}s\mathrm{d}y\mathrm{d}z\sigma_{d-1}(\mathrm{d}w)\mathrm{d}r, \tag{3.63}$$

where σ_{d-1} is the Lebesgue measure on the unit sphere S_{d-1} in $\mathbb{R}^d$. Again, we use (3.49) putting $r' = \log r / \log T$; then it is easy to see that the integrand converges to

$$\mathbb{1}_{[0,1]}(s)\mathbb{1}_{[0,1]}(u)\mathbb{1}_{[0,s]}(r) p_1^2(0)\varphi(y)\varphi(z)\chi^2(0),$$

and is bounded by $C p_1^2(0)\varphi(y)\varphi(z)$. Hence (3.59) follows. By (3.54) and (3.59)–(3.62) we obtain (3.21).

Now we pass to the proof of (3.22). By (3.20) and (3.17) for Φ of the form (3.23), after obvious substitutions we have

$$II(T) \le \frac{C}{(\log T)^{9/2}} \int_{\mathbb{R}^d} \int_0^T \mathcal{T}_s \left(\varphi \int_0^T \mathcal{T}_u \left(\varphi \int_0^T \mathcal{T}_r \varphi \mathrm{d}r\right) \mathrm{d}u\right)(x)\mathrm{d}s \frac{1}{1+|x|^d}\mathrm{d}x.$$

Using (3.46) twice we get

$$II(T) \le \frac{C}{(\log T)^{5/2}} \int_{\mathbb{R}^d} \int_0^T \mathcal{T}_s\varphi(x)\mathrm{d}s \frac{1}{1+|x|^d}\mathrm{d}x$$
$$= A + B,$$

where, for large T,

$$A = \frac{C}{(\log T)^{5/2}} \int_{|x|^d > T} \cdots \le \frac{C_1}{(\log T)^{3/2}}$$

by (3.48), and

$$B = \frac{C}{(\log T)^{5/2}} \int_{|x|^d \le T} \cdots \le \frac{C_2}{(\log T)^{3/2}} \int_{|x|^d \le T} \frac{1}{1+|x|^d}\mathrm{d}x,$$
$$\le \frac{C_3}{(\log T)^{1/2}}. \tag{3.64}$$

In the first estimate in (3.60) we have used (3.46) once more. Hence (3.22) follows.

Passing to the proof of tightness, first observe that the method employed in the proof of (3.11) can be also used to obtain convergence of finite dimensional distributions of X_T. This fact has already been used in [8]; here we repeat briefly the argument. For $\varphi_1, \varphi_2, \ldots, \varphi_k \in \mathcal{S}(\mathbb{R}^d)$, all $\varphi_j > 0$, and $0 \le t_1 \le \cdots \le t_k \le 1$, it is easy to see that $E \exp\{-\sum_{j=1}^k \langle X_T(t_j), \varphi_j\rangle\}$ has the form (3.14) with

$$\Psi(x,t) = \sum_{j=1}^k \varphi_j(x)\mathbb{1}_{[0,t_j]}(t),$$

and the corresponding v_T given by (3.15).

Approximating Ψ by smooth functions we obtain that (3.16) holds, and then we argue as before.

In particular $X_T(\varepsilon)$ converges in law. Therefore, to prove tightness of X_T in $C([\varepsilon,1], \mathcal{S}'(\mathbb{R}^d))$ it suffices to show that $\langle X_T - X_T(\varepsilon), \varphi\rangle \Rightarrow 0$ in $C([\varepsilon,1])$, for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\varphi \ge 0$.

Define

$$w_T(t) = \frac{1}{F_T}\int_{\varepsilon T}^{tT} \langle N_s, \varphi\rangle \mathrm{d}s, \qquad t \geq \varepsilon.$$

By (1.2) it is clear that it is enough to show that w_T and Ew_T converge to 0 in law in $C([\varepsilon, 1])$. Both processes are increasing, so it suffices to prove that

$$\lim_{T\to\infty} Ew_T(1) = 0.$$

By (3.1) and substitution $x' = xs^{-1/\alpha}$,

$$\begin{aligned} Ew_T(1) &= \frac{1}{F_T}\int_{\varepsilon T}^{T}\int_{\mathbb{R}^{2d}} p_1(x - ys^{-1/\alpha})\varphi(y)\frac{1}{1+|x|^{\mathrm{d}}s}\mathrm{d}x\mathrm{d}y\mathrm{d}s \\ &= J_1(T) + J_2(T), \end{aligned}$$

where

$$J_1(T) = \frac{1}{F_T}\int_{\varepsilon T}^{T}\int_{|x|\leq 1}\int_{\mathbb{R}^d}\ldots,$$

$$J_2(T) = \frac{1}{F_T}\int_{\varepsilon T}^{T}\int_{|x|>1}\int_{\mathbb{R}^d}\ldots.$$

We have

$$J_2(T) \leq \frac{C}{F_T}\int_{\varepsilon T}^{T} s^{-1}\mathrm{d}s = C\frac{\log(1/\varepsilon)}{F_T} \to 0,$$

$$\begin{aligned} J_1(T) &\leq \frac{C_1}{F_T}\int_{\varepsilon T}^{T}\int_{|x|\leq 1}\frac{1}{1+|x|^d s}\mathrm{d}x\mathrm{d}s \\ &= \frac{C_2}{F_T}\int_{\varepsilon T}^{T}\int_0^1 \frac{1}{1+rs}\mathrm{d}r\mathrm{d}s \leq \frac{C_3\log(1/\varepsilon)}{(\log T)^{1/2}} \to 0. \quad \square \end{aligned}$$

3.9. Systems with more general intensity measures μ

In this section we consider a measure μ of the form (2.15). We sketch the proof of Proposition 2.9 and we discuss Example 2.10.

Proof of Proposition 2.9. We concentrate on the case $\gamma < d < \alpha$. The other cases will be mentioned later.

First notice that it suffices to assume that $\nu \equiv 0$ in (2.15), since it is easy to see that with our normalization the terms corresponding to ν will vanish in the limit. We repeat the steps of the proof of Theorem 2.2. Observe that boundedness of h implies that (3.37) also holds in the present case, and hence (3.22) is obtained in the same way as before. Also, the tightness is proved without any change.

It remains to show (3.21). Instead of (3.32) we have

$$I(T) = \int_{\mathbb{R}^d} g_T(x)\frac{T^{\gamma/\alpha}}{1+|xT^{1/\alpha}|^{\gamma}} h(T^{1/\alpha}x)\mathrm{d}x, \tag{3.65}$$

where g_T is defined by (3.33). We write

$$I(T) = I_1(T) + I_2(T), \tag{3.66}$$

where

$$I_1(T) = \int_{\mathbb{R}^d} g_\infty(x) \frac{1}{|x|^\gamma} h(T^{1/\alpha} x)\mathrm{d}x, \tag{3.67}$$

with g_∞ given by (3.34), and

$$I_2(T) = \int_{\mathbb{R}^d} \left(g_T(x) \frac{T^{\gamma/\alpha}}{1 + |xT^{1/\alpha}|^\gamma} - \frac{g_\infty(x)}{|x|^\gamma} \right) h(T^{1/\alpha} x)\mathrm{d}x.$$

(3.36) implies that $\lim_{T\to\infty} I_2(T) = 0$.

Note that from assumption (2.16) it follows that

$$\lim_{R\to\infty} \int_{\mathbb{R}^d} a(x)h(Rx)\mathrm{d}x = C \int_{\mathbb{R}^d} a(x)\mathrm{d}x \tag{3.68}$$

for $a(x) = \mathbb{1}_{|x|\le r}$, where C is the limit (2.16) divided by the volume of the unit ball in $\mathbb{R}^d$. Hence, it is easy to see that (3.68) also holds for any spherically symmetric integrable function a. The function $g_\infty(x)|x|^{-\gamma}$ is obviously spherically symmetric and integrable ($\gamma < d$ and (3.36)). Therefore (3.68) implies (3.21). This completes the proof in the case $\gamma < d < \alpha$.

In the remaining cases for $\gamma < d$, tightness and (3.22) follow immediately from the corresponding proofs for μ of the form (2.1). Also, to obtain (3.21) we repeat the same steps, obtaining $I(T)$ in an analogous form to (3.65). And then we apply (3.68). □

For $d = \gamma$ the method described above cannot be applied. For example, in the case $d = \gamma = 1 < \alpha$ the function $g_\infty(x)|x|^{-d}$ is not integrable. To prove (3.21) we would need existence of the limit

$$\lim_{T\to\infty} \frac{1}{\log T} \int_{|x|\le 1} g_\infty(x) \frac{T^{1/\alpha}}{1 + |xT^{1/\alpha}|} h(xT^{1/\alpha})\mathrm{d}x.$$

This limit is easy to obtain for $h \equiv 1$, but it is not clear how to formulate an elegant condition assuring its existence in a more general case.

The case $\gamma = d = \alpha$ is even more complicated because in (3.63) we would have $h(wr^{1/\alpha})$ under the integrals.

Proof of non-existence of the limit in Example 2.10. It is obvious that the only non-trivial normalization is that given by (2.4), since h is bounded and separated from 0. Analogously to in the previous proof, convergence of X_T is equivalent to convergence of $I_1(T)$ defined by (3.67). We will show that $I_1(T)$ does not converge. Let $T_n = n^{n\alpha+\alpha/2}$, $n = 2, 3, \ldots$. On the set $\{x : \frac{1}{\sqrt{n}} \le |x| \le \sqrt{n}\}$ we have

$$h(T_n^{1/\alpha} x) = u(n) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 2 & \text{if } n \text{ is odd.} \end{cases}$$

It is clear that $\lim_{n\to\infty}(I_1(T_n) - I_1'(n)) = 0$, where

$$I_1'(n) = u(n) \int_{\frac{1}{\sqrt{n}} \le |x| \le \sqrt{n}} g_\infty(x)|x|^{-\gamma}\mathrm{d}x,$$

and obviously $I_1'(n)$ does not converge. □

3.10. The finite measure case

Proof of Lemma 2.12. We improve slightly the proof of Lemma 2 of Bingham [2]. Let

$$\alpha' = 1 - \frac{1}{\alpha}. \tag{3.69}$$

It is easy to see using self-similarity that

$$\lim_{\theta\to 0} \theta^{\alpha'} \int_0^\infty \int_{\mathbb{R}} \mathrm{e}^{-\theta s} \varphi(y) p_s(x-y)\mathrm{d}y\mathrm{d}s = K \int_{\mathbb{R}} \varphi(x)\mathrm{d}x \tag{3.70}$$

for any $\varphi \in \mathcal{S}(\mathbb{R})$, but in general the convergence is not uniform in $x \in \mathbb{R}$ if φ is not compactly supported. Therefore condition (A) of Darling–Kac [11] is not satisfied, so, unlike Bingham, we cannot apply their theorem directly. We prove that

$$Z_T \Rightarrow_f KL\lambda \tag{3.71}$$

($\Rightarrow_f$ denotes convergence of finite dimensional distributions), where L is a continuous increasing process whose inverse is an α'-stable subordinator. On the other hand, it is known (see [1], Prop. 4, Ch. V; see also [15]) that such L is the local time process at 0 of ζ.

It is clear that in order to prove (3.71) it suffices to show that

$$(\langle Z(t_1), \varphi_1\rangle, \ldots, \langle Z(t_k), \varphi_k\rangle) \Rightarrow K\left(L(t_1)\int_{\mathbb{R}} \varphi_1(x)\mathrm{d}x, \ldots, L(t_k)\int_{\mathbb{R}} \varphi_k(x)\mathrm{d}x\right) \tag{3.72}$$

for any $t_1, \ldots, t_k \in [0, 1]$, $\varphi_1, \ldots, \varphi_k \in \mathcal{S}(\mathbb{R})$, $\varphi_1, \ldots, \varphi_k \geq 0$, $k = 1, 2, \ldots$.

Fix $\varphi_1, \ldots, \varphi_k$ as above ($\varphi_j \not\equiv 0$). Let M_T be the measure on $\mathbb{R}_+^k$ such that

$$M_T([0, t_1] \times \cdots \times [0, t_k]) = \frac{1}{K^k} E \prod_{j=1}^k \frac{\langle Z_T(t_j), \varphi_j\rangle}{\langle \lambda, \varphi_j\rangle}. \tag{3.73}$$

(3.72) will be proved if we show

$$\lim_{T\to\infty} M_T([0, t_1] \times \cdots \times [0, t_k]) = EL(t_1)\ldots L(t_k) \tag{3.74}$$

for all $t_1, \ldots, t_k \in [0, 1]$. To this end, by Lemma 3 of [2] it suffices to prove that

$$\lim_{T\to\infty} \int_{\mathbb{R}_+^k} \mathrm{e}^{-(\theta_1 t_1 + \cdots + \theta_k t_k)} M_T(\mathrm{d}t_1, \ldots, \mathrm{d}t_k) = \sum_\pi \left[(\theta_{\pi(1)} + \cdots + \theta_{\pi(k)})(\theta_{\pi(2)} + \cdots + \theta_{\pi(k)}) \cdots \theta_{\pi(k)}\right]^{-\alpha'} \tag{3.75}$$

for all $\theta_1, \ldots, \theta_k \geq 0$, the summation being over all permutations π of $\{1, \ldots, k\}$. For simplicity we will show (3.75) for $k = 2$. Without loss of generality we may assume that $\langle \lambda, \varphi_j\rangle = 1$, $j = 1, 2$.

We have

$$\int_0^\infty \int_0^\infty \mathrm{e}^{-\theta t_1 - \theta t_2} M_T(\mathrm{d}t_1, \mathrm{d}t_2) = \frac{1}{K^2}\left(J_1(T) + J_2(T)\right), \tag{3.76}$$

where

$$J_1(T) = \frac{1}{F_T^2}\int_0^\infty \int_0^{t_2} \mathrm{e}^{-\theta_1 t_1-\theta_2 t_2} T^2 E\varphi_1(\zeta_{Tt_1})\varphi_2(\zeta_{Tt_2})\mathrm{d}t_1\mathrm{d}t_2, \tag{3.77}$$

$$J_2(T) = \frac{1}{F_T^2}\int_0^\infty \int_0^{t_1} \ldots \mathrm{d}t_2\mathrm{d}t_1, \tag{3.78}$$

where ... denotes the same integrand as in (3.77).

By the Markov property, self-similarity and (2.19), we have

$$J_1(T) = \int_0^\infty \int_0^{t_2} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{e}^{-\theta_1 t_1-\theta_2 t_2} \varphi_1(y)\varphi_2(z) t_1^{-1/\alpha}(t_2-t_1)^{-1/\alpha}$$
$$\times p_1(T^{-1/\alpha} t_1^{-1/\alpha} y) p_1(T^{-1/\alpha}(t_2-t_1)^{-1/\alpha}(z-y))\mathrm{d}z\mathrm{d}y\mathrm{d}t_1\mathrm{d}t_2,$$

and hence

$$\lim_{T\to\infty} J_1(T) = \int_0^\infty \int_0^{t_2} \frac{\mathrm{e}^{-\theta_1 t_1}}{t_1^{1/\alpha}} \frac{\mathrm{e}^{-\theta_2 t_2}}{(t_2-t_1)^{1/\alpha}} \mathrm{d}t_2\mathrm{d}t_1 p_1^2(0)$$
$$= K^2(\theta_1+\theta_2)^{-\alpha'}\theta_2^{-\alpha'},$$

where $K = \frac{1}{\pi\alpha}\Gamma(\frac{1}{\alpha})\Gamma(1-\frac{1}{\alpha})$. The limit of $J_2(T)$ is calculated identically, so we obtain (3.75).

For $\varphi \geq 0$, (3.71) implies that $\langle Z_T, \varphi\rangle \Rightarrow KL\langle\lambda, \varphi\rangle$ in $C([0, 1])$, since $\langle Z_T, \varphi\rangle$ is an increasing process. From this it follows immediately that $\{\langle Z_T, \varphi\rangle\}_T$ is tight for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Hence the proof of the lemma is complete by Mitoma's theorem [23]. □

Proof of Theorem 2.11. It is easy to see that Lemma 2.12 remains true with the same limit if ζ is replaced by $x + \zeta$.

On the other hand,

$$\langle Y_T(t), \varphi\rangle = \sum_{x_j\in N_0} \frac{1}{F_T}\int_0^{tT} \varphi(x_j+\zeta_s^j)\mathrm{d}s,$$

where $\zeta^1, \zeta^2, \ldots$ are independent copies of ζ, independent of N_0. Now the theorem follows from Lemma 2.12 and the fact that $N_0(\mathbb{R})$ has the same law as ν. □

Proof of Theorem 2.13. Let Z_T be defined by (2.21) with $F_T = \log T$. It suffices to prove that

$$\langle Z_T(1), \varphi\rangle \Rightarrow p_1(0)\rho_1\langle\lambda, \varphi\rangle \tag{3.79}$$

for $\varphi \geq 0$. Indeed, (3.79) implies that $\frac{1}{\log T}\int_{t_1T}^{t_2T}\varphi(\zeta_s)\mathrm{d}s$ converges to 0 in probability for any $0 < t_1 \leq t_2$. Now it is easy to see that

$$(\langle Z_T(t_1), \varphi_1\rangle, \ldots, \langle Z_T(t_k), \varphi_k\rangle) \Rightarrow p_1(0)(\rho_1\langle\lambda, \varphi_1\rangle, \ldots, \rho_1\langle\lambda, \varphi_k\rangle)$$

for $0 < t_1 \leq \cdots \leq t_k$. Hence, proceeding similarly as before we obtain that $Z_T \Rightarrow K\rho_1\lambda$ in $C([\varepsilon, 1], \mathcal{S}'(\mathbb{R}^d))$, and this easily implies (2.22).

Observe that (3.79) has exactly the form as in the Darling–Kac theorem [11], but we cannot apply it directly, since

$$\frac{1}{-\log\theta}\int_0^\infty \int_{\mathbb{R}^d} \mathrm{e}^{-\theta s}\varphi(y)p_s(x-y)\mathrm{d}y\mathrm{d}s$$

may not converge uniformly in x as $\theta \to 0$ (so condition (A) is not satisfied). Nevertheless, the proof of the Darling–Kac theorem can be repeated with some care in the present case yielding the desired result. □

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