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Independence Structures on the Submodules of a Module

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Two definitions of dimension of a module are each shown to be the rank of an independence structure on a certain set of submodules of the module. This applies to Varadarajan's dual Goldie dimension and to Fleury's spanning dimension; the dualization of the latter is also discussed.

1. INTRODUCTION

In [2], it is shown that an independence space may be defined on a certain set of submodules of a given module. We refer the reader to that paper, or to [10, chs. 6, 7] or [1] for definitions and notation of independence theory. In particular, in [2] we point out that an independence space may be defined by its collection of independent sets, or by its collection of circuits. It may also be defined by its span operator, or dependence operator, for which the axioms can be written

D(1) For $A \subseteq E$, $A \subseteq [A]$,
D(2) if $b \in [A]$ then $[A \cup \{b\}] \subseteq [A]$,
D(3) if $b \in [A \cup \{c\}]$ but $b \notin [A]$ then $c \in [A \cup \{b\}]$,
D(4) if $b \in [A]$ then $b \in [A']$ for some finite $A' \subseteq A$.
Then $\mathscr{E} = \{A \subseteq E$: for each $a \in A$, $a \notin [A \backslash \{a\}]\}$.

Throughout this paper R will denote a ring with 1 and all modules will be unitary left R-modules. For a submodule N of M, N is an *essential* submodule of M ($N \leqslant_e M$) if, for $L \leqslant M$, $L \cap N = 0$ implies $L = 0$; N is a *small* submodule of M ($N \leqslant_s M$) if, for $L \leqslant M$, $L + N = M$ implies $L = M$. A non-zero module M is *uniform* if all its non-zero submodules are essential, and M is *hollow* if all its proper submodules are small. Note that a nontrivial submodule (factor module) of a uniform (hollow) module is also uniform (hollow). Let

$$U(M) = \{N \leqslant M\colon N \text{ is uniform}\},$$

$$H(M) = \{N \leqslant M\colon N \text{ is hollow}\},$$

$$\mathrm{Uf}(M) = \{N < M\colon M/N \text{ is uniform}\}, \text{ and}$$

$$\mathrm{Hf}(\mathrm{M}) = \{N < M\colon M/N \text{ is hollow}\}.$$

For $N \in \mathrm{Uf}(M)$, we call N a *uniform-factor* submodule of M; similarly, $N \in \mathrm{Hf}(M)$ is a *hollow-factor* (h.f.) submodule of M. We see that a uniform-factor submodule is a meet-irreducible member of the lattice of submodules of M, and a hollow submodule is a join-irreducible member. A submodule N is uniform if 0 is meet-irreducible in the lattice interval $[0, N]$, and N is a h.f. submodule if M is join-irreducible in the lattice interval $[N, M]$.

For $N \leqslant M$, a *complement* of N in M is a submodule L of M, maximal such that $N \cap L = 0$. Dually, a *supplement* of N in M is L, minimal such that $N + L = M$. By Zorn's lemma, if $N \cap K = 0$ then K is contained in a complement of N; we will say that M has property (S) if, for K, $N \leqslant M$ such that $N + K = M$, K contains a supplement L of N in M (equivalently, there exists a supplement L of $N \cap K$ in K).

In [2] it is shown that if $\mathscr{G}(M) = \{\{M_i\colon i \in I\} \subseteq U(M)$: the sum $\sum_{i \in I} M_i$ is direct$\}$, then $(U(M), \mathscr{G}(M))$ is an independence space. Under certain conditions, which are satisfied when M has finite Goldie dimension, the rank of $\mathscr{G}(M)$ is the Goldie dimension of M.

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Recently a dualization of Goldie dimension has been given by Varadarajan, [8], [6], and Fleury, [4], has defined a further notion of dimension. This latter can also be dualized, and we can define independence structures relating to each of these. Note that the concept of duality here is duality in the abelian category *R-mod*, not duality of independence structures. In fact, this duality manifests itself here primarily as duality in the lattice of submodules of M.

2. The Dual Goldie Structure

The following lemma dualizes the condition for a sum of submodules to be direct. Unfortunately, a finiteness condition is necessary.

Lemma 2.1. (Generalized weak Chinese remainder theorem.) *Let $K_1, \ldots, K_r \leq M$. Then $M/\bigcap_{i=1}^r K_i$ is naturally isomorphic to $\prod_{i=1}^r M/K_i$ (equivalently, the natural map $M \to \prod_{i=1}^r M/K_i$ is an epimorphism) if and only if, for each $l = 1, \ldots, r$, $K_l + \bigcap_{j \neq l} K_j = M$. In this case, for $\varnothing \subset J \subset I = \{1, \ldots, r\}$, $\bigcap_{i \in I \setminus J} K_i + \bigcap_{j \in J} K_j = M$.*

Proof. The equivalence is [8, lemma 1.4], and the last remark is easy to show.

Definition. We define the dual Goldie structure $\mathscr{G}\mathrm{d}(M) \subseteq \mathscr{P}(\mathrm{Hf}(M))$ [the set of subsets of $(\mathrm{Hf}(M)$] by
(a) for $\{K_1, \ldots, K_r\} \subseteq \mathrm{Hf}(M)$, $\{K_1, \ldots, K_r\} \in \mathscr{G}\mathrm{d}(M)$ if $K_1, \ldots, K_r$ satisfy the conditions of Lemma 2.1, and
(b) for $\{K_i : i \in I\} \subseteq \mathrm{Hf}(M)$, $\{K_i : i \in I\}$ is in $\mathscr{G}\mathrm{d}(M)$ if every finite subset of it is, according to (a).

Part (b) of the definition is unfortunately necessary unless it can be shown that the property of (a) is of finite character. In view of (b), $\mathscr{G}\mathrm{d}(M)$ satisfies I(3) automatically, and every dependent set contains a (finite) circuit. We consider these circuits.

Lemma 2.2. *Let $\{M_1, \ldots, M_r\}$ be a circuit of $\mathscr{G}\mathrm{d}(M)$. Then for $\varnothing \subset J \subset I = \{1, \ldots, r\}$, $\bigcap_{i \in I \setminus J} M_i + \bigcap_{j \in J} M_j \neq M$.*

Proof. Suppose the result is false, and consider a specific counter-example. Then as $\bigcap_{i \in I \setminus J} M_i + \bigcap_{j \in J} M_j = M$, $M/\bigcap_{i \in I} M_i$ is naturally isomorphic to

$$\frac{M}{\bigcap_{i \in I \setminus J} M_i} \oplus \frac{M}{\bigcap_{j \in J} M_j}$$

and as $\{M_i : i \in I \setminus J\}$ and $\{M_j : j \in J\}$ are in $\mathscr{G}\mathrm{d}(M)$, this is naturally isomorphic to $(\prod_{i \in I \setminus J} M/M_i \oplus \prod_{j \in J} M/M_j)$. Thus $\{M_1, \ldots, M_r\} \in \mathscr{G}\mathrm{d}(M)$, a contradiction.

Theorem 2.3. $\mathscr{G}\mathrm{d}(M)$ *is an independence space.*

Proof. As $\mathscr{G}\mathrm{d}(M)$ satisfies I(3), the circuits satisfy C(1) and C(3), and the independent sets are those not containing a circuit. It remains to show C(2). Let $\{M_1, \ldots, M_r\}$ and $\{N_1, \ldots, N_s\}$ be distinct circuits of $\mathscr{G}\mathrm{d}(M)$ with $M_1 = N_1$; suppose further that $M_i = N_i$ for $i = 1, \ldots, t$ $(t < r, s)$, and that the two circuits have no other common members. We have, by Lemma 2.2, that $M \neq M_1 + (M_2 \cap \cdots \cap M_r)$ and $M \neq N_1 + (N_2 \cap \cdots \cap N_s)$. We show that $\{M_2, \ldots, M_r, N_{t+1}, \ldots, N_s\} \notin \mathscr{G}\mathrm{d}(M)$ by showing that $M \neq (M_2 \cap \cdots \cap M_r) + (N_{t+1} \cap \cdots \cap N_s)$. Suppose otherwise. Let $m \in M$ be given. Since $\{N_1, \ldots, N_s\}$ is a circuit, $\{N_1, \ldots, N_t\} \in \mathscr{G}\mathrm{d}(M)$ and we have $M = N_1 + (N_2 \cap \cdots \cap N_t)$.

So let $m = n_1 + n^*$, with $n_1 \in N_1$ and $n^* \in N_2 \cap \cdots \cap N_t$; by our supposition we may let $n^* = m' + n'$, with $m' \in M_2 \cap \cdots \cap M_r$ and $n' \in N_{t+1} \cap \cdots \cap N_s$. Thus $n' = n^* - m' \in N_2 \cap \cdots \cap N_t$ $(= M_2 \cap \cdots \cap M_t)$, and $m = n_1 + m' + n' \in N_1 + (M_2 \cap \cdots \cap M_r) + (N_2 \cap \cdots \cap N_s)$. Hence $M = N_1 + (M_2 \cap \cdots \cap M_r) + (N_2 \cap \cdots \cap N_s)$; since M/N_1 is hollow this means that either $M = N_1 + (M_2 \cap \cdots \cap M_r)$ or $M = N_1 + (N_2 \cap \cdots \cap N_s)$, which (as $M_1 = N_1$) is a contradiction. Thus $\{M_2, \ldots, M_r, N_{t+1}, \ldots, N_s\}$ is dependent and so contains a circuit, as required.

Clearly, for $\{K_1, \ldots, K_r\} \in \mathscr{G}\,\mathrm{d}(M)$, if $K_1 \cap \cdots \cap K_r \leq_s M$, then $\{K_1, \ldots, K_r\}$ is maximal independent and so $r = \mathrm{rk}(\mathscr{G}\,\mathrm{d}(M))$. The converse holds under certain conditions. To determine them we first need a lemma.

LEMMA 2.4. *If $\{K_1, \ldots, K_r\} \in \mathscr{G}\,\mathrm{d}(M)$, and $K_1 \cap \cdots \cap K_r \leq K < M$, then K is contained in a h.f. submodule of M.*

PROOF. Let t be minimal such that $N = (K_1 \cap \cdots \cap K_t) + K < M$. (Note that if $t = 0$ we say that $K_1 \cap \cdots \cap K_t = M$.) we show that M/N is hollow. As $\{K_1, \ldots, K_t\} \in \mathscr{G}\,\mathrm{d}(M)$, $(K_1 \cap \cdots \cap K_{t-1}) + K_t = M$ and so

$$\frac{M}{K_t} = \frac{K_1 \cap \cdots \cap K_{t-1}}{K_1 \cap \cdots \cap K_t}$$

Since, by the minimality of t, $(K_1 \cap \cdots \cap K_{t-1}) + K = M$, we have $(K_1 \cap \cdots \cap K_{t-1}) + N = M$ and so

$$\frac{M}{N} \cong \frac{K_1 \cap \cdots \cap K_{t-1}}{(K_1 \cap \cdots \cap K_{t-1}) \cap N}.$$

As $(K_1 \cap \cdots \cap K_{t-1}) \cap N \geq K_1 \cap \cdots \cap K_t$, M/N is isomorphic to a factor module of M/K_t, which is hollow; hence M/N is hollow and N is the required h.f. submodule of M.

We can now prove part of the duals of Lemmas 3 and 4 of [2]. These results relate $\mathscr{G}\,\mathrm{d}(M)$ to the work of Varadarajan [8]. Recall that each basis of $\mathscr{G}\,\mathrm{d}(M)$ has the same cardinal, $\mathrm{rk}(\mathscr{G}\,\mathrm{d}(M))$. We recall from [8, definition 1.9], that M has *corank* r (r a non-negative integer or ∞) if r is minimal such that the following holds: if $K_1, \ldots, K_s < M$ and the natural map $M \to \prod_{i=1}^{s} M/K_i$ is an epimorphism, then $s \leq r$. We also note that in view of the remark preceding Lemma 2.4, condition (b) following is equivalent to: M has weak corank $r < \infty$ ([8, definition 1.18]).

THEOREM 2.5. *For a module M, $(a) \Leftrightarrow (b) \Rightarrow (c) \Rightarrow (d)$, and if $\mathrm{rk}(\mathscr{G}\,\mathrm{d}(M)) = r < \infty$, then $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d)$:*

(a) *M has corank $r < \infty$;*

(b) *$\mathrm{rk}(\mathscr{G}\,\mathrm{d}(M)) = r < \infty$, and for some basis $\{K_1, \ldots, K_r\}$ of $\mathscr{G}\,\mathrm{d}(M)$, $K_1 \cap \cdots \cap K_r \leq_s M$;*

(c) *every non-trivial factor module of M has a hollow factor module;*

(d) *for each basis of $\mathscr{G}\,\mathrm{d}(M)$, the intersection of the modules in it is small in M.*

Further, if (a) holds and the natural map $M \to \prod_{i=1}^{r} M/K_i$ $(K_i < M)$ is an epimorphism, then each M/K_i is hollow.

PROOF. (a)$\Rightarrow$(c). Suppose that $L < M$ and M/L does not have a hollow factor module. Let r be given. Set $L_0 = L$, and for each $i = 0, \ldots, r$ we have that M/L_i is not hollow and we can choose L_i' and L_{i+1} such that $L_i < L_i'$, $L_{i+1} < M$ and $L_i' + L_{i+1} = M$. Thus we have natural epimorphisms $M/L_i \to M/L_i' \oplus M/L_{i+1}$ and hence the natural epimorphism $M \to \prod_{i=0}^{r} M/L_i'$. So M does not satisfy (a) for any $r < \infty$.

(b)⇒(c). Let $\{K_1, \ldots, K_r\}$ be a basis of $\mathscr{G}\mathrm{d}(M)$ such that $K_1 \cap \cdots \cap K_r \leqslant_s M$, and let $L < M$ be given. Then let $K = (K_1 \cap \cdots \cap K_r) + L < M$. Hence, by Lemma 2.4, M/K, and hence M/L, has a hollow factor module.

(c)⇒(d). Suppose (c) holds, and let $\{K_i: i \in I\}$ be a basis of $\mathscr{G}\mathrm{d}(M)$. Suppose that $\bigcap_{i \in I} K_i$ is not small in M. Then let $\bigcap_{i\in I} K_i + L = M$, with $L < M$, and let M/L have hollow factor M/N, so $\bigcap_{i \in I} K_i + N = M$. As $\{K_i: i \in I\}$ is a basis, there is a circuit $\{K_j: j \in J\} \cup \{N\}$ for some $J \subseteq I$. Then, by Lemma 2.2, $\bigcap_{j\in J} K_j + N < M$, contrary to $\bigcap_{i\in I} K_i + N = M$. Thus $\bigcap_{i \in I} K_i \leqslant_s M$.

(a)⇔(b). Suppose that either (a) or (b) holds; we have shown above that (c) and (d) hold. Suppose corank $(M) \geqslant r$. Let $M \to \prod_{i=1}^r M/K_i$ be a natural epimorphism, and let each M/K_i have hollow factor M/L_i. Then $M \to \prod_{i=1}^r M/L_i$ is a natural epimorphism, $\{L_1, \ldots, L_r\} \in \mathscr{G}\mathrm{d}(M)$, and $\mathrm{rk}(\mathscr{G}\mathrm{d}(M) \geqslant r$. On the other hand, it is clear that $\mathrm{rk}(\mathscr{G}\mathrm{d}(M)) \geqslant r$ implies that corank $(M) \geqslant r$. It remains to show that if (a) holds, then the intersection of some basis is small in M; this follows from (d), which we have shown is implied by (a). Also, under the general supposition that $\mathrm{rk}(\mathscr{G}\mathrm{d}(M)) = r < \infty$, clearly (d)⇒(b) and so all the conditions are equivalent.

Finally, suppose (a) holds, with a natural epimorphism $M \to \prod_{i=1}^r M/K_i$. Suppose M/K_j is not hollow, and let $M/K_j = L/K_j + L'/K_j$, where $K_j < L$, $L' < M$. Then as the natural map $M/K_j \to M/L \oplus M/L'$ is an epimorphism (by Lemma 2.1), we have the natural epimorphism $M \to M/L \oplus M/L' \oplus \prod_{i=2}^r M/K_i$, contradicting the maximality of r. Thus each M/K_i is hollow.

An independence space may have the property of being *modular*, i.e. given a circuit $\{e_1, \ldots, e_p\}$ and q such that $2 \leqslant q \leqslant p-2$ there is an element e such that $\{e_1, \ldots, e_q, e\}$ and $\{e_{q+1}, \ldots, e_p, e\}$ are both dependent (in which case they are necessarily circuits).

THEOREM 2.6. $(\mathrm{Hf}(M), \mathscr{G}\mathrm{d}(M))$ *is modular.*

PROOF. Let $\{M_1, \ldots, M_p\}$ be a circuit, and let $2 \leqslant q \leqslant p-2$. By Lemma 2.2, let $L = (M_1 \cap \cdots \cap M_q) + (M_{q+1} \cap \cdots \cap M_p) < M$. Then by Lemma 2.4, we may let $N \in \mathrm{Hf}(M)$ such that $N \geqslant L$. Now $(M_1 \cap \cdots \cap M_q) + N = N < M$, so $\{M_1, \ldots, M_q, N\}$ and similarly $\{M_{q+1}, \ldots, M_p, N\}$ are dependent, as required.

We note that this theorem has the same consequences as the corresponding result for the Goldie structure, as discussed in [2]. That is, if $\mathscr{G}\mathrm{d}(M)$ is 'connected' as an independence space, and of rank at least 3, then it naturally corresponds to a projective geometry, which is (if Desarguesian) coordinatizable over a unique division ring D. In this case the structure of D remains an open question. Since the division rings thus found for the Goldie structure are all of the form $En(E)/J(En(E))$ for E uniform injective, we may conjecture that D is at least sometimes a division ring of the form $En(P)/J(En(P))$ where P is (finitely generated) hollow projective (see [9, 4.1 to 4.3], and also [5]).

This subject is also dealt with in [3].

3. THE FLEURY STRUCTURE

In [4], Fleury develops a notion of dimension on a module whose non-small submodules satisfy the DCC. We use his ideas to develop an independence structure $\mathscr{F}(M)$ on $H(M)$ for arbitrary M. In particular, this does not require that submodules of M have supplements in M, as they do in the modules considered by Fleury.

We define a map f on $\mathscr{P}(H(M))$ which will be related to the dependence operator of $\mathscr{F}(M)$.

DEFINITION. For $N, M_i \in H(M)$, we say $N \in f(\{M_i: i \in I\})$ if

$$\frac{\sum_{i\in I} M_i + N}{\sum_{i\in I} M_i} \leqslant_s \frac{M}{\sum_{i\in I} M_i},$$

i.e. $\sum_{i\in I} M_i + N + X = M \Rightarrow \sum_{i\in I} M_i + X = M$. This is related to Fleury's work by the following result.

LEMMA 3.1. *Let X be a supplement in M of $\sum_{i\in I} M_i + N$ ($M_i, N \in H(M)$). Then $N \in f(\{M_i: i \in I\})$ if and only if $\sum_{i\in I} M_i + X = M$.*

PROOF. The forward implication is clear. Suppose $\sum_{i\in I} M_i + X = M$ but $N \notin f(\{M_i: i \in I\})$, i.e. for some $Y < M$ $\sum_{i\in I} M_i + N + Y = M$ but $\sum_{i\in I} M_i + Y = M' < M$. Then as $M > M' = \sum_{i\in I} M_i + (X \cap M')$, we have that $X > X \cap M'$; but $M = M' + N = \sum_{i\in I} M_i + (X \cap M') + N$, which contradicts X being a supplement of $\sum_{i\in I} M_i + N$.

THEOREM 3.2. *f obeys axioms $D(1)$ to $D(3)$.*

PROOF. D(1) is clear. To show D(2), let $N \in f(\{M_i: i \in I\})$ and let $L \in f(\{M_i: i \in I\} \cup \{N\})$: we show $L \in f(\{M_i: i \in I\})$. Let $X \leqslant M$ such that $\sum_{i\in I} M_i + L + X = M$. Then $\sum_{i\in I} M_i + N + L + X = M$ and so, as $L \in f(\{M_i: i \in I\} \cup \{N\})$, $\sum_{i\in I} M_i + N + X = M$. Likewise, as $N \in f(\{M_i: i \in I\})$, we have $\sum_{i\in I} M_i + X = M$, and it follows that $L \in f(\{M_i: i \in I\})$.

We show D(3) by supposing that $N \in f(\{M_i: i \in I\} \cup \{L\})$ and $L \notin f(\{M_i: i \in I\} \cup \{N\})$ and thence showing that $N \in f(\{M_i: i \in I\})$. Let $\sum_{i\in I} M_i + N + Y = M$; we will show that $\sum_{i\in I} M_i + Y = M$. By our supposition there exists $X \leqslant M$ such that $\sum_{i\in I} M_i + N + L + X = M$ but $\sum_{i\in I} M_i + N + X = M' < M$. As $\sum_{i\in I} M_i + N + Y = M$, $M' = \sum_{i\in I} M_i + N + (M' \cap Y)$ and $\sum_{i\in I} M_i + L + N + (M' \cap Y) = M' + L = M$. Since $N \in f(\{M_i: i \in I\} \cup \{L\})$, it follows that $\sum_{i\in I} M_i + L + (M' \cap Y) = M$, and so $M' = \sum_{i\in I} M_i + (M' \cap L) + (M' \cap Y)$. As $M' \neq M$, $M' \cap L$ is a proper submodule of L, and, as L is hollow, $M' \cap L \leqslant_s L$, from which it follows that $M' \cap L \leqslant_s M$. Now, as $M' + Y = M$, $\sum_{i\in I} M_i + (M' \cap L) + Y = M$ and as $M' \cap L \leqslant_s M$, $\sum_{i\in I} M_i + Y = M$, as required; thus $N \in f(\{M_i: i \in I\})$.

Let us now define f' on $\mathscr{P}(E)$: $N \in f'(\{M_i: i \in I\})$ if $N \in f(\{M_i: i \in J\})$ for some finite $J \subseteq I$. Then it is clear that f' obeys D(1) to D(4), and so is the span operator of an independence structure, $\mathscr{F}(M)$, the Fleury structure, on $H(M)$.

THEOREM 3.3 *$\mathscr{F}(M) = \{\{M_i: i \in I\} \subseteq H(M)$: for $j \in I$, $M_j \notin f'(\{M_i: i \in I \backslash j\})\}$ is an independence structure.*

It is clear that for $M_i \in H(M)$, $\sum_{i\in I} M_i = M$ implies that $f(\{M_i\}) = H(M)$, but the converse does not necessarily hold. We have, however, the following result.

PROPOSITION 3.4. Let $\{N_1, \ldots, N_s\}$ be a finite subset of $H(M)$ such that $\sum_{i=1}^{s} N_i = M$. Then the sum of every basis of $\mathscr{F}(M)$ is M, and the bases are precisely the minimal subsets of $H(M)$ whose sum is M.

PROOF. Suppose that $\{M_1, \ldots, M_r\}$ is a basis of $\mathscr{F}(M)$ such that $\sum_{i=1}^{r} M_i = M' < M$. Then, as each $N_j \in f'(\{M_1, \ldots, M_r\})$, $(M' + N_j)/M' \leqslant_s M/M'$, and so $\sum_{j=1}^{s} (M' + N_j)/M' \leqslant_s M/M'$. That is, $M/M' \leqslant_s M/M'$, which is a contradiction. Hence $\sum_{i=1}^{r} M_i = M$. Thus the spanning subsets of $H(M)$ are precisely those whose sum is M, and so the bases, being the minimal spanning subsets, are the minimal subsets whose sum is M.

We now look at the question of refining a given expression for M as a sum of submodules to a sum of hollow submodules.

LEMMA 3.5.
(a) *Let* $L+N=M$ *(with* $N<M$*). Then* $L\in H(M)\Rightarrow N\in \mathrm{Hf}(M)$.
(b) *Let* $L(\neq 0)$ *be a supplement of* N *in* M. *Then* $L\in H(M)\Leftrightarrow N\in \mathrm{Hf}(M)$.

PROOF. (a) If L is hollow, its factor $L/(L\cap N)$ is hollow, and $L/(L\cap N)\cong M/N$.
(b) Let $N\in \mathrm{Hf}(M)$ and let $A, B<L$. Then as L is a supplement of N, $A+N<M$ and $B+N<M$. As M/N is hollow, $(A+N)+(B+N)<M$, i.e. $(A+B)+N<M$, and so $A+B<L$. Thus L is hollow.

THEOREM 3.6. *For a module* M *with the supplement property* (S), (a)$\Rightarrow$(b)$\Leftrightarrow$(c)$\Rightarrow$(d); *if* $\mathrm{rk}(\mathscr{F}(M))<\infty$, *then* (a) *to* (d) *are equivalent*:
(a) $\mathrm{rk}(\mathscr{F}(M))<\infty$, *and for some basis* $\{L_1,\dots,L_r\}$, $\sum_{i=1}^r L_i=M$;
(b) *every non-trivial factor module of* M *has a hollow factor module*;
(c) *if* N *is a non-small submodule of* M *then there exists* $L\in H(N)$ *such that* $L\not<_s N$;
(d) $M=\sum_{i\in I} L_i$ *for every basis* $\{L_i: i\in I\}$ *of* $\mathscr{F}(M)$.

PROOF. (b)$\Rightarrow$(c). Let a non-small submodule N of M be given, and let $X<M$ such that $N+X=M$. Let M/Y be a hollow factor module of M/X, by (b); so $Y\geqslant X$ and $N+Y=M$. Let $L\leqslant N$ be a supplement of Y in M, so L is hollow by Lemma 3.5 (b). As $L\leqslant N$ and $L+Y=M$, $L+(Y\cap N)=N$; as $N\not\leqslant Y$ (for $N+Y=M$), L is not small in N.
(c)$\Rightarrow$(b). Let $A<M$ be given, and let N be a supplement of A. By (c), Let $N=L+K$, with L hollow and $K<N$. As $K<N$, $K+A<M$ and as $K+A+L=M$, $M/(K+A)\cong L/((K+A)\cap L)$, which is hollow since L is hollow. Thus $M/(K+A)$ is the required hollow factor module of M/A.
(c)$\Rightarrow$(d). Let $\{L_i: i\in I\}$ be a basis of $\mathscr{F}(M)$. If $\sum_{i\in I} L_i<M$, let N be a supplement of $\sum_{i\in I} L_i$ in M. Then by (c), let $L\leqslant N$, $L\in H(M)$ such that $L+K=N$, $K<N$. Then as $K<N$, $\sum_{i\in I} L_i+K<M$ whereas $\sum_{i\in I} L_i+L+K=M$. Hence $L\notin f(\{L_i: i\in I\})$ which contradicts $\{L_i: i\in I\}$ being a basis. Thus $\sum_{i\in I} L_i=M$.
(a)$\Rightarrow$(b) will be proved following Theorem 3.8, and, trivially, (d)$\Rightarrow$(a) when $\mathscr{F}(M)$ is of finite rank.

Note that condition (b) of Theorem 3.6 is just condition (c) of Theorem 2.5.

COROLLARY 3.7 *Let* M *have property* (S), *let property* (c) *of Theorem* 3.6 *hold, and let* $\mathrm{rk}(\mathscr{F}(M))=r<\infty$. *If* $M=M_1+\cdots+M_s$ *is an irredundant sum, then we can write* $M=L_1+\cdots+L_r$, *an irredundant sum, with each* L_i *hollow and contained in some* M_j; *also* $s\leqslant r$.

PROOF. Given the irredundant sum $M=M_1+\cdots+M_s$, choose some M_i which is not hollow, and replace M_i by a submodule of it which is a supplement of $\sum_{j\neq i} M_j$; then if the new M_i is not hollow, replace it by $L+K$, where $L+K=M_i$, L is hollow and $K<M_i$ [which is possible by condition (c) of Theorem 3.6]. Then the sum $M=\sum_{j\neq i} M_i+L+K$ is irredundant. This process can be repeated until it stops, when all the submodules in the sum are hollow; the number of hollow submodules in the sum increases by at least one each time. The process must stop, for the set $\{L_1,\dots,L_t\}$ of hollow submodules in the sum at any stage is independent (by the irredundancy of the sum), and so $t\leqslant r$. Since we get at least one hollow module from each M_i, $s\leqslant r$. Proposition 3.4 then shows that the final expression $M=L_1+\cdots+L_t$ satisfies $t=r$.

Theorem 3.6 suggests a connection between the Fleury and the dual Goldie structures for modules with property (S); this is explored by Varadarajan [8] in the case of modules with finite corank.

THEOREM 3.8. *Let M have property (S). Then if either $\mathscr{G}\mathrm{d}(M)$ or $\mathscr{F}(M)$ is of finite rank, then they are of equal rank; in this case, if $\{M_1, \ldots, M_r\}$ and $\{N_1, \ldots, N_r\}$ are bases of $\mathscr{F}(M)$ and $\mathscr{G}\mathrm{d}(M)$, respectively, then $M_1 + \cdots + M_r = M$ if and only if $N_1 \cap \cdots \cap N_r \leqslant_s M$.*

PROOF. Let I be a finite index set. Let $\mathrm{rk}(\mathscr{F}(M)) \geqslant |I|$, with $\{M_i: i \in I\} \in \mathscr{F}(M)$. Let $\sum_{i \in I} M_i$ have a supplement X in M. Then for $j \in I$, let $N_j = \sum_{i \in I \setminus j} M_i + X$; $N_j < M$ by Lemma 3.1, and as $N_j + M_j = M$, $N_j \in \mathrm{Hf}(M)$ by Lemma 3.5. Since, for $j \in I$, $N_j + \cap_{i \in I \setminus j} N_i \geqslant \sum_{i \in I \setminus j} M_i + X + M_j + X = M$, we have $\{N_i: i \in I\} \in \mathscr{G}\mathrm{d}(M)$ and $\mathrm{rk}(\mathscr{G}\mathrm{d}(M)) \geqslant |I|$.

Conversely, let $\mathrm{rk}(\mathscr{G}\mathrm{d}(M)) \geqslant |I|$, with $\{N_i: i \in I\} \in \mathscr{G}\mathrm{d}(M)$. Then, for $j \in I$, $N_j + \bigcap_{i \in I \setminus j} N_i = M$; let M_j be a supplement in M of N_j, contained in $\bigcap_{i \in I \setminus j} N_i$. Hence $M_j \in H(M)$ by Lemma 3.5. As $\sum_{i \in I \setminus j} M_i + M_j + N_j = M$ but $\sum_{i \in I \setminus j} M_i + N_j \leqslant N_j < M$, we have $M_j \notin f(\{M_i: i \in I \setminus j\})$ and so $\{M_i: i \in I\} \in \mathscr{F}(M)$ and $\mathrm{rk}(\mathscr{F}(M)) \geqslant |I|$.

Thus $\mathscr{F}(M)$ and $\mathscr{G}\mathrm{d}(M)$ have equal or infinite rank. Suppose they have finite rank. Then by Proposition 3.4 if some basis of $\mathscr{F}(M)$ has sum M, then so has every basis; by Theorem 2.5, (b)$\Rightarrow$(d), if the intersection of some basis of $\mathscr{G}\mathrm{d}(M)$ is small in M, then this is so for every basis. If this latter is the case, then by Theorem 2.5 (b)$\Rightarrow$(c) and Corollary 3.7, the sum of each basis of $\mathscr{F}(M)$ is M. Alternatively, this follows from the fact that in either of the two constructions above, $M_1 + \cdots + M_r + (N_1 \cap \cdots \cap N_r) = M$, which can be shown by induction as follows. For $s < r$, $M_{s+1} \leqslant N_1 \cap \cdots \cap N_s$, and so $M_{s+1} + (N_1 \cap \cdots \cap N_s \cap N_{s+1}) = (M_{s+1} + N_{s+1}) \cap (N_1 \cap \cdots \cap N_s) = N_1 \cap \cdots \cap N_s$. Thus $M_1 + \cdots + M_s + M_{s+1} + (N_1 \cap \cdots \cap N_s \cap N_{s+1}) = M_1 + \cdots + M_s + (N_1 \cap \cdots \cap N_s)$, which is equal to M by induction on s.

Conversely, suppose that $N_1 \cap \cdots \cap N_r \not\leqslant_s M$ for some basis $\{N_1, \ldots, N_r\}$ of $\mathscr{G}\mathrm{d}(M)$. Let $(N_1 \cap \cdots \cap N_r) + Y = M$, with $Y < M$. Thus for $j \in I$ $(= \{1, \ldots, r\})$, $\bigcap_{i \in I \setminus j} N_i = (N_1 \cap \cdots \cap N_r) + (\bigcap_{i \in I \setminus j} N_i \cap Y)$, and hence by lemma 2.1, $M = N_j + \bigcap_{i \in I \setminus j} N_i = N_j + (\bigcap_{i \in I \setminus j} N_i \cap Y)$. Thus, in the second construction above, we can choose each M_j to be a supplement of N_j contained in $\bigcap_{i \in I \setminus j} N_i \cap Y$. Then $\{M_1, \ldots, M_r\}$ is a basis of $\mathscr{F}(M)$ whose sum is contained in Y. (It can also be shown, referring to either construction above, that, provided $M_1 + \cdots + M_r = M$, $N_1 \cap \cdots \cap N_r \leqslant_s M$; the method is to show by induction on s that if $(N_1 \cap \cdots \cap N_s) + X = M$, then $M_{s+1} + \cdots + M_r + X = M$).

PROOF OF THEOREM 3.6, (a)$\Rightarrow$(b). Suppose $\{L_1, \ldots, L_r\}$ is a basis of $\mathscr{F}(M)$ such that $L_1 + \cdots + L_r = M$. Then by Theorem 3.8, there is a basis $\{N_1, \ldots, N_r\}$ of $\mathscr{G}\mathrm{d}(M)$ such that $N_1 \cap \cdots \cap N_r \leqslant_s M$, and the result follows by Theorem 2.5 (b)$\Rightarrow$(c).

4. THE DUAL FLEURY STRUCTURE

Section 3 can be dualized directly.

DEFINITION. For N, $M_i \in \mathrm{Uf}(M)$, we say $N \in h(\{M_i: i \in I\})$ if $\bigcap_{i \in I} M_i \cap N \leqslant_e \bigcap_{i \in I} M_i$, i.e. $\bigcap_{i \in I} M_i \cap N \cap X = 0 \Rightarrow \bigcap_{i \in I} M_i \cap X = 0$.

LEMMA 4.1. *Let X be a complement in M of $\bigcap_{i \in I} M_i \cap N$ $(M_i, N \in \mathrm{Uf}(M))$. Then $N \in h(\{M_i : i \in I\} \Leftrightarrow \cap_{i \in I} M_i \cap X = 0$.*

The proof is similar to that of Lemma 3.1; results and proofs of Section 3 are dualized by generally interchanging $\cap$ with $+$, 0 with M, $\leqslant$ with $\geqslant$, and changing other concepts accordingly (e.g. $N \leqslant_e M$ to $N \leqslant_s M$, $N \in H(M)$ to $N \in \mathrm{Uf}(M)$). A result corresponding to Theorem 3.2 holds, and if we define h' from h as we did f' from f we get the dual Fleury independence structure.

THEOREM 4.3. *$\mathscr{F}\mathrm{d}(M) = \{\{M_i: i \in I\} \subseteq Uf(M)$: for each $i \in I$ and finite $J \subseteq I \setminus i$, $M_i \notin h(\{M_j: j \in J\})\}$ is an independence structure on* Uf(M).

Since the dual of property (S) holds in every module M, it is superfluous to quote the dual of Proposition 3.4 as well as that of Theorem 3.6, and we let the reader dualize Lemma 3.5.

THEOREM 4.6. *For a module M,* (a)$\Rightarrow$(b)$\Leftrightarrow$(c)$\Rightarrow$(d); if $\mathrm{rk}(\mathscr{F}\mathrm{d}(M)) \leqslant \infty$, *then* (a) *to* (d) are equivalent:
(a) $\mathrm{rk}(\mathscr{F}\mathrm{d}(M)) < \infty$, *and for some basis* $\{L_1, \ldots, L_r\}$, $\bigcap_{i=1}^r L_i = 0$;
(b) *every non-zero submodule of M has a uniform submodule*;
(c) *if N is a non-essential submodule of M, then there exists $L \in \mathrm{Uf}(M)$ such that $L \geqslant N$ and $L/N \not\leqslant_e M/N$ (i.e. there exists $K > N$ such that $L \cap K = N$)*;
(d) $0 = \bigcap_{i \in I} L_i$ *for every basis* $\{L_i: i \in I\}$ *of* $\mathscr{F}\mathrm{d}(M)$.

We note that condition (b) above is condition (a) of [2, lemma 3(b)]. Again, we leave it to the reader to dualize Corollary 3.7. Part of the dual of Theorem 3.8 is deducible from [7, theorems 4.9 and 4.10].

THEOREM 4.8. *If either $\mathscr{G}(M)$ or $\mathscr{F}\mathrm{d}(M)$ is of finite rank, then they have equal rank; in this case, if* $\{M_1, \ldots, M_r\}$ and $\{N_1, \ldots, N_r\}$ are *bases of* $\mathscr{F}\mathrm{d}(M)$ and $\mathscr{G}(M)$, respectively, then $M_1 \cap \cdots \cap M_r = 0$ *if and only if* $N_1 + \cdots + N_r \leqslant_e M$.

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