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LARGE-TIME ASYMPTOTIC EQUIVALENCE FOR A CLASS OF NON-AUTONOMOUS SEMILINEAR PARABOLIC EQUATIONS

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ABSTRACT. – In this article we prove new results concerning the long-time behavior of solutions to a class of non-autonomous semilinear parabolic Neumann boundary-value problems defined on open bounded connected subsets Ω of $\mathbf{R}^N$. The nature of the equations that we investigate leads us to consider two complementary situations, according to whether the time-dependent lower order terms in the equations possess recurrence properties. If the lower order terms are recurrent, we prove that every solution stabilizes around a spatially homogeneous and recurrent solution of the same Neumann problem in the $\mathcal{C}^1(\overline{\Omega})$-topology. In contrast, if the lower order terms are not recurrent, the asymptotic states need not be solutions to the original problem and we prove that every solution stabilizes around such an asymptotic

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state, again in the $C^1(\overline{\Omega})$-topology. In all cases the dynamics of the asymptotic solutions are governed by a compact and connected set of scalar ordinary differential equations, which are thereby asymptotically equivalent to the original Neumann problem for large times. A major difficulty to be bypassed in the proofs of our theorems stems from the fact that we allow the nonlinearities to depend explicitly on the gradient of the unknown function. Our method of proof rests upon the use of comparison principles and upon the existence of exponential dichotomies for the family of evolution operators associated with the principal part of the equations. It is also based on ideas that stem from the classic reduction methods of non-autonomous finite-dimensional dynamical systems originally devised by MILLER, STRAUSS-YORKE and SELL.

1. Introduction and outline

In this article we investigate the long-time behavior of classical solutions to non-autonomous Neumann boundary-value problems of the form

$$(1.1)\qquad \left\{\begin{array}{l} u_t(x,\,t) = k(t)\,\Delta u\,(x,\,t) + s(t)g\,(u\,(x,\,t),\,\nabla u\,(x,\,t)), \\ \quad (x,\,t) \in \Omega \times \mathbf{R}^+ \\ \mathrm{Ran}\,(u) \subseteq (u_0,\,u_1) \\ \dfrac{\partial u}{\partial n}(x,\,t) = 0, \qquad (x,\,t) \in \partial\Omega \times \mathbf{R}^+ \end{array}\right\}$$

In relation (1.1) Ω denotes an open bounded connected subset of $\mathbf{R}^N$ with a sufficiently regular boundary $\partial\Omega$, $\mathrm{Ran}\,(u)$ denotes the range of u and n the normalized outer normal vector to $\partial\Omega$. We assume that the three functions k, s and g satisfy the following hypotheses, respectively:

(**K**) We have $k \in C^\mu(\mathbf{R}_0^+,\,\mathbf{R}^+)$ with $\mu \in (0,\,1]$ (Hölder-Lipschitz continuity of exponent μ) and

$$\underline{k} = \inf_{t \in \mathbf{R}_0^+} k\,(t) > 0, \qquad \overline{k} = \sup_{t \in \mathbf{R}_0^+} k\,(t) < \infty.$$

(**S**) We have $s \in C^\mu(\mathbf{R},\,\mathbf{R})$ with $\mu \in (0,\,1]$; moreover $t \to s\,(t) = 0\,(1)$ and $t \to \int_0^t d\xi\, s\,(\xi) = 0\,(1)$ as $|\,t\,| \to \infty$.

(**G**) We have $g \in C^1([u_0,\,u_1] \times \mathbf{R}^N)$ where the numbers $u_0,\,u_1 \in \mathbf{R}$ are such that $g\,(u_0,\,0) = g\,(u_1,\,0) = 0$ and $g\,(u,\,0) > 0$ for every $u \in (u_0,\,u_1)$. Moreover, we assume that there exists a bounded function c: $[u_0,\,u_1] \to \mathbf{R}_0^+$ such that $|\,g\,(u,\,q) - g\,(u,\,0)\,| \leq c\,(u)\,|\,q\,|^2$ for every $q \in \mathbf{R}^N$.

Problems of the form (1.1) have a plethora of applications in various areas of science, ranging from theoretical physics to population dynamics, including the theory of heat diffusion, of nerve pulse propagation and of population genetics [3]. For this reason alone it is essential that one investigates the long-time behavior of solutions to Problem (1.1), since for many applications only the knowledge of the ultimate behavior of solutions is relevant.

There have been several recent works devoted to related questions, which pertain to a variety of reaction-diffusion equations that bear some analogy with Problem (1.1) ([8], [10], [11], [12]). However, many of these deal with situations where both k and s are periodic functions of equal period, and where g is independent of ∇u. In such cases the end result depends in a very sensitive way on various hypotheses regarding the average behavior of s. Accordingly, the ultimate behavior of solutions can be described by a global attractor that may consist of either the constant u_0, or of the constant u_1 or of a one-parameter family of time-periodic solutions which are spatially homogeneous. In all of these cases the method of proof is basically the same: one exploits the periodicity of the equation to deduce important spectral information about the corresponding Poincaré map through the Krein-Rutman theory, which one then combines with some fundamental ideas from the theory of monotone dynamical systems. This approach has been thoroughly discussed in the monograph by HESS [12], and further developed by BRUNOWSKI *et al.* [5], TAKAC [26] and by DANCER and HESS [6] for the analysis of more complicated periodic problems. In this context, it is worth observing that the first result concerning the long-time behavior of solutions to problems of the form (1.1) with $k = 1$, s periodic and g dependent on ∇u is as recent as 1991, and is again by DANCER and HESS (see Proposition 5 of [7]) (For another recent work concerning discrete-time dynamical systems, see [17]).

When the functions k and s are no longer periodic and when g depends explicitly on ∇u, the investigation of the long-time behavior of solutions to parabolic problems of the form (1.1) leads to many qualitatively new difficulties which prevent one from applying the method of [12]. On the one hand, there is no natural and useful substitute for the notion of Poincaré map; on the other hand, Liapunov functionals are not readily available, if available at all. This set of difficulties recently prompted some authors to develop new methods to handle certain classes of non-autonomous and non-periodic problems. For instance, SHEN and YI investigated a class of almost-periodic parabolic problems in one space dimension through a detailed analysis of the corresponding skew-product flows ([23], [24]), while one of the present authors developed a stable- and center-manifold theory to analyze problems of the form (1.1) with $k = 1$, s almost-periodic and g independent of ∇u ([28]-[31]). In these four works, we studied the long-time behavior of all classical solutions to such problems, and we also characterized their stabilization properties by explicit rates of decay. Moreover, we also obtained a rather complete description of the geometry of the corresponding flow, of the global attractor and of its Liapunov stability properties.

In this paper our primary purpose is to investigate the stabilization properties of solutions to Problem (1.1) when hypotheses (**K**), (**S**) and (**G**) hold. The nature of Problem (1.1) leads us to consider two complementary situations, according to whether the function s possesses recurrence properties. If s is recurrent, we can prove that each classical solution stabilizes around a spatially homogeneous and recurrent solution of the same Neumann problem in the $C^1(\overline{\Omega})$-topology. In contrast, if s is not recurrent then the asymptotic states are no longer solutions of Problem (1.1) and we exhibit conditions that ensure the stabilization around such asymptotic states, again in the $C^1(\overline{\Omega})$-topology. Our main results concerning both cases are stated and further discussed in Section 2. In each case the dynamics of the asymptotic solutions are governed by scalar ordinary differential equations, which are thereby asymptotically equivalent to Problem (1.1) for large times. Our proofs of the main results are carried out in Section 3, where we first prove a stabilization result for the orbits of an infinite-dimensional dynamical system associated with Problem (1.1) in the strong topology of the Sobolev space $H^{2,p}(\Omega)$ where $p \in (\max(N, 2), \infty)$. Our method of proof there rests upon the existence of exponential dichotomies for the family of linear evolution operators associated with the principal part of Problem (1.1), and upon the use of parabolic comparison principles. A major difficulty to be bypassed there is the explicit dependence of g on ∇u. The stabilization result of Section 3 then directly leads to the proofs of the theorems stated in Section 2. Finally, we devote Section 4 to some concluding remarks and we refer the reader to [4] for a short announcement of the results.

Our work was primarily motivated by the desire to provide some new insights into the qualitative behavior of solutions to boundary-value problems of the form

$$(1.2)\quad \left\{\begin{array}{ll} u_t(x,t) = \operatorname{div}(k(x,t)\nabla u(x,t)) + g(t, u(x,t), \nabla u(x,t)), & \\ \qquad\qquad\qquad\qquad\qquad (x,t) \in \Omega \times \mathbf{R}^+ & \\ \dfrac{\partial u}{\partial n}(x,t) = 0, \qquad (x,t) \in \partial\Omega \times \mathbf{R}^+ & \end{array}\right\}$$

of which Problem (1.1) represents a special separable case. We defer the presentation of our results concerning (1.2) to a separate publication.

2. Statements and discussion of the main theorems

Throughout the remaining part of this article we assume that Ω has a $\mathcal{C}^{3+[N/2]}$-boundary in the sense of [1] in such a way that Ω lies only on one side of $\partial\Omega$, and that it satisfies the interior ball condition for every $x \in \partial\Omega$ [9]. We also assume that all functions that we define on Ω or on $\overline{\Omega}$ are real-valued and make no further mention of the matter. We are primarily interested in the long-time behavior of classical solutions to Problem (1.1). In order to define this notion properly, let $\mathcal{C}^{2,1}(\Omega \times \mathbf{R}^+)$ be the set consisting of all functions $z \in \mathcal{C}(\Omega \times \mathbf{R}^+)$ such that $(x,t) \to D^\alpha \partial_t^\gamma z(x,t) \in \mathcal{C}(\Omega \times \mathbf{R}^+)$ for each $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbf{N}^N$ and each $\gamma \in \mathbf{N}$ satisfying $\sum_{j=1}^N \alpha_j + 2\gamma \le 2$. In a similar way we define $\mathcal{C}^{1,0}(\overline{\Omega} \times \mathbf{R}^+)$ as the set consisting of all $z \in \mathcal{C}(\overline{\Omega} \times \mathbf{R}^+)$ with the property that $D^\alpha z \in \mathcal{C}(\overline{\Omega} \times \mathbf{R}^+)$ for every $\alpha \in \mathbf{N}^N$ such that $\sum_{j=1}^N \alpha_j \le 1$. It will be essential later that both u and ∇u possess Hölder-Lipschitz continuity properties in the time variable. This remark motivates the following.

Definition 2.1. – A function $u \in \mathcal{C}^{2,1}(\Omega \times \mathbf{R}^+) \cap \mathcal{C}(\overline{\Omega} \times \mathbf{R}_0^+) \cap \mathcal{C}^{1,0}(\overline{\Omega} \times \mathbf{R}_0^+)$ is said to be a classical solution to Problem (1.1) if the following conditions are satisfied:

(C_1) There exist $\theta \in (0, 1]$, $\tau \in \mathbf{R}^+$, and a function $c \in L^p(\Omega)$ with $p \in (\max(N, 2), \infty)$ such that $|u(x,t) - u(x,t')| \le c(x)|t - t'|^\theta$ and $|\nabla u(x,t) - \nabla u(x,t')| \le c(x)|t - t'|^\theta$ for almost every $x \in \Omega$ and for every $t, t' \in [\tau, \infty)$.

(C_2) $x \to u(x,t) \in \mathcal{C}^2(\overline{\Omega})$ for every $t \in \mathbf{R}_0^+$.

(C_3) $(x, t) \to u_t(x, t) \in \mathcal{C}(\overline{\Omega} \times \mathbf{R}_0^+)$, $t \to u_t(x, t) \in \mathcal{C}(\mathbf{R}^+)$ uniformly in $x \in \overline{\Omega}$ and u satisfies relations (1.1) identically.

As already noticed in Section 1, the structure of Problem (1.1) makes it convenient to distinguish between the cases where s is recurrent and where it is not. The appropriate notion of recurrence is here the following.

DEFINITION 2.2. – A function s satisfying hypothesis (**S**) is said to be recurrent if there exists at least one real sequence $(t_n) \to \infty$ such that the sequence of translates (s_{t_n}) defined by $s_{t_n}(t) = s(t + t_n)$ converges locally uniformly to s on $\mathbf{R}$ as $n \to \infty$; in other words $\sup_{t \in I} |s(t + t_n) - s(t)| \to 0$ as $n \to \infty$ for every compact interval $I \subset \mathbf{R}$.

We call the sequence (t_n) of the preceding definition a returning sequence for s. Our first result is then the following.

THEOREM 2.1. – *Assume that hypotheses* $(\mathbf{K})$, $(\mathbf{S})$ *and* $(\mathbf{G})$ *hold and let u be any classical solution to Problem* (1.1). *Then the following two statements are equivalent:*

(a) *The function s is recurrent.*

(b) *There exists a unique classical solution $\hat{u}$ of* (1.1), *independent of x and recurrent, such that*

$$\begin{aligned} &\lim_{t \to \infty} \sup_{x \in \overline{\Omega}} |u(x, t) - \hat{u}(t)| = 0 \\ &\lim_{t \to \infty} \sup_{x \in \overline{\Omega}} |\nabla u(x, t)| = 0 \end{aligned} \tag{2.1}$$

Moreover, if either statement (a) *or* (b) *holds, we may choose the same returning sequence for $\hat{u}$ as we do for s.*

Remarks.

1. It is plain that the class of recurrent functions contains the algebra of all (Bohr) almost-periodic functions satisfying property $(\mathbf{S})$: it is sufficient to choose (t_n) as any sequence of almost-periods diverging to infinity. Thus, if s is periodic and if $k = 1$, we recover a result of DANCER and HESS stated in Proposition 5 of [7] in a slighly weaker form. We also note that the method of proof of these two authors is strictly limited to the periodic case, as they rely on the theory of discrete monotone dynamical systems.

2. As a spatially homogeneous solution of Problem (1.1), $\widehat{u}$ satisfies the separable initial value problem $\widehat{u}'(t) = s(t)\, g(\widehat{u}(t), 0)$, $\widehat{u}(0) \in (u_0, u_1)$

for every $t \in \mathbf{R}$. In such a case we say that this separable problem is asymptotically equivalent to (1.1) for large times. In a certain sense, the reduction of Problem (1.1) described in Theorem 2.1. corresponds to a reduction by a center-manifold of dimension one (compare with the proof given in Section 3).

3. By our definition of recurrence, the function $\widehat{u}$ of Theorem 2.1 satisfies $\sup_{t \in I} |\widehat{u}(t + t_n) - \widehat{u}(t)| \to 0$ as $n \to \infty$ for every compact interval $I \subset \mathbf{R}$. This is a very particular situation in that if s is not recurrent, sequences of translates of the form $t \to \widehat{u}(t + t_n)$ where $\widehat{u}$ is any x-independent solution of (1.1) need not converge to any solution of (1.1). In fact, it is quite possible that the solutions of (1.1) stabilize around asymptotic states which are no longer solutions of the same boundary-value problem. The main difficulty here lies in the determination of what those asymptotic states are, and in the determination of their governing equations.

We begin our analysis of the non-recurrent case with the following result.

THEOREM 2.2. – *Assume that hypotheses* $(\mathbf{K})$, $(\mathbf{S})$ *and* $(\mathbf{G})$ *hold and let* u *be any classical solution to Problem* (1.1). *Then the following two statements are equivalent:*

(a) *We have* $\int_0^\infty d\xi\, s(\xi) < \infty$.

(b) *There exists a unique constant* $a_u \in (u_0, u_1)$ *such that*

$$\begin{aligned} &\lim_{t \to \infty} \sup_{x \in \overline{\Omega}} |u(x, t) - a_u| = 0 \\ &\lim_{t \to \infty} \sup_{x \in \overline{\Omega}} |\nabla u(x, t)| = 0 \end{aligned} \tag{2.2}$$

Remarks.

1. The statement of the preceding theorem is intuitively clear for functions s of the form $s(t) = e^{-|t|}$ or $s(t) = (1 + |t|)^{-\alpha}$ where $\alpha \in (1, \infty)$, in that the nonlinear term of (1.1) "rapidly goes to zero" as $t \to \infty$. But it is plain that Theorem 2.2 fails to hold for functions of the form $s(t) = (1 + |t|)^{-\alpha}$ where $\alpha \in [0, 1]$. We also note that condition (a) alone holds for functions like $s(t) = \sin(t^2)$. Although this function is not Hölder continuous and does not go to zero when $t \to \infty$, we still get the stabilization of all classical solutions of (1.1) to some $a_u \in (u_0, u_1)$ in this case, but only in the $L^p(\Omega)$-topology (compare with the arguments of Section 3).

2. Condition (a) is indeed a condition of non-recurrence: it would easily follow from the recurrence of s and condition (a) that $s = 0$ (compare with the arguments of Section 3).

3. With the exception of the trivial case $s = 0$, it follows from hypothesis $(\mathbf{G})$ that the constant a_u is not a solution to Problem (1.1). Thus if condition (a) of Theorem 2.2 holds, Problem (1.1) is asymptotically equivalent to the trivial problem $\widehat{u}'(t) = 0$, $\widehat{u}(0) \in (u_0, u_1)$ for large times! This means that every classical solution to (1.1) stabilizes around an asymptotic state that is spatially and temporally homogeneous.

It remains to consider cases where s is not recurrent and where $-\infty < \liminf_{t\to\infty} \int_0^t d\xi\, s(\xi) < \limsup_{t\to\infty} \int_0^t d\xi\, s(\xi) < \infty$. We begin with the following result, which may be considered as an extension of the first two.

THEOREM 2.3. – *Assume that hypotheses* $(\mathbf{K})$, $(\mathbf{S})$ *and* $(\mathbf{G})$ *hold and let* u *be any classical solution to Problem* (1.1). *If* $s = s_0 + s_1$ *where* s_0 *is continuous and recurrent and if* $\int_0^\infty d\xi\, s_1(\xi) < \infty$, *then there exists a unique classical solution* $\widehat{u}_0$ *of the initial value problem* $\widehat{u}'(t) = s_0(t)\, g(\widehat{u}(t), 0)$, $\widehat{u}(0) \in (u_0, u_1)$, *independent of* x *and recurrent, such that*

$$\begin{aligned} &\lim_{t\to\infty} \sup_{x\in\overline{\Omega}} |u(x, t) - \widehat{u}_0(t)| = 0 \\ &\lim_{t\to\infty} \sup_{x\in\overline{\Omega}} |\nabla u(x, t)| = 0 \end{aligned} \tag{2.3}$$

Moreover, we may choose for $\hat{u}_0$ *the same returning sequence as we do for* s_0.

Remarks. – Theorem 2.3 clearly reduces to Theorem 2.1 if $s_1 = 0$, and to Theorem 2.2 if $s_0 = 0$. In the general case we may summarize the statement of Theorem 2.3 by saying that Problem (1.1) is asymptotically recurrent. For instance, we may take $s(t) = \sum_{j=1}^{n} \sin(\omega_j t) + e^{-|t|}$ where the set of frequencies $\{\omega_1, \ldots, \omega_n\}$ is rationally independent: the function $t \to e^{-|t|}$ is here completely irrelevant to the long-time dynamics, which are quasi-periodic. Finally, notice that the condition on s in Theorem 2.3 is sufficient but not necessary for the statement to hold, unless one strengthens the hypothesis on s_0 a bit (compare with the proofs given in Section 3).

The above three theorems show that the ultimate behavior of all classical solutions to Problem (1.1) depends in a very sensitive way on the ultimate behavior of s. In order to handle the general case and to encode the long-time behavior of s into our theory, we shall now invoke methods from topological dynamics that can be traced back to the original works of MILLER [16], STRAUSS-YORKE [25] and SELL [20]-[22]). These works concern the reduction of finite-dimensional non-autonomous dynamical systems and the underlying theory of limiting equations. Write momentarily $C(\mathbf{R})$ for the Fréchet space that consists of all continuous functions on $\mathbf{R}$ endowed with the topology of uniform convergence on compact sets. For any $s \in C(\mathbf{R})$ satisfying hypothesis $(\mathbf{S})$, let $\mathbf{E}(s) \subset C(\mathbf{R})$ be the set consisting of all functions s^* for which there exists a sequence $(t_n^*) \to \infty$ such that $\sup_{t \in I} |s(t + t_n^*) - s^*(t)| \to 0$ as $n \to \infty$ for every compact interval $I \subset \mathbf{R}$. It is clear that $\mathbf{E}(s)$ coincides with the ω-limit set of s relative to the dynamical system induced by the time-translation operator in $C(\mathbf{R})$, so that much of the information needed regarding the long-time behavior of s is already contained in $\mathbf{E}(s)$. Our next result describes a reduction of Problem (1.1) to ordinary differential equations whose dynamics are governed by the elements of $\mathbf{E}(s)$. It is exclusively based on hypotheses $(\mathbf{K})$, $(\mathbf{S})$ and $(\mathbf{G})$ and thereby includes the general case where

$$-\infty < \liminf_{t \to \infty} \int_0^t d\xi\, s(\xi) < \limsup_{t \to \infty} \int_0^t d\xi\, s(\xi) < \infty \tag{2.4}$$

The trade-off for this degree of generality is, however, that we shall have to settle for a slightly weaker convergence result.

THEOREM 2.4. – *Assume that hypotheses* $(\mathbf{K})$, $(\mathbf{S})$ *and* $(\mathbf{G})$ *hold and let* u *be any classical solution to Problem* (1.1). *Then the following statements hold:*

(a) *The set* $\mathbf{E}(s)$ *is a non-empty, compact, connected and invariant subset of* $C(\mathbf{R})$.

(b) *For any* $s^* \in \mathbf{E}(s)$, *there exists a sequence* $(t_n) \to \infty$ *and a classical solution* u^* *of the initial-value problem* $(u^*)'(t) = s^*(t)\, g(u^*(t), 0)$, $u^*(0) \in (u_0, u_1)$, $t \in \mathbf{R}$, *such that the relation*

$$\lim_{n \to \infty} \sup_{t \in I} \sup_{x \in \overline{\Omega}} |u(x, t + t_n) - u^*(t)| = 0 \tag{2.5}$$

holds for every compact interval $I \subset \mathbf{R}$. Moreover, we have

$$\lim_{t\to\infty} \sup_{x\in\overline{\Omega}} |\nabla u(x,\, t)| = 0 \tag{2.6}$$

Remarks.

1. The invariance property of the set $\mathbf{E}(s)$ stated in Theorem 2.4 refers to the invariance under the flow generated by the time-translation operator in $C(\mathbf{R})$. In other words, every translate of $s^* \in \mathbf{E}(s)$ also belongs to $\mathbf{E}(s)$, a fact that can be easily verified.

2. If s satisfies hypothesis $(\mathbf{S})$ and is periodic of minimal period $T \in \mathbf{R}^+$, then $\mathbf{E}(s)$ consists exclusively of the translates $(s_t)_{t\in[0,\,T)}$. More generally, if s satisfies hypothesis $(\mathbf{S})$ then s is recurrent if, and only if, $s \in \mathbf{E}(s)$ and we may choose $s^* = s$, $u^* = \hat{u}$ in the second statement of Theorem 2.4 so that relation (2.5) reduces to a weaker variant of relation (2.1).

3. If s satisfies hypothesis $(\mathbf{S})$ and condition (a) of Theorem 2.2 then $\mathbf{E}(s) = \{0\}$ and $s(t) \to 0$ as $t \to \infty$. In fact, the first relation (2.2) of Theorem 2.2 implies that

$$\lim_{n\to\infty} \sup_{t\in I} \sup_{x\in\overline{\Omega}} |u(x,\, t+t_n) - a_u| = 0 \tag{2.7}$$

for every sequence $(t_n) \to \infty$ and every compact interval $I \subset \mathbf{R}$, while relation (2.5) holds whenever $s^* \in \mathbf{E}(s)$. Relations (2.5) and (2.7) then imply that $\sup_{t\in I} |u^*(t) - a_u| = 0$ for every compact interval $I \subset \mathbf{R}$, so that

$$u^*(t) = G^{-1}\left\{\int_0^t d\xi\, s^*(\xi) + G(u^*(0))\right\} = a_u \tag{2.8}$$

for every $t \in \mathbf{R}$. In relation (2.8), the function $G : (u_0,\, u_1) \to \mathbf{R}$ stands for any primitive of the function $u \to 1/g(u,\, 0)$ and G^{-1} denotes the monotone inverse of G. From relation (2.8) we infer that the function $t \to \int_0^t d\xi\, s^*(\xi)$ is a constant, and hence that $s^* = 0$. So $\mathbf{E}(s) = \{0\}$ and we conclude from a classic result of topological dynamics that $s(t) \to 0$ as $t \to \infty$ (invoke, for instance, Theorem VII.6 of [22]). Here again, Theorem 2.4 reduces to a weaker form of Theorem 2.2.

4. There are evidently bounded functions s that satisfy condition (a) of Theorem 2.2 and which do not go to zero as $t \to \infty$. According to the

preceding remark such functions cannot be Hölder continuous. A case in point is $s(t) = \sin(t^2)$ for which we have $\mathbf{E}(s) = \varnothing$. This is hardly a surprise in view of the first remark following the statement of Theorem 2.2. One possible interpretation of this fact is that the locally convex topology of uniform convergence on compact sets in $C(\mathbf{R})$ is inadequate to describe the corresponding stabilization phenomena.

5. If s satisfies hypothesis (**S**) and if $s = s_0 + s_1$ where both s_0 and s_1 are bounded and uniformly continuous (a stronger hypothesis than the corresponding one in Theorem 2.3), then for each $s^* \in \mathbf{E}(s)$ there exists $s_0^* \in \mathbf{E}(s_0)$ and $s_1^* \in \mathbf{E}(s_1)$ such that $s^* = s_0^* + s_1^*$. In fact, on the one hand there exists a sequence $(t_n) \to \infty$ such that $\sup_{t\in I} |s_0(t+t_n) + s_1(t+t_n) - s^*(t)| \to 0$ as $n \to \infty$ for every compact interval $I \subset \mathbf{R}$. On the other hand, owing to the relative compactness of the sets of translates of s_0 and s_1, there exists a subsequence $(t_{n'}) \subseteq (t_n)$ such that $\sup_{t\in I} |s_0(t+t_{n'}) - s_0^*(t)| \to 0$ for some $s_0^* \in \mathbf{E}(s_0)$, and there exists a further subsequence $(t_{n''}) \subseteq (t_{n'})$ such that $\sup_{t\in I} |s_1(t+t_{n''}) - s_1^*(t)| \to 0$ for some $s_1^* \in \mathbf{E}(s_1)$. Consequently, the three relations

$$\begin{aligned} &\sup_{t\in I} | s_0(t+t_{n''}) + s_1(t+t_{n''}) - s^*(t) | \to 0 \\ &\sup_{t\in I} | s_0(t+t_{n''}) - s_0^*(t) | \to 0 \\ &\sup_{t\in I} | s_1(t+t_{n''}) - s_1^*(t) | \to 0 \end{aligned} \tag{2.9}$$

hold simultaneously, from which we infer that $\sup_{t\in I} | s^*(t) - s_0^*(t) - s_1^*(t) | = 0$ for every compact interval $I \subset \mathbf{R}$. Hence $s^* = s_0^* + s_1^*$ or, symbolically, $\mathbf{E}(s_0 + s_1) \subseteq \mathbf{E}(s_0) + \mathbf{E}(s_1)$. Thus if, in addition, we have $\int_0^\infty d\xi\, s_1(\xi) < \infty$ then $\mathbf{E}(s_1) = \{0\}$ from Remark 3 so that $\mathbf{E}(s_0 + s_1) \subseteq \mathbf{E}(s_0)$: the dynamics of the limiting equations are entirely governed by the elements of $\mathbf{E}(s_0)$; in case s_0 is recurrent, we retrieve in this way a weaker variant of Theorem 2.3.

In statement (b) of Theorem 2.4 the function $s^* \in \mathbf{E}(s)$ is given and we prove the existence of a sequence $(t_n) \to \infty$ so that relation (2.5) holds. We can in fact turn things around a bit and prove a dual result whereby the sequence $(t_n) \to \infty$ is given and the existence of $s^* \in \mathbf{E}(s)$ is proven. We complete this section by stating such a dual result.

THEOREM 2.5. – *Assume that hypotheses* $(\mathbf{K})$, $(\mathbf{S})$ *and* $(\mathbf{G})$ *hold and let* u *be any classical solution to Problem* (1.1). *Let* $(t_n) \to \infty$ *be any sequence diverging to infinity. Then there exist a function* $s^* \in \mathbf{E}(s)$ *and a classical solution* u^* *of the initial-value problem* $(u^*)'(t) = s^*(t)\, g(u^*(t), 0)$, $u^*(0) \in (u_0, u_1)$, $t \in \mathbf{R}$ *such that the relation*

$$\lim_{n\to\infty} \sup_{t\in I} \sup_{x\in\overline{\Omega}} |u(x, t+t_n) - u^*(t)| = 0 \tag{2.10}$$

holds for every compact interval $I \subset \mathbf{R}$. *Moreover, we have*

$$\lim_{t\to\infty} \sup_{x\in\overline{\Omega}} |\nabla u(x, t)| = 0 \tag{2.11}$$

3. A stabilization result in the Sobolev space $H^{2,p}(\Omega)$ and proof of the main theorems

In the first part of this section we prove a stabilization result for the orbits of an infinite-dimensional dynamical system associated with Problem (1.1). Write $\Delta_{p,\mathcal{N}}$ for the $L^p(\Omega)$-realization of Laplace's operator defined on the domain $H^{2,p}_{\mathcal{N}}(\Omega)$ consisting of all functions $z \in H^{2,p}(\Omega)$ satisfying Neumann's boundary condition $\frac{\partial z}{\partial n}(x) = 0$ for every $x \in \partial\Omega$, where $p \in (\max(N, 2), \infty)$ is the constant of Definition 2.1. Let $u : \mathbf{R}_0^+ \to L^p(\Omega)$ be the map defined by $u(t)(x) = u(x, t)$ where $(x, t) \to u(x, t)$ is a classical solution to Problem (1.1). In a similar way we define $\nabla u : \mathbf{R}_0^+ \to L^p(\Omega, \mathbf{R}^N)$ by $\nabla u(t)(x) = \nabla u(x, t)$. From condition (C_1) of Definition 2.1, it follows that

$$\| u(t) - u(t') \|_p \le \| c \|_p \, | t - t' |^{\theta} \tag{3.1}$$

$$\| \nabla u(t) - \nabla u(t') \|_p \le \| c \|_p \, | t - t' |^{\theta} \tag{3.2}$$

for every $t, t' \in [\tau, \infty)$. We further notice that the function $(u(t), \nabla u(t))$: $\Omega \to [u_0, u_1] \times \mathbf{R}^N$ is bounded for every $t \in \mathbf{R}^+$; furthermore, the nonlinearity in (1.1) induces a mapping $\mathfrak{g}$ going from the set of all bounded functions in $L^p(\Omega) \times L^p(\Omega, \mathbf{R}^N)$ into $L^p(\Omega)$ because of the continuity of g on $[u_0, u_1] \times \mathbf{R}^N$. Conditions (C_2) and (C_3) of Definition 2.1 then imply that $u \in \mathcal{C}(\mathbf{R}_0^+, L^p(\Omega)) \cap \mathcal{C}^1(\mathbf{R}^+, L^p(\Omega))$, and that u provides a classical solution to the evolution equation

$$(3.3) \quad \begin{cases} u'(t) = k(t)\,\Delta_{p,\mathcal{N}}\,u(t) + s(t)\,\mathfrak{g}(u(t), \nabla u(t)), & t \in \mathbf{R}^+ \\ u(t) \in (u_0, u_1) \end{cases}$$

in $L^p(\Omega)$. Now write $\|\cdot\|_{2,p}$ for the Sobolev norm in $H_{\mathcal{N}}^{2,p}(\Omega)$ defined by $\|z\|_{2,p} = \|(\lambda_0 - \Delta_{p,\mathcal{N}})\,z\|_p$ where $\lambda_0 \in \mathbf{R}^+$ is chosen in such a way that $\|z_0\|_{2,p} = |z_0|$ for every $z_0 \in \mathbf{R}$. The main result of this section is then the following.

THEOREM 3.1. – *Assume that hypotheses* $(\mathbf{K})$, $(\mathbf{S})$ *and* $(\mathbf{G})$ *hold and let* u *be any classical solution to Problem* (1.1). *Then there exists a unique classical solution* $\hat{u}$ *of* (1.1), *independent of* x *and bounded, such that*

$$(3.4) \qquad \lim_{t\to\infty} \|u(t) - \hat{u}(t)\|_{2,p} = 0$$

Referring back to the definition of the Sobolev norm, it is plain that statement (3.4) is equivalent to having $\|u(t) - \hat{u}(t)\|_p \to 0$ and $\|\Delta_{p,\mathcal{N}}\,u(t)\|_p \to 0$ as $t \to \infty$. Our arguments to prove these two statements rest upon the simultaneous use of parabolic comparison principles and of exponential dichotomies for the family of evolution operators generated by $k(t)\,\Delta_{p,\mathcal{N}}$.

We begin by outlining briefly our strategy. Our analysis is based on the introduction of an auxiliary function $v_\alpha : \overline{\Omega} \times \mathbf{R}_0^+ \to \mathbf{R}^+$ that satisfies a certain parabolic differential inequality when α is sufficiently large. The existence of such a function amounts to changing the dependent variable in Problem (1.1). To show what kind of function we are looking for we first notice that every x-independent solution $\widehat{u}$ to Problem (1.1) satisfies the scalar separable initial-value problem $\widehat{u}'(t) = s(t)\,g(\widehat{u}(t), 0)$, $\widehat{u}(0) = \widehat{\nu} \in (u_0, u_1)$ for every $t \in \mathbf{R}$. Such a solution $\widehat{u}$ is necessarily of the form

$$(3.5) \qquad \widehat{u}(t) = G^{-1}\left\{\int_0^t d\xi\, s(\xi) + G(\widehat{\nu})\right\}$$

where $G : (u_0, u_1) \to \mathbf{R}$ stands for any primitive of the function $u \to 1/g(u,0)$ and where G^{-1} denotes the monotone inverse of G. For any α, $v_0 \to \mathbf{R}^+$, we can always rewrite relation (3.5) as

$$\text{(3.6)} \qquad \widehat{u}\,(t) = G^{-1}\left\{\int_0^t d\xi\, s\,(\xi) + \alpha^{-1}\mathrm{Ln}(v_0) + G\,(\widehat{\mu})\right\}$$

for some $\widehat{\mu} \in (u_0, u_1)$. Of course, there is a one-to-one correspondence between $\widehat{\mu}$ and $\widehat{\nu}$ since G is strictly monotone. Then, given any classical solution u to Problem (1.1) as in Theorem 3.1, we define v_α as the function whose relationship to u is formally identical to the relationship between v_0 and $\widehat{u}$ in (3.6). This gives

$$\text{(3.7)} \qquad u\,(x,t) = G^{-1}\left\{\int_0^t d\xi\, s\,(\xi) + \alpha^{-1}\mathrm{Ln}(v_\alpha(x,t)) + G\,(\hat{\mu})\right\}$$

or equivalently

$$\text{(3.8)} \qquad v_\alpha(x,t) = \exp\left[\alpha\left\{\int_{\hat{\mu}}^{u(x,t)} \frac{d\xi}{g(\xi,0)} - \int_0^t d\xi\, s\,(\xi)\right\}\right]$$

We then proceed by showing that $v_\alpha$stabilizes around some $v_0 \in \mathbf{R}^+$ in $L^1(\Omega)$ as $t \to \infty$; this determines v_0 uniquely, which in turn determines the unique $\widehat{u}$ of Theorem 3.1 through relation (3.6). From this we infer that $||u(t) - \widehat{u}(t)||_p \to 0$ as $t \to \infty$, and finally that $||\Delta_{p,\mathcal{N}}u(t)||_p \to 0$ from the use of exponential dichotomies for the linear part of (3.3).

The derivation of a parabolic differential inequality for v_α requires the control of the dependence of g on ∇u. This is accomplished by using the quadratic growth estimate of hypothesis $(\mathbf{G})$. The precise result is

LEMMA 3.1. – *Let u be any classical solution to Problem* (1.1) *and let v_α be given by relation* (3.8) *where we assume $\alpha \in \mathbf{R}^+$ and $\widehat{\mu} \in (u_0, u_1)$. Then $v_\alpha \in \mathcal{C}^{2,1}(\Omega \times \mathbf{R}^+) \cap \mathcal{C}(\overline{\Omega} \times \mathbf{R}_0^+) \cap \mathcal{C}^{1,0}(\overline{\Omega} \times \mathbf{R}^+)$ and, for α sufficiently large, we have*

$$\text{(3.9)} \qquad \left\{\begin{array}{ll} (v_\alpha)_t(x,t) \leq k(t)\Delta v_\alpha(x,t), & (x,t) \in \Omega \times \mathbf{R}^+ \\ \dfrac{\partial v_\alpha}{\partial n}(x,t) = 0, & (x,t) \in \partial\Omega \times \mathbf{R}^+ \end{array}\right\}$$

In addition, there exists a positive constant c *such that* $v_\alpha(x,t) \leq c$ *for every* $(x,t) \in \overline{\Omega} \times \mathbf{R}_0^+$.

Proof. – The regularity statement follows from the corresponding properties of u. As for relations (3.9), we first note that

$$\nabla v_\alpha(x,t) = \alpha \frac{v_\alpha(x,t)}{g(u(x,t),0)} \nabla u(x,t) \tag{3.10}$$

so that v_α satisfies the homogeneous Neumann boundary condition since u does. This proves the second relation in (3.9); in order to prove that $k(t)\Delta v_\alpha(x,t) - (v_\alpha)_t(x,t) \geq 0$, we first calculate each term separately from relation (3.8) by making use of the first equation in (1.1). After regrouping the various contributions we obtain

$$\begin{aligned} k(t)\Delta v_\alpha(x,t) - (v_\alpha)_t(x,t) &= \\ = \alpha k(t) v_\alpha(x,t) &\left\{ \frac{\alpha - \frac{\partial g}{\partial u}(u(x,t),0)}{g^2(u(x,t),0)} \right\} |\nabla u(x,t)|^2 \\ - \alpha &\left\{ \frac{s(t)g(u(x,t),\nabla u(x,t))}{g(u(x,t),0)} - s(t) \right\} v_\alpha(x,t) \end{aligned} \tag{3.11}$$

Since both α and v_α are positive, we see that the right-hand side of (3.11) is non-negative if, and only if, the inequality

$$\begin{aligned} k(t)\left(\alpha - \frac{\partial g}{\partial u}(u(x,t),0)\right)|\nabla u(x,t)|^2 &\geq \\ \geq s(t)g(u(x,t),0)(g(u(x,t),\nabla u(x,t)) &- g(u(x,t),0)) \end{aligned} \tag{3.12}$$

holds. In order to prove this last inequality for α large enough, we construct an upper bound for the right-hand side and a lower bound for the left-hand side of (3.12) which still satisfy the above inequality for α sufficiently large. Since both s and $u \to g(u,0)$ are bounded and since g satisfies the quadratic growth condition of hypothesis (**G**), we first note that there exists $c_1 > 0$, depending only on u_0, u_1 and on the uniform norm of s, such that the estimate

$$s(t)g(u(x,t),0)(g(u(x,t),\nabla u(x,t)) - g(u(x,t),0)) \leq c_1 |\nabla u(x,t)|^2 \tag{3.13}$$

holds pointwise on $\Omega \times \mathbf{R}^+$. As for the left-hand side of (3.12), we write momentarily $m = \max_{u\in[u_0,u_1]} \frac{\partial g}{\partial u}(u,0)$ and choose $\alpha \in \mathbf{R}^+ \cap (m,\infty)$. Since $\underline{k} > 0$ by one of the requirements of hypothesis $(\mathbf{K})$, we get the lower bound

$$(3.14)\quad k(t)\left(\alpha - \frac{\partial g}{\partial u}(u(x,t),0)\right)|\nabla u(x,t)|^2 \geq \underline{k}(\alpha - m)|\nabla u(x,t)|^2$$

pointwise on $\Omega \times \mathbf{R}^+$. The comparison of the right-hand side of (3.14) with the right-hand side of (3.13) now shows that we always have

$$(3.15)\qquad \underline{k}(\alpha - m)|\nabla u(x,t)|^2 \geq c_1|\nabla u(x,t)|^2$$

for $\alpha \in \mathbf{R}^+ \cap [m + c_1\underline{k}^{-1},\infty)$; this proves that the first relation in (3.9) holds. Finally, we infer from relation (3.8) that

$$(3.16)\qquad v_\alpha(x,0) = \exp\left[\alpha \int_{\widehat{\mu}}^{u(x,0)} \frac{d\xi}{g(\xi,0)}\right]$$

Moreover, $u(x,0)$ is a positive distance away from u_0 and u_1 for every $x \in \overline{\Omega}$ since $x \to u(x,0)$ is continuous on $\overline{\Omega}$ and $\overline{\Omega}$ is compact. From this and relation (3.16), we infer that there exists $c > 0$ such that $v_\alpha(x,0) \leq c$ for every $x \in \overline{\Omega}$. The last statement of the lemma then follows from the parabolic maximum principle applied to Problem (3.9). □

The result of Lemma 3.1 and a simple consideration of symmetry imply the following result, which will also be useful.

LEMMA 3.2. – *Let u be any classical solution to Problem* (1.1). *Then the function* $(x,t) \to G(u(x,t))$ *is bounded on* $\overline{\Omega} \times \mathbf{R}_0^+$.

Proof. – From relation (3.7) we have

$$(3.17)\qquad G(u(x,t)) = \int_0^t d\xi s(\xi) + \alpha^{-1}\mathrm{Ln}(v_\alpha(x,t)) + G(\widehat{\mu})$$

so that this function is bounded from above by the last part of hypothesis $(\mathbf{S})$ and the last statement of Lemma 3.1. In order to get a bound from below, we cannot proceed directly from (3.17) since there is no known strictly positive lower bound for v_α. We avoid this difficulty by observing that there exists a constant k_1 such that $u_1 - u(x,t) \geq k_1 > 0$ for each $(x,t) \in \overline{\Omega} \times \mathbf{R}_0^+$; if this were not the case, there would exist a sequence $((x_n, t_n)) \subseteq \overline{\Omega} \times \mathbf{R}_0^+$ such that $u(x_n, t_n) \to u_1$ as $n \to \infty$, and hence such that $G(u(x_n, t_n)) \to \infty$, since the first part of hypothesis $(\mathbf{G})$ implies that $G(u) \to \infty$ as $u \to u_1$. This would contradict the fact that (3.17) is bounded from above. By symmetry, the existence of the above constant k_1 then immediately implies the existence of a constant $k_0 \in (0, u_1 - u_0)$ such that

$$u(x,t) - u_0 \geq k_0 > 0 \tag{3.18}$$

for every $(x,t) \in \overline{\Omega} \times \mathbf{R}_0^+$. In order to see this it is sufficient to define the function $\widetilde{u}$ by $\widetilde{u}(x,t) = u_0 + u_1 - u(x,t)$ and to notice that $\widetilde{u}$ provides a classical solution to a problem of the form (1.1) with a new function $\widetilde{s}$ and a new nonlinearity $\widetilde{g}$ which still satisfy hypothesis $(\mathbf{S})$ and $(\mathbf{G})$, respectively. From relation (3.18), we infer that $u(x,t) \geq u_0 + k_0 \in (u_0, u_1)$ for every $(x,t) \in \overline{\Omega} \times \mathbf{R}_0^+$, and hence that $G(u(x,t)) \geq G(u_0 + k_0)$ since G is monotone increasing on (u_0, u_1). $\square$

Lemma 3.2 immediately implies the following result through relation (3.17).

LEMMA 3.3. – *We have* $\inf_{(x,t)\in\overline{\Omega}\times\mathbf{R}_0^+} v_\alpha(x,t) > 0$ *for every* $\alpha > 0$ *sufficiently large.*

Our last preliminary result concerns the properties of exponential dichotomies for the family of evolution operators $\{U(t,r)\}_{t\geq r}$ generated by $k(t)\Delta_{p,\mathcal{N}}$. Let $\{W_{\Delta_{p,\mathcal{N}}}(\sigma)\}_{\sigma\geq 0}$ be the diffusion semigroup on $L^p(\Omega)$ whose infinitesimal generator is $\Delta_{p,\mathcal{N}}$. Since $k > 0$ on $\mathbf{R}_0^+$, the function $t \to \int_r^t d\xi k(\xi)$ is strictly monotone increasing and it is easily verified that

$$U(t,r) = W_{\Delta_{p,\mathcal{N}}}\left(\int_r^t d\xi k(\xi)\right) \tag{3.19}$$

From relation (3.19), it follows that the exponential dichotomies for $\{U(t,r)\}_{t\geq r}$ are determined by those of $\{W_{\Delta_{p,\mathcal{N}}}(\sigma)\}_{\sigma\geq 0}$. Let $(\lambda_k)_{k=1}^\infty \cup \{0\}$ be the pure point spectrum of $\Delta_{p,\mathcal{N}}$ where $(\lambda_k) \subseteq \mathbf{R}^-$. There is a natural separation between the subspaces of $L^p(\Omega)$ corresponding to the negative eigenvalues (λ_k) and the eigenspace corresponding to $\lambda = 0$, which implies that $\{W_{\Delta_{p,\mathcal{N}}}(\sigma)\}_{\sigma\geq 0}$ decays exponentially on the former at a rate determined by the largest negative eigenvalue λ_1. Write P and Q for the corresponding projection operators; that is, Q is the projection from $L^p(\Omega)$ onto the eigenspace of $\Delta_{p,\mathcal{N}}$ corresponding to $\lambda = 0$ while P is the complementary projection. Owing to relation (3.19) and to the fact $\{W_{\Delta_{p,\mathcal{N}}}(\sigma)\}_{\sigma \geq 0}$ is an analytic semigroup, the preceding remarks lead to the following result (We also refer the reader to [13] and [19] for recent investigations of exponential dichotomies for linear evolutionary equations in infinite dimensions).

LEMMA 3.4. – *There exists $c > 0$ such that the two estimates*

$$\|U(t,r)P\varphi\|_p \leq c\exp[-|\lambda_1|\underline{k}(t-r)]\|\varphi\|_p \tag{3.20}$$

$$\|\Delta_{p,\mathcal{N}}U(t,r)P\varphi\|_p \leq c\underline{k}^{-1}(t-r)^{-1}\exp[-|\lambda_1|\underline{k}(t-r)]\|\varphi\|_p \tag{3.21}$$

hold for every t, r with $t - r \in \mathbf{R}^+$, and for every $\varphi \in L^p(\Omega)$. In relation (3.20) *and* (3.21), *λ_1 denotes the largest negative eigenvalue of $\Delta_{p,\mathcal{N}}$ and $\underline{k}$ denotes the positive constant of hypothesis* ($\mathbf{K}$).

Proof. – From relation (3.19) and the properties of exponential dichotomies for the diffusion semigroup on $L^p(\Omega)$, we immediately infer that

$$\|U(t,r)P\varphi\|_p \leq c\exp\left[-|\lambda_1|\int_r^t d\xi k(\xi)\right]\|\varphi\|_p$$

and that

$$\|\Delta_{p,\mathcal{N}}U(t,r)P\varphi\|_p \leq c\left(\int_r^t d\xi k(\xi)\right)^{-1}\exp\left[-|\lambda_1|\left(\int_r^t d\xi k(\xi)\right)\right]\|\varphi\|_p$$

for some $c > 0$. Relations (3.20) and (3.21) then follow from the definition of $\underline{k}$ in hypothesis ($\mathbf{K}$). □

With Lemmata 3.1–3.4 we can now prove the following

PROPOSITION 3.1. – *Given any classical solution to Problem* (1.1), *there exists a unique classical solution $\widehat{u}$ of* (1.1), *independent of x and bounded, such that*

$$\lim_{t\to\infty}\|u(t) - \widehat{u}(t)\|_p = 0 \tag{3.22}$$

Proof. – Let $v_\alpha(t) : \overline{\Omega} \to \mathbf{R}^+$ be defined by $v_\alpha(t)(x) = v_\alpha(x,t)$ where v_α is given by relation (3.8). The function $t \to Qv_\alpha(t)$ is monotone decreasing because of inequality (3.9), which implies that $Qv_\alpha(t) \to v_0 = \inf_{t\in\mathbf{R}_0^+} Qv_\alpha(t)$ as $t \to \infty$. By Lemma (3.3) we have $v_0 > 0$. In addition $Pv_\alpha(t) \to 0$ strongly in $L^1(\Omega)$ as $t \to \infty$, so that $v_\alpha(t) \to v_0$ strongly in $L^1(\Omega)$ as $t \to \infty$. Given this v_0, let $\widehat{u}$ be the unique classical solution to Problem (1.1) given by relation (3.6). From relations (3.6), (3.7), the fact that G^{-1} has a uniformly bounded derivative on $\mathbf{R}$ and Lemma 3.3, we infer the sequence of estimates

$$\int_\Omega dx|u(t)(x) - \widehat{u}(t)| \leq 0(1) \int_\Omega dx|Ln(v_\alpha(x,t)) - Ln(v_0)| \leq$$
$$\leq 0(1) \max\left((\inf v_\alpha(x,t))^{-1},\, v_0^{-1}\right) \int_\Omega dx|v_\alpha(t)(x) - v_0| \to 0$$

as $t \to \infty$, so that $u(t) - \widehat{u}(t) \to 0$ in $L^1(\Omega)$. By classical interpolation, it is now sufficient to prove that $u(t) - \widehat{u}(t)$ remains bounded in $L^\infty(\Omega)$ for large times in order to get relation (3.22). But this follows immediately from the boundedness of the derivative of G^{-1}, Lemma 3.2 and the boundedness of $t \to \int_0^t d\xi s(\xi)$, for

$$\|u(t) - \widehat{u}(t)\|_\infty = \sup_{x\in\overline{\Omega}} |G^{-1}(G(u(x,t))) - G^{-1}(G(\widehat{u}(t)))| \leq$$
$$\leq 0(1) \sup_{x\in\overline{\Omega}} |G(u(x,t)) - G(\widehat{u}(t))| \leq c < \infty$$

for some $c > 0$, uniformly in t. □

In order to complete the proof of Theorem 3.1, it remains to show that $\Delta_{p,\mathcal{N}} u(t) \to 0$ strongly in $L^p(\Omega)$ as $t \to \infty$. This is obviously equivalent to proving that $k(t)\Delta_{p,\mathcal{N}} u(t) \to 0$ in $L^p(\Omega)$, because of the last part of hypothesis (**K**). Write $y(t) = u(t) - \widehat{u}(t)$; referring back to equation (3.3) we obtain

$$y'(t) = k(t)\Delta_{p,\mathcal{N}} y(t) + s(t)(\mathfrak{g}(\widehat{u}(t) + y(t), \nabla y(t)) - \mathfrak{g}(\widehat{u}(t), 0)) \tag{3.23}$$

for the corresponding evolution equation. Since $k(t)\Delta_{p,\mathcal{N}} u(t) = k(t)\Delta_{p,\mathcal{N}} y(t)$, it is then sufficient to prove that

$$s(t)(\mathfrak{g}(\widehat{u}(t) + y(t), \nabla y(t)) - \mathfrak{g}(\widehat{u}(t), 0)) \to 0 \tag{3.24}$$

$$y'(t) \to 0 \tag{3.25}$$

strongly in $L^p(\Omega)$. The complication here is again due to the gradient in equation (3.23), and we shall see that the Hölder continuity properties of k, s, as well as relations (3.1), (3.2), Lemma 3.4 and Proposition 3.1 are all important to control it. We proceed by stating the existence of a uniform bound for $\nabla u(t) = \nabla y(t)$.

LEMMA 3.5. – *Let u be any classical solution to Problem* (1.1). *Then there exists a constant $c \in (0, \infty)$ such that the estimate*

$$\sup_{(x,t)\in\overline{\Omega}\times\mathbf{R}_0^+} |\nabla y(t)(x)| \leq c \tag{3.26}$$

holds.

Proof. – Because of the range condition in (1.1), there exists $\overline{c} \in (0, \infty)$ such that $\sup_{(x,t)\in\overline{\Omega}\times\mathbf{R}_0^+} |u(t)(x)| \leq \overline{c}$. The fact that (3.26) holds then follows from the standard *a priori* estimates for semilinear parabolic equations ([2], [14]). □

According to right-hand side of (3.23), we now define $f : \mathbf{R}_0^+ \to L^p(\Omega)$ by

$$f(t) = s(t)(\mathfrak{g}(\widehat{u}(t) + y(t), \nabla y(t)) - \mathfrak{g}(\widehat{u}(t), 0)) \tag{3.27}$$

This means that

$$f(t)(x) = s(t)(g(\widehat{u}(t) + y(x,t), \nabla y(x,t)) - g(\widehat{u}(t), 0))$$

for every $(x, t) \in \overline{\Omega} \times \mathbf{R}^+$. We proceed by proving some essential pointwise and L^p-bounds for f.

LEMMA 3.6. – *There exists $c \in (0, \infty)$ such that the pointwise estimate*

$$|f(t)(x)| \leq c\{|y(x,t)| + |\nabla y(x,t)|\} \tag{3.28}$$

holds for every $(x, t) \in \overline{\Omega} \times \mathbf{R}_0^+$. Furthermore

$$\|f(t)\|_p^p \leq 0(1)\{\|y(t)\|_p^p + \|\nabla y(t)\|_p^p\} \tag{3.29}$$

for each $t \in \mathbf{R}_0^+$. Finally, the mapping $f : \mathbf{R}_0^+ \to L^p(\Omega)$ is Hölder-Lipschitz continuous on $[\tau, \infty)$.

Proof. – From the smoothness hypothesis concerning g and relation (3.27), we may perform a Taylor expansion around $(\hat{u}(t), 0)$; we obtain

$$(3.30)\quad f(t)(x) = s(t)\int_0^1 d\mathfrak{s}\left\{\frac{\partial g}{\partial u}(\widehat{u}(t) + \mathfrak{s}y(x,t), \mathfrak{s}\nabla y(x,t))y(x,t) + \right.$$
$$\left. + \nabla_q g(\widehat{u}(t) + \mathfrak{s}y(x,t), \mathfrak{s}\nabla y(x,t)) \cdot \nabla y(x,t)\right\}$$

Now the first part of hypothesis (**G**) implies that g and all its first-order partial derivatives are bounded on bounded subsets of $\mathbf{R} \times \mathbf{R}^N$. Estimate (3.28) then follows from (3.30) upon using the boundedness of s, together with the boundedness of $(x, t, \mathfrak{s}) \to \widehat{u}(t) + \mathfrak{s}y(x,t)$ and that of $(x, t, \mathfrak{s}) \to \mathfrak{s}\nabla y(x,t)$, the latter being an immediate consequence of Lemma 3.5. Estimate (3.29) is then a direct consequence of inequality (3.28) and of the convexity of $\xi \to |\xi|^p$. As for the last statement of the Lemma, we start again from relation (3.28) and use an appropriate Taylor expansion as well as the *a priori* estimates for $u(t)$ and $\nabla u(t)$; we obtain

$$\| f(t) - f(t')\|_p^p \leq$$
$$\leq 0(1)\{\, |s(t) - s(t')|^p + |\widehat{u}(t) - \widehat{u}(t')|^p + ||u(t) - u(t')||_p^p + ||\nabla u(t) - \nabla u(t')||_p^p\}$$

and the assertion follows from the Hölder-Lipschitz property of hypothesis (**S**), relations (3.1) and (3.2) and the fact that $\widehat{u}$ is Lipschitz continuous. □

The comparison of relation (3.29) with relation (3.27) now shows that it is sufficient to prove that $||\nabla y(t)||_p \to 0$ in order to get (3.24) since we already know that $||y(t)||_p \to 0$ by Proposition 3.1. We achieve this in Proposition 3.2 below . We first need an *a priori* bound on the strong derivative $y'(t)$.

LEMMA 3.7. – *There exists* $c \in (0, \infty)$ *such that*

$$(3.31)\qquad \sup_{t \in \mathbf{R}_0^+} ||y'(t)||_p \leq c$$

Proof. – Let P and Q be the projection operators defined before the statement of Lemma 3.4. Relation (3.31) then amounts to proving an L^p-bound for both $Py'(t)$ and $Qy'(t)$ since P and Q are complementary projectors in $L^p(\Omega)$. We first note that $Q\Delta_{p,\mathcal{N}} = \Delta_{p,\mathcal{N}}Q = 0$ on $H^{2,p}_{\mathcal{N}}(\Omega)$ as a consequence of Gauss' divergence theorem. Applying then the operator Q on both sides of equation (3.23) and using relation (3.27) we obtain $Qy'(t) = Qf(t)$. But from relation (3.29), Proposition 3.1 and lemma 3.5 we infer that $t \to ||f(t)||_p$ is bounded, so that $Qy'(t)$ is bounded in the L^p-topology as well. It remains to prove that $t \to ||Py'(t)||_p$ is bounded. For this we invoke the variation of constants formula along with relation (3.19). For every $t \in [\tau, \infty)$ we obtain

(3.32)

$$\begin{aligned}
Py'(t) &= k(t)\Delta_{p,\mathcal{N}}U(t,\tau)Py(\tau) + \int_\tau^t d\xi k(t)\Delta_{p,\mathcal{N}}U(t,\xi)Pf(\xi) + Pf(t) \\
&= k(t)\Delta_{p,\mathcal{N}}U(t,\tau)Py(\tau) + \int_\tau^t d\xi k(t)\Delta_{p,\mathcal{N}}U(t,\xi)P(f(\xi) - f(t)) + \\
&\quad + Pf(t) + \int_\tau^t d\xi k(t)\Delta_{p,\mathcal{N}}U(t,\xi)Pf(t) = \\
&= k(t)\Delta_{p,\mathcal{N}}U(t,\tau)Py(\tau) + \int_\tau^t d\xi k(t)\Delta_{p,\mathcal{N}}U(t,\xi)P(f(\xi) - f(t)) + \\
&\quad + U(t,\tau)Pf(t) + \int_\tau^t d\xi (k(t) - k(\xi))\Delta_{p,\mathcal{N}}U(t,\xi)Pf(t)
\end{aligned}$$

We conclude the proof by showing that each term in (3.32) remains bounded in the L^p-topology as time becomes large. This is evident for the first and third terms since k and f are bounded and since relations (3.20) and (3.21) hold. In fact, these two terms converge exponentially rapidly to zero as $t \to \infty$. As for the second term in relation (3.32) we get the estimate

$$\begin{aligned}
(3.33) \quad & \int_\tau^t d\xi ||k(t)\Delta_{p,\mathcal{N}}U(t,\xi)P(f(\xi) - f(t))||_p \leq \\
& \leq c\underline{k}^{-1}\overline{k} \int_\tau^t d\xi (t-\xi)^{-1} \exp[-|\lambda_1|\underline{k}(t-\xi)] ||f(\xi) - f(t)||_p \leq \\
& \leq 0(1) \int_\tau^t d\xi (t-\xi)^{\beta} \exp[-|\lambda_1|\underline{k}(t-\xi)]
\end{aligned}$$

for some $\beta \in (-1, 0]$, upon using the boundedness of k, relation (3.21) and the Hölder continuity of f. It is now plain that the last integral in (3.33) is uniformly bounded in t as $t \to \infty$. Because of the boundedness of f and the Hölder continuity of k, the fourth term in (3.32) has an identical structure and can be handled in a similar way. □

We can now prove the desired result.

PROPOSITION 3.2. – *Given any classical solution to Problem* (1.1) *we have*

$$\lim_{t \to \infty} ||\Delta_{p,\mathcal{N}} u(t)||_p = 0 \tag{3.34}$$

Proof. – From the above considerations the proof of relation (3.34) is reduced to proving that $||\nabla u(t)||_p = ||\nabla y(t)||_p \to 0$ and that $||y'(t)||_p \to 0$. We first notice that the *a priori* estimate (3.26) implies the L^∞-bound $||\nabla y(t)||_\infty \leq c$ for each $t \in \mathbf{R}_0^+$. Again by classical interpolation, this means that it is sufficient to prove the L^2-convergence $||\nabla y(t)||_2 \to 0$ as $t \to \infty$ in order to get $||\nabla y(t)||_p \to 0$. To this end, define the function $Y : \mathbf{R}_0^+ \to \mathbf{R}_0^+$ by

$$Y(t) = 1/2 \int_\Omega dx y^2(t)(x) = 1/2||y(t)||_2^2 \tag{3.35}$$

This function is clearly differentiable and, upon using equation (3.23) and relation (3.27), we obtain

$$\begin{aligned} Y'(t) &= \int_\Omega dx y(t)(x) y'(t)(x) = \\ &= \int_\Omega dx y(t)(x) f(t)(x) - k(t) \int_\Omega dx |\nabla y(t)(x)|^2 \end{aligned} \tag{3.36}$$

after one integration by parts to account for Neumann's boundary condition. Since k is uniformly bounded away from zero by hypothesis (**K**), relation (3.36) then allows us to write successively

$$\begin{aligned} \underline{k}||\nabla y(t)||_2^2 &\leq k(t) \int_\Omega dx |\nabla y(t,x)|^2 \leq \\ &\leq \int_\Omega dx |y(t)(x)||f(t)(x)| + \int_\Omega dx |y(t)(x)||y'(t)(x)| \leq \\ &\leq 0(1)||y(t)||_2(||y(t)||_2 + ||\nabla y(t)||_2 + ||y'(t)||_2) \end{aligned} \tag{3.37}$$

as a consequence of the pointwise estimate (3.28) and Schwarz inequality. We now observe that the first factor in the last expression converges to zero as $t \to \infty$ as a consequence of proposition 3.1, while the second factor remains bounded as a consequence of lemmata 3.5 and 3.7. These facts and estimate (3.37) then imply that $||\nabla y(t)||_2 \to 0$ as $t \to \infty$, and hence that $||\nabla y(t)||_p \to 0$ as already noticed. We conclude by proving that $||y'(t)||_p \to 0$ as $t \to \infty$. Our point of departure for this is again the observation that $||y'(t)||_p \to 0$ if, and only if, $||Py'(t)||_p \to 0$ and $||Qy'(t)||_p \to 0$. Since $Qy'(t) = Qf(t)$ and since $||f(t)||_p \to 0$ by the first part of the proof and relation (3.29), we already have $||Qy'(t)||_p \to 0$. In order to handle $Py'(t)$, our starting point is once more relation (3.32). We already know that the first and third terms of (3.32) converge exponentially rapidly to zero as a consequence of the exponential dichtomies of lemma 3.4. It remains to show that the two integrals in (3.32) also converge to zero in the L^p-topology as $t \to \infty$. For the last integral it is immediate since we have the sequence of estimates

$$\begin{aligned}(3.38)\qquad &\int_T^t d\xi ||(k(t) - k(\xi))\Delta_{p,\mathcal{N}} U(t,\xi) Pf(t)||_p \leq \\ &\leq 0(1) \int_T^t d\xi (t-\xi)^\mu ||\Delta_{p,\mathcal{N}} U(t,\xi) Pf(t)||_p \leq \\ &\leq 0(1) \int_T^t d\xi (t-\xi)^\beta \exp[-|\lambda_1|\underline{k}(t-\xi)] ||f(t)||_p\end{aligned}$$

as a consequence of the Hölder-Lipschitz continuity of k and relation (3.21), where $\beta \in (-1, 0]$. The assertion then follows from the fact that the last integral in (3.38) remains bounded and that $||f(t)||_p \to 0$ when $t \to \infty$. As for the remaining integral in (3.32), we first define function $\chi : \mathbf{R}^+ \to \mathbf{R}_0^+$ by

$$(3.39)\qquad \chi(\omega) = \sup\{||f(t) - f(t')||_p : t, t' \in [\omega, \infty)\}$$

From this definition and the fact that $||f(t)||_p \to 0$, it follows that χ is monotone decreasing and that $\chi(\omega) \to 0$ as $\omega \to \infty$. Furthermore, the Hölder-Lipschitz continuity of f and relation (3.39) both imply that

$$(3.40)\qquad ||f(t) - f(t')||_p \leq 0(1)\sqrt{\chi(\omega)}|t - t'|^\sigma$$

for $t, t' \in [\omega, \infty)$ and some $\sigma \in (0,1)$. The third integral in (3.32) can then be estimated from above by

$$(3.41)\ \int_{\tau}^{t} d\xi ||k(t)\Delta_{p,\mathcal{N}} U(t,\xi) P(f(\xi) - f(t))||_p \leq$$
$$\leq 0(1) \int_{\tau}^{t} d\xi (t-\xi)^{-1} \exp[-|\lambda_1| \underline{k}(t-\xi)] ||f(\xi) - f(t)||_p \leq$$
$$\leq 0(1)\sqrt{\chi(\omega)} \int_{\tau}^{t} d\xi (t-\xi)^{\beta} \exp[-|\lambda_1| \underline{k}(t-\xi)] \leq$$
$$\leq 0(1)\sqrt{\chi(\omega)}$$

whenever $t, \tau \in [\omega, \infty)$, since the last integral in (3.41) is bounded. This immediately implies that for each $\varepsilon > 0$, there exists $\omega_\varepsilon > 0$ such that

$$\left\| \int_{\tau}^{t} d\xi k(t) \Delta_{p,\mathcal{N}} U(t,\xi) P(f(\xi) - f(t)) \right\|_p < \varepsilon$$

for $t, \tau \in [\omega_\varepsilon, \infty)$. We conclude that $Py'(t) \to 0$, and hence that $y'(t) \to 0$ in $L^p(\Omega)$ as $t \to \infty$. □

Propositions 3.1 and 3.2 prove Theorem 3.1 entirely. Owing to the geometric interpretation of the projection operator Q, the stabilization result of Theorem 3.1 corresponds in a certain sense to a reduction of problem (3.3) by a center-manifold of dimension one, since the large-time dynamics are eventually governed by the scalar ordinary differential equation $\widehat{u}'(t) = s(t)g(\widehat{u}'(t), 0)$, $\widehat{u}(0) = \widehat{\nu} \in (u_0, u_1)$ where $t \in \mathbf{R}$. our proofs of the main theorems of Section 2 then amount to investigating the possibility of having further reductions of the solutions $\{\widehat{u}\}_{\widehat{\nu} \in (u_0,u_1)}$ when $t \to \infty$. If s is recurrent, then each $\widehat{u}$ is also recurrent and no further reduction is possible. This is part of the content of Theorem 2.1 whose proof is given first.

Proof of Theorem 2.1. – Let u be any classical solution to Problem (1.1). By Theorem 3.1 there exists a unique classical solution $\widehat{u}$ of (1.1), independent of x and bounded, such that $||u(t) - \widehat{u}(t)||_{2,p} \to 0$ as $t \to \infty$. This immediately implies relations (2.1) since there exists the continuous embedding $H^{2,p}(\Omega) \to C^1(\overline{\Omega})$. In order to prove that (a) implies (b), it remains then to prove that the recurrence of s implies the recurrence of $\widehat{u}$, in fact with the same returning sequence (t_n). Thus assume that (a) holds with the returning sequence (t_n); since $t \to \int_0^t d\xi s(\xi)$ is bounded, this implies that $t \to \int_0^t d\xi s(\xi)$ is recurrent as well with the same returning sequence (t_n) [15]. Since $\widehat{u}$ is necessarily of the form (3.5) and G^{-1} has a uniformly bounded derivative on $\mathbf{R}$, we then infer that the estimate

$$\begin{aligned}(3.42)\qquad & \sup_{t\in I}|\widehat{u}(t+t_n)-\widehat{u}(t)|\leq \\ & \leq 0(1)\sup_{t\in I}\left|\int_0^{t+t_n}d\xi s(\xi)-\int_0^t d\xi s(\xi)\right|\to 0\end{aligned}$$

holds as $n\to\infty$, for every compact interval $I\subset\mathbf{R}$. This and the preceding remark prove (b). Conversely, assume that there exists an x-independent and recurrent solution $\widehat{u}$ to Problem (1.1). Then $\widehat{u}$ is of the form (3.5) and there exists $(t_n)\to\infty$ such that

$$(3.43)\qquad \sup_{t\in I}|\widehat{u}(t+t_n)-\widehat{u}(t)|\to 0$$

holds as $n\to\infty$, for every compact interval $I\subset\mathbf{R}$. From relation (3.5) or from the proof of Lemma 3.2, we further infer that there exist constants $k_{0,1}\in(u_0,u_1)$ such that $u_0<k_0\leq\widehat{u}(t)\leq k_1<u_1$ for every $t\in\mathbf{R}$. This and the smoothness of g then allow us to conclude that

$$\begin{aligned}&\sup_{t\in I}\left|\int_0^{t+t_n}d\xi s(\xi)-\int_0^t d\xi s(\xi)\right|=\\ &=\sup_{t\in I}\left|\int_{\widehat{u}(t)}^{\widehat{u}(t+t_n)}\frac{d\xi}{g(\xi,0)}\right|\leq 0(1)\sup_{t\in I}|\hat{u}(t+t_n)-\widehat{u}(t)|\to 0\end{aligned}$$

as $n\to\infty$, so that $t\to\int_0^t d\xi s(\xi)$ is recurrent. Now write momentarily F for this primitive of s and define the sequence of functions $(\Phi_N)_{N\in\mathbf{N}}$ by $(\Phi_N)(t)=N\{F(t+1/N)-F(t)\}$. Obviously Φ_N is recurrent for each N with the same returning sequence (t_n). Furthermore, the Hölder-Lipschitz continuity of s implies that s is uniformly continuous on $\mathbf{R}$, which immediately implies that $\Phi_N\to s$ uniformly as $N\to\infty$. Hence s is recurrent as well with the same returning sequence (t_n), which proves (a) and the very last statement of the theorem. □

If s is not recurrent, there exists a very simple reduction of $\widehat{u}$ if $\int_0^\infty d\xi s(\xi)<\infty$. In fact, we see from the explicit from (3.5) that

$\lim_{t\to\infty}\widehat{u}(t)$ exists if, and only if, $\int_0^\infty d\xi s(\xi) < \infty$. This and Theorem 3.1 lead to the proof of the second theorem of Section 2.

Proof of Theorem 2.2. – Let u be any classical solution to Problem (1.1). As before, we infer that there exists a classical solution $\widehat{u}$ of (1.1), independent of x and bounded, such that $||u(t) - \widehat{u}(t)||_{2,p} \to 0$ as $t \to \infty$. Now assume that condition (a) holds; from relation (3.5) we infer that $\lim_{t\to\infty}\widehat{u}(t) = a_u$ for some $a_u \in (u_0, u_1)$, and consequently that $||u(t) - a_u||_{2,p} \to 0$ as $t \to \infty$. This and the continuous embedding $H^{2,p}(\Omega) \to C^1(\overline{\Omega})$ imply statement (b). Conversely, assume that statement (b) holds. This means that $||u(t) - a_u||_{C^1(\overline{\Omega})} \to 0$ as $t \to \infty$. Since we also have $||u(t) - \widehat{u}(t)||_{2,p} \to 0$ for some $\widehat{u}$ of the form (3.5), we get *a fortiori* $||u(t) - \widehat{u}(t)||_{C^1(\overline{\Omega})} \to 0$ and hence $|\widehat{u}(t) - a_u| \to 0$. But this last relation and relation (3.5) imply that $\int_0^\infty d\xi s(\xi) < \infty$. □

We now turn to the proof of the third theorem of Section 2 where a more subtle kind of reduction takes place. That theorem illustrates the simplest case of a reduction when s is not recurrent and when $-\infty < \liminf_{t\in\mathbf{R}} \int_0^t d\xi s(\xi) < \limsup_{t\in\mathbf{R}} \int_0^t d\xi s(\xi) < \infty$.

Proof of Theorem 2.3. – Let u be any classical solution to Problem 1.1. Then $||u(t) - \widehat{u}(t)||_{2,p} \to 0$ as before, where $\widehat{u}$ is necessarily of the form

$$\widehat{u}(t) = G^{-1}\left\{\int_0^t d\xi s_0(\xi) + \int_0^t d\xi s_1(\xi) + G(\widehat{\nu})\right\}$$

for some $\widehat{\nu} \in (u_0, u_1)$. Now define the function $\widehat{u}_0$ by

$$\widehat{u}_0(t) = G^{-1}\left\{\int_0^t d\xi s_0(\xi) + G(\widehat{\nu}_0)\right\} \tag{3.44}$$

where $\widehat{\nu}_0 = G^{-1}\{\int_0^\infty d\xi s_1(\xi) + G(\widehat{\nu})\}$. Clearly $\widehat{u}_0$ is recurrent and satisfies the initial-value problem $\widehat{u}'(t) = s_0(t)g(\widehat{u}(t), 0)$, $\widehat{u}(0) = \widehat{\nu}_0$ for $t \in \mathbf{R}$. Then

$$|\widehat{u}(t) - \widehat{u}_0(t)| \le 0(1)\left|\int_0^t d\xi s_1(\xi) + G(\widehat{\nu}) - G(\widehat{\nu}_0)\right| \to 0$$

as $t \to \infty$ by the above choice of $\widehat{\nu}_0$, from which we conclude that $||u(t) - \widehat{u}(t)||_{C^1(\overline{\Omega})} \to 0$ holds as $t \to \infty$. The fact that the returning sequence for $\widehat{u}_0$ may be chosen to be the same as that of s_0 follows from the arguments given in the proof of Theorem 2.1. □

As already noticed in Section 2, the structure of the set $\mathbf{E}(s)$ can be very complicated when s is not recurrent and when relation (2.4) holds. Speaking very loosely, Theorem 2.4 asserts that there are as many relevant limiting equations of the form $(u^*)'(t) = s^*(t)g(u^*(t), 0)$ as there are elements in $\mathbf{E}(s)$, which eventually describe the dynamics of all solutions to (1.1) along appropriate sequences (t_n). We now turn to the proof of that result.

Proof of Theorem 2.4. – Hypothesis $(\mathbf{S})$ first implies that s is bounded and uniformly continuous. It then follows from Ascoli's theorem that the set of all translates of s is relatively compact in $\mathcal{C}(\mathbf{R})$. This and the general principles of topological dynamics immediately prove statement (a), since $\mathbf{E}(s)$ is the ω-limit set of s relative to the dynamics generated by the time-translation operator in $\mathcal{C}(\mathbf{R})$ [22]. Now let u be any classical solution to Problem (1.1); again by Theorem 3.1 and the continuous embedding $H^{2,p}(\Omega) \to C^1(\overline{\Omega})$, there exists a unique solution $\widehat{u}$ to (1.1) of the form (3.5) such that the relations

$$\lim_{t\to\infty} \sup_{x\in\overline{\Omega}} |u(x, t) - \widehat{u}(t)| = 0 \tag{3.45}$$

$$\lim_{t\to\infty} \sup_{x\in\overline{\Omega}} |\nabla u(x, t)| = 0 \tag{3.46}$$

hold. Pick an $s^* \in \mathbf{E}(s)$ and let $(t_n^*) \to \infty$ be any sequence such that $\sup_{t\in I} |s(t + t_n^*) - s^*(t)| \to 0$ as $n \to \infty$ for every compact interval $I \subset \mathbf{R}$. Since $\widehat{u}$ is bounded and a positive distance away from u_0 and u_1 (compare with the proof of Lemma 3.2), there exists a subsequence $(t_n) \subseteq (t_n^*)$ such that $\widehat{u}(t_n) \to a^* \in (u_0, u_1)$ as $n \to \infty$. We then consider the initial-value problem

$$\left\{\begin{array}{ll} (u^*)'(t) = s^*(t)g(u^*(t), 0), & t \in \mathbf{R} \\ u^*(0) = a^* \in (u_0, u_1) & \end{array}\right\} \tag{3.47}$$

whose unique solution is

$$u^*(t) = G^{-1}\left\{\int_0^t d\xi s^*(\xi) + G(a^*)\right\} \tag{3.48}$$

On the one hand, we have $G(\widehat{u}(t_n)) - G(a^*) \to 0$ as $n \to \infty$ by definition of a^*. On the other hand, for every compact interval $I \subset \mathbf{R}$ there exists a positive constant $c(I)$ such that the estimate

$$\sup_{t\in I}\left|\int_0^t d\xi(s(\xi+t_n)-s^*(\xi))\right|\leq c(I)\sup_{t\in[-c(I),c(I)]}|s(t+t_n)-s^*(t)|\to 0 \tag{3.49}$$

holds as $n\to\infty$. From the above considerations, relations (3.5), (3.48), (3.49) and the fact that G^{-1} has a uniformly bounded derivative, we infer the sequence of estimates

$$\begin{aligned}
&\sup_{t\in I}|\widehat{u}(t+t_n)-u^*(t)|\leq \\
&\leq 0(1)\sup_{t\in I}\left|\int_0^{t+t_n} d\xi s(\xi)+G(\widehat{\nu})-\int_0^t d\xi s^*(\xi)-G(a^*)\right|= \\
&\leq 0(1)\sup_{t\in I}\left|\int_0^t d\xi(s(\xi+t_n)-s^*(\xi))+G(\widehat{u}(t_n))-G(a^*)\right|\leq \\
&\leq 0(1)\sup_{t\in I}\left|\int_0^t d\xi(s(\xi+t_n)-s^*(\xi))\right|+0(1)|G(\widehat{u}(t_n))-G(a^*)|\to 0
\end{aligned} \tag{3.50}$$

as $n\to\infty$, for every compact interval $I\subset\mathbf{R}$. Relation (2.5) now follows from relations (3.45) and (3.50), while relation (2.6) is relation (3.46). □

The proof of Theorem 2.5 follows from similar arguments and is therefore omitted.

Remark. – There is of course a very close connection between the above method of proof of Theorem 2.1-2.5 and the language of skew-product flows developed by SACKER and SELL in [18]. Thus for any $s\in\mathcal{C}(\mathbf{R})$ satisfying hypothesis $(\mathbf{S})$, let $\mathbf{H}(s)\subset\mathcal{C}(\mathbf{R})$ be the hull of s relative to the Fréchet topology of $\mathcal{C}(\mathbf{R})$. Then it is clear that $\mathbf{E}(s)\subseteq\mathbf{H}(s)$, and that the scalar equation $\widehat{u}'(t)=s(t)g(\widehat{u}(t),0)$ generates a skew-product flow π on $\mathbf{R}\times\mathbf{H}(s)$ in the usual sense. Let $\mathfrak{p}$: $\mathbf{R}\times\mathbf{H}(s)\mapsto\mathbf{H}(s)$ be the associated canonical projection; let $\widehat{u}$ be any solution of the form (3.5) and let $\mathbf{E}(\widehat{u},s)$ be the ω-limit set of the pair $(\widehat{u},s)$ in π. Using the fact that for any two distinct solutions $\widehat{u}$ and $\widehat{u}^*$ of the form (3.5) we have $\inf_{t\in\mathbf{R}}|\widehat{u}(t)-\widehat{u}^*(t)|>0$, we can then easily infer from the results and methods of Chapters 2 and 3 of [18] that the set $\mathbf{E}(\widehat{u},s)\cap\mathfrak{p}^{-1}(s^*)$ is a singleton for every $s^*\in\mathbf{E}(s)$, namely $\mathbf{E}(\widehat{u},s)\cap\mathfrak{p}^{-1}(s^*)=\{(u^*,s^*)\}$ where u^* is of the form (3.48). This fact and Theorem 3.1 then also imply the statement of theorem 2.4 (or 2.5), and hence also those of Theorems

2.1–2.3. Of course, the use of this skew-product flow formalism becomes indispensable to the qualitative analysis of more general equations such as (1.2) or such as those investigated in ([23], [24]), for which explicit solutions such as (3.5) are not available.

4. Some concluding remarks

In this paper we have investigated the long-time behaviour of solutions to the class of non-autonomous parabolic boundary-value problems given by relations (1.1) and hypotheses (**K**), (**S**) and (**G**). By using the properties of exponential dichotomy associated with the principal part of the equations along with some general principles of topological dynamics, we have shown that the long-time dynamics of those solutions are governed by a compact, connected and invariant set of completely integrable ordinary differential equations. In particular, we have exhibited a necessary and sufficient condition for Problem (1.1) to possess a global recurrent attractor, as well as a necessary and sufficient condition for Problem (1.1) to possess an attractor that consists exclusively of spatially and temporally homogeneous solutions of some limiting equations. Finally, we would like to mention the recent works of VISHIK and CHEPYZHOV (see, for instance, [27] and some of its references) who have devised a completely different method to investigate the existence of global attractors of non-autonomous partial differential equations and their properties. Their method is, however, chiefly focused on the almost-periodic and the quasi-periodic cases.

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