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The extremal function for partial bipartite tilings

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ABSTRACT

For a fixed bipartite graph H and given $\alpha \in (0, 1)$, we determine the threshold $T_H(\alpha)$ which guarantees that any n-vertex graph with at least $T_H(\alpha)\binom{n}{2}$ edges contains $(1 - o(1))\frac{\alpha}{v(H)}n$ vertex-disjoint copies of H. In the proof, we use a variant of a technique developed by Komlós [J. Komlós, Tiling Turán theorems, Combinatorica 20 (2) (2000) 203–218].

1. Introduction

The Turán theorem [12], one of the most important results in extremal graph theory, gives a sharp threshold, denoted by $\mathrm{ex}(n, K_r)$, for the maximum number of edges of an n-vertex graph with no copy of K_r. Even though the Turán theorem applies to any pair of values n and r, the interesting instances are rather those when n is large compared to r. Erdős and Stone [2] extended the result by determining the asymptotic behavior of the function $\mathrm{ex}(n, H)$ for a fixed non-bipartite graph H. The same problem in the case that H is a fixed bipartite graph is – despite considerable effort – wide open for most graphs H. This is known as the *Zarankiewicz problem*. Let us recall that when H has color classes of sizes s and t, $s \le t$, then the Kövari–Sós–Turán theorem [8] asserts that

$$\mathrm{ex}(n, H) \le O(n^{2-1/s}) = o(n^2). \quad (1)$$

On the other hand, a standard random graph argument gives that $\mathrm{ex}(n, K_{s,t}) \ge \Omega(n^{2-(s+t-2)/(st-1)})$.

It is natural to extend the above *existential questions* to *tiling questions*. In such a setting, one asks for the maximum number of edges of an n-vertex graph which does not contain ℓ vertex-disjoint copies of a graph H. This quantity is denoted by $\mathrm{ex}(n, \ell \times H)$. Erdős and Gallai [3] gave a complete solution to the problem in the case when $H = K_2$.

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Theorem 1 (*Erdős–Gallai [3]*). *Suppose that $\ell \le n/2$. Then*

$$\mathrm{ex}(n, \ell \times K_2) = \max\left\{(\ell-1)(n-\ell+1) + \binom{\ell-1}{2}, \binom{2\ell-1}{2}\right\}.$$

Given $n, x \in \mathbb{N}, x \le n$, we define two graphs $M_{n,x}$ and $L_{n,x}$ as follows. The graph $M_{n,x}$ is an n-vertex graph whose vertex set is split into sets A and B, $|A| = x$, $|B| = n - x$, A induces a clique, B induces an independent set, and $M_{n,x}[A, B] \simeq K_{x,n-x}$. The graph $L_{n,x}$ is the complement of $M_{n,n-x}$, i.e., it is an n-vertex graph whose edges induce a clique of order x. Obviously, $e(M_{n,\ell-1}) = (\ell-1)(n-\ell+1)+\binom{\ell-1}{2}$, and $e(L_{n,2\ell-1}) = \binom{2\ell-1}{2}$. Moreover, it is easy to check that there are no ℓ vertex-disjoint edges in either of the graphs $M_{n,\ell-1}, L_{n,2\ell-1}$. Therefore, when $\ell < \frac{2}{5}n + O(1)$, the graph $M_{n,\ell-1}$ is (the unique) graph showing that $\mathrm{ex}(n, \ell \times K_2) \ge (\ell-1)(n-\ell+1) + \binom{\ell-1}{2}$. The graph $L_{n,2\ell-1}$ is the unique extremal graph for the problem otherwise.

Moon [10] started the investigation of $\mathrm{ex}(n, \ell \times K_r)$. Allen et al. [1] only recently determined the behavior of $\mathrm{ex}(n, \ell \times K_r)$ for the whole range of ℓ in the case $r = 3$, and they made a substantial progress for larger values of r. Simonovits [11] determined the value $\mathrm{ex}(n, \ell \times H)$ for a non-bipartite graph H, fixed value of ℓ and large n.

An equally important density parameter which can be considered in the context of tiling questions is the minimum degree of the host graph. That is, we ask what is the largest possible minimum degree of an n-vertex graph which does not contain ℓ vertex-disjoint copies of H. In the case $H = K_r$, the precise answer is given by the Hajnal–Szemerédi theorem[1] [4]. An asymptotic threshold for a general fixed graph H was determined by Komlós [5]. In this case, the threshold depends on a parameter which Komlós calls the *critical chromatic number*. The critical chromatic number of H is a real between $\chi(H) - 1$ and $\chi(H)$, defined by

$$\chi_{\mathrm{cr}}(H) := \frac{(\chi(H)-1)v(H)}{v(H)-\sigma}, \tag{2}$$

where σ is the size of the smallest possible color-class in any $\chi(H)$-coloring of H. Let us also note that Komlós' result [5] gives an asymptotic min-degree threshold even in the case when H is bipartite. In this case the near-extremal graphs for the problem are complete bipartite graphs.

In the present paper we use a variation of the technique developed by Komlós to determine the asymptotic behavior of the function $\mathrm{ex}(n, \ell \times H)$ for a fixed bipartite graph H. Let H be an arbitrary bipartite graph. Suppose that $b: V(H) \to [2]$ is a proper coloring of H which minimizes $|b^{-1}(1)|$. We define quantities $s(H) := |b^{-1}(1)|, t(H) := |b^{-1}(2)|$. Obviously, $s(H) \le t(H)$, and $s(H)+t(H) = v(H)$. Furthermore, we define $V_1(H) := b^{-1}(1)$ and $V_2(H) := b^{-1}(2)$. The sets $V_1(H)$ and $V_2(H)$ are uniquely defined provided that H does not contain a balanced bipartite graph as one of its components; in this other case we fix a coloring b satisfying the above conditions and use it to define uniquely $V_1(H)$ and $V_2(H)$.

Given $s, t \in \mathbb{N}$, we define a function $T_{s,t}: (0, 1) \to (0, 1)$ by setting

$$T_{s,t}(\alpha) := \max\left\{\frac{2s\alpha}{s+t}\left(1 - \frac{s\alpha}{2(s+t)}\right), \alpha^2\right\}, \tag{3}$$

for $\alpha \in (0, 1)$. Note that $T_{s',t'} = T_{s,t}$ when $s' = ks$ and $t' = kt$. Also, note that

$$T_{s,s}(\alpha)\binom{n}{2} = \mathrm{ex}\left(n, \frac{\alpha n}{2} \times K_2\right) + o(n^2), \tag{4}$$

[1] In its original formulation, the Hajnal–Szemerédi theorem asserts that an n-vertex graph G with minimum-degree at least $\frac{r-1}{r}n$ contains a K_r-tiling missing at most $r - 1$ vertices of G, thus giving an answer only to the question of almost perfect tilings. When the minimum-degree of G is lower, we can however add auxiliary vertices which are complete to G and obtain an n'-vertex graph G' such that the Hajnal–Szemerédi theorem applies to G'. The restriction of the almost perfect K_r-tiling of G' to G gives a K_r-tiling which is optimal in the worst case.

and, in general, for $s \le t$, the number $T_{s,t}(\alpha)\binom{n}{2}$ is asymptotically the maximum between the number of edges of $M_{n,\frac{\alpha s}{s+t}n}$ and $L_{n,\alpha n}$.

Our main result is the following.

Theorem 2. *Suppose that H is a bipartite graph with no isolated vertices, $s := s(H), t := t(H)$. Let $\alpha \in (0, 1)$ and $\varepsilon > 0$. Then there exists an $n_0 = n_0(s, t, \alpha, \varepsilon)$ such that for any $n \ge n_0$, any graph G with n vertices and at least $T_{s,t}(\alpha)\binom{n}{2}$ edges contains more than $(1-\varepsilon)\frac{\alpha}{s+t}n$ vertex-disjoint copies of the graph H.*

After some remarks below, we introduce the tools needed for our proof of Theorem 2 in Section 2. The proof is then given in Section 3.

(1) Let H, s and t be as in the hypothesis of the theorem, $\varepsilon' > 0$ and $\beta \in (0, 1)$. Then we may find an $\alpha > \beta(s+t)$ and an $\varepsilon < \varepsilon'$ sufficiently small, such that for n large enough, by Theorem 2, any graph G with n vertices and at least $T_{s,t}(\alpha)\binom{n}{2} < (T_{s,t}(\beta(s+t)) + \varepsilon')\binom{n}{2}$ edges contains at least βn vertex-disjoint copies of H. Hence $\mathrm{ex}(n, \beta n \times H) \le T_{s,t}(\beta(s+t))\binom{n}{2} + \varepsilon' n^2$. This asymptotically matches the lower bound which comes – as in Theorem 1 – from graphs $M_{n,\beta sn-1}$ and $L_{n,\beta(s+t)n-1}$. Indeed, neither of these graphs contains βn vertex-disjoint copies of H, as any such copy would require at least s vertices in the clique subgraph of $M_{n,\beta sn-1}$, and at least $s + t = v(H)$ non-isolated vertices in $L_{n,\beta(s+t)n-1}$, respectively. Note however that for most values of H, the graphs $M_{n,\beta sn-1}$ and $L_{n,\beta(s+t)n-1}$ are not extremal for the problem. For example, we can replace the independent set in the graph $L_{n,\beta(s+t)n-1}$ by any H-free graph. This links us to the Zarankiewicz problem, and suggests that an exact result is not within the reach of current techniques.

(2) Note that Theorem 2 could be restated using the notion of critical chromatic number. Indeed, formula Eq. (2) simplifies in the bipartite setting to

$$\chi_{\mathrm{cr}}(H) = \frac{v(H)}{t(H)}.$$

In other words, the value $T_{s(H),t(H)}(\alpha)$ defined by Eq. (3) can be determined from $\chi_{\mathrm{cr}}(H)$, without knowing $s(H)$ and $t(H)$.

(3) The assumption on H to contain no isolated vertices in Theorem 2 is made just for the sake of compactness of the statement. Indeed, let H' be obtained from H by removing all the isolated vertices. Then there is a simple relation between the sizes of optimal coverings by vertex disjoint copies of H and H' in an n-vertex graph G. Let x and x' be the number of vertices covered by a maximum family of vertex-disjoint copies of H and H' in G, respectively. We have that

$$x = \min\left\{ v(H)\left\lfloor \frac{n}{v(H)} \right\rfloor, \frac{x' v(H)}{v(H')} \right\}.$$

(4) One can attempt to obtain an analogue of Theorem 2 for graphs with higher chromatic number. This however appears to be substantially more difficult. To indicate the difficulty, let us recall that there are two types ($M_{n,x}$ and $L_{n,x}$) of extremal graphs for the H-tiling problem for bipartite H. The graphs $M_{n,x}$ and $L_{n,x}$ have a *block structure*, i.e., their vertex set can be partitioned into *blocks* (two, in this case), such that any two vertices from the same block have almost the same neighborhoods. These two graphs appear even in the simplest case of $H = K_2$ (cf. Theorem 1). However, when H is not balanced, if we let α go from 0 to 1, the transition between the two extremal structures which determine the threshold function occurs at a different time in the evolution. On the other hand, there are five types of extremal graphs for the problem of determining $\mathrm{ex}(n, \ell \times K_3)$ as shown in [1]. All the five types have a block structure. It is plausible that when H is a general 3-colorable graph, the same five types of extremal graphs determine the threshold function for H-tilings. However, the transitions between them occur at different times and the block sizes depend on various structural properties of H. In particular, we have indications that the critical chromatic number alone does not determine $\mathrm{ex}(n, \alpha n \times H)$ in this situation.

If $\mathcal{F}$ is a family of graphs, and G is a graph, an *$\mathcal{F}$-tiling* in G is a set of vertex-disjoint subgraphs of G, each of them isomorphic to a graph in $\mathcal{F}$. If $\mathcal{F} = \{H\}$ then we simply say H-tiling. $V(F)$ denotes the vertices of G covered by an $\mathcal{F}$-tiling F, and $|F| = |V(F)|$ is the *size* of the tiling F. If F is a collection of bipartite graphs, we let $V_1(F) = \bigcup_{H\in F} V_1(H)$ and $V_2(F) = \bigcup_{H\in F} V_2(H)$. For $n \in \mathbb{N}$, we write $[n]$ to denote the set $\{1, 2, \ldots, n\}$.

2. Tools for the proof of the main result

Our main tool is Szemerédi's Regularity Lemma (see [7,9] for surveys). To state it we need some more notation.

Let $G = (V, E)$ be an n-vertex graph. If A, B are disjoint nonempty subsets of $V(G)$, the *density* of the pair (A, B) is $d(A, B) = e(A, B)/(|A| \| B|)$. We say that (A, B) is an *ε-regular pair* if $|d(X, Y) - d(A, B)| < \varepsilon$ for every $X \subset A$, $|X| > \varepsilon|A|$ and $Y \subset B$, $|Y| > \varepsilon|B|$.

The following statement asserts that large subgraphs of regular pairs are also regular.

Lemma 3. *Let (A, B) be an ε-regular pair with density d, and let $A' \subset A$, $|A'| \geq \alpha|A|$, $B' \subset B$, $|B'| \geq \alpha|B|$, $\alpha \geq \varepsilon$. Then (A', B') is an ε'-regular pair with $\varepsilon' = \max\{\varepsilon/\alpha, 2\varepsilon\}$, and for its density d' we have $|d' - d| < \varepsilon$.*

Let $\varepsilon > 0$ and $d \in [0, 1]$. An *(ε, d)-regular partition* of G with *reduced graph* $R = (V', E')$ is a partition $V_0 \dot\cup V_1 \dot\cup \cdots \dot\cup V_k$ of V with $|V_0| \leq \varepsilon n$, $|V_i| = |V_j|$ for any $1 \leq i < j \leq k$, $V(R) = \{V_1, V_2, \ldots, V_k\}$, such that (V_i, V_j) is an ε-regular pair in G of density greater than d whenever $V_iV_j \in E(R)$, and the subgraph $G' \subset G$ induced by the ε-regular pairs corresponding to the edges of R has more than $e(G) - (d + 3\varepsilon)n^2/2$ edges. In this case, we also say that G has an (ε, d)-reduced graph R, and call the sets V_i, $1 \leq i \leq k$, the *clusters* of G.

The following lemma is a consequence of the so-called degree version of the Regularity Lemma [7, Theorem 1.10].

Lemma 4 (*Regularity Lemma*)**.** *For every $\varepsilon > 0$ and $m \in \mathbb{N}$ there is an $M = M(\varepsilon, m)$ such that, if G is any graph with more than M vertices and $d \in [0, 1]$ is any real number, then G has an (ε, d)-reduced graph R on k vertices, with $m \leq k \leq M$.*

Given four positive numbers a, b, x, y we say that the pair a, b *dominates* the pair x, y, if $\max\{x, y\}/\min\{x, y\} \geq \max\{a, b\}/\min\{a, b\}$. The following easy lemma states that $K_{a,b}$ has an almost perfect $K_{s,t}$-tiling provided that a, b dominates s, t.

Lemma 5. *For any $s, t \in \mathbb{N}$ there exists a constant C such that the following holds. Suppose that the pair $a, b \in \mathbb{N}$ dominates s, t. Then the graph $K_{a,b}$ contains a $K_{s,t}$-tiling containing all but at most C vertices of $K_{a,b}$.*

Proof. If $s = t$ then necessarily $a = b$. There obviously exists a $K_{s,t}$-tiling containing all but at most $C := 2(s - 1)$ vertices of $K_{a,b}$.

With no loss of generality, we may suppose that $a \leq b$ and $s < t$. Then $as \leq bt$ and $bs \leq at$. A tiling with $\lfloor (bt - as)/(t^2 - s^2) \rfloor$ copies of $K_{s,t}$ with the s-part of the $K_{s,t}$ placed in the a-part of the $K_{a,b}$ and $\lfloor (at - bs)/(t^2 - s^2) \rfloor$ copies placed the other way misses at most $C := 2(s + t - 1)$ vertices of $K_{a,b}$. □

The next lemmas, versions of the Blow-up Lemma [6], assert that regular pairs have almost as good tiling properties as complete bipartite graphs.

Lemma 6. *For every $d > 0$, $\gamma \in (0, 1)$ and any two graphs R and H, there is an $\varepsilon = \varepsilon(H, d, \gamma) > 0$ such that the following holds for all positive integers s. Let R_s be the graph obtained from R by replacing every vertex of R by s vertices, and every edge of R by a complete bipartite graph between the corresponding s-sets. Let G be any graph obtained similarly from R by replacing every vertex of R by s vertices, and every edge of R with an ε-regular pair of density at least d. If R_s contains an H-tiling of size at least $\gamma v(R_s)$ then so does G.*

Lemma 7. *For every bipartite graph H and every $\gamma, d > 0$ there exists an $\varepsilon = \varepsilon(H, d, \gamma) > 0$ such that the following holds. Suppose that there is an H-tiling in $K_{a,b}$ of size x. Let (A, B) be an arbitrary ε-regular pair with density at least d, $|A| = a$, $|B| = b$. Then the pair (A, B) contains an H-tiling of size at least $x - \gamma(a + b)$.*

Finally, let us state a straightforward corollary of the König matching theorem.

Fact 8. *Let $G = (A \dot\cup B, E)$ be a bipartite graph with color classes A and B. If G has no matching with $l+1$ edges, then $e(G) \le l \max\{|A|, |B|\}$.*

3. The proof

In this section, we first state and prove the main technical result, Lemma 9. Then, we show how it implies Theorem 2.

For $s, t \in \mathbb{N}$, we set $\mathcal{F}_1 := \{K_{s,t}, K_{s,t-1}, K_2\}$ and $\mathcal{F}_2 := \{K_{st,t^2}, K_{st-1,(t-1)t}, K_{st,(t-1)t}, K_2\}$. Let us note that when $s < t$, the sizes of the two color classes of any graph from $\mathcal{F}^* := \mathcal{F}_1 \cup \mathcal{F}_2$ dominate s and t.

Let F be a $K_{s,t}$-tiling in a graph G, $s < t$. Suppose E_0 and E_1 are matchings in $G[V(G) - V(F), V_1(F)]$ and $G[V_2(F)]$, respectively, such that each copy K of $K_{s,t}$ in F has at most one vertex matched by E_0 and at most one vertex matched by E_1. If any $K \in F$ which has a vertex matched by E_0, also has a vertex matched by E_1, then we call the pair (E_0, E_1) an *F-augmentation*. Note that in this case E_0 and E_1 are vertex disjoint, as $V_1(F) \cap V_2(F) = \emptyset$.

The main step in our proof of Theorem 2 is the following lemma.

Lemma 9. *Let $t > s \ge 1, \alpha \in (0, 1)$ and $\varepsilon > 0$. Then there exists an $\varepsilon' = \varepsilon'(s, t, \alpha, \varepsilon) > 0$ and an $h = h(s, t, \alpha, \varepsilon) > 0$ such that the following holds. Suppose G is an n-vertex graph with $n \ge h$ and $e(G) \ge T_{s,t}(\alpha)\binom{n}{2}$, and F is a $K_{s,t}$-tiling in G of maximum size with $|F| \le (1-\varepsilon)\alpha n$. Then one of the following is true:*

(i) *there exists an $\mathcal{F}_1$-tiling F' in G with $|F'| \ge |F| + \varepsilon' n$, or*
(ii) *there exists an F-augmentation (E_0, E_1) such that E_0 contains at least $\varepsilon' n$ edges.*

Proof. Set

$$\varepsilon' := \frac{1}{4} \min\left\{\frac{\varepsilon\alpha^2}{3t+1}, \frac{\varepsilon s\alpha}{(3t+1)(s+t)}\right\},$$

and let h be sufficiently large.

Suppose for a contradiction that the assertions of the lemma are not true.

Set $L := V(G) - V(F)$ and $m := |L|$. Let $\mathcal{C} := \{V_1(K): K \in F\}$, $\mathcal{D} := \{V_2(K) : K \in F\}$ and $C := \bigcup \mathcal{C}$, $D := \bigcup \mathcal{D}$. We call members of $\mathcal{C}$ *lilliputs* while members of $\mathcal{D}$ are *giants*. We say that giant $V_2(K) (K \in F)$ is *coupled* with lilliput $V_1(K)$.

As F is a maximum size $K_{s,t}$-tiling in G, by (1) we have that

$$e(G[L]) = o(n^2). \tag{5}$$

Let r be the number of copies of $K_{s,t}$ in F. Then $r \le (1-\varepsilon)\alpha n/(s+t)$. Moreover, we have

$$m = n - (s+t)r. \tag{6}$$

Let us define an auxiliary graph $H = (V', E')$ as follows. The vertex-set of H is $V' := \mathcal{C} \cup \mathcal{D} \cup L$. For any $x \in L$ and $K \in F$ the edge $xV_1(K)$ belongs to E' iff $\mathrm{N}_G(x) \cap V_1(K) \neq \emptyset$. Similarly, the edge $xV_2(K)$ belongs to E' iff $\mathrm{N}_G(x) \cap V_2(K) \neq \emptyset$. Finally, for any distinct $K, K' \in F$ the edge $V_2(K)V_2(K')$ belongs to E' iff $E_G(V_2(K), V_2(K')) \neq \emptyset$. The vertices L and the vertices $\mathcal{C}$ induce two independent sets in H.

As (i) does not hold, $H[L, \mathcal{D}]$ does not contain a matching with at least $\varepsilon' n$ edges. It follows from Fact 8 that

$$e_G(L, D) \le \varepsilon' nt \max\{m, r\} \le t\varepsilon' n^2. \tag{7}$$

Let M be a maximum matching in $H[L, \mathcal{C}]$ with l edges. Obviously, $l \le r$. By Fact 8, we have that

$$e_G(L, C) \le ls \max\{m, r\}. \tag{8}$$

Let $\mathcal{C}' \subseteq \mathcal{C}$ be the lilliputs matched by M. We write $\mathcal{D}' \subseteq \mathcal{D}$ for the giants coupled with $\mathcal{C}'$. Set $D' = \bigcup \mathcal{D}'$.

Suppose for a moment that $H[\mathcal{D}'] \cup H[\mathcal{D}', \mathcal{D} - \mathcal{D}']$ contains a matching T with at least $\varepsilon' n$ edges. Let $\mathcal{D}''$ be the giants in $\mathcal{D}'$ matched by T and M' the set of edges in M matching the lilliputs coupled with $\mathcal{D}''$. Then M' and T give rise to an F-augmentation (E_0, E_1) in G with $|E_0| = |M'| \geq |T| \geq \varepsilon' n$, contradicting our assumption that (ii) does not hold.

So $H[\mathcal{D}'] \cup H[\mathcal{D}', \mathcal{D} - \mathcal{D}']$ does not contain a matching with at least $\varepsilon' n$ edges. Applying Theorem 1 and passing to the graph G, we get

$$e(G[D'] \cup G[D', D - D']) \leq t^2 \mathrm{ex}(r, \varepsilon' n \times K_2) + r\binom{t}{2} \leq 2t^2 \varepsilon' nr + r\binom{t}{2}.$$

Therefore,

$$\begin{aligned} e(G[C \cup D]) &= e(G[D'] \cup G[D', D - D']) + e(G[D - D']) + e(G[C]) + e_G(C, D) \\ &\leq 2t^2 \varepsilon' nr + r\binom{t}{2} + \binom{(r-l)t}{2} + \binom{rs}{2} + r^2 st. \end{aligned} \tag{9}$$

Summing up the bounds (5) and (7)–(9) we get:

$$\begin{aligned} e(G) &= e(G[L]) + e_G(L, D) + e_G(L, C) + e(G[C \cup D]) \\ &\leq o(n^2) + t\varepsilon' n^2 + ls \max\{m, r\} + 2\varepsilon' nrt^2 + r\binom{t}{2} + \binom{(r-l)t}{2} + \binom{rs}{2} + r^2 st. \end{aligned}$$

Using the convexity of $f(l) := ls \max\{m, r\} + \binom{(r-l)t}{2}$ on $[0, r]$, and the fact that $rt \leq n$, we get:

$$e(G) \leq o(n^2) + 3t\varepsilon' n^2 + r\binom{t}{2} + r^2 st + \binom{rs}{2} + \max\left\{\binom{rt}{2}, rs \max\{m, r\}\right\}.$$

However, $r^2 s \leq \binom{rt}{2} + o(n^2)$, and hence from (6) we get:

$$\begin{aligned} e(G) &\leq o(n^2) + 3t\varepsilon' n^2 + r\binom{t}{2} + r^2 st + \binom{rs}{2} + \max\left\{\binom{rt}{2}, rs(n - (s+t)r)\right\} \\ &< \max\left\{\binom{(s+t)r}{2}, \binom{rs}{2} + rs(n - rs)\right\} + (3t+1)\varepsilon' n^2, \end{aligned}$$

where in the last inequality we have majorized the term $r\binom{t}{2} + o(n^2)$ by $\varepsilon' n^2$. But

$$\binom{(s+t)r}{2} + (3t+1)\varepsilon' n^2 \leq \binom{(1-\varepsilon)\alpha n}{2} + \frac{\varepsilon \alpha^2 n^2}{4} < \left(1 - \frac{\varepsilon}{2}\right)\frac{\alpha^2 n^2}{2},$$

and

$$\begin{aligned} \binom{rs}{2} + rs(n - rs) + (3t+1)\varepsilon' n^2 &< rsn - \frac{r^2 s^2}{2} + \frac{\varepsilon s \alpha n^2}{4(s+t)} \\ &\leq \frac{2s\alpha}{s+t}\left(1 - \frac{\alpha s}{2(s+t)} + \frac{\varepsilon(2-\varepsilon)\alpha s}{2(s+t)} - \frac{3\varepsilon}{4}\right)\frac{n^2}{2} \\ &< \frac{2s\alpha}{s+t}\left(1 - \frac{\alpha s}{2(s+t)} - \frac{\varepsilon}{4}\right)\frac{n^2}{2}. \end{aligned}$$

Consequently for large enough n,

$$e(G) < T_{s,t}(\alpha)\binom{n}{2},$$

a contradiction. □

Suppose $G = (V, E)$ is a graph and $r \in \mathbb{N}$. The *r-expansion* of G is the graph $G' = (V', E')$ defined as follows. The vertex set of G' is $V \times [r]$. For $a, b \in [r]$, an edge $((u, a), (v, b))$ belongs to E' iff uv

belongs to E. Note that there is a natural projection $\pi_{G'}: V' \to V$ that maps every vertex (u, a) from G' to the vertex u in G. We are interested in the following property of r-expansions. Suppose that K is a copy of any graph from $\mathcal{F}^*$ in G. Then $\pi_{G'}^{-1}(V(K))$ contains a complete bipartite graph B with color classes of sizes $s(K)r$ and $t(K)r$. By Lemma 5 we can tile B almost perfectly with copies of $K_{s,t}$. If F is an $\mathcal{F}^*$-tiling in G, we can apply the above operation on each member $K \in F$ and obtain a new tiling F' – which we call *retiling* – in the graph G'.

We are now ready to prove Theorem 2.

Proof of Theorem 2. Note that it suffices to prove the theorem for $H \simeq K_{s,t}$.

We first deal with the particular case $t = s$. Set $\alpha' := (1 - \varepsilon/4)\alpha$. Let $\varepsilon_1 := \frac{1}{5}(T_{s,t}(\alpha) - T_{s,t}(\alpha'))$, and ε_2 be given by Lemma 7 for input parameters H, $d := \varepsilon_1$ and $\gamma := \alpha\varepsilon/8$. Suppose that k_0 is sufficiently large. Let M be the bound from Lemma 4 for precision $\varepsilon_R := \min\{\varepsilon_1, \varepsilon_2\}$ and minimal number of clusters k_0. Let C be given by Lemma 5 for the input parameters s, t. Fix $n_0 \gg MC$. Suppose that G is an n-vertex graph, $n \ge n_0$, with at least $T_{s,t}(\alpha)\binom{n}{2}$ edges. We apply Lemma 4 on G to obtain an (ε_R, d)-reduced graph R with k clusters, $k_0 \le k \le M$. We have that

$$\begin{aligned} e(R) &\ge (T_{s,t}(\alpha) - d - 3\varepsilon_1)\binom{k}{2} \\ &= \left(T_{s,t}(\alpha') + \frac{1}{5}(T_{s,t}(\alpha) - T_{s,t}(\alpha'))\right)\binom{k}{2} \overset{(4)}{>} \mathrm{ex}\left(k, \frac{\alpha' k}{2} \times K_2\right). \end{aligned}$$

Therefore, R contains at least $\frac{\alpha' k}{2}$ independent edges. These edges correspond to regular pairs in G which can be tiled almost perfectly with copies of $K_{s,t}$, by means of Lemmas 5 and 7. Elementary calculations give that in this way we get a tiling of size at least $(1 - \varepsilon)\alpha n$.

Consequently we may suppose that $t > s$. We first define a handful of parameters. Set

$$\alpha' := \frac{6 - 4\varepsilon}{6 - 3\varepsilon}\alpha, \qquad \gamma := (1 - \varepsilon/2)\alpha', \qquad d := \frac{2}{5}(T_{s,t}(\alpha) - T_{s,t}(\alpha')).$$

Note that $\gamma = (1 - 2\varepsilon/3)\alpha$.

Let ε_R be given by Lemma 6 for input graph $K_{s,t}$, density $d/2$ and approximation parameter γ. We may suppose that ε_R is sufficiently small such that $\gamma(1 - \varepsilon_R) > (1 - \varepsilon)\alpha$ and $\varepsilon_R < d/2$. Let C be given by Lemma 5 for input s, t. Further, let ε' and h be given by Lemma 9 for input parameters α' and $\varepsilon/4$. We may assume that $\varepsilon' < \varepsilon$. Set

$$p := t^2 \left\lceil \frac{4C}{\varepsilon'} \right\rceil, \qquad q := \left\lceil \frac{2t}{\varepsilon'} \right\rceil.$$

Let M be the upper bound on the number of clusters given by Lemma 4 for input parameters h (for the minimal number of clusters) and $\varepsilon_R p^{-q}/2$ (for the precision). Let $n_0 > Mp^q$ be sufficiently large.

Suppose now that G is a graph with $n > n_0$ vertices and at least $T_{s,t}(\alpha)\binom{n}{2}$ edges. We first apply Lemma 4 to G with parameters $\varepsilon_R p^{-q}/2$ and h. In this way we obtain an $(\varepsilon_R p^{-q}/2, d)$-reduced graph R with at least h vertices.

Let us now define a sequence of graphs $R^{(i)}$ by setting $R^{(0)} = R$ and letting $R^{(i)}$ be the p-expansion of $R^{(i-1)}$, $i = 1, 2, \ldots, q$. Note that $e(R^{(i)}) \ge T_{s,t}(\alpha')\binom{v(R^{(i)})}{2}$ for every $i \in \{0, 1, \ldots, q\}$.

Let $F^{(i)}$ be a maximum size $K_{s,t}$-tiling in $R^{(i)}$ for $i = 0, 1, \ldots, q$. We claim that

$$|F^{(i)}| \ge \min\left\{\frac{i\varepsilon' v(R^{(i)})}{2t}, \left(1 - \frac{\varepsilon}{2}\right)\alpha' v(R^{(i)})\right\}. \tag{10}$$

To this end it suffices to show that for any $i \ge 1$,

(C1) if $|F^{(i-1)}| > (1 - \varepsilon/4)\alpha' v(R^{(i-1)})$, then $\frac{|F^{(i)}|}{v(R^{(i)})} \ge \frac{|F^{(i-1)}|}{v(R^{(i-1)})} - \frac{\varepsilon\alpha'}{4}$, and

(C2) if $|F^{(i-1)}| \le (1 - \varepsilon/4)\alpha' v(R^{(i-1)})$, then $\frac{|F^{(i)}|}{v(R^{(i)})} \ge \frac{|F^{(i-1)}|}{v(R^{(i-1)})} + \frac{\varepsilon'}{2t}$.

In the case (C1), according to Lemma 5, the retiling of $F^{(i-1)}$ in $R^{(i)}$ has size at least $|F^{(i-1)}|(p-C) > (1-\varepsilon/2)\alpha' v(R^{(i)})$, thus proving the statement.

Consequently we may suppose that we are in case (C2). Apply Lemma 9 to the graph $R^{(i-1)}$ and the tiling $F^{(i-1)}$, with parameters α' and $\varepsilon/4$.

Suppose first that assertion (i) of the lemma holds. Then $R^{(i-1)}$ contains an $\mathcal{F}_1$-tiling F with $\frac{|F|}{v(R^{(i-1)})} \geq \frac{|F^{(i-1)}|}{v(R^{(i-1)})} + \varepsilon'$. By retiling F, we get a $K_{s,t}$-tiling in $R^{(i)}$ with size at least

$$|F|(p-C) > i\varepsilon' v(R^{(i)})/(2t),$$

thus proving the statement.

Suppose now that assertion (ii) of Lemma 9 is true. Then $R^{(i-1)}$ contains an $F^{(i-1)}$-augmentation (E_0, E_1) with $|E_0| \geq \varepsilon' v(R^{(i-1)})$. Let $r = p/t$. We shall denote by T the t-expansion of $R^{(i-1)}$ and by T' the r-expansion of T. Note that T' is isomorphic to $R^{(i)}$.

Let us build an $\mathcal{F}_2$-tiling in T in the following way.

For every edge $e = (u, v) \in E_0$ with $u \in V(F^{(i-1)})$ we choose an edge $e' = (u', v')$ in T with $\pi_T(u') = u$ and $\pi_T(v') = v$. We shall denote by w_e the vertex u' corresponding to u.

For every edge $e = (u, v) \in E_1$ we choose a set S_e of t independent edges in $\pi_T^{-1}(e)$.

For every $K \in F^{(i-1)}$ we shall also choose a subgraph K' of T. We distinguish the following cases. If K has no vertex matched by E_0 or E_1, then we let $K' := T[\pi_T^{-1}(K)]$. If K has a vertex u matched by E_1 but no vertex matched by E_0, we let $K' := T[\pi_T^{-1}(K-u)]$. Then $K' \simeq K_{st,(t-1)t}$. Finally, if K has a vertex u matched by an edge $e \in E_0$ and a vertex v matched by an edge in E_1, we let $K' := T[\pi_T^{-1}(K-v)] - w_e$. Note that in this last case $K' \simeq K_{st-1,(t-1)t}$.

It is easy to see that

$$F := \{e' : e \in E_0\} \cup \{K' : K \in F^{(i-1)}\} \cup \left(\bigcup_{e \in E_1} S_e\right)$$

is an $\mathcal{F}_2$-tiling in T. Moreover, we have that $\frac{|F|}{v(T)} \geq \frac{|F^{(i-1)}|}{v(R^{(i-1)})} + \frac{\varepsilon'}{t}$. So the retiling of F in T' has size at least $|F|(r-C) \geq i\varepsilon' v(R^{(i)})/(2t)$. This proves (C2) and also (10).

Using Lemma 3, we may subdivide every cluster corresponding to a vertex of R into p^q equal-sized parts, by discarding some vertices if necessary. This gives us an $(\varepsilon_R, d/2)$-reduced graph R'. By construction $R' \simeq R^{(q)}$. By (10), there is a $K_{s,t}$-tiling F in R' with size at least $(1-\varepsilon/2)\alpha' v(R')$. Let G' be the subgraph of G induced by the clusters corresponding to the vertices of R'. By applying Lemma 6 to R', we see that G' has a $K_{s,t}$-tiling of size at least $\gamma v(G') \geq \gamma(1-\varepsilon_R)v(G) > (1-\varepsilon)\alpha v(G)$, and so does G.

This finishes the proof of Theorem 2. □

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