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Adapted solutions of backward stochastic differential equations with non-Lipschitz coefficients

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Abstract

In this paper we shall establish a new theorem on the existence and uniqueness of the adapted solution to a backward stochastic differential equation under a weaker condition than the Lipschitz one.

Keywords: Backward stochastic differential equation; Adapted solution; Bihari's inequality

1. Introduction

The equation for the adjoint process in optimal stochastic control (see Bensoussan, 1982: Bismut, 1973; Haussmann, 1986; Kushner, 1972) is a linear version of the following backward stochastic differential equation:

$$x(t) + \int_t^1 f(s, x(s), y(s))\,\mathrm{d}s + \int_t^1 [g(s, x(s)) + y(s)]\,\mathrm{d}w(s) = X \tag{1.1}$$

on $0 \leqslant t \leqslant 1$. Here $\{w(t): 0 \leqslant t \leqslant 1\}$ is a q-dimensional Brownian motion defined on the probability space $(\Omega, \mathscr{F}, P)$ with the natural filtration $\{\mathscr{F}_t: 0 \leqslant t \leqslant 1\}$ (i.e. $\mathscr{F}_t = \sigma\{w(s): 0 \leqslant s \leqslant t\}$), and X is a given $\mathscr{F}_1$-measurable $\mathbb{R}^d$-valued random variable such that $E|X|^2 < \infty$. Moreover, f is a mapping from $\Omega \times [0,1] \times \mathbb{R}^d \times \mathbb{R}^{d\times q}$ to $\mathbb{R}^d$ which is assumed to be $\mathscr{P} \otimes \mathscr{B}_d \otimes \mathscr{B}_{d\times q}/\mathscr{B}_{d\times q}$-measurable, where $\mathscr{P}$ denotes the σ-algebra of $\mathscr{F}_t$-progressively measurable subsets of $\Omega \times [0,1]$. Also g is a mapping from $\Omega \times [0,1] \times \mathbb{R}^d$ to $\mathbb{R}^{d\times q}$ which is assumed to be $\mathscr{P} \otimes \mathscr{B}_d/\mathscr{B}_{d\times q}$-measurable. In the field of control, we usually regard $y(\cdot)$ as an adapted control and $x(\cdot)$ as the state of the system. We are allowed to choose an adapted control $y(\cdot)$ which drives the state $x(\cdot)$ of the system to the given target X at time $t = 1$. This is the so-called reachability problem. So in fact we are looking for a pair of stochastic processes $\{x(t), y(t): 0 \leqslant t \leqslant 1\}$ with values in $\mathbb{R}^d \times \mathbb{R}^{d\times q}$ which is $\mathscr{F}_t$-adapted and satisfies

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Eq. (1.1). Such a pair is called an adapted solution of the equation. Pardoux and Peng (1990) showed the existence and uniqueness of the adapted solution under the condition that $f(t,x,y)$ and $g(t,x)$ are uniformly Lipschitz continuous in (x,y) or in x respectively. More recently, Pardoux and Peng (1992) and Peng (1991) gave the probabilistic representation for the given solution of a certain system of quasilinear parabolic partial differential equations in terms of the solutions of the backward stochastic differential equations. In other words, they obtained a generalization of the well-known Feynman–Kac formula. In view of the powerfulness of the Feynman–Kac formula in the study of partial differential equations, e.g. the KPP equation (cf. Freidlin, 1985), one may expect that the Pardoux–Peng generalized formula will play an important role in the study of quasilinear parabolic partial differential equations. Hence from both viewpoints of the control theory and the study of partial differential equations, it is useful to study the backward stochastic differential equations in more detail.

Pardoux and Peng (1990) established the existence and uniqueness of the solution to Eq. (1.1) under the uniform Lipschitz condition, that is there exists a constant $K > 0$ such that

$$|f(t,x,y) - f(t,\bar{x},\bar{y})|^2 \leqslant K(|x-\bar{x}|^2) + |y-\bar{y}|^2) \quad \text{a.s.,} \tag{1.2a}$$

$$|g(t,x) - g(t,\bar{x})|^2 \leqslant K|x-\bar{x}|^2 \quad \text{a.s.} \tag{1.2b}$$

for all $x,\bar{x} \in \mathbb{R}^d, y,\bar{y} \in \mathbb{R}^{d\times q}$ and $0 \leqslant t \leqslant 1$. On the other hand, it is somehow too strong to require the uniform Lipschitz continuity in applications, e.g. in dealing with quasilinear parabolic partial differential equations. So it is important to find some weaker conditions than the Lipschitz one under which the backward stochastic differential equation has a unique solution. In the first instance, perhaps one would like to try the local Lipschitz condition plus the linear growth condition, as these conditions guarantee the existence and uniqueness of the solution for a (forward) stochastic differential equation. To be precise, let us state these conditions as follows: For each $n = 1, 2, \ldots$, there exists a constant $c_n > 0$ such that

$$|f(t,x,y) - f(t,\bar{x},\bar{y})|^2 \leqslant c_n(|x-\bar{x}|^2 + |y-\bar{y}|^2) \quad \text{a.s.,} \tag{1.3a}$$

$$|g(t,x) - g(t,\bar{x})|^2 \leqslant c_n|x-\bar{x}|^2 \quad \text{a.s.} \tag{1.3b}$$

for all $0 \leqslant t \leqslant 1$, $x,\bar{x} \in \mathbb{R}^d$, $y,\bar{y} \in \mathbb{R}^{d\times q}$ with $\max\{|x|,|\bar{x}|,|y|,|\bar{y}|\} < n$; and moreover there exists a constant $c > 0$ such that

$$|f(t,x,y)|^2 \leqslant c(1+|x|^2+|y|^2) \text{ and } |g(t,x)|^2 \leqslant c(1+|x|^2) \quad \text{a.s.} \tag{1.3c}$$

for all $0 \leqslant t \leqslant 1$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^{d\times q}$. Unfortunately, it is still open whether (1.3a–c) guarantee the existence and uniqueness of the solution to the backward stochastic differential equation (1.1). The difficulty here is that the technique of stopping time and localization seems not to work for backward stochastic differential equations. Now the question is: Are there any weaker conditions than the Lipschitz continuity under which the backward stochastic differential equation has a unique solution?

In this paper we shall give a positive answer. We shall propose the following condition:

For all $x, \bar{x} \in \mathbb{R}^d$, $y, \bar{y} \in \mathbb{R}^{d\times q}$ and $0 \leqslant t \leqslant 1$,

$$|f(t,x,y) - f(t,\bar{x},\bar{y})|^2 \leqslant \kappa(|x-\bar{x}|^2) + c|y-\bar{y}|^2 \quad \text{a.s.}, \tag{1.4a}$$

$$|g(t,x) - g(t,\bar{x})|^2 \leqslant \kappa(|x-\bar{x}|^2) \quad \text{a.s.}, \tag{1.4b}$$

where $c > 0$ and κ is a concave nondecreasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\kappa(0) = 0$, $\kappa(u) > 0$ for $u > 0$ and

$$\int_{0+} \frac{\mathrm{d}u}{\kappa(u)} = \infty.$$

The main aim of this paper is to show that under this condition the backward equation (1.1) has a unique solution.

To see the generality of our result, let us give a few examples of the function $\kappa(\cdot)$. Let $K > 0$ and let $\delta \in (0,1)$ be sufficiently small. Define

$$\kappa_1(u) = Ku, \quad u \geqslant 0.$$

$$\kappa_2(u) = \begin{cases} u\log(u^{-1}), & 0 \leqslant u \leqslant \delta, \\ \delta\log(\delta^{-1}) + \kappa_2'(\delta-)(u-\delta), & u > \delta. \end{cases}$$

$$\kappa_3(u) = \begin{cases} u\log(u^{-1})\log\log(u^{-1}), & 0 \leqslant u \leqslant \delta, \\ \delta\log(\delta^{-1})\log\log(\delta^{-1}) + \kappa_3'(\delta-)(u-\delta), & u > \delta. \end{cases}$$

They are all concave nondecreasing functions satisfying

$$\int_{0+} \frac{\mathrm{d}u}{\kappa_i(u)} = \infty.$$

In particular, we see that the Lipschitz condition (1.2a, b) is a special case of our proposed condition (1.4a, b). In other words, in this paper we obtain a more general result than that of Pardoux and Peng (1990).

On the other hand, we should also bring the reader's attention to a recent paper of Pardoux and Peng (1994), in which somewhat different studies of the non-Lipschitz backward stochastic differential equations were presented, Pardoux and Peng (1994) considered a slightly special case of Eq. (1.1), i.e. the case when $g(t,x) \equiv 0$. In other words, they considered the following backward stochastic differential equation:

$$x(t) + \int_t^1 f(s, x(s), y(s))\,\mathrm{d}s + \int_t^1 y(s)\,\mathrm{d}w(s) = X \tag{1.5}$$

on $0 \leqslant t \leqslant 1$. They showed the existence and uniqueness of a solution to Eq. (1.5) under the following conditions: $f(t,x,y)$ is locally Lipschitz continuous in x but uniformly Lipschitz continuous in y; $f(t,x,y)$ satisfies the linear growth condition; and the final value X is bounded. Besides, they also gave some other non-Lipschitz conditions. Essentially speaking, they assumed that not only $f(t,x,y)$ is continuously differentiable in (x, y) with locally bounded first-order derivatives but also X satisfies certain conditions, e.g. X is a bounded random variable belonging to the Wiener space and its derivatives on the Wiener space are bounded. A common feature in these results by Pardoux and Peng is that X needs to be bounded. However, X is generally in L^2 in applications. Compared with their results, our result requires X in L^2 only and is for a more general equation (1.1) than (1.5). Of course the techniques used and conditions proposed in our paper are different from those in Pardoux and Peng (1994) and the Bihari inequality will play a key role in our paper.

We shall organize this paper as follows. We first formulate our main result in Section 2. We then prepare several lemmas in Section 3 and finally prove the main result in Section 4.

2. Statements of main results

Let us first introduce some notations. In this paper, let $|x|$ denote the Eulclidean norm of $x \in \mathbb{R}^d$ and $(x, \bar{x})$ denote the inner product of $x, \bar{x} \in \mathbb{R}^d$. An element $y \in \mathbb{R}^{d\times q}$ will be considered as a $d \times q$ matrix and its Euclidean norm is defined by $|y| = (\mathrm{trace}(yy^{\mathrm{T}}))^{1/2}$. Denote by $M^2(0,1;\mathbb{R}^d)$ (resp. $M^2(0,1;\mathbb{R}^{d\times q})$) the family of $\mathbb{R}^d$-valued (resp. $\mathbb{R}^{d\times q}$-valued) processes which are $\mathscr{F}_t$-progressively measurable and are square integrable on $\Omega\times[0,1]$ with respect to $P\times\lambda$ (here λ denotes the Lebesgue measure on $[0,1]$). To state our main result, let us propose the standing hypotheses:

(H1) $f(\cdot,0,0) \in M^2(0,1;\mathbb{R}^d)$ and $g(\cdot,0) \in M^2(0,1;\mathbb{R}^{d\times q})$.

(H2) For all $x, \bar{x} \in \mathbb{R}^d$, $y, \bar{y} \in \mathbb{R}^{d\times q}$ and $0 \leqslant t \leqslant 1$,

$$|f(t,x,y) - f(t,\bar{x},\bar{y})|^2 \leqslant \kappa(|x-\bar{x}|^2) + c|y-\bar{y}|^2) \quad \text{a.s.,}$$

$$|g(t,x) - g(t,\bar{x})|^2 \leqslant \kappa(|x-\bar{x}|^2) \quad \text{a.s.,}$$

where $c > 0$ and κ is a concave increasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\kappa(0) = 0$, $\kappa(u) > 0$ for $u > 0$ and

$$\int_{0+} \frac{\mathrm{d}u}{\kappa(u)} = \infty. \tag{2.1}$$

Since κ is concave and $\kappa(0) = 0$, one can find a pair of positive constants a and b such that

$$\kappa(u) \leqslant a + bu \quad \text{for all } u \geqslant 0. \tag{2.2}$$

We therefore see that under hypotheses (H1) and (H2), $f(\cdot, x(\cdot), y(\cdot)) \in M^2(0,1;\mathbb{R}^d)$ and $g(\cdot, x(\cdot), y(\cdot)) \in M^2(0,1;\mathbb{R}^{d\times q})$ whenever $x(\cdot) \in M^2(0,1;\mathbb{R}^d)$ and $y(\cdot) \in M^2(0,1;\mathbb{R}^{d\times q})$.

Theorem 2.1. *Assume* (H1) *and* (H2) *hold. Then there exists a unique solution* $(x(\cdot), y(\cdot))$ *to* Eq. (1.1) *in* $M^2(0,1;\mathbb{R}^d)\times M^2(0,1;\mathbb{R}^{d\times q})$.

As an immediate application we obtain the following corollary, which is the main result of Pardoux and Peng (1990).

Corollary 2.2. *Let* (H1) *hold. Assume there exists a* $c > 0$ *such that*

$$|f(t,x,y) - f(t,\bar{x},\bar{y})|^2 \leqslant c(|x-\bar{x}|^2 + |y-\bar{y}|^2) \quad \text{a.s.},$$

$$|g(t,x) - g(t,\bar{x})|^2 \leqslant c|x-\bar{x}|^2 \quad \text{a.s.}$$

for all $x, \bar{x} \in \mathbb{R}^d$, $y, \bar{y} \in \mathbb{R}^{d\times q}$ and $0 \leqslant t \leqslant 1$. *Then there exists a unique solution* $(x(\cdot), y(\cdot))$ *to* Eq. (1.1) *in* $M^2(0,1;\mathbb{R}^d) \times M^2(0,1;\mathbb{R}^{d\times q})$.

3. Lemmas

In order to prove the main result we need to prepare a number of lemmas. We first introduce a lemma due to Pardoux and Peng (1990).

Lemma 3.1 (Pardoux and Peng, 1990). *Let* $g(\cdot) \in M^2(0,1;\mathbb{R}^{d\times q})$ *and* $f: \Omega \times [0,1] \times \mathbb{R}^{d\times q} \to \mathbb{R}^d$ *be a mapping such that* $f(\cdot,0) \in M^2(0,1;\mathbb{R}^d)$ *and*

$$|f(t,y) - f(t,\bar{y})|^2 \leqslant c|y-\bar{y}|^2, \quad y, \bar{y} \in \mathbb{R}^{d\times q},$$

for some $c > 0$. *Then there exists a unique solution* $(x(\cdot), y(\cdot))$ *in* $M^2(0,1;\mathbb{R}^d) \times M^2(0,1;\mathbb{R}^{d\times q})$ *to the following backward stochastic differential equation*:

$$x(t) + \int_t^1 f(s, y(s))\,\mathrm{d}s + \int_t^1 [g(s) + y(s)]\,\mathrm{d}w(s) = X, \quad 0 \leqslant t \leqslant 1.$$

We now construct an approximate sequence using a Picard-type iteration with the help of Lemma 3.1. Let $x_0(t) \equiv 0$, and let $\{x_n(t), y_n(t): 0 \leqslant t \leqslant 1\}_{n\geqslant 1}$ be a sequence in $M^2(0,1;\mathbb{R}^d) \times M^2(0,1;\mathbb{R}^{d\times q})$ defined recursively by

$$x_n(t) + \int_t^1 f(s, x_{n-1}(s), y_n(s))\,\mathrm{d}s + \int_t^1 [g(s, x_{n-1}(s)) + y_n(s)]\,\mathrm{d}w(s) = X \tag{3.1}$$

on $0 \leqslant t \leqslant 1$.

Lemma 3.2. *Under hypotheses* (H1) *and* (H2), *for all* $0 \leqslant t \leqslant 1$ *and* $n \geqslant 1$,

$$E|x_n(t)|^2 \leqslant C_1 \quad \text{and} \quad E\int_0^1 |y_n(s)|^2\,\mathrm{d}s \leqslant C_2, \tag{3.2}$$

where C_1 *and* C_2 *are both positive constants.*

Proof. Applying Itô's formula to $|x_n(t)|^2$ one can derive that

$$E|x_n(t)|^2 + E\int_t^1 |y_n(s)|^2\,\mathrm{d}s$$

$$= E|X|^2 - 2E\int_t^1 (x_n(s), f(s, x_{n-1}(s), y_n(s)))\,ds$$

$$- E\int_t^1 (|g(s, x_{n-1}(s))|^2 + 2\,\mathrm{trace}[g(s, x_{n-1}(s))y_n(s)^T])\,ds.$$

Using the elementary inequality $2|uv| \leqslant u^2/\alpha + \alpha v^2$ for any $\alpha > 0$, one sees

$$E|x_n(t)|^2 + E\int_t^1 |y_n(s)|^2\,ds$$

$$= E|X|^2 + \alpha^{-1}E\int_t^1 |x_n(s)|^2\,ds + \alpha E\int_t^1 |f(s, x_{n-1}(s), y_n(s))|^2\,ds$$

$$+ \alpha^{-1}E\int_t^1 |g(s, x_{n-1}(s))|^2\,ds + \alpha E\int_t^1 |y_n(s)|^2\,ds. \tag{3.3}$$

But by hypotheses (H1), (H2) and (2.2) one can easily show that

$$|f(s, x_{n-1}(s), y_n(s))|^2 \leqslant 2|f(s,0,0)|^2 + 2a + 2b|x_{n-1}(s)|^2 + 2c|y_n(s)|^2$$

and

$$|g(s, x_{n-1}(s))|^2 \leqslant 2|g(s,0,0)|^2 + 2a + 2b|x_{n-1}(s)|^2.$$

Substituting these into (3.3) gives

$$E|x_n(t)|^2 + E\int_t^1 |y_n(s)|^2\,ds$$

$$\leqslant C_3(\alpha) + \alpha^{-1}\int_t^1 E|x_n(s)|^2\,ds + 2b(\alpha + \alpha^{-1})\int_t^1 E|x_{n-1}(s)|^2\,ds$$

$$+ \alpha(2c+1)E\int_t^1 |y_n(s)|^2\,ds,$$

where

$$C_3(\alpha) = E|X|^2 + 2a(\alpha + \alpha^{-1}) + 2\alpha E\int_0^1 |f(s,0,0)|^2\,ds + 2\alpha^{-1}E\int_0^1 |g(s,0,0)|^2\,ds.$$

In particular, choosing $\alpha = 1/2(2c+1)$ and setting $\beta = (2(2c+1))^{-1} + 2(2c+1)]$ we get

$$E|x_n(t)|^2 + \tfrac{1}{2}E\int_t^1 |y_n(s)|^2\,ds$$

$$\leqslant C_4 + 2(2c+1)\int_t^1 E|x_n(s)|^2\,ds + 2b\beta\int_t^1 E|x_{n-1}(s)|^2\,ds$$

$$\leqslant C_4 + C_5 \int_t^1 \left(\sup_{1 \leqslant k \leqslant n} E|x_k(s)|^2 \right) ds, \tag{3.4}$$

where $C_4 = C_3(1/2(2c+1))$ and $C_5 = 2(2c+1) + 2b\beta$. Now let m be any integer. If $1 \leqslant n \leqslant m$, (3.4) gives (recalling $x_0(t) \equiv 0$)

$$E|x_n(t)|^2 \leqslant C_4 + C_5 \int_t^1 \left(\sup_{1 \leqslant k \leqslant m} E|x_k(s)|^2 \right) ds.$$

Therefore

$$\left(\sup_{1 \leqslant k \leqslant m} E|x_k(t)|^2 \right) \leqslant C_4 + C_5 \int_t^1 \left(\sup_{1 \leqslant k \leqslant m} E|x_k(s)|^2 \right) ds.$$

An application of the well-known Gronwall inequality implies

$$\left(\sup_{1 \leqslant k \leqslant m} E|x_k(t)|^2 \right) \leqslant C_4 e^{C_5(1-t)} \leqslant C_4 e^{C_5}.$$

Since m is arbitrary, the first inequality of (3.2) follows by setting $C_1 = C_4 e^{C_5}$. Finally it follows from (3.4) that

$$E \int_0^1 |y_n(s)|^2 \, ds \leqslant 2(C_4 + C_5 C_1) := C_2.$$

The proof is complete. □

Lemma 3.3. *Under hypotheses* (H1) *and (H2), there exists a constant* $C_6 > 0$ *such that*

$$E|x_{n+m}(t) - x_n(t)|^2 \leqslant C_6 \int_t^1 \kappa(E|x_{n+m-1}(s) - x_{n-1}(s)|^2) \, ds \tag{3.5}$$

for all $0 \leqslant t \leqslant 1$ *and* $n, m \geqslant 1$.

Proof. Applying Itô's formula to $|x_{n+m}(t) - x_n(t)|^2$ we have

$$\begin{aligned} &-E|x_{n+m}(t) - x_n(t)|^2 \\ &\quad = 2E \int_t^1 (x_{n+m}(s) - x_n(s), f(s, x_{n+m-1}(s), y_{n+m}(s)) - f(s, x_{n-1}(s), y_n(s))) \, ds \\ &\qquad + E \int_t^1 |g(s, x_{n+m-1}(s)) + y_{n+m}(s) - g(s, x_{n-1}(s)) - y_n(s)|^2 \, ds. \end{aligned} \tag{3.6}$$

By Jensen's inequality we can then deduce in the same way as the proof of Lemma 3.2 that

$$\begin{aligned} &E|x_{n+m}(t) - x_n(t)|^2 + \tfrac{1}{2} E \int_t^1 |y_{n+m}(s) - y_n(s)|^2 \, ds \\ &\quad \leqslant 2(2c+1) \int_t^1 E|x_{n+m}(s) - x_n(s)|^2 \, ds + 2\beta \int_t^1 \kappa(E|x_{n+m-1}(s) - x_{n-1}(s)|^2) \, ds. \end{aligned} \tag{3.7}$$

Now fix $t \in [0, T]$ arbitrarily. If $t \leqslant r \leqslant T$, then

$$E|x_{n+m}(r) - x_n(r)|^2 \leqslant 2(2c+1)\int_r^1 E|x_{n+m}(s) - x_n(s)|^2\,ds + 2\beta\int_t^1 \kappa(E|x_{n+m-1}(s) - x_{n-1}(s)|^2)\,ds.$$

In view of the Gronwall inequality we see that

$$E|x_{n+m}(t) - x_n(t)|^2 \leqslant 2\beta e^{2(2c+1)(1-t)}\int_t^1 \kappa(E|x_{n+m-1}(s) - x_{n-1}(s)|^2)\,ds.$$

So the required (3.6) follows by setting $C_6 = 2\beta e^{2(2c+1)}$. The proof is complete. □

Lemma 3.4. *Under hypotheses* (H1) *and* (H2), *there exists a constant* $C_7 > 0$ *such that*

$$E|x_{n+m}(t) - x_n(t)|^2 \leqslant C_7(1-t) \quad \textit{for all } 0 \leqslant t \leqslant 1 \textit{ and } n, m \geqslant 1.$$

Proof. By Lemmas 3.3 and 3.2,

$$E|x_{n+m}(t) - x_n(t)|^2 \leqslant C_6\int_t^1 \kappa(4C_1)\,ds = C_6\kappa(4C_1)(1-t)$$

and the conclusion follows by letting $C_7 = C_6\kappa(4C_1)$. The proof is complete. □

We now start to prepare a key lemma. To do so, let us introduce some new notations. Choose $T_1 \in [0, 1)$ such that

$$\bar{\kappa}(C_7(1-t)) \leqslant C_7 \quad \text{for all } T_1 \leqslant t \leqslant 1, \tag{3.8}$$

where $\bar{\kappa}(u) = C_6\kappa(u)$. Fix $m \geqslant 1$ arbitrarily and define two sequences of functions $\{\varphi_n(t)\}_{n\geqslant 1}$ and $\{\tilde{\varphi}_{n,m}(t)\}_{n\geqslant 1}$ as follows:

$$\varphi_1(t) = C_7(1-t),$$

$$\varphi_{n+1}(t) = \int_t^1 \bar{\kappa}(\varphi_n(s))\,ds, \quad n = 1, 2, \ldots,$$

$$\tilde{\varphi}_{n,m}(t) = E|x_{n+m}(t) - x_n(t)|^2, \quad n = 1, 2, \ldots,$$

Lemma 3.5. *Under hypotheses* (H1) and (H2), *for any* $m \geqslant 1$ *and all* $n \geqslant 1$,

$$0 \leqslant \tilde{\varphi}_{n,m}(t) \leqslant \varphi_n(t) \leqslant \varphi_{n-1}(t) \leqslant \cdots \leqslant \varphi_1(t) \quad \text{if } t \in [T_1, 1]. \tag{3.9}$$

Moreover, the value $1 - T_1$ *depends only on the function* κ *and not on the final value* X.

Proof. First of all, by Lemma 3.4,

$$\tilde{\varphi}_{1,m}(t) = E|x_{1+m}(t) - x_1(t)|^2 \leqslant C_7(1-t) = \varphi_1(t).$$

Now by Lemma 3.3,

$$\tilde{\varphi}_{2,m}(t) = E|x_{2+m}(t) - x_2(t)|^2 \leqslant C_6 \int_t^1 \kappa(E|x_{1+m}(s) - x_1(s)|^2)\,\mathrm{d}s$$

$$= \int_t^1 \bar{\kappa}(\tilde{\varphi}_{1,m}(s))\,\mathrm{d}s \leqslant \int_t^1 \bar{\kappa}(\varphi_1(s))\,\mathrm{d}s = \varphi_2(t).$$

But by (3.8) we also have

$$\varphi_2(t) = \int_t^1 \bar{\kappa}(C_7(1-s))\,\mathrm{d}s \leqslant \int_t^1 C_7\,\mathrm{d}s = C_7(1-t) = \varphi_1(t).$$

In other words, we have already showed that

$$\tilde{\varphi}_{2,m}(t) \leqslant \varphi_2(t) \leqslant \varphi_1(t) \quad \text{if } t \in [T_1, 1].$$

We next assume that (3.9) holds for some $n \geqslant 2$. Then by Lemma 3.3 again,

$$\tilde{\varphi}_{n+1,m}(t) \leqslant \int_t^1 \bar{\kappa}(\tilde{\varphi}_{n,m}(s))\,\mathrm{d}s \leqslant \int_t^1 \bar{\kappa}(\varphi_n(s))\,\mathrm{d}s = \varphi_{n+1}(t)$$

$$\leqslant \int_t^1 \bar{\kappa}(\varphi_{n-1}(s))\,\mathrm{d}s = \varphi_n(t),$$

that is, (3.9) holds for $n+1$ as well. So, by induction, (3.9) must hold for all $n \geqslant 1$.

To show the fact that the value $1 - T_1$ depends only on the function κ and not on the final value X, note that (3.8) holds if

$$C_6\kappa(C_7(1-T_1)) \leqslant C_6\kappa(4C_1) \quad \text{or} \quad C_7(1-T_1) = C_6\kappa(4C_1)(1-T_1) \leqslant 4C_1.$$

But, by (2.2), this holds if

$$C_6(a + 4bC_1)(1-T_1) \leqslant 4C_1$$

and so if

$$C_6(1+4b)(1-T_1) \leqslant 4$$

since $C_1 \geqslant a$. In other words, if we choose $1 - T_1 = 4/[C_6(1+4b)] < 1$, then (3.8) holds. Recalling the definition of C_6, one sees clearly that $1 - T_1$ depends only on the function κ and not on the final value X. The proof is complete. □

Furthermore let us now state the Bihari inequality (cf. Bihari, (1956); Mao, 1991) which will be a key tool in the proof of Theorem 2.1.

Lemma 3.6 (Bihari's inequality). *Let $T > 0$ and $u_0 \geqslant 0$. Let $u(t), v(t)$ be continuous functions on $[0, T]$. Let $H: \mathbb{R}_+ \to \mathbb{R}_+$ be continuous and nondecreasing such that $H(r) > 0$ for $r > 0$. If*

$$u(t) \leqslant u_0 + \int_0^t v(s)H(u(s))\,\mathrm{d}s \quad \textit{for all } 0 \leqslant t \leqslant T,$$

then

$$u(t) \leqslant G^{-1}\left(G(u_0) + \int_0^t v(s)\,\mathrm{d}s\right)$$

for all such $t \in [0, T]$ *that*

$$G(u_0) + \int_0^t v(s)\,\mathrm{d}s \in \mathrm{Dom}(G^{-1}),$$

where

$$G(r) = \int_1^r \frac{\mathrm{d}s}{H(s)} \quad \textit{for } r \geqslant 0$$

and G^{-1} *is the inverse function of* G. *In particular, if, moreover,* $u_0 = 0$ *and*

$$\int_{0+} \frac{\mathrm{d}s}{H(s)} = \infty$$

then $u(t) = 0$ *for all* $0 \leqslant t \leqslant T$.

At last we can now start to prove our main result, Theorem 2.1.

4. Proof of Theorem 2.1

Existence: We claim that

$$\sup_{T_1 \leqslant t \leqslant 1} E|x_n(t) - x_k(t)|^2 \to 0 \quad \text{as } n, k \to \infty. \tag{4.1}$$

Note that φ_n is continuous on $[T_1, 1]$ due to an obvious uniform bound on $\mathrm{d}\varphi_n(t)/\mathrm{d}t$. Note also that for each $n \geqslant 1$, $\varphi_n(t)$ is decreasing on $[T_1, 1]$, and for each t, $\varphi_n(t)$ is nonincreasing monotonically as $n \to \infty$. Therefore we can define the function $\varphi(t)$ by $\varphi_n(t) \downarrow \varphi(t)$. It is easy to verify that $\varphi(t)$ is continuous and nonincreasing on $[T_1, 1]$. By the definition of $\varphi_n(t)$ and $\varphi(t)$ we get

$$\varphi(t) = \lim_{n\to\infty} \varphi_{n+1}(t) = \lim_{n\to\infty} \int_t^1 \bar{\kappa}(\varphi_n(s))\,\mathrm{d}s = \int_t^1 \bar{\kappa}(\varphi(s))\,\mathrm{d}s, \quad t \in [T_1, 1].$$

Since

$$\int_{0+} \frac{\mathrm{d}u}{\bar{\kappa}(u)} = \infty,$$

Bihari's inequality implies that $\varphi(t) \equiv 0$ on $t \in [T_1, 1]$. Consequently, $\varphi_n(T_1) \downarrow 0$ as $n \to \infty$. So for any $\varepsilon > 0$, one can find an integer $N \geqslant 1$ such that $\varphi_n(T_1) < \varepsilon$ whenever $n \geqslant N$. Now for any $m \geqslant 1$ and $n \geqslant N$, by Lemma 3.5,

$$\sup_{T_1 \leqslant t \leqslant 1} E|x_{n+m}(t) - x_n(t)|^2 = \sup_{T_1 \leqslant t \leqslant 1} \tilde{\varphi}_{n,m}(t) \leqslant \sup_{T_1 \leqslant t \leqslant 1} \varphi_n(t) = \varphi_n(T_1) < \varepsilon.$$

So (4.1) must hold. Applying (4.1) to (3.7) we see immediately that $\{x_n(\cdot)\}$ is a Cauchy sequence in $M^2(T_1, 1; \mathbb{R}^d)$ (which can be defined similarly as $M^2(0,1;\mathbb{R}^d)$) and $\{y_n(\cdot)\}$ is a Cauchy sequence in $M^2(T_1, 1; \mathbb{R}^{d\times q})$. Define their limits by $x(\cdot)$ and $y(\cdot)$ respectively. Now letting $n\to\infty$ in (3.1) we obtain

$$x(t) + \int_t^1 f(s, x(s), y(s))\,ds + \int_t^1 [g(s, x(s)) + y(s)]\,dw(s) = X$$

on $T_1 \leqslant t \leqslant 1$. In other words, we have showed the existence of the solution on $[T_1, 1]$. Note from Lemma 3.5 that the value $1 - T_1$ depends only on the function κ and not on the final value X. Hence one can deduce by iteration the existence on $[1 - k(1 - T_1), 1]$, for each k, and therefore the existence on the whole $[0, 1]$. The existence has been proved.

Uniqueness: To show the uniqueness, let both $(x(\cdot), y(\cdot))$ and $(\bar{x}(\cdot), \bar{y}(\cdot))$ be the solutions of Eq. (1.1). Then, in the same way as the proof of Lemma 3.2 one can show that

$$\begin{aligned} &E|x(t) - \bar{x}(t)|^2 + \tfrac{1}{2}E\int_t^1 |y(s) - \bar{y}(s)|^2\,ds \\ &\quad\leqslant 2[2(2c+1) + (2(2c+1))^{-1}]\int_t^1 [E|x(s) - \bar{x}(s)|^2 + \kappa(E|x(s) - \bar{x}(s)|^2)]\,ds \end{aligned} \tag{4.2}$$

for $0 \leqslant t \leqslant 1$. Since $\kappa(\cdot)$ is a concave function and $\kappa(0) = 0$, we have

$$\kappa(u) \geqslant \kappa(1)u \quad \text{for } 0 \leqslant u \leqslant 1.$$

So

$$\int_{0+} \frac{du}{u + \kappa(u)} \geqslant \frac{\kappa(1)}{\kappa(1) + 1}\int_{0+} \frac{du}{\kappa(u)} = \infty.$$

Therefore one can apply the Bihari inequality to (4.2) to obtain

$$E|x(t) - \bar{x}(t)|^2 = 0 \quad \text{for all } 0 \leqslant t \leqslant 1.$$

So $x(t) = \bar{x}(t)$ for all $0 \leqslant t \leqslant 1$ almost surely. It then follows from (4.2) that $y(t) = \bar{y}(t)$ for all $0 \leqslant t \leqslant 1$ almost surely as well. The uniqueness has been proved and the proof of the theorem is then complete.

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