The Cauchy Problem for an Axially Symmetric Equation and the Schwarz Potential Conjecture for the Torus

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We present our results in this paper in two parts. In the first part, we consider the Cauchy problem for the axially symmetric equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{k}{x} \frac{\partial u}{\partial x} = 0$$

with entire Cauchy data given on an initial plane (see Eq. (2.1)). We solve the Cauchy problem and obtain its solutions in two cases, depending on whether $k$ is a positive even integer or $k$ is a positive odd integer. For $k$ odd, we demonstrate that the solution has more singularities due to the propagation of the singularities of the coefficients. In the second part, the Cauchy problem for the same equation is considered, but instead, its entire Cauchy data are given on an initial sphere (see Eq. (3.1)). Whenever $k$ is a positive even integer, we obtain the global existence of the solution and determine all possible singularities. Whenever $k$ is a positive odd integer, we discuss both local and global solutions. As a consequence of our results in this paper, we show that the Schwarz Potential Conjecture (see the Introduction) for the even dimensional torus is true.

1. INTRODUCTION

Two of the cornerstones for the Cauchy problem of holomorphic partial differential equations are the Cauchy–Kovalevskaya theorem and Zerner’s theorem (see [7, 8, 10, 12]). The Cauchy–Kovalevskaya theorem gives us the local existence and uniqueness of analytic solutions at any noncharacteristic point of the initial hypersurface, and Zerner’s theorem (Theorem 2.1) allows us to extend the local solutions analytically across any point at a regular-analytic real hypersurface, as long as this point is noncharacteristic with respect to the PDE (see Definition 2.1). Globally, if we consider the
Cauchy problem of Laplace's equation with data on a noncharacteristic real hyperplane, these two theorems can yield entire solutions if the Cauchy data are entire. For a general regular-analytic real hypersurface \( \Gamma \), the Schwarz Potential Conjecture (see [11–13]) says that any solutions of Laplace's equation with entire Cauchy data given on \( \Gamma \) can be analytically continued (in \( \mathbb{C}^n \) or \( \mathbb{R}^n \)) as far as the Schwarz potential of \( \Gamma \) (see Definition 3.2) can be continued. In 1992, G. Johnsson [11] showed that the Schwarz Potential Conjecture is true whenever \( \Gamma \) is a quadratic analytic hypersurface. The primary objective of this paper\(^1\) is to determine whether this conjecture holds or not when we take our initial hypersurface as a quartic surface, a torus \( \Gamma \), with axially symmetric data given on \( \Gamma \). If

\[
\Gamma = \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid \left( \sqrt{x_1^2 + \cdots + x_{n-1}^2} - \rho \right)^2 + x_n^2 = 1, \; \rho > 1 \right\}
\]

(1.1)

and the axially symmetric data are \( f(\sqrt{x_1^2 + \cdots + x_{n-1}^2}, x_n) \), then the Schwarz Potential Conjecture suggests that the solutions of Laplace's equation with axially symmetric data on the torus \( \Gamma \) are harmonic on \( \mathbb{R}^n \) except at the possible singularities on the caustic hypersphere \((x_1, \ldots, x_{n-1}, 0) \mid x_1^2 + x_2^2 + \cdots + x_{n-1}^2 = \rho^2\) and on the real line \((0, \ldots, 0, x_n)\). If we let \( x = \sqrt{x_1^2 + \cdots + x_{n-1}^2} \) and \( y = x_n \), then we get the axially symmetric Laplace equation \( \Delta u + \frac{2}{\lambda} \partial_\lambda u = 0 \) with the Cauchy data \( f(x, y) \) on the unit circle \( \Gamma \), where \( f(x, y) \) is real analytic in two variables \( k = n - 2 \) and

\[
\Gamma := \left\{ (x, y) \mid (x - \rho)^2 + y^2 = 1 \right\}.
\]

(1.2)

Therefore, now we only need to show that the Cauchy problem (3.1) has analytic solutions on \( \mathbb{R}^2 \) except at the center of \( \Gamma \) (i.e., at \((\rho, 0)\)) and on the real line \( x = 0 \). Alternatively, after the complexification of \( x \) and \( y \), we only need to show (3.1) has a holomorphic solution on \( \mathbb{C}^2 \) except at the two isotropic complex lines \( y = \pm i(x - \rho) \) and the other complex line \( x = 0 \). Notice that in \( \mathbb{C}^2 \) the singularities of our coefficients \((x = 0)\) intercept the initial hypersphere \( \Gamma \) (given by Eq. (1.2)) at the points \((0, \pm i\sqrt{\rho^2 - 1})\); hereafter, we will refer to these two points at these two "bad points." In order to examine the local behavior of the solution at these two "bad points," we observe (see Section 4) that there is some "similarity" between the local behavior of (2.1) at the point \( x = 0 \) and the local behavior of (3.1) at the two "bad points."

In Section 2, we begin by discussing the Cauchy problem (2.1) where \( \Gamma \) is simply the complex plane. In particular, we find the solutions of (2.1) in

\(^1\) We became aware of this problem from Dr. D. Khavinson.
two cases, depending on whether $k$ is an even or an odd positive integer (where $k = n - 2$, and $n$ is the dimension of the torus mentioned above—later we refer to this $k$ as the dimensional index). Surprisingly, the nature of the solutions differs dramatically in these two cases. It turns out that the solutions when $k$ is odd have singularities not only at the complex line $x = 0$, but also along the isotropic complex lines $x = \pm yi$ (see Remark below). On the other hand, when $k$ is even the solutions have only singularities at $x = 0$ and are globally holomorphic elsewhere.

In Section 3, we mainly discuss the Cauchy problem (3.1). When $k$ is a positive even integer, we show in Theorem 3.2 the existence of the global solution and determine all its possible singularities. We then study the Cauchy problem of the adjoint equation of (3.1), and the Cauchy problem of some other related equations. Moreover, the explicit expressions of the Schwarz potentials of the 4-dimensional torus and the 6-dimensional torus are produced in Examples 3.3 and 3.4, respectively. In Proposition 3.1, we give an induction method to calculate the Schwarz potential for higher even dimensional tori and an algorithm to solve Eq. (3.1). We then conclude that the Schwarz Potential Conjecture is true when $\Gamma$ is a torus of an even dimensional space. On the other hand, when $k$ is a positive odd integer, we reduce the higher dimensional Cauchy problem into a 3-dimensional case (i.e., $k = 1$), and we conjecture that the solutions of the odd case may have more singularities than that of the even case due to propagation of the singularities at the two “bad points” along the two characteristic directions of the Laplace operator.

We close our paper with a few final remarks (Section 4). In these remarks, we discuss the “similarity” of the local solutions at the “bad points” of problems (2.1) and (3.1), extend results of Section 3 with some observations, and raise an open problem and some remaining unsolved questions related to our research.

Notation. We also use $f = g$ to denote $f = g$ and $\nabla f = \nabla g$ as two variable analytic functions. We also use $\Gamma$ to stand for any hypersurface where the Cauchy data are defined. For the rest of this paper, and for $X, Y, x, y \in \mathbb{C}^2$, we let $T = \{(X, Y) : X = 0\}$, $\Pi = \{(X, Y) : Y = \pm X\}$, $\Lambda = \{(X, Y) : \text{Re}(Y \pm iX) = 0\}$, $\Omega = \{(X, Y) : |Y| < |X|\}$, $\Theta = \{(x, y) : y = \pm i(x - \rho)\}$, and $\mathcal{U}_k = \Lambda + \frac{k}{\rho} \partial_x$.

2. SOLUTIONS OF THE CAUCHY PROBLEM WITH ENTIRE DATA ON THE COMPLEX PLANE

Problems related to the singular equation in (2.1) date back to Laplace’s time. We refer the readers to the work of A. Weinstein, R. P. Gilbert, A. Huber, B. Brelot-Collin, and M. Brelot, and the work of R. Millar for some
weakly related results in this area. In [19–21] (also see Note 4), A. Weinstein mainly investigated the potential theory in terms of general solutions to this type of singular equations and addressed some applications in fluid dynamics. In [4, 5], R. P. Gilbert studied the operators that map single variable analytic functions to general solutions of this singular PDE and also presented some analytic properties of the solutions. In [1], H. Begehr and R. P. Gilbert (also see [6]) presented a list of results on singularities of PDE. In [9], A. Huber explored the uniqueness and obtained a Green type formula in \( \mathbb{R}^n \) to the boundary value problem in a hemisphere bounded by the singular plane. This boundary setting is actually one of the most common type found in the early literature (see [2, 6, 9, 19, 20]). In [2], B. Brelot-Collin and M. Brelot continued A. Huber’s work and discussed the integral representations of the positive solutions of singular equations in \( \mathbb{R}^n \) on the upper half-space bounded by the singular plane. In [16, 17], R. Millar addressed the Cauchy problem, but his setting and interest were quite different from ours (see Note 3). More closely related investigations, namely, on the global solution of the Cauchy problem, are those of M. Miyake [15] and G. Johnsson [11]. If we change our singular coefficient term slightly, say to \( \frac{k}{y^p} \) instead of \( \frac{k}{t} \), our problem becomes a special case of Miyake’s theory. Miyake’s theorem (see [11, Theorem 4.3; 15, Theorem 2]) shows that the modified problem (2.1) has a holomorphic solution in all of \( \mathbb{C}^2 \) except at the obvious singularities along the complex line \( y = -p \). However, the difference in our case is that the singular coefficient intercepts the data plane instead of being parallel to it. Therefore, our problem does not fit into either Miyake’s or Johnsson’s theory. In this section, we develop a rather elementary method to derive our results.

**Definition 2.1.** Assume that the \( a_\alpha(z) \) are holomorphic functions in a neighborhood of \( z_0 \) for all \( \alpha \), where \( \alpha = (\alpha_1, \ldots, \alpha_n) \) are multiple indexes. A real hyperplane \( \Sigma = \sum_{i=1}^n \lambda_i x_i = t \) (\( t \in \mathbb{R} \)) passing through a point \( z_0 \) in \( \mathbb{C}^n \) is called Zerner characteristic with respect to the operator \( \mathcal{Q} : \sum_{|\alpha|=m} a_{\alpha}(z_0) \partial^{\alpha_1} \ldots \partial^{\alpha_n} \) at \( z_0 \) if \( \Gamma \) satisfies

\[
\sum_{|\alpha|=m} a_{\alpha}(z_0) \lambda_1^{\alpha_1} \ldots \lambda_n^{\alpha_n} = 0, \quad \text{where} \ |\alpha| = \alpha_1 + \cdots + \alpha_n.
\]

A smooth real hypersurface is called Zerner characteristic with respect to \( \mathcal{Q} \) at point \( z_0 \) if its tangent real hyperplane at \( z_0 \) is a Zerner characteristic with respect to \( \mathcal{Q} \) at point \( z_0 \).

It is easy to check that \( \Lambda \) are the only two characteristic real planes of the Laplace operator at the origin and that \( \Pi \) are two complex planes contained in \( \Lambda \) which are uniquely determined by \( \Lambda \) (see [12, Chap. 6]).
In our analysis below, we basically have two types of solutions defined as follows:

**Definition 2.2.** (1) Solution of type 1 is a solution which is a meromorphic function with only singularities at $\xi_0$ (that is, where the coefficient of the equation is singular). (2) Solution of type 2 is a solution which is holomorphic in $\Omega$ with nonextendable singularities at $\Pi$ and $\Gamma$.

**Remark.** Notice that the data of (2.1) are entire. Singularities of solutions of type 2 are propagated from the origin of the $xy$ plane, where the singular coefficient $x = 0$ intercepts the $x$-$y$ plane, along the two characteristic directions of the Laplace operator.

**Theorem 2.1 (Zerner).** Let $\mathcal{L}u = f$ be a linear, holomorphic equation in a domain $D \subset \mathbb{C}^n$ ($\mathcal{L}$ is defined in Definition 2.1). If $u$ is a holomorphic solution of this equation in an open set $X \subset D$, $z_0 \in \partial X \cap D$, and $\Gamma = \partial X$ is smooth and noncharacteristic at $z_0$ with respect to $\mathcal{L}$, then $u$ extends holomorphically into a neighborhood of $z_0$.

**Theorem 2.2.** The Cauchy problem

$$
\begin{align*}
\mathcal{L}_k u(x, y) &= 0, \quad 0 < k \in \mathbb{N} \\
u_{|y=0} &= f(x, y) \quad (x, y \in \mathbb{C}^2 \setminus \Gamma)
\end{align*}
$$

has a unique entire solution when the data $f(x, y)$ are an even entire function of $x$. Furthermore, whenever the data $f(x, y)$ are an arbitrary entire function, the Cauchy problem has a unique solution of type 1 for any even positive integer $k$ and a unique solution of type 2 for any odd positive integer $k$.

**Remark.** Although the initial plane is $y = 0$, we still have to consider data as a function of two variables, since we need to consider the gradient of the data, which includes $\partial_y f(x, y)$.

For any point $x \neq 0$ on the initial plane $y = 0$, the Cauchy–Kovalevskaya theorem tells us that we have a unique locally holomorphic solution in a small neighborhood of $(x, 0)$ in $\mathbb{C}^2$ provided that the data $f(x, y)$ are analytic as a two variable function of $x, y$ in the neighborhood of $(x, 0)$. Notice that the coefficient $\frac{i}{2}$ of the lower differential term is singular at the complex line $x = 0$ which intercepts the initial plane $y = 0$ at the origin $(0, 0)$. Even for polynomial data, the solution in the neighborhood of the origin is still rather complicated.

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2 R. P. Gilbert addressed a similar problem about general solutions of $\mathcal{L}_k u = 0$ in [5, Theorem (ii)] (or [6, Theorem 4.2.7]). The author expanded $u$ as a Gegenbauer series and gave the radius of the convergence of the series by its coefficients. His result covers part of our conclusion about type 2 solutions.
In order to prove Theorem 2.2, we need the following:

**Theorem 2.3 (Pringsheim).** Suppose that \( \sum_{n=0}^{\infty} a_n z^n \) has convergence radius 1, and all \( a_n \geq 0 \). Then \( z = 1 \) must be a nonextendable singularity.

In the next lemma, we recursively solve the difference equation (2.2) and obtain the solutions of (2.1) in series form for all cases depending on whether \( k \) is even or odd, and whether the exponent of \( x \) is even or odd.

**Lemma 2.1.** When \( f(x, y) = x^{2n} \) (i.e., an even function), the solution of Cauchy Problems (2.1) is the 2n degree polynomial (2.3) for any positive integer \( k \). When \( f(x, y) = x^{2n+1} \) (i.e., \( f \) is an odd function) and \( k \) is a positive even integer, the solution is the rational function (2.4) with single pole of order \( k \) at the origin. When \( f(x, y) = x^{2n+1} \) and \( k \) is a positive odd integer, the solution is the function (2.5), which is of type 2 with singularities at both II and T.

**Proof.** Considering the series solution in the punctured neighborhood of the origin, and since all equations has no singularities of variable \( y \) in the data and coefficients, we can suppose that \( u(x, y) = \sum_{i=0}^{\infty} u_i(x) y^i \), where \( u_i(x) \) is a meromorphic function of \( x \) in any case. Substituting \( u(x, y) \) into Eq. (2.1) and comparing the coefficients of \( y^i \), we get the following difference equation about \( x \):

\[
(l + 1)(l + 2)u_{l+2}(x) + \frac{k}{x} u'_i(x) + u''_i(x) = 0, \quad 0 \leq l < \infty. \tag{2.2}
\]

Then solving problem (2.1) is equivalent to solving (2.2) with \( u_0 = x^{2n} \), or \( u_0 = x^{2n+1} \) whenever \( k \) is either even or odd, respectively. Observe that at each recursive step, the exponent of \( x \) decreases by 2. The question now is how one can terminate the series solutions in the first two cases. Regardless of whether \( k \) is even or odd, when the exponent of \( x \) is even, say \( 2n \), then the series terminates after \( n + 1 \) steps since we get a constant \( u_n(x) \) at the \( n \)th step. Hence, its derivatives vanish, and so does \( u_{n+2}(x) \) at the next step. When the exponent of \( x \) is odd, but \( k \) is an even integer, from the coefficients \( b_{2j} \) of the solution we can easily see that the numerator becomes 0 after \( k/2 + n + 1 \) terms of recursion. So we obtain the solutions of the first 2 cases. The last case, when both the exponent of \( x \) is odd and \( k \) is odd, we obtain an infinite series. Therefore, the resulting 3
solutions are
\[ U_{2n}(x, y) = \sum_{0}^{n} a_{2l}x^{2n-2l}y^{2l}, \quad a_0 = 1, \]
\[ a_{2l} = (-1)^l \frac{2n(2n-2) \cdots (2n-2l+2) \cdot (2n+k-1)(2n+k-3) \cdots (2n+k-2l+1)}{(2l)!} \] (2.3)
\[ V_{2n+1}(x, y) = x \sum_{0}^{n} a_{2l+1}x^{2n-2l}y^{2l} + a_{2n+1}xy^{2n} \sum_{1}^{m} b_{2j} \left( \frac{y}{x} \right)^{2j}, \]
\[ m = \frac{k}{2}, a_1 = 1 \]
\[ a_{2l+1} = (-1)^l \frac{(2n+1)(2n-1) \cdots (2n-2l+3) \cdot (2n+k)(2n+k+2) \cdots (2n+k+2l)}{(2l)!}, \]
\[ b_{2j} = -\frac{(2n)! \cdot (2j-3)(2j-5) \cdots 3 \cdot 1 \cdot k(k-2) \cdots (k-2j+2)}{(2n+2j)!} \] (2.4)
\[ W_{2n+1}(x, y) = x \sum_{0}^{n} a_{2l+1}x^{2n-2l}y^{2l} + a_{2n+1}xy^{2n} \sum_{1}^{\infty} b_{2j} \left( \frac{y}{x} \right)^{2j}, \quad a_0 = 1, \] (2.5)

where \( a_{2l+1} \) and \( b_{2j} \) are defined in (2.4) for \( k \) odd. Now the conclusions about (2.3) and (2.4) are obvious, so we only need to show that (2.5) is a solution of type 2. First, notice that \( k \) is fixed. For any \( j > k \), if \( j \) increases by 1, then \( b_{2j} \) alternates in sign. Let \( c_{2j} = (-1)^l b_{2j} \), \( z = \pm \frac{y}{x} \), and \( T = \sum_{k+1}^{\infty} c_{2j} \left( \frac{z}{x} \right)^{2j} \). We can see that \( T = \sum_{k+1}^{\infty} c_{2j} z^{2j} \) and \( c_{2j} \) has a fixed sign; thus without loss of generality, we can suppose that all coefficients of \( T \) are positive. It is also easy to see that \( b_{2j} = o(j^{-2(l-1)}) \); thus we have \( \sqrt[2j]{c_{2j}} = 1 \), and consequently, \( T \) is analytic in the unit disk. Therefore, \( W_{2n+1}(x, y) \) is holomorphic in the region \( \Omega = \{(X, Y) : |Y| < |X|\} \). On the other hand, as we know that \( c_{2j} > 0 \), then Pringsheim’s theorem shows that \( T \) must have a nonextendable singularity at \( z = 1 \). Therefore, \( W_{2n+1}(x, y) \) must have nonextendable singularities at \( y = \pm x \). Hence \( W_{2n+1}(x, y) \) is a solution of type 2. 

**Lemma 2.2.** The solution of Cauchy Problem (2.1) for any positive integer \( k \) with data \( f(x, y) = yx^{2n} \) is the polynomial \( \int_{0}^{t} U_{2n}(x, t) \, dt \). For any even
integer $k$ with data $f(x, y) = yx^{2n+1}$, the solution of (2.1) is a rational function $\int_0^1 V_{2n+1}(x, t) \, dt$ with a single pole at the origin. The solution for odd $k$ with data $f(x, y) = yx^{2n+1}$ is the function $\int_0^1 W_{2n+1}(x, t) \, dt$ with singularities at both $\Pi$ and $\Sigma$.

**Proof.** We consider $\partial_t u$ instead of $u$ in Eq. (3.1); the situation is the same as that of Lemma 2.1. Simply integrating the solutions of Lemma 2.1, we obtain the solutions of Lemma 2.2.

Now we are ready to give the proof of Theorem 2.2.

**Proof of Theorem 2.2.** First notice that we are considering the data on the plane $y = 0$. Suppose we expand the entire data in a Taylor series as $f(x, y) = \sum_{n=0}^\infty f_n(x)y^n$, where $f_n(x)$ is an entire function of $x$. If we consider (2.1) with data $f_n(x)y^n$ for $n \geq 2$, then the data are actually identically 0, consequently, (2.1) has only the trivial solution $u(x, y) \equiv 0$ (by the Cauchy–Kovalevskaya theorem). Therefore, we only need to consider two types of data: $f(x)$ and $yf(x)$. (When considering (3.1) with data on a sphere, we lose this advantage, so this method does not work in that case—see Section 3).

**Case 1.** We consider the solution of (2.1) with data of type $f(x)$. Suppose that $f_0(x) = \sum_0^\infty c_{2n}x^{2n}$ is an even entire function and that $f_2(x) = \sum_{n=0} c_{2n}x^n$ is an arbitrary entire function. If $U(x, y)$, $V(x, y)$, and $W(x, y)$ are the solutions of Eq. (2.1) for any $k$ with data $f_k(x)$, for even $k$ with data $f_2(x)$, and for odd $k$ with data $f_2(x)$, respectively, then by Lemma 2.2, we have

\[
U(x, y) = \sum_{0}^\infty c_{2n}U_{2n}(x, y) \tag{2.6}
\]

\[
V(x, y) = \sum_{0}^\infty c_{2n}U_{2n}(x, y) + \sum_{0}^\infty c_{2n+1}V_{2n+1}(x, y) \tag{2.7}
\]

\[
W(x, y) = \sum_{0}^\infty c_{2n}U_{2n}(x, y) + \sum_{0}^\infty c_{2n+1}W_{2n+1}(x, y) \tag{2.8}
\]

So we only need to show that $U(x, y)$ is an entire function, $V(x, y)$ is a solution of type 1, and $W(x, y)$ is a solution of type 2.

Let $\mathcal{B}$ be any compact set in $\mathbb{C}^2$, $\mathcal{C}$ be any compact set of $\mathbb{C}^2 \setminus \Sigma$ where solution (2.4) is holomorphic, and $\mathcal{D}$ be any compact set of $\Omega$ where solution (2.5) is holomorphic. With this assumption, we claim that $U(x, y)$, $V(x, y)$, and $W(x, y)$ are uniformly convergent in $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$, respectively. Then Theorem 2.2 follows from Hartogs’ theorem and Weierstrass’s theorem. Hartogs’ theorem says that the analyticity of two variables is equivalent to analyticity in each variable, which is obvious since the series
annihilate operator $\partial / \partial \overline{x}$ and $\partial / \partial \overline{y}$; and Weierstrass’s theorem allows us to take derivatives termwise and preserves uniform convergence.

What is left to prove the claim of uniform convergence of $U(x, y)$, $V(x, y)$, and $W(x, y)$ on compacta $\mathcal{R}$, $\mathcal{S}$, and $\mathcal{D}$.

We omit the simple proof of the first two cases and prove this claim for $W(x, y)$. We estimate $W(x, y)$ first. Since $\mathcal{D}$ is a compact subset of $\mathbb{C}$, then $x$ and $y$ in (2.5) are all bounded and $|x| < r < 1$. Also, it is easy to see that $a_i = o(l^{-k/2-n}) < 1$, and $b_j = o(j^{-k/2-1}) < 1$ for any $l$ and $j$, and so the first term of $W_{2n+1}(x, y)$ is bounded by $n \cdot x^{2n}$ and also the last term of $W_{2n+1}$ is bounded by $\frac{1}{r^n} \cdot xy^{2n}$. Hence, there exist two positive constants $c$ and $M$ such that $W_{2n+1}(x, y) < c \cdot n \cdot M^{2n+1}$. Similarly we have $U_{2n}(x, y) < c \cdot n \cdot M^n$ (c and $M$ may need to be adjusted slightly). Therefore,

$$|W(x, y)| = \left| \sum_{0}^{\infty} c_{2n} U_{2n}(x, y) + \sum_{0}^{\infty} c_{2n+1} W_{2n+1}(x, y) \right| < c \sum_{0}^{\infty} c_n \cdot n \cdot M^n < \infty$$

since $f(x)$ is entire, and so our claim holds.

**Case 2.** We now consider the solution of (2.1) with the data of type $y \cdot f(x)$. Suppose that $f_1(x) = y \sum_{0}^{\infty} c_{2n} x^{2n}$ and that $f_2(x) = y \sum_{0}^{\infty} c_{2n} x^n$. We simply use Lemma 2.3 to replace Lemma 2.2; the proof is an analogue of the proof of Case 1.

Combining the proofs of Case 1 and Case 2 completes our proof.

**Example 2.1.** The solution of the Cauchy problem (2.1) for any positive integer $k$ with meromorphic data $f(x, y) = x^{-k}$ is $u = (x^2 + y^2)^{-k/2}$.

**Remark.** We can extend the result of Theorem 2.2 to the case when $k$ is any positive real number, and the solution of Eq. (2.1) with any entire data is also holomorphic in $\Omega$. A simple observation shows that $\Omega$ and $\Lambda$ are tangent to each other along the two complex lines $\Pi$. For any boundary point of $\Omega \setminus \Pi$, we can extend the local solutions onto the two real characteristic plane $\Lambda$ by using Zerner’s theorem and the continuity method of F. John (see [12]).

3. THE CAUCHY PROBLEM WITH ENTIRE DATA ON THE SPHERE AND SCHWARZ POTENTIALS FOR THE TORUS

Now we turn our attention to our main problem and try to answer if the Schwarz Potential Conjecture is also true when $\Gamma$ is a torus. Notice that we take the initial surface as the unit sphere after changing variables. The
Cauchy–Kovalevskaya theorem says that the local solution of the sphere exists anywhere in the small neighborhoods of each point on our sphere except at the two “bad points.” On the other hand, Zerner’s theorem allows us to extend the local solution into a small hull. Yet, at the two “bad points,” no existence theorem can even help us obtain a local solution. However, when $k$ is a positive even integer, we develop a rather elementary approach again to derive our result, see Theorem 3.2. By applying indirectly a theorem of P. Davis and Johnsson, see Theorem 3.1, we obtain the global existence of the solution of Eq. (3.1) and determine all its possible singularities. Moreover, in Proposition 3.1, we develop an algorithm to solve the Cauchy problem (3.1) and find the Schwarz potentials in this case.

Now, consider the Cauchy problem
\[
\begin{cases}
\partial_x u(x, y) = 0 & (x, y \in \mathbb{C}^2 \setminus \mathbb{T}) \ 0 \neq k \in \mathbb{Z} \\
u|_{\Gamma} = f(x, y), & \Gamma = \{(x, y) : (x - \rho)^2 + y^2 = 1\},
\end{cases}
\]  
(3.1)

where $\mathbb{Z}$ is the set of integers. Except when stated otherwise, all data in this section are entire and are defined on this $\Gamma$.

$\Gamma := \{x : x \in \mathbb{R}^n \mid F(x) = 0 \text{ and } \nabla F \neq 0\}$ is called a regular-analytic hypersurface provided that $F(x)$ is a real analytic function on $\mathbb{R}^n$.

**Definition 3.1** (see [3]). Suppose that $\Gamma := \{(x, y) : x, y \in \mathbb{R} \mid \phi(x, y) = 0\}$ is a real analytic curve in $\mathbb{R}^2$. By letting $z = x + iy$ and $\bar{z} = x - iy$, we obtain $\Phi(z, \bar{z}) = \phi(x, y) = 0$. If $\Phi(z, \bar{z}) = 0 \Rightarrow z \mapsto \bar{z} = S(z)$, then the function $S(z)$ is called the Schwarz function for $\Gamma$.  

**Example 3.1.** If $S(z)$ is the Schwarz function for $\Gamma$, then for $\Gamma = \{(x, y) : y = 0\}$, $S(z) = z$, for $\Gamma = \{(x, y) : x^2 + y^2 = 1\}$, $S(z) = \frac{1}{z}$, and for $\Gamma = \{(x, y) : (x - \rho)^2 + y^2 = 1\}$, $S(z) = \frac{1}{z-\rho} + \rho$.

**Definition 3.2** (see [13]). The Schwarz potential of a regular-analytic hypersurface $\Gamma$ in $\mathbb{R}^n \ (n \geq 2)$ is the unique solution of the Cauchy problem
\[
\begin{cases}
\Delta u = 0 \text{ “near” } \Gamma \\
u|_{\Gamma} = \frac{1}{2}|x|^2 = \frac{1}{2}\sum_{i=1}^{n} x_i^2.
\end{cases}
\]

\(^3\) In [17], R. Millar considered a similar Cauchy problem. He obtained an integral representation to his Cauchy problem with hypergeometric functions. Since his fundamental domain must be symmetric with respect to the $y$-axis, his theory does not fit our Cauchy problem (3.1).
The following theorem is a theorem of P. Davis and is also a simple version of Johnsson’s theorem (see [11, Lemma 5.2, Corollaries 5.3, 5.7, Theorem 4.8] and also see [3, 13]).

**THEOREM 3.1.** The solution of Cauchy problem

$$\begin{cases}
\Delta u = 0, & \quad u|_{\Gamma} = f(x, y), \quad x, y \in \mathbb{C}^2 \\
\Gamma = \{(x, y) : (x - \rho)^2 + y^2 = 1\}
\end{cases} \tag{3.2}$$

is $u(x, y) = \text{Re} F(z)$, where

$$\frac{dF}{dz} = \frac{\partial f}{\partial x} \left( \frac{z + S(z)}{2}, \frac{z - S(z)}{2i} \right) - i \frac{\partial f}{\partial y} \left( \frac{z + S(z)}{2}, \frac{z - S(z)}{2i} \right),$$

$S(z)$ is the Schwarz function defined above, and the constant of $F(z)$ is determined by $f(x, y)$. Consequently, the solution is holomorphic in $\mathbb{C}^2 \setminus \Theta$.

**LEMMA 3.1.** Suppose $k > 0$, and let $v = x^{-1} \cdot \partial_x u_k(x, y)$, where $u_k(x, y)$ is a solution of $\mathcal{L}_k u = 0$. Then $v(x, y)$ is a general solution of $\mathcal{L}_{k+2} u = 0$.

**Proof.** By hypothesis, we have

$$\mathcal{L}_k u_k = \Delta u_k + \frac{k}{x} \partial_x u_k = 0, \quad \text{so} \quad \Delta u_k = -\frac{k}{x} \partial_x u_k.$$

$$\Delta v = \Delta \left( \frac{1}{x} \partial_x u_k \right) = \frac{2}{x^2} \left( \frac{1}{x^2} \partial_x^2 u_k - \frac{1}{x} \partial_x u_k \right) + \frac{1}{x} \partial_x (\Delta u_k)$$

$$= \frac{2}{x} \partial_x \left( -\frac{1}{x} \partial_x u_k \right) + \frac{1}{x} \partial_x \left( -\frac{k}{x} \partial_x u_k \right) = -\frac{k + 2}{x} \partial_x \left( \frac{1}{x} \partial_x u_k \right)$$

$$= -\frac{k + 2}{x} \partial_x v.$$

**LEMMA 3.2 (Weinstein [19]).** Let $\Psi_k$ stand for a general solution of $\mathcal{L}_k u = 0$ for any integer $k$. Then $\Psi_k = cx^{1-k} \Psi_{2-k}$, i.e., $\Psi_{2-k} = cx^{k-1} \Psi_k$ satisfies $\mathcal{L}_{2-k} u = 0$.

The proof of this lemma is simple (see [19]). Notice that we do not consider initial conditions here; we are just considering general solutions. This lemma helps us obtain the next lemma and Corollary 3.2, where we consider the equation along with the initial data.

---

The author referred to [14] to construct the substitution. Actually, A. Weinstein (see [20]) performed this transformation in 1962.
**Lemma 3.3.** The solution of the Cauchy problem (3.1) with $k = 2$ is holomorphic in $\mathbb{C}^2 \setminus (T \cup \Theta)$.

**Proof.** Notice that when $k = 2$, then $2 - k = 0$, and $\mathcal{L}_0 u = \Delta$. Suppose that $v(x, y)$ is the solution of $\mathcal{L}_2 u$ with entire data $f(x, y)$ and that $u(x, y) = x v(x, y)$. Then by Lemma 3.2, $u(x, y)$ satisfies the Laplace equation and has entire data $xf(x, y)$. By Theorem 3.1, $u(x, y)$ is holomorphic in $\mathbb{C}^2 \setminus \Theta$. Therefore, $v(x, y) = \frac{1}{2}u(x, y)$ is holomorphic in $\mathbb{C}^2 \setminus (T \cup \Theta)$.

**Theorem 3.2.** If $k$ is any even positive integer, then the solution of the Cauchy problem of (3.1) with any entire data is holomorphic in $\mathbb{C}^2 \setminus (T \cup \Theta)$.

**Proof.** Our proof is by induction with respect to the dimensional index $k$. Let $\Psi_k$ stand for the solution of Cauchy problem $\mathcal{L}_k u = 0$ with entire data on $\Gamma$. We know the result of the theorem is true for $\Psi_2$ by Lemma 3.3. Assume that the solution $\Psi_k$ is holomorphic in $\mathbb{C}^2 \setminus (T \cup \Theta)$ whenever its data are entire. We only need to show that $\Psi_{k+2}$ is also holomorphic in $\mathbb{C}^2 \setminus (T \cup \Theta)$ whenever its data are also entire.

By the assumption above, we know that the solution of $\mathcal{L}_k u_k = 0$ with data $\int_0^1 t f(t, y) dt$ is holomorphic in $\mathbb{C}^2 \setminus (T \cup \Theta)$ whenever $f(x, y)$ is entire. If $v = x^{-1} \partial_y u_k$, then obviously $v$ is also holomorphic in $\mathbb{C}^2 \setminus (T \cup \Theta)$. Following Lemma 3.1, it is easy to check that $v$ is the solution of Cauchy problem $\mathcal{L}_k u_k = 0$ with entire data $f(x, y)$ on $\Gamma$. Hence $\Psi_{k+2}$ is also holomorphic in $\mathbb{C}^2 \setminus (T \cup \Theta)$.

**Corollary 3.1.** The solution of the Cauchy problem in Theorem 3.2, for any positive odd integer $k$, has the same holomorphic domain as that of the case when $k = 1$.

This is the same as the proof of Theorem 3.2.

**Remark.** Note that we do not have Lemma 3.3 for the case $k = 1$; therefore, we cannot answer the same question as Theorem 3.2 for the case when $k$ is a positive odd integer. We are not able to find the Schwarz potential of torus in this case too. However, if we modify our Cauchy problem (3.1) a little bit, suppose that the data are given on $\Gamma := \{(x, y) \in \mathbb{C}^2 : x^2 + y^2 = 1\}$ instead, notice that now $\Gamma$ is symmetric with respect to the $y$-axis. Then the modified Cauchy problem is actually a special case considered in [17] by R. Millar. We can easily obtain the following solution for any $k$.

**Example 3.2.** For any positive integer $k$, $\frac{1}{2}(x^2 + y^2)^{-k/2}$ is the solution of the Equation $\mathcal{L}_k u(x, y) = 0$ with data $f(x, y) = \frac{1}{2}(x^2 + y^2)$ given on the unit sphere $x^2 + y^2 = 1$ (compare with Example 3.3).
Corollary 3.2. The Cauchy problem (3.1) with $k = -2m$, $m \in N$, and with data $g(x, y) = x^{2m-1}f(x, y)$ where $f$ is entire, has a holomorphic solution in $C^2 \setminus (T \cup \Theta)$.

Check the data and its gradients and use Lemma 3.2.

Now, let $L_k^*$ denote the adjoint operator of $L_k$, that is, $L_k^* u = \Delta u = \partial_x L_k u = \partial_x (u_k) = \Delta u - (k/x) \partial_x u + (k/x^2) u$.

Lemma 3.4. If $u_k$ is a general solution of $L_k u = 0$, then $v_{-k} = \partial_x u_k$ is a general solution of $L_{-k}^* v = 0$.

Proof. $\Delta v_{-k} = \partial_x \Delta u_k = \partial_x ((-k/x) u_k) = -(k/x) v_{-k} + (k/x^2) v_{-k}$.

Corollary 3.3. The solution of Cauchy problem,

\[
\begin{cases}
L_k u(x, y) = 0 & (x, y \in C^2 \setminus T) \quad k = 2m, m \in N \\
u|_\Gamma = x^k f(x, y), & \Gamma = \{(x, y) : (x - \rho)^2 + y^2 = 1\}
\end{cases}
\]  

(3.3)

is holomorphic in $C^2 \setminus (T \cup \Theta)$ for entire $f(x, y)$.

Proof. Notice that $\int_0^s x^{-k} \int_0^t x^{-k} f(s, y) ds \, dt$ has a removable singularity at 0, and this is true of all other data functions in this paragraph. Define the value at $x = 0$ of all these functions as their limit as 0. So if we expand $f(x, y)$ as a Taylor series, we can see that $\int_0^s x^{-k} \int_0^t x^{-k} f(s, y) ds \, dt$ is an entire function. Similarly, we can check all other data considered in this paragraph. Let $u_k$ be a solution of $L_k u = 0$ with data as above. Then by Theorem 3.2, $u_k$ is holomorphic in $C^2 \setminus (T \cup \Theta)$ and so is $v = \partial_x u_k$. By Lemma 3.4, we have

$$L_{-k}^* v = 0,$$

and

$$v|_\Gamma \equiv \left. x^{-k} \int_0^t x^{-k} f(t, y) dt \right|_\Gamma.$$

It is easy to see that the data of $v$ are also entire. Hence $v$ is holomorphic in $C^2 \setminus (T \cup \Theta)$. Let $w = x^k v$. Then $w$ is holomorphic in $C^2 \setminus (T \cup \Theta)$, and $w$ satisfies $L_{-k} w = 0$ with entire data $\int_0^s x^{-k} \int_0^t x^{-k} f(t, y) ds \, dt$. Let $u = \partial_x w$. Then $u$ is holomorphic in $C^2 \setminus (T \cup \Theta)$ and satisfies (3.3).

Example 3.3. The Schwarz potential for the 4-dimensional torus

$$\Gamma := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid (x_1^2 + x_2^2 + x_3^2 - \rho)^2 + x_4^2 = 1, \ \rho > 1\}$$

is

$$\frac{1}{2x} \left[ \rho \left( x - \rho \right)^2 - y^2 \right] + (2 + 3\rho^2) x + 2 \rho \log \left( (x - \rho)^2 + y^2 \right)
- \frac{x - \rho}{(x - \rho)^2 + y^2 - 2\rho^3},$$

where $x = \sqrt{x_1^2 + x_2^2 + x_3^2}, \ y = x_4$. 


EXAMPLE 3.4. The Schwarz potential for the 6-dimensional torus $\Gamma$

\[
\left\{(x_1, \ldots, x_6) \in \mathbb{R}^6 : \left(\sqrt{x_1^2 + \cdots + x_5^2} - \rho\right)^2 + x_6^2 = 1, \rho > 1 \right\}
\]

is $\frac{1}{x} \partial_x \left(1 - \Re F(z) \right)$,

where $z = x - \rho + iy$, $x = \sqrt{x_1^2 + \cdots + x_5^2}$, $y = x_6$, and $F(z) = \int_0^1 \frac{1}{172}(t + t^{-1} + 2\rho)^2(1 - 17t^2 - 12pt + 22 + 20\rho^2 + 42t^{-1} + 15t^{-2})dt$.

Remark. Theorem 3.2 shows that the solution of the Cauchy problem (3.1) is holomorphic elsewhere in $\mathbb{C}^2$ except on possibly $\mathbb{T} \cup \Theta$. Actually this depends on the data if such possible singularities exist. For instance, if the data are $cy + d$ for some constants $c$ and $d$, then we get a trivial solution $cy + d$, which is obviously holomorphic in the whole of $\mathbb{C}^2$. Notice that the Schwarz potential, as the unique solution of the Cauchy problem of a specific data, is a fixed function. It is easy to see that the two Schwarz potentials in Example 3.3 and Example 3.4 have singularities at precisely where we expected (see the Introduction). We obtain these two Schwarz potentials from Theorem 3.1, Lemma 3.1, and Lemma 3.2. More detail will be given in the proof of the next proposition.

**PROPOSITION 3.1.** The solution of (3.1) for $k$ even with data $f(x, y) = \frac{1}{2}(x^2 + y^2)$ on $\Gamma := ((x, y) : (x - \rho)^2 + y^2 = 1)$ must have singularities on $\mathbb{T} \cup \Theta$.

**Proof.** Let $u_{2m}$ stand for the solution of (3.1) with data $\frac{1}{2}(x^2 + y^2)$ for $k = 2m$. We need to show that $u_{2m}$ has singularities on $\mathbb{T} \cup \Theta$.

Examples 3.2 and 3.4 actually show that $u_2(x, y)$ and $u_4(x, y)$ have singularities at $\mathbb{T} \cup \Theta$. Notice that Theorem 3.1 leads us to the solution of Eq. (3.2), Lemma 3.2 gives us a road to solve Eq. (3.1) for $k = 2m$, if we know how to solve Eq. (3.1) for $k = 2m - 2$. Moreover, Lemma 3.2 can help us to reduce the problem of solving Eq. (3.1) for $k = 2$ into the problem of solving Eq. (3.2). Hence we can inductively solve Eq. (3.1) for any even $k$. Considering the data together, this procedure is like a round trip ($m$ stops each direction). In order to solve for $u_{2m}$, we start at $k = 2m$ with data $\frac{1}{2}(x^2 + y^2)$. Let $v_{2m-2} = x \cdot \int_0^1 u_{2m}(t, y)dt$ (i.e., $u_{2m} = x^{-1} \partial_x v_{2m-2}$); by Lemma 3.1, we only need to solve Eq. (3.1) for $k = 2m - 2$ with data $x^2/6 + x^2y^2/2$. Keep on reducing $k$ in this manner and calculating the corresponding data until $k = 2$. Then we change our direction a little bit and use Lemma 3.2 instead. Suppose now the data for $v_2$ is $f_2$; then we let $v = xv_2$ and the data of $v$ be $x \cdot f_2$. Next, we use Theorem 3.1 to solve for $v$, and so we have finished one way of our round trip. It is easy to see from the formula of $F(z)$ in Theorem 3.1 that the
solution $u$ must have singularities at $\Theta$ as long as the data $f(x, y)$ are not trivial; i.e., $\partial_x f$ and $\partial_y f$ are not at all 0, since the Schwarz function $S(z)$ of $\Gamma$ on the right hand of the formula of $F(z)$ is singular at $\Theta$. Hence, $v$ is singular at $\Theta$, so is $v_2$, so is $v_4$, etc., and consequently, so is $u_{2m}$. Notice that by now that $k = 0$. The other half of the round trip is to substitute back to get $u_{2m} \cdot v_2 = x^{-1} v$, $u_k = x^{-1} \partial_x v_2$, etc., and so finally we obtain $u_{2m} = x^{-1} \partial_x v_{2m-2}$. It is not hard to see that $v$ does not have a common factor $x^{2m}$ since it has a constant related to $\rho$. Thus, $u_{2m}$ is singular also at $\mathcal{T}$ (this is natural since the coefficient of (3.1) is singular). Therefore, our theorem follows from this algorithm. Indeed, this is the way we obtained the results in Example 3.3 and Example 3.4.

Remark. Following the argument in the Introduction, with Theorem 3.2 and Proposition 3.1, we conclude that the Schwarz Potential Conjecture is true for any Cauchy problem of Laplace’s equation with axially symmetric entire data on an even dimensional torus. This means that the solution of Laplace’s equation with any axially symmetric entire data does not introduce any additional singularities to those of its Schwarz potential.

4. FINAL REMARKS

Remark 1. Recall in this paper that we had two kinds of singular isotropic complex lines $\Pi$ and $\Theta$. $\Pi$ mainly relates to problem (2.1) and propagates from the “bad point” due to the singularity of the coefficients. We showed that $\Pi$ exists only for odd $k$. $\Theta$ mainly relates to problem (3.1); it propagates from the singular data at infinity and is tangent to the initial complex sphere at infinity. In [11, p. 29], G. Johnsson defined this type of singularities as $L_A$, asymptotic singularities. $\Theta$ is located at the foci of $\Gamma$ and it exists for any $k$ whenever $\Gamma$ is a sphere. Notice that neither the hyperplane of Eq. (2.1) nor the hypersphere of Eq. (3.1) has any Zerner characteristic points on it, and so that the solutions of these two Cauchy problems do not have any other type of singularities (i.e., $K$, $L_L$, or $L_I$ type of singularities defined in [11]). The singularities of these solutions of both equations at $\mathcal{T}$ directly result from the singular coefficients; they are different from any type of singularities defined in [11].

Remark 2. The “similarity” in the local behavior between problem (2.1) and problem (3.1) is due to the similar properties of the “bad points” of these two problems. Since the Cauchy problems (2.1) and (3.1) are defined on different initial surfaces $\Gamma$ (namely, one was on the plane and the other was on the sphere), we had to transform the sphere into the complex plane. As a result, we can compare the local properties of the two equations at the “bad points.” For convenience, we shift the $x$ coordinate
\( \rho \) unit to the left. Then Eq. (2.1) becomes

\[
\begin{cases}
\Delta u + \frac{k}{x + \rho} \partial_z u = 0, & 0 < k \in \mathbb{N}, \rho > 1 \\
u|_\Gamma = f(x, y), & \Gamma = \{(x, y) \in \mathbb{C}^2 : x^2 + y^2 = 1\}.
\end{cases}
\]

Suppose \( Z = x + iy, \ W = s + it, \) where \( x, y, s, t \in \mathbb{C} \). Take the Mobius transformation

\[
W = \frac{i(Z - 1)}{Z + 1}
\]

and suppose the data of \( u \) on the \( s - t \) plane are \( g(s, t) \). Separating the real and the imaginary parts of the Mobius transformation above and substituting \( s \) and \( t \) for \( x \) and \( y \), we obtain

\[
\begin{cases}
\Delta u + C\left(2(1-t)s\right) \partial_s u + \left((1-t)^2 - s^2\right) \partial_t u = 0 \\
u|_\Gamma = g(s, t), & \Gamma = \{(s, t) \in \mathbb{C}^2 : t = 0\},
\end{cases}
\]

where \( C = k/((\rho - 1)(s^2 + (t - \rho/\rho - 1))2 - (1/\rho - 1)^2)(1 - t^2 + s^2)^2\) and \( g(s, t) \) has only two singularities at \( s = \pm i \) on \( y = 0 \). Then we find that the coefficient has two sets of singular isotropic lines; one interprets the initial plane at points \( s = \pm i \), which correspond to infinity of the original space. The other intercepts the initial plane at two so-called “bad points,” \( s = \pm i \sqrt{\frac{\rho + 1}{\rho - 1}} \), which is an analog to \( x = 0 \) in (2.1). Now if we restrict our attention to a sufficiently small neighborhood, say \( \Xi \), of one of these two bad points, and after a translation to move the singular point into origin, we only need to discuss the local solution of the Cauchy problem

\[
\begin{cases}
\Delta u + \frac{k}{s} (p \partial_s u + r \partial_t u) = 0 \\
u|_{s=0} = h(s, t), & (h(s, t) \text{ is holomorphic in } \Xi),
\end{cases}
\]

where \( p \) and \( r \) can be regarded as nonzero constants in \( \Xi \).

Now the data of (4.1) are also given on a plane \( t = 0 \), and although (4.1) is slightly more complicated due to the existence of the additional term \( \partial_s u \), we can see that the singular terms of (2.1) and (4.1) are both a first order pole at their corresponding bad points and both intercept the initial surface at the perpendicular direction.

Theorem 2.2 and the similarity of local behavior between (2.1) and (3.1) suggest that we may find more singularities for the solution of (3.1) when \( k \)
is odd than we would when \( k \) is even, especially when the data of (3.1) are an odd function. Corollary 2.9 in [10] shows that if the local solution of a holomorphic PDE is singular at a proper subset of an irreducible algebraic variety, then the solution must be singular at the whole algebraic variety. Hence in our case, the solution of (3.1) for \( k \) odd is either singular on the whole isotropic lines originating at the two bad points in addition to \( T \cup \Theta \), or only singular at \( T \cup \Theta \). Besides this dimensional index \( k \), the data of course also make a difference to the solution. For instance, the solutions of Eq. (2.1) between even data and odd data also differ dramatically. Also remember that the data of the Schwarz potential are an even function. Therefore, for the odd-dimensional torus, if one is able to show that for \( k = 1 \), the local solution of (3.1) at the two bad points has extra singularities other than \( T \cup \Theta \) when the data are odd, and that the local solution only has singularities at \( T \cup \Theta \) when the data are even, then it follows that the Schwarz Potential Conjecture would not hold in this case.

**Remark 3.** In order to investigate the Schwarz Potential Conjecture for the torus, we restrict our discussion to the operator \( \mathcal{L}_k = \Delta + \frac{k}{x} \partial_x \) with only two variables so far. We may easily extend our results to more general axially symmetric equations and provide a partial answer to G. Johnsson’s first question [11, p. 46], where he asked if his results about the elliptic Cauchy problem on quadratic hypersurfaces can be extended into higher order initial hypersurfaces. Let \( \Delta_n u = \sum_{i=1}^m \partial_i^2 u \), \( \mathcal{X}_n u = \Delta_n u + \sum_{i=1}^m a_i(x) \partial_i u + c(x) u \), and \( \Psi_k u = \mathcal{X}_{n-1} u + \partial_x^2 u + \frac{k}{x} \partial_x u \), where \( 0 < k, m, n \in \mathbb{N} \), \( a_i(x), c(x) \) are entire. Observe that all terms in \( \mathcal{X}_{n-1} u \) except the entire coefficients are independent to the axially symmetric variable \( x_n \), so Lemmas 3.1, 3.2, and 3.3 all hold if we replace the operator \( \mathcal{L}_k \) by \( \Psi_k \). Notice that in [11], G. Johnsson has fully described all singularities of the solutions of the Cauchy problem of \( \mathcal{X}_{n-1} u + \partial_x^2 u = 0 \) on quadratic hypersurfaces. Using [11, Theorem 4.8] to replace our Theorem 3.1, we can locate all the singularities of the Cauchy problem of the more general equation \( \Psi_k u = 0 \) on quadratic hypersurfaces as we did in Theorem 3.2 above. To be more specific, we obtain the following: If \( k \) is even and the entire data are set on any quadratic hypersurface \( \Gamma (T \cap \Gamma = \emptyset \text{ in } \mathbb{R}^n) \), then the singularities of the Cauchy problem of the more general equation \( \Psi_k u = 0 \) are the union of the singularities of the solutions of the Cauchy problem of \( \mathcal{X}_{n-1} u + \partial_x^2 u = 0 \) and the singularities of the coefficients \( T \).

For example, we can easily see that Theorem 3.2 is still true if we replace the Laplace equation by the Helmholtz equation. We can also obtain the explicit solutions as our Examples 3.3 and 3.4 to the Helmholtz equations except that they are in integral form with the integrands involving some Bessel functions. Moreover, in the same way as we did for the Laplace operator above, the extended result about operator \( \Psi_k \) can be
transformed to the corresponding result to locate the singularities of the Cauchy problem of axially symmetric elliptic equations on more general tori. Recall that the torus we considered in (1.1) has only one non-axially symmetric variable $x_n$. Noticed that, instead of two variables in $\mathbb{R}_k$, $\mathbb{R}^k$ has any number of variables. Therefore, our results extend to the more general tori that may have any number of non-axially symmetric variables $x_n, x_{n+1}, \ldots, x_m$, provided that the number of axially symmetric variable $k = n - 2$ is even.

**Remark 4.** In Corollary 3.3, we briefly considered the adjoint equation of (3.1). It is possible to extend this result further. An open question is the following: Given that the data are entire on the same regular analytic hypersurface $\Gamma$, do the solutions of the two adjoint equations have the same analytic domain?

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