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Generalized-Function Solutions of Linear Systems

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1. INTRODUCTION

It is well known [1] that normal linear homogeneous systems of differential equations with infinitely smooth coefficients have no solutions in the space of generalized functions other than the classical solutions. In contrast to this case, for equations with singularities in the coefficients there may appear new solutions in generalized functions and in addition some classical solutions may disappear. Research in this direction, still developed insufficiently, discovers new aspects and properties in the theory of differential and functional differential equations. We note, in particular, papers [2–5], which contain references to previous work. Thus, in [3] the author proved that the system

$$x'(t) = Ax(t) + tBx(\lambda t), \qquad -1 < \lambda < 1, \tag{1.1}$$

has a solution in the class of distributions—an impossible phenomenon for ordinary differential equations without singularities. A more general result was obtained in [4], where it was shown that, under certain conditions, the system

$$x'(t) = \sum_{j=0}^{\infty} A_j(t)\, x(\lambda_j t)$$

has a solution

$$x(t) = \sum_{n=0}^{\infty} c_n \delta^{(n)}(t) \tag{1.2}$$

in the generalized function space $(S_0^\beta)'$ with arbitrary $\beta > 1$.

In this paper we study solutions to functional differential systems in the form of finite linear combinations of the delta distribution and its derivatives. Two new theorems are established also for solutions in the space $(S_0^\beta)'$. The results are applied to some important second order ordinary differential

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equations of mathematical physics. The basic ideas in the method of proof are employed to investigate a special class of integral equations.

2. Functional Differential Equations

The number m is called the order of the distribution

$$x = \sum_{k=0}^{m} c_k \delta^{(k)}(t), \qquad c_m \neq 0. \tag{2.1}$$

In the sequel C^m denotes the space of m times continuously differentiable functions of the real variable t and $\langle f, \phi \rangle$ is the value of the functional f on the test function $\phi(t)$. The norm of a matrix is defined to be

$$\|A\| = \max_i \sum_j |a_{ij}|$$

and E is the identity matrix.

THEOREM 2.1. *The criterion for the existence of solutions* (2.1) *to the system*

$$tx'(t) = \sum_{j=0}^{N} A_j(t)\, x(\lambda_j t) \tag{2.2}$$

with matrices $A_j(t) \in C^m$ in a neighborhood of $t = 0$ and constants $\lambda_j \neq 0$ is that some roots μ of the equation

$$\det\left(\sum_{j=0}^{N} |\lambda_j|^{-1} \lambda_j^{-\mu} A_j(0) + (\mu + 1)E \right) = 0 \tag{2.3}$$

be nonpositive integers. If m is the smallest of their absolute values there exists a solution of order m.

Proof. In the Taylor expansions

$$A_j(t) = \sum_{k=0}^{m} A_{jk} t^k + R_{jm}(t)$$

the remainders and all their derivatives up to the order m vanish at $t = 0$. Therefore

$$R_{jm}(t)\, x(t) = 0,$$

for any distribution of the form (2.1), and the sets of solutions of order not exceeding m to (2.2) and to the system

$$tx'(t) = \sum_{j=0}^{N} \sum_{k=0}^{m} A_{jk} t^k x(\lambda_j t) \tag{2.4}$$

are coincident. The Fourier transformation of Eq. (2.4) yields

$$-(sF(s))' = \sum_{j=0}^{N} \sum_{k=0}^{m} |\lambda_j|^{-1} (i\lambda_j)^{-k} A_{jk} F^{(k)}(s/\lambda_j). \tag{2.5}$$

The necessary and sufficient condition for the distribution $x(t)$ to have the order m is that its transform $F(s)$ be a polynomial of degree m. Differentiating relation (2.5) n times and putting $s = 0$, we obtain

$$\left(\sum_{j=0}^{N} |\lambda_j|^{-1} \lambda_j^{-n} A_{j0} + (n+1)E \right) F_n + \sum_{j=0}^{N} \sum_{k=1}^{m} (-i)^k |\lambda_j|^{-1} \lambda_j^{-n-k} A_{jk} F_{n+k} = 0 \qquad (n = 0, 1, \ldots), \tag{2.6}$$

where $F_n = F^{(n)}(0)$. The requirement $F_n = 0$, $n > m$, reduces (2.6) to a finite system of matrix equations, the last of which,

$$\left(\sum_{j=0}^{N} |\lambda_j|^{-1} \lambda_j^{-m} A_{j_0} + (m+1)E \right) F_m = 0,$$

possesses a nontrivial solution F_m. Its substitution into the foregoing equations allows one to find F_n $(n < m)$ successively since the matrices

$$\sum_{j=0}^{N} |\lambda_j|^{-1} \lambda_j^{-n} A_{j0} + (n+1)E, \qquad n < m,$$

are nondegenerate.

COROLLARY 2.1. *The system*

$$tx'(t) = A(t)\, x(t) + \sum_{j=1}^{N} A_j(t)\, x(\lambda_j t) \tag{2.7}$$

has a solution of order m with support $t = 0$, if $A_j(0) = 0$ $(j \geqslant 1)$ and $m+1$ is the smallest modulus of the negative integer eigenvalues of the matrix $A(0)$.

THEOREM 2.2. *The system*

$$tx'(t) = \sum_{j=0}^{N} A_j(t)\, x(\phi_j(t)), \tag{2.8}$$

in which $A_j(t) \in C^m$, $\phi_j(t) \in C^1$, has a solution (2.1) of order m, if the following hypotheses are satisfied: (1) the real zeroes t_{jn} of the functions $\phi_j(t)$ are simple and form a finite or countable set; (2) $A^{(k)}(t_{jn}) = 0$ $(k = 0, \ldots, m)$, for $t_{jn} \neq 0$; (3) m is the smallest modulus of the nonpositive integer roots of equation (2.3) with $\lambda_j = \phi_j'(0)$.

Proof. From the representation $\phi_j(t) = (t - t_{jn})\,\lambda_j(t)$ it follows that $\lambda_j(t_{jn}) = \phi_j'(t_{jn}) \neq 0$. In some neighborhood D_{jn} of the point t_{jn} that does not contain other points of $\{t_{jn}\}$,

$$\delta^{(k)}(\lambda_j(t)(t - t_{jn})) = \delta^{(k)}(\lambda_j(t_{jn})(t - t_{jn})) = 0, \qquad t \neq t_{jn}.$$

But, for $t = t_{jn}$, $\lambda_j(t)(t - t_{jn}) = \lambda_j(t_{jn})(t - t_{jn})$. Therefore

$$\delta^{(k)}(\phi_j(t)) = \delta^{(k)}(\phi_j'(t_{jn})(t - t_{jn})), \qquad t \in D_{jn}. \tag{2.9}$$

Since for the distribution (2.1) $\operatorname{supp} x(\phi_j(t)) = \{t_{jn}\}$ there will be, by virtue of (2.9),

$$x(\phi_j(t) = \sum_n x(\phi_j'(t_{jn})(t - t_{jn})). \tag{2.10}$$

It is easy to show that

$$\delta^{(k)}(\phi_j'(t_{jn})(t - t_{jn})) = \delta^{(k)}(t - t_{jn})/(\phi_j'(t_{jn}))^{k+1} \operatorname{sgn} \phi_j'(t_{jn}).$$

Hence

$$x(\phi_j(t)) = \sum_n \sum_{k=0}^{m} c_k \delta^{(k)}(t - t_{jn})/(\phi_j'(t_{jn}))^{k+1} \operatorname{sgn} \phi_j'(t_{jn}).$$

Owing to

$$A_j(t)\, \delta^{(k)}(t - t_{jn}) = 0, \qquad t_{jn} \neq 0, \quad k = 0, \ldots, m,$$

we make the conclusion that

$$A_j(t)\, x(\phi_j(t)) = 0, \qquad 0 \bar{\in} \{t_{jn}\}$$

and the problem of existence of a solution (2.1) to system (2.8) is brought to the same question for (2.2), where $\lambda_j = \phi_j'(0)$.

Now we shall study solutions of the form (1.2) of some functional differential equations in the space $(S_0^\beta)'$ with arbitrary $\beta > 1$. To ensure the convergence of the series in (1.2) it is sufficient to require that for $n \to \infty$ the vectors c_n satisfy the inequalities

$$\|c_n\| \leqslant ac^n n^{-n\rho}, \qquad \rho > 1. \tag{2.11}$$

In fact, since the test functions $\phi(t) \in S_0^\beta$ are subject to restrictions [6, p. 175]

$$|\phi^{(n)}(t)| \leqslant bd^n n^{n\beta},$$

then

$$\left\| \sum_{n=0}^{\infty} \langle c_n \delta^{(n)}(t), \phi(t) \rangle \right\|$$
$$= \left\| \sum_{n=0}^{\infty} (-1)^n \phi^{(n)}(0)\, c_n \right\|$$
$$\leqslant \sum_{n=0}^{\infty} |\phi^{(n)}(0)|\, \|c_n\| \leqslant ab \sum_{n=0}^{\infty} (c\, dn^{\beta-\rho})^n < \infty,$$

for $\beta < \rho$. If series (1.2) converges, its sum is the general form of a linear functional in the space $(S_0^\beta)'$ concentrated on the point $t = 0$ [7].

THEOREM 1.4. *The system*

$$tx'(t) = Ax(t) + tBx(\lambda t) \qquad (\lambda = \text{const}) \tag{2.12}$$

with constant matrices A and B has a solution in the space $(S_0^\beta)'$ *concentrated on* $t = 0$, *if B is nonsingular and* $-1 < \lambda < 1$, $\lambda \neq 0$.

Proof. The Fourier transformation changes (2.12) to the equation

$$-(sF(s))' = AF(s) - \frac{i}{\lambda\,|\lambda|} BF'(s/\lambda),$$

from which there follow relations

$$F_{n+1} = -i\lambda^{n+1}\, |\lambda|\, B^{-1}(A + (n+1)E)\, F_n,$$
$$F_n = F^{(n)}(0), \qquad n = 0, 1,...$$

and inequalities

$$\|F_{n+1}\| \leqslant |\lambda|^{n+2}\, (n + q + 1)\, \|B^{-1}\| \cdot \|F_n\|$$

that imply estimates

$$\frac{1}{n!}\|F_n\| \leqslant |\lambda|^{n(n+3)/2}\|B^{-1}\|^n \binom{n+q}{n} \|F_0\|, \tag{2.13}$$

where q is some fixed natural number such that $\|A\| \leqslant q$. Since for the coefficients c_n of series (1.2),

$$\|c_n\| = \frac{1}{n!}\|F_n\|$$

the bounds (2.13) prove the theorem, as the condition $|\lambda| < 1$ makes them more restrictive than (2.11).

Relation (2.12) provides an interesting example of a system that may have two essentially different solutions in the space $(S_0^\beta)'$ with support $t = 0$. According to the previous results (2.12) has a solution of finite order if the matrix A assumes negative integer eigenvalues. At the same time there exists an infinite order solution (1.2) if $A \neq -nE$, for all $n \geqslant 1$.

The particular importance of the system

$$\sum_{j=0}^{\infty} \sum_{k=0}^{m} (A_{jk} + tB_{jk})\, x^{(k)}(\lambda_j t) = tx(\lambda t) \tag{2.14}$$

is that depending on the coefficients it combines either equation with a singular or regular point at $t = 0$ and in both cases there exists, under certain conditions, a solution of the form (1.2).

THEOREM 1.5. *Let system* (2.14) *in which $x(t)$ is an r-dimensional vector and A_{jk}, B_{jk} are constant matrices of order $(r \times r)$, satisfy the following hypotheses*:

(i) *λ_j and λ are real numbers such that*

$$0 < |\lambda| < 1, \qquad |\lambda_j| \geqslant 1, \quad j \geqslant 0;$$

(ii) *the series $\sum_{j=0}^{\infty} |\lambda_j|^{-1} A^{(j)}$ and $\sum_{j=0}^{\infty} |\lambda_j|^{-2} B^{(j)}$ where*

$$A^{(j)} = \max_{0 \leqslant k \leqslant m} \|A_{jk}\|, \qquad B^{(j)} = \max_{0 \leqslant k \leqslant m} \|B_{jk}\|$$

are convergent.

(iii) $\displaystyle\sum_{j=0}^{\infty} \lambda_j^{-2} \|B_{j0}\| < \lambda^{-2}.$

Then in the space of generalized functions $(S_0^\beta)'$ with arbitrary $\beta > 1$ there exists a solution $x(t)$, concentrated on $t = 0$.

Proof. Substituting the unknown solution (1.2) into (2.14) we obtain, by virtue of

$$\delta^{(n)}(\lambda_j t) = |\lambda_j|^{-1} \lambda_j^{-n} \delta^{(n)}(t),$$

the equality

$$\sum_{j=0}^{\infty} \sum_{k=0}^{m} \sum_{n=0}^{\infty} |\lambda_j|^{-1} \lambda_j^{-n-k}(A_{jk} c_n \delta^{(n+k)}(t) - (n+k) B_{jk} c_n \delta^{(n+k-1)}(t))$$
$$= -\sum_{n=1}^{\infty} n |\lambda|^{-1} \lambda^{-n} c_n \delta^{(n-1)}(t),$$

from which it follows that

$$(n+1)|\lambda|^{-1} \lambda^{-n-1} c_{n+1} = \sum_{j=0}^{\infty} \sum_{k=0}^{m} (n+1) |\lambda_j|^{-1} \lambda_j^{-n-1} B_{jk} c_{n+1-k}$$
$$- \sum_{j=0}^{\infty} \sum_{k=0}^{m} |\lambda_j|^{-1} \lambda_j^{-n} A_{jk} c_{n-k}, \qquad n \geqslant 0, \qquad (2.15)$$

and, hence,

$$\left(E - |\lambda| \lambda^{n+1} \sum_{j=0}^{\infty} |\lambda_j|^{-1} \lambda_j^{-n-1} B_{j0}\right) c_{n+1}$$
$$= |\lambda| \lambda^{n+1} \left(\sum_{j=0}^{\infty} \sum_{k=1}^{m} |\lambda_j|^{-1} \lambda_j^{-n-1} B_{jk} c_{n+1-k} - \frac{1}{n+1} \sum_{j=0}^{\infty} \sum_{k=0}^{m} |\lambda_j|^{-1} \lambda_j^{-n} A_{jk} c_{n-k} \right).$$

Inequalities (i) and (iii) ensure the existence of inverse matrices to the coefficients of c_{n+1} for all n:

$$\left(E - |\lambda| \lambda^{n+1} \sum_{j=0}^{\infty} |\lambda_j|^{-1} \lambda_j^{-n-1} B_{j0}\right)^{-1}$$
$$= \sum_{i=0}^{\infty} |\lambda|^i \lambda^{(n+1)i} \left(\sum_{j=0}^{\infty} |\lambda_j|^{-1} \lambda_j^{-n-1} B_{j0}\right)^i,$$
$$\left\| \left(E - |\lambda| \lambda^{n+1} \sum_{j=0}^{\infty} |\lambda_j|^{-1} \lambda_j^{-n-1} B_{j0}\right)^{-1} \right\|$$
$$\leqslant \left(1 - \lambda^2 \sum_{j=0}^{\infty} \lambda_j^{-2} \|B_{j0}\|\right)^{-1}.$$

Therefore formulas (2.15) determine the values c_n uniquely with the exactness to an arbitrary factor c_0 and because of conditions (i) and the convergence of series (ii) provide the bounds

$$\|c_{n+1}\| \leqslant \mu q^n \sum_{k=0}^{m} \|c_{n-k}\|, \qquad 0 < q < 1, \tag{2.16}$$

μ is some constant. We set

$$M_n = \max_{0 \leqslant i \leqslant n} \|c_i\|. \tag{2.17}$$

Then from (2.16),

$$\|c_{n+1}\| \leqslant \mu(m+1)\, q^n M_n.$$

For large values of n,

$$\mu(m+1)\, q^n < 1.$$

Hence $\|c_{n+1}\| \leqslant M_n$ and $M_{n+1} = M_n$. Thus, we arrive at the conclusion that, starting with some N,

$$M_n = M_N, \qquad n \geqslant N. \tag{2.18}$$

The application of (2.18) to (2.16) successively yields

$$\|c_{N+1}\| \leqslant \mu(m+1)\, q^N M_N,$$
$$\|c_{N+2}\| \leqslant \mu(m+1)\, q^{N+1} M_N,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\|c_{N+m+1}\| \leqslant \mu(m+1)\, q^{N+m} M_N.$$

Therefore

$$\|c_{N+k+1}\| \leqslant \mu(m+1)\, q^N M_N, \qquad 0 \leqslant k \leqslant m. \tag{2.19}$$

Putting $n = N+m+1, \ldots, N+2m+1$ in (2.16) and using (2.19), we obtain

$$\|c_{N+1+(m+1)+k}\| \leqslant \mu^2(m+1)^2\, q^N q^{N+m} M_N, \qquad 0 \leqslant k \leqslant m. \tag{2.20}$$

By employing (2.20) we can establish that

$$\|c_{N+1+2(m+1)+k}\| \leqslant \mu^3(m+1)^3\, q^N q^{N+m} q^{N+2m} M_N, \qquad 0 \leqslant k \leqslant m,$$

and continuation of the iteration process allows to assume that, for all n and $0 \leqslant k \leqslant m$,

$$\|c_{N+1+n(m+1)+k}\| \leqslant \mu^{n+1}(m+1)^{n+1}\, q^{n(n+1)m/2}. \tag{2.21}$$

Replacing n in (2.16) by $N+1+n(m+1)+m, \ldots, N+1+n(m+1)+2m$ we find that (2.21) holds for $\|c_{N+1+(n+1)(m+1)+k}\|$; i.e., these inequalities have been established by induction. Inequalities (2.21) prove the theorem since the condition $0<q<1$ makes them more stringent than (2.11). It remains to observe that equations (1.1) and (2.12) are special cases of (2.14).

3. Ordinary Differential Equations

The results of the previous section are applied to investigate finite order solutions of some important equations of mathematical physics.

THEOREM 3.1. *The equation*

$$\sum_{j=0}^{n} t^j a_j(t)\, y^{(j)}(t)=0$$

has a solution of order m concentrated on $t=0$ if $a_j(t) \in C^m$, $a_n(0) \neq 0$ and m is the smallest nonnegative integer root of the relation

$$\sum_{j=0}^{n} (-1)^j\, a_j(0)(m+j)!=0.$$

Proof. We define the vector x with components

$$x_j=t^{j-1}y^{(j)}, \qquad j=1,\ldots, n$$

and use Corollary 2.1.

COROLLARY 3.1. *The Bessel equation*

$$t^2y''+ty'+(t^2-p^2)y=0$$

has a solution of order m with the support $t=0$ iff

$$p^2=(m+1)^2$$

and it is given by the formula

$$y=C\sum_{k=0}^{[m/2]} \frac{(m-k)!}{4^k k!(m-2k)!}\,\delta^{(m-2k)}(t), \qquad C=\text{const.}$$

THEOREM 3.2. *The criterion for the existence of an m order solution* (2.1) *of the equation*

$$tx''+p(t)\,x'+q(t)x=0$$

with $p(t) \in C^{m+1}$ *and* $q(t) \in C^m$ *in some neighborhood of* $t=0$ *is the implementation of the following conditions*:

(i) $p(0) = m+2$,

(ii) *the system of equations*

$$n(p_0 - n - 1)\, x_{n-1} + \sum_{k=1}^{m+1} (-1)^k \left(p_k(n+k) - q_{k-1}\right) x_{n+k-1} = 0$$

$$\left(n = 0,\dots, m+1;\, p_k = \frac{p^{(k)}(0)}{k!},\, q_k = \frac{q^{(k)}(0)}{k!}\right)$$

has a nonzero solution $(x_0,\dots, x_m)$.

COROLLARY 3.2. *The degenerate hypergeometric equation*

$$tx'' + (b-t)\, x' - ax = 0 \tag{3.1}$$

has a finite order solution iff a and b are positive integers and $b \geqslant a+1$. *This solution is given by the formula*

$$x = C d^{a-1}/dt^{a-1} (d/dt - 1)^{b-a-1}\, \delta(t)$$

and its order is $m = b-2$.

Proof. Relations (i), (ii) of the previous theorem will be as follows for Eq. (3.1):

$$b = m+2,$$

$$n(b-n-1)\, x_{n-1} + (n+1-a)\, x_n = 0, \qquad n = 0,\dots, m+1,$$

whence $b \geqslant 2$ is an integer and if a is not a positive integer $\leqslant b-1$, all $x_n = 0$. On the contrary, when the hypotheses are observed then $x_k = 0$, $k \leqslant a-2$, and taking any $x_m \neq 0$ we find

$$x_{m-k} = (-1)^k \binom{b-a-1}{k} (m-k)!C,\, C \neq 0, \qquad 0 \leqslant k \leqslant b-a-1.$$

COROLLARY 3.3. *The equation* [8]

$$tx'' + ax' + btx = 0, \qquad b \neq 0, \tag{3.2}$$

has a finite order solution iff the coefficient a is a positive even integer. This solution is given by the formula

$$x = C(d^2/dt^2 + b)^{(a-2)/2}\, \delta(t)$$

and its order is $m = a-2$.

Proof. For Eq. (3.2) conditions (i), (ii) of Theorem 3.2 become as

$$a = m + 2,$$

$$n(a-n-1)\, x_{n-1} = bx_n, \qquad n = 0,..., m+1,$$

from which it follows that $x_{2k-1} = 0$, $k \geqslant 1$, and the order m is even. Choosing any $x_m \neq 0$ we determine

$$x_{m-2k} = b^k \binom{m/2}{k} (m-2k)!\, x_m, \qquad 0 \leqslant k \leqslant m/2.$$

COROLLARY 3.4. *The equation* [8]

$$tx'' + (t+a+b)\, x' + ax = 0$$

has a finite order solution iff a and b are positive integers. This solution is given by the formula

$$x = Cd^{a-1}/dt^{a-1}(d/dt+1)^{b-1}\, \delta(t)$$

and its order is $m = a + b - 2$.

4. AN INTEGRAL EQUATION IN THE SPACE OF TEMPERED DISTRIBUTIONS

Recently Kreinovič suggested the following query [9]. Is black-body 3°K radiation really of cosmological origin or is it a mixture of radiation of many bodies as some physicists suggest? Of course, since the particular law is currently known only approximately, we cannot answer for sure. But in case we know precisely that the spectrum is subject to Planck's law, will it mean that the second case is disproved? In mathematical terms:

(1) If $\forall \omega > 0$ $(\int_0^\infty A(\beta)(e^{\beta\omega}-1)^{-1}\, d\beta = (e^{\beta_0\omega}-1)^{-1})$ and $A(\beta) \geqslant 0$, is $A(\beta)$ equal to $\delta(\beta-\beta_0)$? A positive answer to this question will follow, if one can prove that

(2) If $\forall \omega > 0$ $(\int_0^\infty A(\beta)(e^{\beta\omega}-1)^{-1}\, d\beta = 0)$ then $A(\beta) \equiv 0$. Denoting $B = \ln \beta$, $W = \ln \omega$ and turning to Fourier transforms, this can be reduced to the question

(3) Is the Fourier transform of $(\exp(\exp z)-1)^{-1}$ everywhere different from 0?

As it appeared from the replies presented to [10] the conjecture in statements (2) and (3) had been discussed and proved in [11, p. 41] under certain assumptions on A, e.g., if $e^{-c\beta}A(\beta)$ is integrable on $(0, \infty)$ for some

$c > 0$. However, since the solution of (1) is not an ordinary function, it seems appropriate to consider the problem from the standpoint of the distribution theory. Our purpose is to establish a general theorem which includes the foregoing results as a particular case. Throughout the exposition we employ distributions of slow growth arising naturally in the generalized Fourier and Laplace transformations. Let $t \in E^1$ be one-dimensional real variable; S is the linear space of all functions $\phi(t)$ that are infinitely smooth and are such that, for any $m \geqslant 0$, $k \geqslant 0$,

$$\lim_{t \to \pm\infty} t^m \phi^{(k)}(t) = 0. \tag{4.1.}$$

The elements of S are called testing functions of rapid descent. If ϕ is in S, every one of its derivatives is again in S. A sequence of functions $\phi_n \in S$ is said to converge in S, if for each set of nonnegative integers m and k the sequence $\{|t|^m \phi_n^{(k)}(t)\}$ converges uniformly over all of E_1. A distribution f is said to be of slow growth if it is a linear functional on the space S. Such generalized functions are also called tempered distributions. The space of all distributions of slow growth is denoted by S' and $\langle f, \phi \rangle$ is the value of the functional f applied to $\phi \in S$. The support of a testing function $\phi(t)$ is the closure of all points where $\phi(t)$ is different from zero. Two distributions f and g are said to be equal over an open set G if

$$\langle f, \phi \rangle = \langle g, \phi \rangle,$$

for every testing function $\phi(t)$ whose support is contained in G. The support of a distribution f is the smallest set outside of which f equals zero. If a set X contains the support of a distribution f, it is said that f is concentrated on X. We denote by S_+ the space of all functions defined on $t \in (0, \infty)$ which are infinitely differentiable and satisfy (4.1), for $t \to +\infty$, and S'_+ is the space of all tempered distributions concentrated on $(0, \infty)$. Now we introduce the

DEFINITION. Let $\phi \in S_+$, $f \in S'_+$. Then

$$\langle f, \phi \rangle = \lim_{\Delta \to 0^-} \langle f_\Delta(t), \phi(t) \rangle, \tag{4.2}$$

for

$$f\Delta = f, \qquad \Delta \leqslant t < \infty,$$
$$f_\Delta = 0, \qquad t < \Delta.$$

THEOREM 4.1. *The equation*

$$\langle f(t), \phi(t, \omega) = \phi(t_0, \omega) \qquad (0 < \omega < \infty, t_0 > 0) \tag{4.3}$$

has a unique solution

$$f(t) = \delta(t - t_0)$$

if the following conditions are satisfied:

(i)
$$\phi(t, \omega) = \sum_{n=1}^{\infty} a_n e^{-\omega\lambda_n t}, \qquad t > 0, \tag{4.4}$$

with positive parameters a_n, λ_n *such that*

$$a_{n+1} \leqslant a_n, \qquad \lambda_n < \lambda_{n+1}, \qquad \lim_{n\to\infty} n/\lambda_n < \infty;$$

(ii) $e^{-ct}f(t) \in S'_+$, *for some* $c > 0$;

(iii) *the Laplace transform* $F(p) = \langle f(t), e^{-pt}\rangle$ *is nonnegative on the real half-axis* $c < p < \infty$.

Proof. Equation (4.3) can be written as

$$\lim_{\Delta\to 0^+} \langle f_\Delta, \phi\rangle = \phi(t_0, \omega)$$

and, by virtue of hypothesis (i), series (4.4) converges on any interval $0 < \Delta \leqslant t < \infty$ in the sense of S. Hence

$$\lim_{\Delta\to 0^+} \langle f_\Delta, \phi\rangle = \lim_{\Delta\to 0^+} \sum_{n=1}^{\infty} a_n \langle f_\Delta(t), e^{-\omega\lambda_n t}\rangle = \phi(t_0, \omega). \tag{4.5}$$

Considering the series in (4.5) and taking into account (iii), one concludes that, for Δ sufficiently small and $\omega > c/\lambda_1$, all its terms are nonnegative and continuous with respect to ω. Since the sum $\langle f_\Delta, \phi\rangle$ is also continuous this series converges uniformly and

$$\sum_{n=1}^{\infty} a_n \lim_{\Delta\to 0^+} \langle f_\Delta, e^{-\omega\lambda_n t}\rangle = \sum_{n=1}^{\infty} a_n \langle f, e^{-\omega\lambda_n t}\rangle = \phi(t_0, \omega).$$

Thus,

$$\sum_{n=1}^{\infty} a_n F(\omega\lambda_n) = \phi(t_0, \omega). \tag{4.6}$$

The function $F(p)$ of the complex variable p is analytic in the half-plane $\operatorname{Re} p > c$ as the Laplace transform of the distribution $f(t)$ and, in view of uniform convergence, the left side of (4.6) is analytic for $\omega > c/\lambda_1$. Since

$\phi(t_0, \omega)$ is analytic for all $\omega > 0$ Eq. (4.6) is continued analytically onto $(0, \infty)$. Moreover, inasmuch as the right side of

$$\sum_{n=1}^{\infty} a_n(F(\omega\lambda_n) - e^{-\omega\lambda_n t_0}) = 0$$

exists also for $\omega = 0$ and the convergence in this relation is uniform, one may differentiate it and approach ω to zero:

$$\sum_{n=1}^{\infty} a_n \lambda_n^k (F^{(k)}(0+) - (-t_0)^k) = 0, \qquad k = 0, 1, \ldots .$$

Therefore

$$F^{(k)}(0+) = (-t_0)^k, \qquad F(p) = e^{-pt_0}, \operatorname{Re} p \geqslant 0,$$

and

$$\langle f(t), e^{-pt} \rangle = \langle \delta(t - t_0), e^{-pt} \rangle,$$

where $\delta(t - t_0)$ is the delta functional. Finally,

$$f(t) = \delta(t - t_0)$$

and the uniqueness of this solution follows from the fact that if the Laplace transforms of two distributions f, $g \in S'_+$ coincide on some vertical line in their regions of convergence, then $f = g$ [12, p. 225]. This concludes the proof.

For the homogeneous equation $\langle f(t), \phi(t, \omega) \rangle = 0$, we obtain, under the previous assumptions,

$$\sum_{n=1}^{\infty} a_n F(\omega\lambda_n) = 0$$

and hence $F^{(k)}(0+) = 0$, $k \geqslant 0$. The result $F(p) = 0$ implies $f(t) = 0$. Obviously, series (4.4) transforms into $(e^{\omega t} - 1)^{-1}$ when $a_n = 1$, $\lambda_n = n$ and $t, \omega > 0$.

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