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Embedding in Brownian motion with drift and the Azéma–Yor construction

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Abstract

We consider the embedding of a probability distribution in Brownian motion with drift. We first give a sufficient condition on the target measure, under which a variant of the Azéma–Yor (1979a, Séminaire de Probabilités XIII, Lecture Notes in Mathematics, Vol. 721, Springer, Berlin, pp. 90–115) construction for this problem works. A necessary and sufficient condition for embeddability by means of some stopping time, not necessarily finite, is also provided. This latter condition is then analyzed in some detail. © 2000 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

Skorokhod's problem of embedding a given law with mean zero in Brownian motion, which has been studied by many authors, found a remarkable solution in the paper of Azéma and Yor (1979a), see also Zaremba, 1985. Their approach has the advantage that it gives an explicit formula for a stopping time embedding a given law. In their paper, Azéma and Yor also give a generalization to embedding in a recurrent diffusion which starts at zero (see their Proposition 4.1).

Throughout this note, (B_t) denotes standard Brownian motion on the line starting from zero, κ is a strictly positive real constant, and $X_t = B_t + \kappa t$ is Brownian motion with drift rate κ. (Since $(-X_t)$ is Brownian motion with drift rate $-\kappa$, results for negative values of κ can easily be read off from ours.) Since the diffusion (X_t) is transient, not recurrent, Proposition 4.1 of Azéma and Yor (1979a) is not directly applicable. Nevertheless, we show that if μ is a probability measure on $\mathbb{R}$ satisfying

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$\int \mathrm{e}^{-2\kappa x}\,\mathrm{d}\mu(x) = 1$, then μ is embeddable in (X_t) by means of a variant of the Azéma–Yor stopping time.

Secondly, we apply existing results on Skorokhod's problem for general Markov processes to show that a probability measure μ on $]-\infty,\infty]$ is embeddable in (X_t) by means of some stopping time, not necessarily finite, iff $\int \mathrm{e}^{-2\kappa x}\,\mathrm{d}\mu(x) \leqslant 1$. (We set $X_\infty = \infty$, which makes sense since $X_t \to \infty$ as $t \to \infty$.) Thirdly, we characterize those target measures μ such that there exists $\kappa > 0$ for which μ is embeddable in Brownian motion with drift rate κ, and we give conditions on μ under which there is a largest such value of κ. Finally, we give some equivalent conditions for the uniform integrability of the stopped process $(X_{T\wedge t})$ and deduce that the largest value of κ mentioned above minimizes the expected value of the corresponding stopping time.

2. Results

The embedding problems for the continuous processes $X_t = B_t + \kappa t$ and $Y_t = \mathrm{e}^{-2\kappa X_t} - 1$ are equivalent via the obvious change of the space variable. To apply the arguments of Azéma and Yor (1979a), it is more convenient to work with (Y_t) than with (X_t), since the former is a martingale starting from zero. Hence, we shall obtain the embedding result for (X_t) as a corollary from an embedding result for (Y_t).

But first let us make some elementary observations. We set $Y_\infty = -1$, which makes sense since $Y_t \to -1$ as $t \to \infty$. Let T be any stopping time for (Y_t). Then $\mathbf{E}(Y_{T\wedge t}) = 0$ for $0 \leqslant t < \infty$ by the optional stopping theorem, so, on applying Fatou's lemma to $1 + Y_{T\wedge t}$ as $t \to \infty$, we get $\mathbf{E}(Y_T) \leqslant 0$. Equality holds iff $(Y_{T\wedge t})$ is uniformly integrable. The reverse implication is obvious. To prove the forward implication, suppose $\mathbf{E}(Y_T) = 0$. Then as $n \to \infty$, $\mathbf{E}(1 + Y_{T\wedge n}) \to \mathbf{E}(1 + Y_T) < \infty$, and $0 \leqslant 1 + Y_{T\wedge n} \to 1 + Y_T$ a.s., so $1 + Y_{T\wedge n} \to 1 + Y_T$ in L^1 by a standard generalization of the dominated convergence theorem. Hence, $\{Y_{T\wedge n}\colon n = 1,2,3,\ldots\}$ is uniformly integrable. Therefore $(Y_{T\wedge t})$ is uniformly integrable, because it is a martingale. To sum up, if ν is the law of Y_T, then ν is a probability measure on $[-1,\infty[$, $\int y\,\mathrm{d}\nu(y) \leqslant 0$ with equality iff $(Y_{T\wedge t})$ is uniformly integrable, and $T < \infty$ a.s. iff ν lives on $]-1,\infty[$. Expressing this in terms of (X_t), if μ is the law of X_T, then μ is a probability measure on $]-\infty,\infty]$, $\int \mathrm{e}^{-2\kappa x}\,\mathrm{d}\mu(x) \leqslant 1$ with equality iff $(\exp(-2\kappa X_{T\wedge t}))$ is uniformly integrable, and $T < \infty$ a.s. iff μ lives on $\mathbb{R}$.

Now we recall the definition of the barycenter function Ψ_ν of a probability measure ν on $\mathbb{R}$, as given by Azéma and Yor (1979a):

$$\Psi_\nu(y) = \begin{cases} \dfrac{1}{\nu([y,\infty[)} \displaystyle\int_{[y,\infty[} z\,\mathrm{d}\nu(z) & \text{if } \nu([y,\infty[) > 0, \\ y & \text{if } \nu([y,\infty[) = 0. \end{cases}$$

Theorem 2.1. *Let ν be a probability measure on the open interval $]-1,\infty[$ satisfying $\int_{-1}^{\infty} y\,\mathrm{d}\nu(y) = 0$ and let $T = \inf\{t\colon Y_t^* \geqslant \Psi_\nu(Y_t)\}$, where $Y_t^* = \sup_{s\leqslant t} Y_s$. Then Y_T has distribution ν.*

Proof. In Theorem 3.4 of Azéma and Yor (1979a), with their (X_t) and μ replaced by our (Y_t) and ν, the proof that the law of Y_T is ν does not use the assumption in the statement of that theorem that $\int y^2\,\mathrm{d}\nu(y) < \infty$, as Azéma and Yor themselves point out in the companion paper (1979b). But even on taking this into account, the statement of their Theorem 3.4 does not apply directly because we do not have $\liminf_{t\to-\infty} Y_t = -\infty$ and $\limsup_{t\to\infty} Y_t = \infty$. However, their proof does not require this either. Any condition which suffices to guarantee that $T < \infty$ a.s. will do. Now, from our assumptions on ν, we get $\Psi_\nu(y) \to 0$ as $y \to -1$. As (Y_t^*) almost surely becomes strictly positive immediately after $t = 0$ and since $Y_t \to -1$ as $t \to \infty$, the almost sure finiteness of $T = \inf\{t\colon Y_t^* \geqslant \Psi_\nu(Y_t)\}$ follows. Hence Y_T has distribution ν.

It is worth observing that by time-changing (Y_t) in a suitable way, we can prove the theorem without delving into the details of the proof of Theorem 3.4 of Azéma and Yor. By Ito's formula, we have $\mathrm{d}Y_t = \sigma(Y_t)\,\mathrm{d}B_t$, where $\sigma(y) = -2\kappa(y+1)$, so $\mathrm{d}\langle Y,Y\rangle_t = \sigma(Y_t)^2\,\mathrm{d}t$. Let $A_t = \int_0^t \sigma(Y_u)^2\,\mathrm{d}u$ for $0 \leqslant t \leqslant \infty$, and let $\zeta = A_\infty$. Then for each ω, the map $t \mapsto A_t(\omega)$ is an increasing homeomorphism from $[0,\infty]$ onto $[0,\zeta(\omega)]$. For $0 \leqslant s \leqslant \infty$, let $\tau_s = \inf\{t\colon A_t > s\}$. Then for each ω, $\tau_0(\omega) = 0$, the map $s \mapsto \tau_s(\omega)$ is continuous and strictly increasing on $[0,\zeta(\omega)]$, and if $\zeta(\omega) \leqslant s \leqslant \infty$, then $\tau_s(\omega) = \infty$. In particular, as $s \uparrow \zeta$, $\tau_s \uparrow\uparrow \infty$, so $Y_{\tau_s} \to -1$ almost surely. Let (B_t') be a Brownian motion on the line, starting from 0, independent of (B_t). For $0 \leqslant s < \infty$, let

$$\beta_s = \begin{cases} Y_{\tau_s} & \text{if } 0 \leqslant s < \zeta, \\ -1 + B'_{s-\zeta} & \text{if } \zeta \leqslant s < \infty. \end{cases}$$

Then (β_s) is a Brownian motion on the line, starting from 0, (see for instance, Revuz and Yor, 1994, p. 174). Clearly ζ is the first time (β_s) hits -1. Letting $S = \inf\{s\colon \sup_{r\leqslant s} \beta_r \geqslant \Psi_\nu(\beta_s)\}$, we find that the law of β_S is ν, by the result of Azéma and Yor applied to the process (β_s). But the same argument that showed that $T < \infty$ a.s. shows that $S < \zeta$ a.s., and then it is clear that $Y_T = \beta_S$ a.s. □

Corollary 2.1. *Let μ be a probability measure on $\mathbb{R}$ satisfying $\int \mathrm{e}^{-2\kappa x}\,\mathrm{d}\mu(x) = 1$. Let $u\colon \mathbb{R} \to \mathbb{R}$ be the scale function $u(x) = \mathrm{e}^{-2\kappa x} - 1$, let ν be the image of the measure μ under the map u, let $\Psi_\mu^\kappa = u^{-1} \circ \Psi_\nu \circ u$, let $m_t = \inf_{s\leqslant t} X_s$, and let $T = \inf\{t\colon m_t \leqslant \Psi_\mu^\kappa(X_t)\}$. Then X_T has distribution μ.*

Remark. (1) Let $\tilde Y_t = -Y_t$, let $\tilde\nu$ be a probability measure on the open interval $]-\infty,1[$ satisfying $\int_{-\infty}^1 \tilde y\,\mathrm{d}\tilde\nu(\tilde y) = 0$, let $\tilde T = \inf\{t\colon \tilde Y_t^* \geqslant \Psi_{\tilde\nu}(\tilde Y_t)\}$, where $\tilde Y_t^* = \sup_{s\leqslant t} \tilde Y_s$. Then $\tilde Y_{\tilde T}$ has distribution $\tilde\nu$. To see this, let the notation be as in the second paragraph of the proof of Theorem 2.1, let $\tilde\beta_t = -\beta_t$, and let $\tilde S = \inf\{r\colon \sup_{r\leqslant s} \tilde\beta_r \geqslant \Psi_{\tilde\nu}(\tilde\beta_s)\}$. By the result of Azéma and Yor, the law of $\tilde\beta_{\tilde S}$ is $\tilde\nu$. Now $\Psi_{\tilde\nu}(\tilde y) = \tilde y$ for $\tilde y \geqslant 1$, so $\tilde S \leqslant \zeta$. Since $\tilde\nu(\{1\}) = 0$, $\tilde S < \zeta$ a.s., so $\tilde Y_{\tilde T} = \tilde\beta_{\tilde S}$ a.s.

(2) Let μ be as in Corollary 2.1. Let $\tilde u\colon \mathbb{R} \to \mathbb{R}$ be the scale function $\tilde u(x) = 1 - \mathrm{e}^{-2\kappa x}$, let $\tilde\nu$ be the image of the measure μ under the map $\tilde u$, let $\tilde\Psi_\mu^\kappa = \tilde u^{-1} \circ \Psi_{\tilde\nu} \circ \tilde u$, let $M_t = \sup_{s\leqslant t} X_s$, and let $\tilde T = \inf\{t\colon M_t \geqslant \tilde\Psi_\mu^\kappa(X_t)\}$. Then $X_{\tilde T}$ has distribution μ. This follows from the preceding remark just as Corollary 2.1 follows from Theorem 2.1.

(3) Let $\tilde T$ and (M_t) be as in (2), and let T and (m_t) be as in Corollary 2.1, so that X_T and $X_{\tilde T}$ both have distribution μ. Then, by Section 3 of Azéma and Yor (1979b),

T and $\tilde{T}$ have the following dual extremality properties: Let T' be any stopping time such that $X_{T'}$ has distribution μ; then for each $x \in \mathbb{R}$, $\mathbf{P}(M_{\tilde{T}} \geqslant x) \geqslant \mathbf{P}(M_{T'} \geqslant x)$ and $\mathbf{P}(m_T \leqslant x) \geqslant \mathbf{P}(m_{T'} \leqslant x)$.

Theorem 2.2. *A measure v is the law of Y_T for some stopping time T, not necessarily finite, iff v is a probability measure on the closed interval $[-1,\infty[$ and $\int_{[-1,\infty[} y \, dv(y) \leqslant 0$.*

Proof. We have already observed that the forward implication holds. Let us prove the reverse implication. We use the same notation as in the second paragraph of the proof of Theorem 2.1. Clearly (Y_{τ_s}) is Brownian motion in $]-1,\infty[$, killed at -1, which is a Markov process whose potential density with respect to Lebesgue measure on $]-1,\infty[$ is $v(y,\hat{y}) = 2\min\{y+1, \hat{y}+1\}$. The state space of (Y_{τ_s}) is $]-1,\infty[$. The point -1 plays the role of a cemetery point. Let v be a probability measure on $[-1,\infty[$ such that $\int_{[-1,\infty[} y \, dv(y) \leqslant 0$. The potential of v is the measure vV on $]-1,\infty[$ whose density with respect to Lebesgue measure is the function $v\hat{V}$ on $]-1,\infty[$ defined by $v\hat{V}(\hat{y}) = \int_{-1}^{\infty} v(y,\hat{y}) \, dv(y)$. We claim that $vV \leqslant \delta_0 V$, where δ_0 denotes the unit point mass at 0. Now $\delta_0 \hat{V}(\hat{y}) = 2\min\{\hat{y}+1, 1\}$ for $-1 < \hat{y} < \infty$. If $0 \leqslant \hat{y} < \infty$, then $v\hat{V}(\hat{y}) \leqslant 2\int_{-1}^{\infty} y+1 \, dv(y) \leqslant 2 = \delta_0 \hat{V}(\hat{y})$. If $-1 < \hat{y} < 0$, then $v\hat{V}(\hat{y}) = 2\int_{-1}^{\hat{y}} y+1 \, dv(y) + 2(\hat{y}+1)\int_{\hat{y}}^{\infty} dv(y) \leqslant 2(\hat{y}+1) = \delta_0 \hat{V}(\hat{y})$. This proves the claim. Hence by general results on Skorokhod's embedding problem for Markov processes, v is the law of Y_{τ_S} where S is some stopping time for (Y_{τ_s}), (see the remark below). We may take $T = \tau_S$. □

Remark. The fact that v is the law of Y_{τ_S}, where S is some stopping time for (Y_{τ_s}), follows from Rost (1970), provided we are willing to accept a randomized stopping time. If a non-randomized stopping time is desired, then since $v\hat{V}$ is continuous, this follows from the methods of Baxter and Chacon (1974), or more simply, since we are in one dimension, from the methods of Chacon and Walsh (1976). (In either case, the proofs must be modified slightly since we are not dealing with Brownian motion on the whole line. For a result which applies directly, see Falkner (1980, Theorem 2.2). For more general results, see Falkner and Fitzsimmons (1991) and the references therein.

Corollary 2.2. *A measure μ is the law of (X_T) for some stopping time T, not necessarily finite, iff μ is a probability measure on $]-\infty,\infty]$ and $\int e^{-2\kappa x} \, d\mu(x) \leqslant 1$.*

Given a probability measure μ on $]-\infty,\infty]$, it is interesting to ask whether we can find a value of $\kappa > 0$ such that μ can be embedded in $X_t = B_t + \kappa t$. By the previous corollary, μ can be so embedded for a particular $\kappa > 0$ iff $\varphi(\kappa) \leqslant 1$, where φ is the function on $\mathbb{R}$ defined by $\varphi(\theta) = \int e^{-2\theta x} \, d\mu(x)$. The following result tells us when such a κ exists.

Proposition 2.1. *Let μ be a probability measure on $]-\infty,\infty]$. To avoid trivialities, assume that $\mu \neq \delta_0$. If $\varphi(\kappa) \leqslant 1$ for some $\kappa > 0$, then $\int x \, d\mu(x) > 0$. Conversely, if*

$\int x\,\mathrm{d}\mu(x) > 0$ and $\varphi(a) < \infty$ for some $a > 0$, then for all sufficiently small $\kappa > 0$, $\varphi(\kappa) < 1$.

Proof. Suppose that $\kappa > 0$ and $\varphi(\kappa) \leqslant 1$. Then $\int_{-\infty}^{0} x\,\mathrm{d}\mu(x)$ is finite. If μ is not a point mass, then by Jensen's inequality, $\exp(-2\kappa \int x\,\mathrm{d}\mu(x)) < \varphi(\kappa)$, so $\int x\,\mathrm{d}\mu(x) > 0$. If μ is a point mass, then $\mu = \delta_a$ for some $a > 0$, so again $\int x\,\mathrm{d}\mu(x) > 0$.

Conversely, suppose $\int x\,\mathrm{d}\mu(x) > 0$ and $\varphi(a) < \infty$ for some $a > 0$. Define measures μ_1 and μ_2 by $\mu_1(\mathrm{d}x) = \mu(]-\infty, 0[\cap \mathrm{d}x)$ and $\mu_2(\mathrm{d}x) = \mu([0,\infty] \cap \mathrm{d}x)$. Let φ_1 and φ_2 be the functions on $\mathbb{R}$ defined by $\varphi_1(\theta) = \int \mathrm{e}^{-2\theta x}\,\mathrm{d}\mu_1(x)$ and $\varphi_2(\theta) = \int \mathrm{e}^{-2\theta x}\,\mathrm{d}\mu_2(x)$. By the dominated convergence theorem, $\varphi_1'(\theta) = \int -2x\mathrm{e}^{-2\theta x}\,\mathrm{d}\mu_1(x)$ for $\theta < a$. By the monotone convergence theorem, we have $[\varphi_2(\theta) - \varphi_2(0)]/\theta \to \int -2x\,\mathrm{d}\mu_2(x)$ as $\theta \downarrow 0$. Thus, the right-hand derivative of φ at 0 is $\int -2x\,\mathrm{d}\mu(x)$, which is strictly negative. (It may be $-\infty$.) Since $\varphi(0) = 1$, it follows that for all sufficiently small $\kappa > 0$, $\varphi(\kappa) < 1$. □

Remark. (1) The set $\{\kappa\colon \varphi(\kappa) < \infty\}$ is an interval and since $\mu \neq \delta_0$, φ is strictly convex on this interval, so there can be at most one value of $\kappa > 0$ for which $\varphi(\kappa) = 1$. Also, if $\int x\,\mathrm{d}\mu(x) > 0$ and there exists $a > 0$ such that $1 \leqslant \varphi(a) < \infty$, then the restriction of φ to $[0,a]$ is continuous by the dominated convergence theorem, and there is exactly one value of $\kappa > 0$ for which $\varphi(\kappa) = 1$.

(2) In general, however, there may be no value of $\kappa > 0$ such that $\varphi(\kappa) = 1$. For instance, if $\mu(]-\infty, 0[) = 0$, then since $\mu \neq \delta_0$, $\varphi(\kappa) < 1$ for all $\kappa > 0$.

(3) Suppose, on the other hand, that $\mu(]-\infty, 0[) > 0$. Then $\varphi(\theta) \to \infty$ as $\theta \to \infty$. Hence the non-empty interval $\Theta = \{\theta \geqslant 0\colon \varphi(\theta) \leqslant 1\}$ is bounded above. Let $\kappa = \sup \Theta$. Then $\kappa \in \Theta$, by Fatou's lemma. Thus, under the present assumptions, there is a largest value of $\kappa > 0$ such that μ can be embedded in $(B_t + \kappa t)$. This largest value of κ minimizes the expected value of the corresponding stopping time — see Proposition 2.2 below.

(4) However, even if $\mu(]-\infty, 0[) > 0$, there may be no value of $\kappa > 0$ such that $\varphi(\kappa) = 1$. For instance, let $\mu = (1-p)\lambda_1 + p\lambda_2$, where $1/2 < p < 1$ and where the measures λ_1 and λ_2 are defined by

$$\lambda_1(\mathrm{d}x) = \frac{c\mathrm{e}^{-2|x|}}{1+x^2} 1_{(x<0)}\,\mathrm{d}x \quad \text{and} \quad \lambda_2(\mathrm{d}x) = \frac{c\mathrm{e}^{-2|x|}}{1+x^2} 1_{(x\geqslant 0)}\,\mathrm{d}x,$$

where the positive constant c is chosen so that λ_1 and λ_2 are probability measures. Then μ is a probability measure on $\mathbb{R}$, $\int x\,\mathrm{d}\mu(x) > 0$, $\varphi(\theta) < \infty$ if $-1 \leqslant \theta \leqslant 1$, and $\varphi(\theta) = \infty$ if $|\theta| > 1$. It is easy to check that for p close enough to 1, $\varphi(1) < 1$. Now φ is convex on $[-1,1]$ and $\varphi(0) = 1$. Hence $\varphi(\theta) < 1$ for $0 < \theta \leqslant 1$, when p is close enough to 1.

Proposition 2.2. *Let T be any stopping time for (X_t). Then the following conditions are equivalent.*

(a) *$(X_{T\wedge t})$ is L^1-bounded.*

(b) *$(X_{T\wedge t})$ is uniformly integrable.*

(c) *$(X_{T\wedge t})$ is L^1-dominated.*

(d) $X_T \in L^1$.

(e) $E(T) < \infty$.

If these equivalent conditions hold, then $E(T) = E(X_T)/\kappa$ *by the optional sampling theorem.*

Proof. It is trivial that (c) $\Rightarrow$ (b) $\Rightarrow$ (a). That (a) $\Rightarrow$ (d) follows from Fatou's lemma. To finish, we show that (e) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e). Since $(B_t^2 - t)$ is a martingale, $E(B_{T\wedge t}^2) = E(T \wedge t)$, so $E(\sup_t B_{T\wedge t}^2) \leqslant 2E(T)$ by Doob's L^2 maximal inequality applied to the non-negative submartingale $(|B_{T\wedge t}|)$. Thus, if $E(T) < \infty$, then $E(\sup_t |X_{T\wedge t}|) < \infty$, so in particular $X_T \in L^1$. Conversely, suppose that $X_T \in L^1$. Now, $X_T \geqslant M + (\kappa/2)T$ where $M = \inf_t (B_t + (\kappa/2)t)$. But $M \in L^1$. In fact, $-M$ is exponentially distributed with parameter κ; see Revuz and Yor (1994, p. 70). Hence $T \in L^1$, because $0 \leqslant T \leqslant (2/\kappa)(X_T - M)$. $\square$

Remark. It follows that for Brownian motion with drift, the expectation of a stopping time embedding a given distribution depends only on the distribution being embedded. This is a reflection of the transience of Brownian motion with drift and stands in sharp contrast to what happens for Brownian motion on the line without drift. (Consider for instance the first time (B_t) hits 0 after hitting 1.)

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