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Nonlinear Dynamic Structures with Developing Discontinuity

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Abstract

Discontinuity is one of the most important nonlinear factors challenging both methods of nonlinear analysis and its applications. The problem becomes more complicated when the discontinuity is developing under the influence of dynamic processes in the structures. This paper presents the methodology for analysis and simulation of the systems with the developing discontinuity. It is based on the combination of analytical technique using nonlinear integral equations and the Matlab-Simulink computation. This methodology is applied to analysis of nonlinear dynamics of the cracked bar. Some new nonlinear phenomena have been revealed and validated.

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Keywords: structure with discontinuity; vibro-impact; virtual linear system; latent defect

1. Introduction

Arising and development of the discontinuity (crack) in the structure leads to its gradual modification influenced strongly by additional nonlinear forces of impact and friction between the contact surfaces of the discontinuous elements as well as a general transformation of continuum surrounding the discontinuity. This brings an essential change in dynamic response of the structure to dynamic loading which can be traced for diagnostic of the structural health to prevent the possible failure. Several technological problems address to consideration of the structures with developing discontinuity. Among them are: emergence of crack or delamination of material, slackening of joints, machining etc. Few researchers have tried to address the issue of the effect of discontinuity (crack) on the dynamics of the structure.

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The behavior of a cracked cantilever beam under harmonic loading was studied using a two dimensional finite element analyses[1]. In this study, they showed that frequency did not change with oscillation amplitude, but the steady state response obtained was rich in multiple sub-harmonics and super harmonic components. They also found that in super-harmonic resonance, due to the impacts between cracks faces there were significant wiggles in phase-portraits. A piecewise linear system of cracked bar was modelled using perturbation method to obtain nonlinear vibration associated with periodic motion[2]. In this, they found that the bilinear frequency formula is a good approximation for the effective natural frequency of cracked beam. Equations of motion and boundary conditions[3] were developed by considering the cracked bar as one dimensional continuum based on Hu-Washizu–Barr formulation[4]. They found that breathing cracks result in a smaller drop in the dominant system frequency as compared to the natural frequencies of linear system with open cracks. The cracked beam was modelled as an oscillator with a bilinear restoring force to study the effect of crack closure on the dynamics of beam[5]. Bispectral analysis was performed on the obtained response, which showed high sensitivity to the nonlinear behaviour of the system compared to other techniques. An integrated technique[6] was used to non-destructively identify two discrete cracks in a simply supported beam based on vibration theory. They investigated the two different types of model for crack induced damage. For the first model, they used a massless infinitesimal spring to represent discrete cracks and for the second model they used a continuum damage concept. Their findings suggested that the continuum damage model can be used first to identify the discrete elements of a structure that contains cracks. Then the spring model can be used to quantify the location and severity of discrete crack in each damaged element. Nonlinear Non-Destructive Evaluation (NDE) was proposed to detect small fractured defects invisible by linear NDE techniques[7]. It was found that contact vibrations led to threshold nonlinear distortion due to the "clapping" and "kissing" (grazing) mechanism of crack faces interactions. Vibro-impact interactions of these types in structures with discontinuities were described earlier [8]. Self-modulation mode of nonlinear acoustic vibrations of cracked defects was analyzed and observed experimentally [9]. The self-modulation occurred due to simultaneous excitation of two sub-harmonics with close frequencies. It was observed from the spectral patterns that both self-modulation and sub-harmonic modes demonstrate a high localization around the defect which could be used for early detection and recognition of damaged areas.

The paper is devoted to the consideration of the structures which develop a discontinuity during their functioning under influence of variable loads. Introduction of a new substructure, called as a *virtual linear system*, was used to distinguish clearly the influence of geometrical and dynamical factors on the behavior of the structure. Gradual transformation of the structure from a continuous to a discontinuous one is modelled with implantation of an additional structure referred to as a *latent defect*. It includes both a nonlinear operator describing the discontinuity, and a linear elastic element regulating the dynamic response of the discontinuity. Progressive weakening of the linear element stiffness controls the nonlinear influence of the discontinuity.

A method of simulation for studying the dynamics of the structures with the developing discontinuities is described based on the combination of analytic technique and the Matlab-Simulink computation developed[10]. These simulation results were validated by performing experiments[11]. It was found that the crack-induced nonlinearity leads to the generation of higher harmonics, whose intensity is a function of a distance from the crack. Sideband frequencies were clearly revealed leading to the modulation of exciting frequency due to systematic crack faces interaction. This nonlinear transformation of modulated vibration by crack led to generation of a low frequency periodic component whose intensity is proportional to the forced response of the cracked bar at the exciting frequency. The phenomenon revealed can be used as a diagnostic tool for structural health monitoring.

As an example, a model of a cracked bar subjected to longitudinal excitation is used to analyze a nonlinear response as a way to monitor structural health.

2. Dynamic model of structure with the developing discontinuity

Consider a stepped bar of length $2l$ fixed – fixed at both ends as shown in Fig .1. To develop the model of structure with developing discontinuity it is assumed that there is a propagating crack at the interface between the stepped bar under the influence of longitudinal excitation. This crack splits the stepped fixed-fixed bar into two subsystems, one is the cantilever bar fixed at the left of length l $(i = 1)$ and other is the cantilever bar fixed at the right end of length l $(\mathrm{i} = 2)$. This new double element structure is called a *virtual linear system*. As shown in Fig.2 the splitted cross sections of semi-rods are capable to move without obstruction preserving their linear behavior.

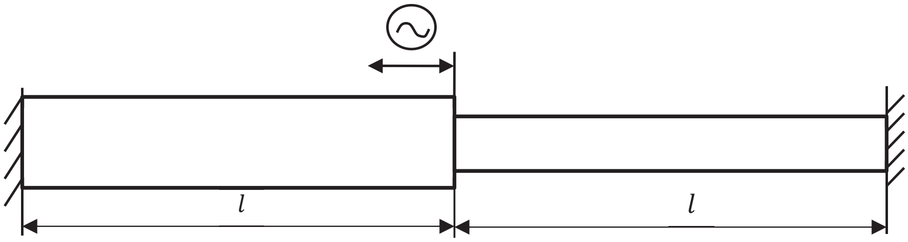


Fig.1 Schematic of fixed – fixed stepped bar

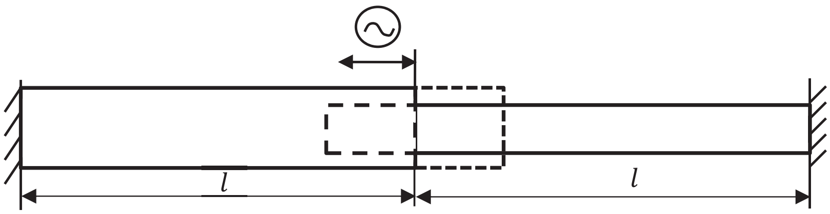


Fig. 2. Schematic of virtual linear system

Under the influence of the longitudinal excitation the real split structure demonstrates an interaction of the crack faces. This interaction between crack faces transforms the continuous process (Fig. 3a) into successive impulses (Fig. 3c) modulated by the velocity of the input process at the instant when the process reaches the threshold value (Fig. 3b) (closing of the crack). Fig. 3 shows an example of temporal transformations of deformation $u(x, t)$ into the force $F[u(x, t)]$ by impact element with force characteristic $F(u)$.

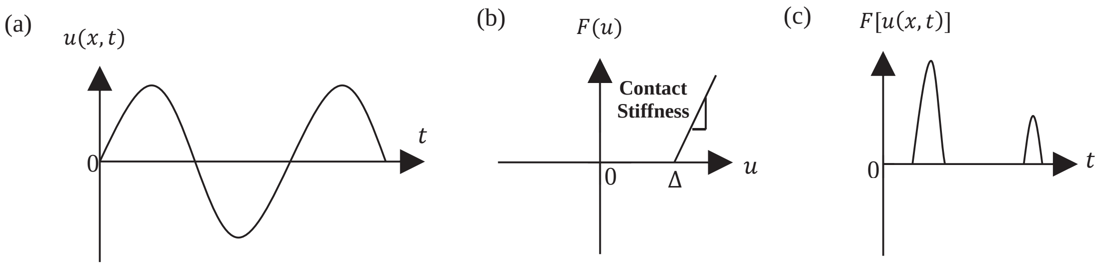


Fig 3. Contact force characteristics

This force characteristic of contact interaction is given by an expression called static force characteristics of the impact pair[8]:

$$F(u) = \psi(u - \Delta)\eta(u - \Delta) \tag{1}$$

where ψ is linearized coefficient of contact stiffness

$$\eta(u) = \begin{cases} 0, & u < 0 \\ 1, & u \geq 0 \end{cases} \tag{2}$$

Some additional complicating factors like thermo-plasticity and fatigue can be taken into consideration in the more sophisticated models of interaction. To model the discontinuous element like developing crack in continuous structure a special mathematical technique is used for proper matching of distributed ($i = 1\,, 2$) elements and local discontinuous of the resulting structure. This is the force characteristics of contact interaction for the discontinuous elements like developing crack and Green's functions calculated in the contact areas for the media [8, 10]. To describe the motion of interaction between the continuous and discontinuous elements the integral equations are used.

The equations of motion of faces (cross sections with a coordinate $x=l$) for discontinuous element in the structure with the developing discontinuity are described as follows:

$$u_1(l,t) = \int_0^t h_1(t-\tau)F_\epsilon[u(l,\tau), \dot{u}(l,\tau)]d\tau - \int_0^t h_1(t-\tau)P_1(\tau)d\tau \tag{3}$$

$$u_2(l,t) = -\int_0^t h_2(t-\tau)F_\epsilon[u(l,\tau), \dot{u}(l,\tau)]d\tau + \int_0^t h_2(t-\tau)P_2(\tau)d\tau \tag{4}$$

where $F_\epsilon[u(l,\tau), \dot{u}(l,\tau)]$ – force characteristics of contact interaction treated as a slowly changing function, $P_i(\tau)$ – reduced external forces and $h_i(t)$ – Greens function of substructures, $(i = 1{,}2)$.

The relative motion between the faces of discontinuous element at the developing discontinuity is given as

$$u(l,t) = u_2(l,t) - u_1(l,t) \tag{5}$$

Substituting Eq. (3) and Eq. (4) in Eq. (5) we have

$$u(l,t) = \int_0^t h(t-\tau)P(\tau)d\tau - \int_0^t h(t-\tau)F_\epsilon[u(l,\tau), \dot{u}\,(l,\tau)]d\tau \tag{6}$$

where

$h(t) = h_1(t) + h_2(t)$ which is the resultant Green's function
$P_1 = P_2 = P$ which is the external force.

Fourier transformation of Eq. (6) produces an operator form which will be used for the simulation purpose.

$$u(l,t) = L(j\omega)(P(t) - F_\epsilon[u(l,t), \dot{u}(l,t)]) \tag{7}$$

where

$L(j\omega) = L_1(j\omega) + L_2(j\omega)$ – dynamic compliance operator or receptance operator of the continuous elements i.e. the cantilever bars, ω is angular frequency and $j^2 = -1$.

3. Modeling of uniform bar with a crack

For simulation purpose, a uniform fixed – fixed bar of length $x = l$ under longitudinal harmonic excitation is considered as shown in Fig.4. In this case, it is assumed that the discontinuity, i.e. crack, develops itself at the right fixed end of the bar. This crack splits the fixed-fixed bar into two subsystems, one is the cantilever bar fixed at the left end which is as virtual linear system for the case and other is an impact pair describing interacting faces of the crack. Fig. 5 shows the schematic of *virtual linear system* under the longitudinal excitation leading to longitudinal displacement causing cracks faces interaction. To model the structure with developing discontinuity two factors are considered: one is the well-established fact that developing crack change the stiffness of the fixation and other is the generation of contact force due to cracks faces interaction.

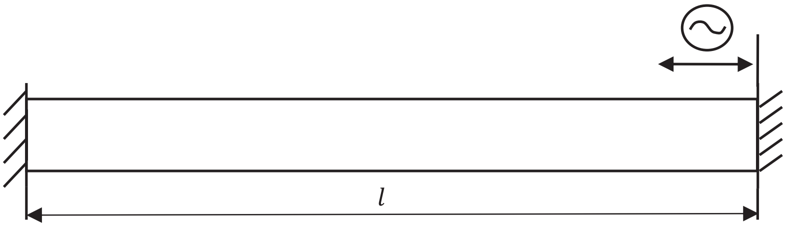


Fig 4. Schematic of fixed-fixed bar under longitudinal excitation

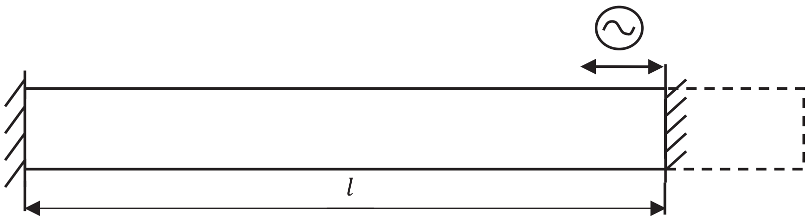


Fig 5. Schematic of virtual linear system

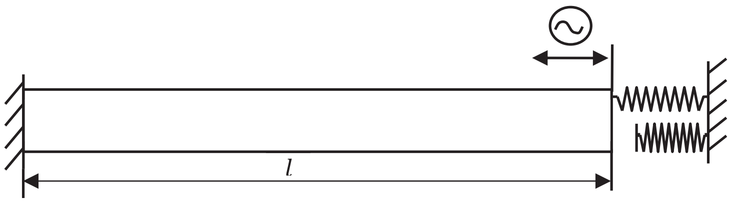


Fig 6. Schematic model of a latent defect (propagating crack) in cracked bar

Fig.6 shows the schematic model of the fixed-fixed cracked bar with the developing crack. A change in stiffness of fixation is modeled as a linear variable spring and the contact forces between crack faces are modelled as a limiter. For this case, in order to consider the linear spring force (F_k) Eq. (7) can be modified as

$$u(l,t) = L(j\omega)(P(t) - (F_\epsilon[u(l,t), \dot{u}(l,t)] + F_k) \tag{8}$$

According to the modal analysis for straight rods, there is a bilinear form for the representation of receptance[12] which is given as in Eq. (9)

$$L(j\omega)=\sum_{v=1}^{\infty}\frac{A_v(x)A_v(x)}{(j\omega)^2+\Omega_v^2} \tag{9}$$

where $A_v(x)$ are the modal shape functions of the virtual linear system, Ω_v its natural frequencies and ω is the angular frequency.

This bilinear representation of receptance in Eq. (9) can be extended to systems with energy dissipation in the form of internal friction in the material; $r_v=\chi\Omega_v/4\pi\omega$ parameter[13] is introduced in the expression of receptance in which χ is absorption coefficient responsible for internal damping of material. Hence, Eq. (9) can be rewritten as:

$$L(s)=L_{xx}(s)=\sum_{v=1}^{\infty}\frac{A_v(x)A_v(x)}{s^2+2r_v\Omega_v s+\Omega_v^2} \tag{10}$$

where s is a complex variable.

The expression in Eq. (10) consists of infinite number of vibration modes and the coefficients of each mode remain functions of the continuous coordinate x which represents the position of the section under consideration. The coefficient $A_v(x)$ representing the transferring action to the section at $x=l$ from the concentrated harmonic force applied at the section $x=l$ can be written as:

$$L(s)=L_{ll}(s)=\sum_{v=1}^{\infty}\frac{A_v(l)A_v(l)}{s^2+2r_v\Omega_v s+\Omega_v^2} \tag{11}$$

From the schematic shown in Fig. 6, under the influence longitudinal harmonic load, the bar starts to interact with a limiter which is responsible for the generation of contact force. Propagation of the crack is modelled as changes in the stiffness of the connecting spring as the crack propagates. The contact force generated is implanted as the nonlinear feedback and changes in stiffness of the connecting spring as linear feedback as shown in Fig. 7. When the cantilever bar interacts with the limiter and as the crack propagates, vibration displacement of cantilever bar at any arbitrary section x from the fixed end of the bar is defined as a function of $u(x,t)$.

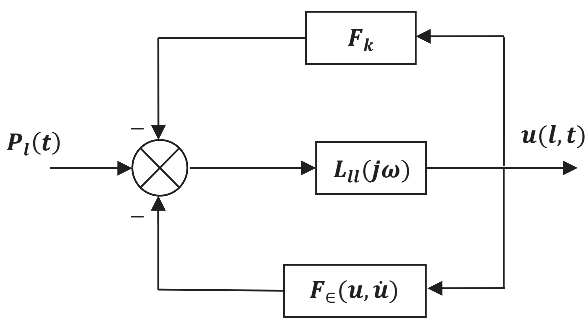


Fig 7. Block diagram of Eq. (8) with account of Eq. (11)

By introducing the slowly changing contact force characteristics $F_{\in}(u(l,t),\dot{u}(l,t))$, the changes in stiffness due to propagating crack and receptance operator $L_{lx}(s)$, coupling displacement $u(x,t)$ of the arbitrary section to the force acting at $x = l$, the operator equation for the unknown function $u(x,t)$ can be written in the following form:

$$u(x,t) = L_{lx}(s)P_l(t) - L_{lx}(s)[F_{\in}(u(l,t),\dot{u}(l,t)) + F_k] \qquad (12)$$

The simulation of system is performed using Matlab–Simulink on the cantilever bar taking Eq. (11) and Eq. (12) into account. For simulation purposes only, the first ten modes of vibration are taken into consideration which gives the accurate numerical results. For simulations, the material properties of steel rod are considered having the dimensions of 300 mm × 25 mm × 10 mm.

4. Simulation results

To obtain the linear response of the system shown in Fig. 6 the crack considered in the system is proposed to be small enough to have any influence on the dynamics of the system. So the contact force generated due to the cracks faces interaction will be negligible. To simulate this, contact stiffness is considered as 1% of stiffness of bar and the

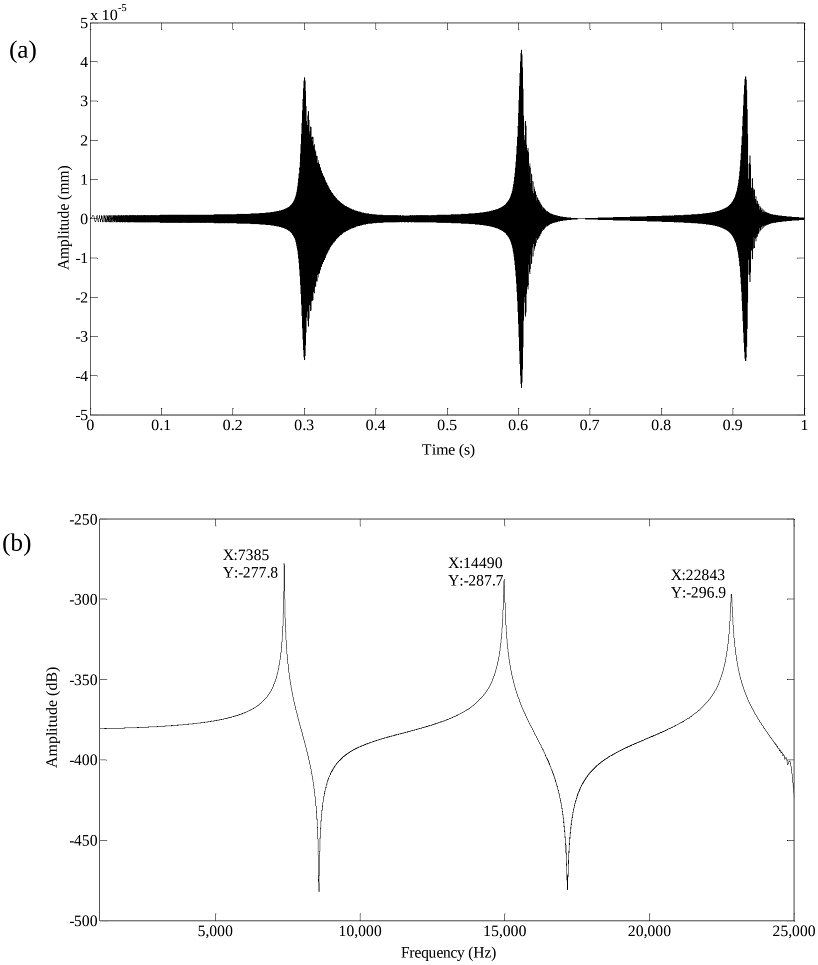


Fig. 8 Linear response of the system (a) Time response (b) Frequency response

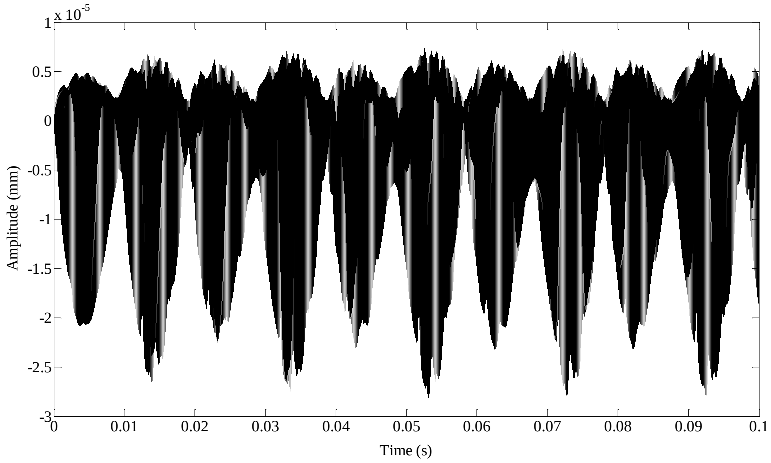
spring stiffness is considered as 600% of the stiffness of the bar. The response (Fig. 8) obtained using these stiffness parameters is taken as the reference response and the frequency of the first mode obtained is used as an excitation frequency for further simulations.

As the crack length increases, spring stiffness decreases which accounts for the reduction in stiffness of cracked bar and contact force function changes due crack faces interaction leading to generation of contact forces which is the function of contact stiffness. Different cases are considered depending on different contact stiffness and change in a stiffness of bar as a function of crack length. For simulation purposes, spring stiffness and contact stiffness is varied between 240% to 900% of the stiffness of the bar for the given value of spring stiffness and contact stiffness. Below are two examples of simulation results. Table 1 show the parameters used in these two simulations. The effect of different spring stiffness parameter is considered to understand the dynamics of the cracked bar. Decrease in the spring stiffness is attributed to the softening effect in connection of the rod with the base due to crack propagation.

Cases	Spring stiffness	Contact stiffness
1	420%	600%
2	240%	600%

Table 1: Simulation parameters (as percentage of the bar stiffness).

(a)



(b)

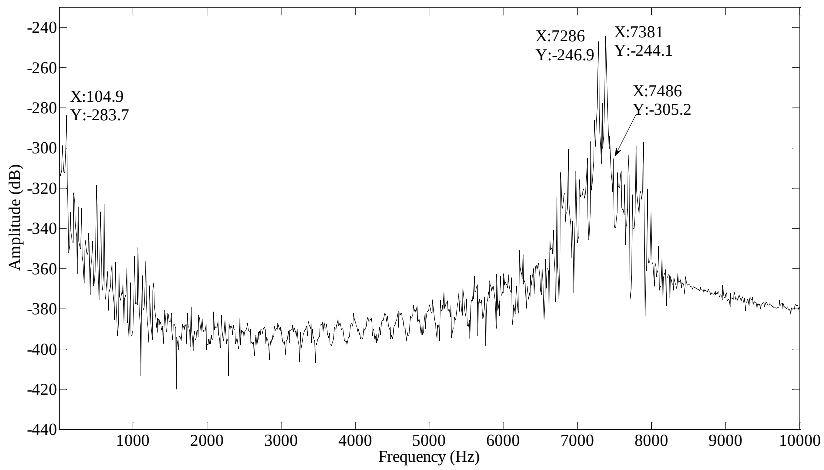
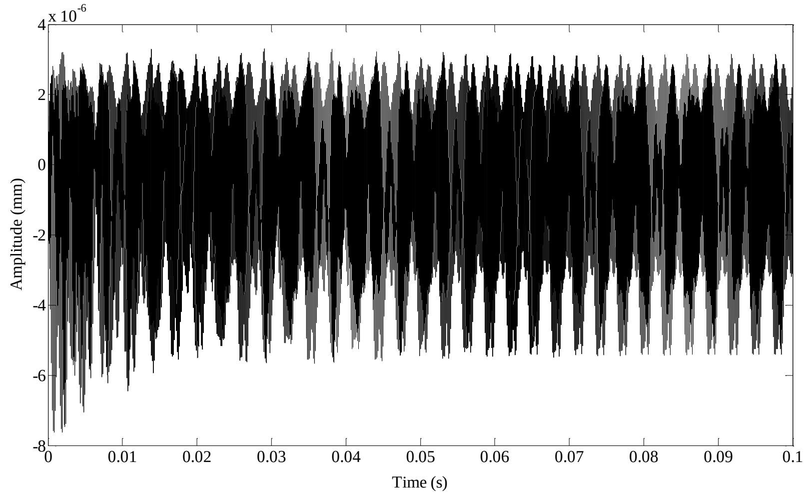


Fig. 9. Nonlinear response of the system for case 1 (a) Time response (b) Frequency response.

(a)



(b)

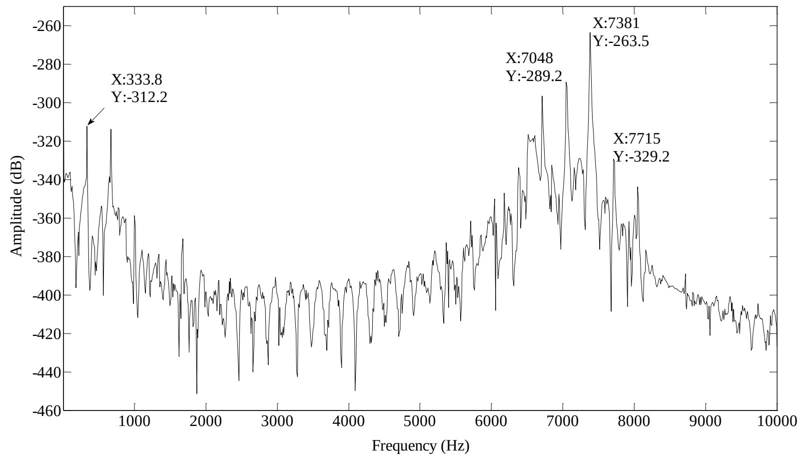


Fig. 10. Nonlinear response of the system for case 2 (a) Time response (b) Frequency response.

In these cases, it is assumed that the crack is propagating, i.e. spring stiffness is kept varying but is always less than contact stiffness. For these cases, system responses (Fig. 9 and Fig. 10) are obtained by exciting at the frequency of the first mode of linear system using the simulation parameter given in table 1. In time responses, for case 1 (Fig. 9a) and case 2 (Fig.10a) intense modulation was observed as the crack propagates. The frequency responses obtained for case 1 (Fig. 9b) and case 2 (Fig. 10b) shows that as the spring stiffness is varied , along with the frequency of excitation there is also the presence of the side band frequencies and low frequency component. It is also observed that as the spring stiffness decreases there is a shift in the low frequency component (Fig.10b).

5. Conclusion

Concept of receptance operator along with the theory of modal analysis to generate the Matlab-Simulink model of the structures with developing discontinuities is evolved. The methodology is applied to analysis of nonlinear dynamics for the cracked bar. It is found that there is occurrence of the intense modulation in the time response as the crack propagates. This indicates that nonlinear transformation of the modulated signal by the crack generates a low-frequency component. Along with side band frequencies this can be a good indicator of crack detection.

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