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Gossiping with multiple sends and receives[☆]

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Abstract

We consider the problem of gossiping in several important networks in as few rounds as possible. During a single round, each processor may send an unlimited size message to k neighbors, or receive messages from k neighbors, but a processor cannot both send and receive during the same round. The network architectures we consider are trees, cycles, grids, hypercubes, and toroidal (or "wrap-around") grids. As an interesting corollary of several of our main results, we obtain an optimal $(d + 1)$-round gossiping algorithm for the d-dimensional hypercube when $k = 2$ and show that gossiping in d rounds is impossible regardless of the size of k.

1. Introduction

Consider a distributed system in which each processor possesses a unique piece of information which must be disseminated to each of the other processors in the system. This fundamental communication problem has generally been referred to in the literature as "gossiping". For an excellent survey of previous work on gossiping and related information dissemination problems under various models of communications see [4].

In the model of communication used in this paper, the processors are allowed to communicate with each other only by sending messages over a fully-synchronized, reliable, point-to-point network G. Each edge of G will represent a direct two-way communication link between the two endprocessors. The communication will progress in well-defined discrete time units called "rounds". During a single round,

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each processor will be allowed to send a message to, or receive a message from, at most k ($1 \leqslant k \leqslant \Delta(G)$) adjacent processors; however, a processor cannot both send and receive during the same round. The size of an individual message is assumed to be unlimited, and thus we may assume that the only message a processor ever sends is everything it already knows. Given $k \geqslant 1$ and a network G, we use $r_k(G)$ to denote the minimum number of rounds needed to complete gossiping in G in the above model; obviously $r_k(G) \geqslant \mathrm{diam}(G)$ for any $k \geqslant 1$. On the other hand, it can be shown that $r_k(G) \leqslant 2 \cdot \mathrm{radius}(G)$ when $k \geqslant \Delta(G)$.

There has been much previous study of $r_k(G)$ when $k = 1$, and essentially optimal gossiping algorithms when $k = 1$ are known for complete graphs [2, 3, 6, 7], trees [1], cycles [6], and higher-dimensional grids [6]. Two interesting classes of networks which have been considered for $k = 1$, but for which optimal algorithms are not yet known, are hypercubes [5, 6] and more generally higher-dimension toroidal grids [6]. We will say more about gossiping in these latter two architectures when $k = 1$ later in this paper.

On the other hand, gossiping when $k \geqslant 2$ has been previously considered only for complete graphs. Entringer and Slater [3] conjectured the following result, which has recently been proven by Sundaram and Winkler [7].

Theorem A. *Let $k \geqslant 1$. Then we have*

$$\lceil \log_{\lambda(k)} n \rceil \leqslant r_k(K_n) \leqslant \lceil \log_{\lambda(k)} n \rceil + 3,$$

where $\lambda(k) = \frac{1}{2}(k + \sqrt{k^2 + 4})$.

Our goal in this paper is to consider $r_k(G)$ for several important networks when $k \geqslant 2$. The architectures to be discussed are trees, cycles, grids, hypercubes, and toroidal grids. We will be especially interested in toroidal grids, since, as we will see, both cycles and hypercubes are special cases of toroidal grids.

Our main result is that if every dimension of a toroidal grid G is even, then gossiping can be completed in G in $\mathrm{diam}(G) + 1$ rounds when $k = 2$, but cannot be completed in $\mathrm{diam}(G)$ rounds regardless of the size of k. As an important and immediate corollary, we obtain an optimal $(d + 1)$-round algorithm for gossiping in a d-dimensional hypercube with $k = 2$.

The remainder of the paper is organized as follows. In Section 2, we give an easy proof that gossiping in a toroidal grid G cannot be done in $\mathrm{diam}(G)$ rounds, regardless of the size of k. In Section 3, in order to motivate and facilitate the results in Section 4, we present optimal gossiping algorithms for cycles. In Section 4.1, we describe and prove a $\mathrm{diam}(G) + 1$-round algorithm for toroidal grids when every dimension is even. In the remainder of Section 4, we consider toroidal grids in which at least one dimension is odd, and conjecture the value of $r_k(G)$ for any toroidal grid G and $k \geqslant 2$. In Section 5, we discuss gossiping in trees and higher-dimensional grids when $k \geqslant 2$.

2. A lower bound for gossiping

We now prove that $r_k(G) \geqslant \mathrm{diam}(G) + 1$ holds for many networks G, regardless of the size of k. Though easy to prove, this bound will apply to all cycles, hypercubes, and toroidal grids, and will be useful in the sequel.

Theorem 1. *Suppose every vertex of G is of distance* $\mathrm{diam}(G)$ *from at least one other vertex in G. Then $r_k(G) \geqslant \mathrm{diam}(G) + 1$, for all $k \geqslant 1$.*

Proof. Consider any vertex $x \in V(G)$, and suppose $\mathrm{dist}(x, y) = \mathrm{diam}(G)$. Then in any $\mathrm{diam}(G)$-round gossiping algorithm, x must be first receiving y's message during round $\mathrm{diam}(G)$. But since x was any vertex in G, it would follow that every vertex in G must be receiving during round $\mathrm{diam}(G)$. But then no one is left to send during round $\mathrm{diam}(G)$, an impossibility. $\square$

3. Gossiping in cycles

We now give an optimal gossiping algorithm for cycles.

Theorem 2. *Let $n \geqslant 4$ and $k \geqslant 2$. Then $r_k(C_n) = \lceil n/2 \rceil + 1$.*

Proof. We assume $V(C_n) = \{1, 2, \ldots, n\}$. To show that $r_k(C_n) \leqslant \lceil n/2 \rceil + 1$, we use the following easy gossiping algorithm.

Algorithm $\{n\text{-cycle}\}$
  $S \leftarrow [1, 3, 5, \ldots, 2\lfloor n/2 \rfloor - 1\}$;
  repeat $\lceil n/2 \rceil + 1$ times
    begin
      each vertex in S sends its current message to its two neighbors on C_n;
      $S \leftarrow S + 1$ {addition is modulo n}
    end.

We leave the easy verification that the algorithm is correct to the reader.

We now turn to a lower bound for $r_k(C_n)$, noting that we may assume $k = 2$ since $\Delta(C_n) = 2$. Since $r_2(C_n) \geqslant \lceil n/2 \rceil + 1 = \mathrm{diam}(C_n) + 1$ when n is even by Theorem 1, it suffices to show that $r_2(C_n) \geqslant \lceil n/2 \rceil + 1 = (n + 3)/2$, when $n \geqslant 5$ is odd.

Suppose to the contrary that $r_2(C_n) = (n + 1)/2 = \mathrm{diam}(C_n) + 1$. We may assume without loss of generality that vertex n does not send during round 1. But then n's message must proceed unimpeded during rounds $2, 3, \ldots, (n + 1)/2$ toward the diametrically opposite vertices $(n \pm 1)/2$; in particular, the consecutive vertices $(n \pm 1)/2$ must both be receiving during round $(n + 1)/2$. This implies that vertex $(n + 3)/2$ must be sending during round $(n+1)/2$ (else no one sends to vertex $(n+1)/2$ during that round), and so vertex $(n+3)/2$ must have received all messages by round $(n-1)/2$. But in order for vertex $(n+3)/2$ to be finished receiving by round $(n-1)/2$, its diametrically opposite vertices, vertices 1 and 2, must have both sent during the first round. Thus, vertex 3 could not have sent during the first round. Since neither vertex n nor vertex 3 sends during the first round, the messages of these two vertices must move unimpeded towards each other during the second and third rounds. This in turn forces vertices 1 and 2 to be both sending and receiving during the third round, a contradiction.

This proves Theorem 2. □

4. Gossiping in toroidal grids

A toroidal grid is obtained from an ordinary grid by adding connections that "wrap around" to connect the end points in each dimension. Consider an $a_1 \times a_2 \times \cdots \times a_d$ d-dimensional toroidal grid T with each $a_i \geqslant 2$. Clearly, $\operatorname{diam}(T) = \sum_{i=1}^{d} \lfloor a_i/2 \rfloor$. We call $a_1, a_2, \ldots, a_d$ the *dimensions* of T. We will denote the number of odd dimensions of T by $\#\mathrm{odd}(T)$. Note that a cycle is just a 1-dimensional toroidal grid, while the d-dimensional hypercube is just a $2 \times 2 \times \cdots \times 2$ (d times) toroidal grid.

We begin with a trivial upper bound for $r_2(T)$, for any toroidal grid T.

Theorem 3. *If T is any d-dimensional toroidal grid, then*

$$r_2(T) \leqslant \operatorname{diam}(T) + d + \#\mathrm{odd}(T).$$

Proof. Simply gossip in the wrap-around a_1-cycles in parallel, then in the a_2-cycles, etc. By Theorem 2, this takes just $\sum_{a_i \text{even}} (\lfloor a_i/2 \rfloor + 1) + \sum_{a_i \text{odd}} (\lfloor a_i/2 \rfloor + 2) = \operatorname{diam}(T) + d + \#\mathrm{odd}(T)$ rounds. □

In contrast to Theorem 3, the best known upper bound for $r_1(T)$ is $\operatorname{diam}(T) + 18d + 35$; this result is due to Krumme et al. [6]. A tight upper bound for $r_1(T)$ remains an interesting open problem.

4.1. All dimensions even

Let T be an $a_1 \times a_2 \times \cdots \times a_d$ toroidal grid, with all a_i even. Suppose we properly 2-color the vertices of T with, say, black and white.

Definition. A *two-way send along dimension i* is a round in which every vertex of one color sends its current message to both its neighbors along dimension i. A *one-way send along dimension i* is a round in which every vertex of one color sends its message

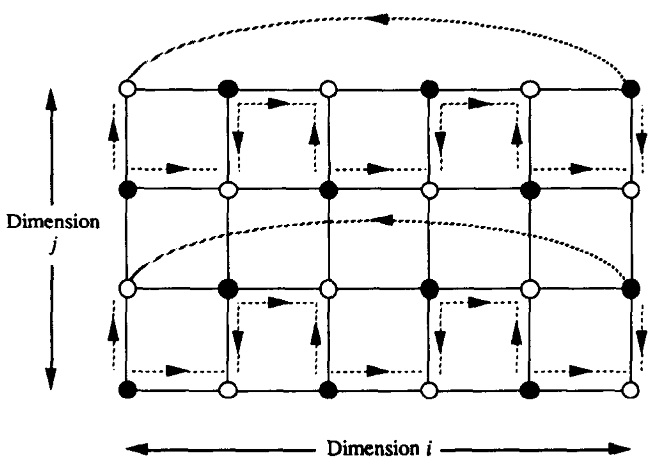

Fig. 1. An (i,j)-cross-dimensional send along dimension i.

to one neighbor along dimension i, and each vertex of the opposite color receives exactly one message.

Definition. Let $1 \leqslant i,j \leqslant d$. An *$(i,j)$-cross-dimensional send along dimension i* is a round in which every vertex of one color (say black) sends its current message to two vertices as shown in Fig. 1.

We are now in a position to prove the following.

Theorem 4. *If T is a toroidal grid in which every dimension is even, then* $r_2(T) = \mathrm{diam}(T) + 1$.

Proof. We have $r_2(T) \geqslant \mathrm{diam}(T) + 1$ by Theorem 1, and so it suffices to describe a $\mathrm{diam}(T) + 1$-round gossiping algorithm for T with $k = 2$. The vertices sending during the odd (even) numbered rounds will be the black (white) vertices.

Algorithm {toroidal grid with all dimensions even}
  do a 1-way send along dimension 1;
  for $j = 1$ to $d - 1$ do
    begin
      do $(a_j/2) - 1$ 2-way sends along dimension j;
      do a $(j, j + 1)$-cross-dimensional send along dimension j;
    end;
  do $(a_d/2) - 1$ 2-way sends along dimension d;
  do a 1-way send along dimension d.

To prove the algorithm is correct, consider the first $a_1/2$ rounds consisting of a 1-way send and $(a_1/2) - 1$ 2-way sends along dimension 1. It is easy to verify that each vertex which received during round $a_1/2$ will have learned all the messages of the a_1-cycle to which it belongs. The (1, 2)-cross-dimensional send then accomplishes, in a single round, two things:

(i) every vertex knows the messages of all the vertices in the a_1-cycle to which it belongs;

(ii) a 1-way send along dimension 2.

The above pattern is then repeated along dimension 2, etc., to complete gossiping. □

Let us now consider gossiping in hypercubes; of course, hypercubes are a very important special case of toroidal grids. Since each dimension is 2, the algorithm in Theorem 4 can be simplified slightly as follows.

Algorithm $\{d$-dimensional hypercube$\}$
  do a 1-way send along dimension 1;
  for $j = 1$ to $d - 1$ do
    a$(j, j + 1)$-cross-dimensional send along dimension j;
  do a 1-way send along dimension d.

Letting Q_d denote the d-dimensional hypercube, we obtain

Corollary 5. $r_2(Q_d) = d + 1 = \mathrm{diam}(Q_d) + 1$.

Of course, $r_k(Q_d) = d + 1$ for every $k \geqslant 2$ by Theorem 1 and Corollary 5. By contrast, the tightest known bounds for $r_1(Q_d)$ are $1.44d \leqslant r_1(Q_d) \leqslant 1.88d$ when $d \geqslant 8$; this result is due to Krumme [5]. It is perhaps surprising how much $r_k(Q_d)$ improves when k merely increases from 1 to 2. A tight asymptotic bound for $r_1(Q_d)$ remains an apparently difficult open problem.

4.2. Exactly one odd dimension

Theorem 6. *Let T be an $a_1 \times a_2 \times \cdots \times a_d$ toroidal grid with exactly one odd dimension, say a_d. Then*

(i) $r_2(T) = \mathrm{diam}(T) + 1$ *if* $a_d = 3$;

(ii) $\mathrm{diam}(T) + 1 \leqslant r_2(T) \leqslant \mathrm{diam}(T) + 2$, *if* $a_d \geqslant 5$.

Proof. (i) Consider an $a_1 \times a_2 \times \cdots \times a_{d-1} \times 2$ toroidal subgrid T' of T, noting that in effect we retain the "wrapping effect" in the final dimension. We then use the algorithm in Theorem 4 to gossip in T', except that during the first (final) round, a 1-way send, each vertex in $T - T'$ sends to (resp. receives from) its neighbor in the 3-cycle along dimension d who is receiving (resp. sending) at that round in T'.

(ii) It suffices to show that $r_2(T) \leqslant \operatorname{diam}(T) + 2$. Properly 2-color an $a_1 \times a_2 \times \cdots \times a_{d-1} \times (a_d - 1)$ subgrid of T. For each $a_1 \times \cdots \times a_{d-1}$ toroidal subgrid, gossip as in the algorithm of Theorem 4 for $r = a_1/2 + \cdots + a_{d-1}/2$ rounds. The vertices which receive at round r each know the cumulative message of the $a_1 \times \cdots \times a_{d-1}$ toroidal subgrid to which they belong. Then, gossip during round $r + 1$ is shown in Fig. 2, where the darkened vertices denote the receivers during round r, and the $a_1 \times a_2 \times \cdots \times a_{d-1} \times (a_d - 1)$ subgrid used all but the last column of Fig. 2.

At the completion of round $r + 1$, two things have been accomplished:

(i) every vertex knows the cumulative message of the $a_1 \times \cdots \times a_{d-1}$ toroidal subgrid to which it belongs;

(ii) the first round of the algorithm in Theorem 2 for the a_d-cycles along dimension d has been completed.

We then use $\lfloor a_d/2 \rfloor + 1$ additional rounds to complete gossiping in the a_d-cycles along dimension d. □

We put forth the following conjecture.

Conjecture 7. Let T be an $a_1 \times \cdots \times a_d$ toroidal grid with exactly one odd dimension, say a_d. Then

$$r_2(T) = \begin{cases} \operatorname{diam}(T) + 1 & \text{if } a_d = 3, \\ \operatorname{diam}(T) + 2 & \text{if } a_d \geqslant 5. \end{cases}$$

Regarding Conjecture 7, we have verified only that if T is the 2×5 toroidal grid, then $r_2(T) = \operatorname{diam}(T) + 2$.

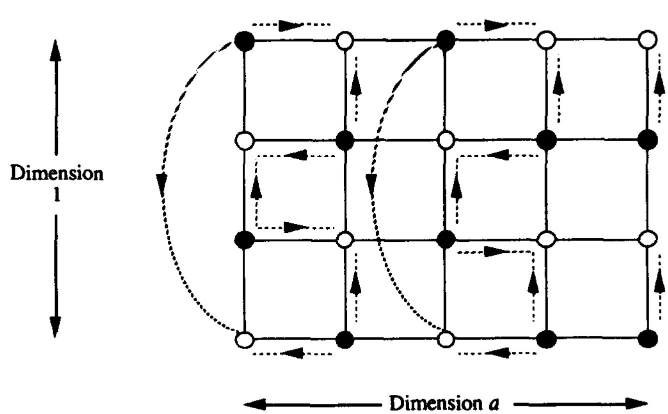

Fig. 2. Round $r + 1$.

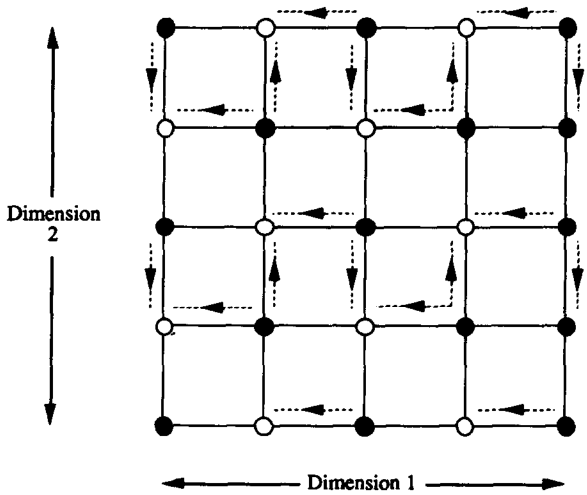

Fig. 3.

4.3. Two or more odd dimensions

We immediately put forth the following conjecture regarding toroidal grids having two or more odd dimensions.

Conjecture 8. Let T be a toroidal grid with two or more odd dimensions. Then for any $k \geqslant 2$,

$$r_k(T) = \begin{cases} \operatorname{diam}(T) + \#\operatorname{odd}(T), & \text{if every odd dimension is 3,} \\ \operatorname{diam}(T) + \#\operatorname{odd}(T) + 1 & \text{otherwise.} \end{cases}$$

Conjecture 8 appears quite difficult (but see Section 6, for a very natural related conjecture which would imply $r_k(T) \leqslant \operatorname{diam}(T) + \#\operatorname{odd}(T) + 1$). Thus we content ourselves with establishing bounds for 2-dimensional toroidal grids when both dimensions are odd.

Theorem 9. *Let T be an $a_1 \times a_2$ toroidal grid with a_1, a_2 odd.* Then

$$\operatorname{diam}(T) + 2 \leqslant r_2(T) \leqslant \operatorname{diam}(T) + 3.$$

Proof. We first establish the upper bound. Let the a_1-cycles (resp. a_2-cycles) be indexed using $1, 2, \ldots, a_2$ (resp. $1, 2, \ldots, a_1$). Gossip in the a_1-cycles for $\lfloor a_1/2 \rfloor + 1$ rounds as in Theorem 2 so that in the a_1-cycles with odd (resp. even) index, the vertices in columns $1, 3, 5, \ldots, a_2$ (resp. $2, 4, 6, \ldots, a_2 - 1, a_2$) learn the cumulative message of the vertices in their a_1-cycle. Then gossip for one round as shown in Fig. 3, where the darkened vertices are the ones which learned the cumulative message of their rows during the first $\lfloor a_1/2 \rfloor + 1$ rounds.

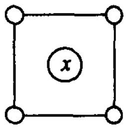

Fig. 4. The x-square.

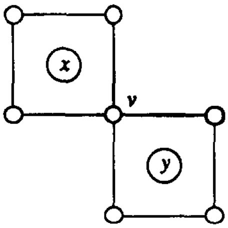

Fig. 5.

This accomplishes two things:

(i) every vertex now knows all the messages of the a_1-cycle to which it belongs;

(ii) the first round of the algorithm in Theorem 2 has been completed in the a_2-cycles.

We then do $\lfloor a_2/2 \rfloor + 1$ additional rounds to complete gossiping in the a_2-cycles, a total of $\lfloor a_1/2 \rfloor + \lfloor a_2/2 \rfloor + 3 = \mathrm{diam}(T) + 3$ rounds.

We next show that $r_2(T) \geqslant \mathrm{diam}(T) + 2$. Suppose to the contrary that $r_2(T) = \mathrm{diam}(T) + 1$. Consider a vertex x which does not send at round 1. Then there exist four vertices which are diametrically distant from x which form the corners of a "small square" in T; we will term this set of four vertices the *x-square* in T, and denote it as shown in Fig. 4. Note that since x does not send at round 1, all the vertices in the x-square must be receiving at round $\mathrm{diam}(T) + 1$.

Claim 1. *If x, y are nonadjacent vertices in a small square of T, then at least one of these must be sending at round* 1.

Proof. If neither x nor y sends at round 1, then by the observation above, all the vertices shown in Fig. 5 must be receiving during round $\mathrm{diam}(T) + 1$. But then no one is available to send to the vertex v during round $\mathrm{diam}(T) + 1$, an impossibility. This proves Claim 1.

Claim 2. *If x sends at round* 1, *then exactly one* (*resp. three*) *neighbor*(*s*) *of x must be receiving from x* (*resp. sending*) *during round* 1.

Proof. By Claim 1, x has at most two neighbors who are not sending during round 1, with equality demanding that these two neighbors both lie in the same row, or both in the same column, as x. Suppose without loss of generality that x has two neighbors, y, z in the same vertical column which do not send at round 1 (see Fig. 6). Consider the vertex a. By Claim 1, a must send during round 1. If a sends to b_1 during round 1, then

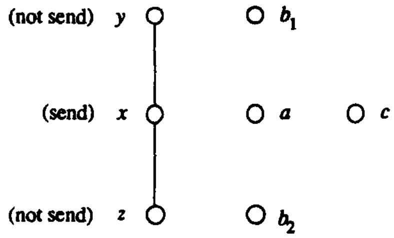

Fig. 6.

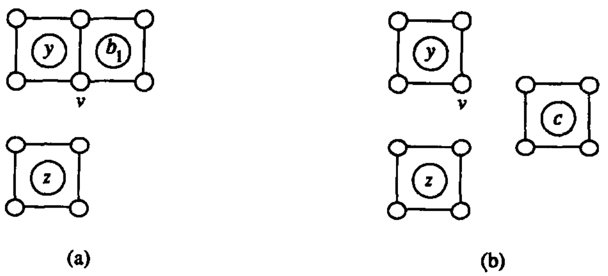

Fig. 7.

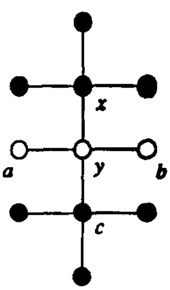

Fig. 8.

during round diam$(T) + 1$, all the vertices in Fig. 7(a) must be receiving, leaving no one to send to v. A symmetric argument shows that a cannot send to b_2 at round 1. Finally, if a sends to c at round 1, then all the vertices in Fig. 7(b) must be receiving during round diam$(T) + 1$, leaving no one to send to v.

This proves Claim 2.

Suppose now that x sends to y, as depicted in Fig. 8 in which black (white) vertices represent vertices which send (receive) during round 1. If any of the other neighbors a, b, c of y send during round 1, they must send to y, by Claim 2. But since $k = 2$, y can receive from at most one of a, b, c; by Claims 1 and 2, it follows easily that a, b (resp. c) must be receiving (resp. sending) during round 1. This justifies Fig. 8. Note that both a and b must be receiving from the two vertices above and below them in Fig. 8.

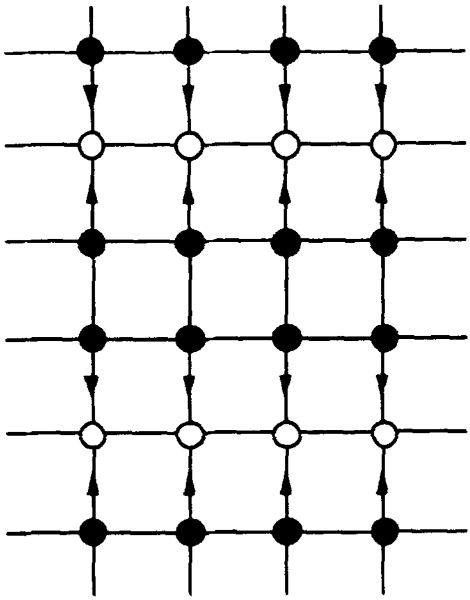

Fig. 9. Round 1.

The above implies that we may assume without loss of generality that the pattern of sending and receiving during round 1 is as shown in Fig. 9.

Claim 3. *Each vertex x which received during round* 1 *must send during round* 2 *to both its neighbors which were sending during round* 1.

Proof. Otherwise x's message cannot reach all the vertices in the x-square by the end of round $\mathrm{diam}(T) + 1$. This proves Claim 3.

Now the darkened nodes which received x's message during round 2 must send them to other darkened nodes during round 3. However, *every* darkened node received during round 2, and so must be sending during round 3. This contradiction completes the proof of Theorem 9. □

In line with Conjecture 8, we conjecture that the upper bound $\mathrm{diam}(T) + 3$ in Theorem 9 is always correct except for the 3×3 grid.

5. Gossiping in trees and grids

We observe that algorithms developed previously for trees and grids when $k = 1$ can easily be adapted to give optimal gossiping algorithms for these architectures when $k \geqslant 2$. Since gossiping in almost all d-dimensional grids G can be done in $\mathrm{diam}(G)$ rounds when $k = 1$, obviously no reduction is possible for these grids for larger k. By contrast, since almost all trees T require substantially more than $\mathrm{diam}(T)$

rounds when $k = 1$, increasing the value of k can lead to a significant reduction in the number of rounds required for gossiping.

(i) An optimal gossiping algorithm for trees when $k = 1$ was given in [1]. Given a tree T, let $B_k(T)$ denote the set of vertices from which broadcasting with k transmitter/receivers at each vertex takes the minimum number of rounds (when $k = 1$, $B_k(T)$ is simply the *broadcast center* [1,4] of the tree). Let $b_k(T)$ denote the number of rounds needed to broadcast a message from a vertex in $B_k(T)$ to all of T. It is easy to find $B_k(T)$ and $b_k(T)$ in linear time (cf. [1]). It is also easy to show that $r_k(T) = 2b_k(T)$ using the argument of Theorem 2 in [1].

(ii) For higher-dimensional grids G, Krumme [6] has shown that if each dimension is at least 9, then gossiping in G can be completed in $\operatorname{diam}(G)$ rounds when $k = 1$, and afortiori when $k \geqslant 2$. On the other hand, it is known that gossiping in $\operatorname{diam}(G)$ rounds is not possible if some dimension of G is a 2 or 3. The optimal number of rounds for gossiping in grids with one or more small dimensions remains open.

6. Concluding remarks

We conclude by putting forth a very natural conjecture which we have not been able to settle. It is closely related to Conjecture 8.

Conjecture 10. Let T_1 (T_2) be an $a_1 \times a_2 \times \cdots \times a_d$ ($b_1 \times b_2 \times \cdots \times b_d$) toroidal grid with $a_i \geqslant b_i$ for all i (briefly, the dimensions of T_1 majorize the dimensions of T_2). Then for any $k \geqslant 1$, we have $r_k(T_1) \geqslant r_k(T_2)$.

We observe that if Conjecture 10 is true, then for any toroidal grid T we would have (cf. Conjecture 8) $r_2(T) \leqslant \operatorname{diam}(T) + \#\mathrm{odd}(T) + 1$. To see this, simply increase each odd dimension of T by 1 to obtain a toroidal grid T' whose dimensions are all even and majorize those of T. It then follows by Conjecture 10 and Theorem 4 that $r_2(T) \leqslant r_2(T') = \operatorname{diam}(T') + 1 = \operatorname{diam}(T) + \#\mathrm{odd}(T) + 1$.

References

[1] A. Bagchi, S.L. Hakimi, J. Mitchem and E.F. Schmeichel, Parallel algorithms for gossiping by mail, Inform. Process. Lett. 34 (1990) 197–202.

[2] S. Even and B. Monien, On the number of rounds necessary to disseminate information, in: Proceedings of ACM Symposium on Parallel Algorithms and Architectures, Santa Fe (1989) 318–327.

[3] R.C. Entringer and P.J. Slater, Gossips and telegraphs, J, Franklin Inst. 307 (1979) 353–360.

[4] S.M. Hedetniemi, S.T. Hedetniemi and A.L. Liestman, A survey of gossiping and broadcasting in communication networks, Networks 18 (1988) 319–349.

[5] D.M. Krumme, Fast gossiping for the hypercube, SIAM J. Comput. 21 (1992) 365–380.

[6] D.W. Krumme, G. Cybenko and K.N. Venkataraman, Gossiping in minimal time, SIAM J. Comput. 21 (1992) 111–139.

[7] V.S. Sundaram and P. Winkler, Fast information sharing in a complete network, Discrete Appl. Math. 42 (1993) 75–86.