Oscillation Criteria for PDE with $p$-Laplacian via the Riccati Technique

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The conditions on the function $c(x) : \mathbb{R}^n \to \mathbb{R}$ are derived, which ensure that the PDE $\text{div}(|\nabla u|^{p-2} \nabla u) + c(x)|u|^{p-2}u = 0$, $p > 1$, is oscillatory; i.e., $\infty$ belongs to the closure of the set of zeros of every solution defined on the domain $\Omega = \{x \in \mathbb{R}^n : ||x|| > 1\}$. The main tool is the Riccati technique combined with suitable a priori bounds.

Key Words: $p$-Laplacian; oscillatory solution; Riccati equation.

1. INTRODUCTION

Let us consider the partial differential equation with $p$-Laplacian and the nonlinearity of the Emden–Fowler type

$$\text{div}(|\nabla u|^{p-2} \nabla u) + c(x)\Phi(u) = 0, \quad (1.1)$$

where $p > 1$, $\Phi(u) = |u|^{p-2}u = |u|^{p-1} \text{sgn} u$, $x = (x_1, x_2, \ldots, x_n)$, $||\cdot||$ is the usual Euclidean norm in $\mathbb{R}^n$, and $\nabla$ is the usual nabla operator. Define the sets $\Omega(a) = \{x \in \mathbb{R}^n : a \leq ||x||\}$, $\Omega(a, b) = \{x \in \mathbb{R}^n : a \leq ||x|| \leq b\}$.

The function $c(x)$ is assumed to be integrable on every compact subset of $\Omega(1)$. The solution of Eq. (1.1) is every function of the class $W^{1,p}_{\text{loc}}(\Omega(1))$ which satisfies Eq. (1.1) almost everywhere on $\Omega(1)$.

Definition 1.1. The function $u : \mathbb{R}^n \to \mathbb{R}$ is said to be oscillatory if $u$ has zero on $\Omega(a)$ for every $a \in \mathbb{R}^+$. Equation (1.1) is said to be oscillatory if every solution (if any exists) is oscillatory. Conversely, Eq. (1.1) is nonoscillatory if there exists a solution which is not oscillatory.

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The special cases of Eq. (1.1) are the linear Schrödinger equation
\[ \Delta u + c(x)u = 0 \] (1.2)
if \( p = 2 \) and the half-linear ordinary differential equation
\[ (\Phi(u'))' + c(x)\Phi(u) = 0, \quad ' = \frac{d}{dx} \] (1.3)
if \( n = 1 \).

Concerning the linear PDE (1.2) there exists well-elaborated oscillation theory. Definition 1.1 is, according to this theory, the definition of weak oscillation. A somewhat different approach in the oscillation theory makes use of nodal domains. A bounded domain \( \Omega \subseteq \mathbb{R}^n \) is said to be the nodal domain of a nontrivial solution of Eq. (1.2) if \( u|_{\partial \Omega} = 0 \). Equation (1.2) is said to be nodally (strongly) oscillatory if for every \( a \in \mathbb{R}^+ \) there exists a nontrivial solution of Eq. (1.2) having nodal domain in \( \Omega(a) \). These two concepts of oscillation are equivalent, provided the function \( c(x) \) is locally Hölder continuous; see [1, 18]. For further references about oscillation theory of linear PDE see [11, 19, 23, 26].

Basic features of oscillation theory of linear PDE have been developed also for several types of nonlinear equations, which own similar behavior as the linear equations. The reader is referred to [12, 16, 21, 22, 24]. For more comprehensive discussion about the oscillation theory of Eq. (1.1) see [8]. For references concerning the ordinary half-linear equation (1.3) see [3, 5–7, 10, 13–15].

The equations with \( p \)-Laplacian have wide applications in various physical and biological problems. These equations describe the behavior of the systems whose energetic functional is of degree \( p \). Further let us mention the equation
\[ \text{div}(\|\nabla u\|^{p-2} \nabla u) + \lambda u = 0 \] (1.4)
which appears in the study of non-Newtonian fluids. (Non-Newtonian fluids are typically the fluids which are a suspension of particles, deformed by the acting of forces, in liquid. Most biological fluids, like blood, have this property.) Other relevant applications of the equations with \( p \)-Laplacian are in the glaciology and slow diffusion problems (the flow through porous media, e.g., a dam filled by rocks). For more examples of applications the reader is referred to [4] and the references therein.

Among the equations with \( p \)-Laplacian Eq. (1.1) plays a special role. Its importance lies in the fact that many of the results of qualitative theory of linear ODE extend also for Eqs. (1.1) and (1.3); see the references given
above and also [9] for references about eigenvalue problems. These equations are called half-linear, since the set of solutions preserves homogeneity (a constant multiple of any solution is again a solution of (1.1), (1.3)). A good knowledge of the properties of half-linear Eq. (1.1) allows us to study also the equations of the type (1.4) as the “sub-half-linear” or “supper-half-linear” case. A similar procedure in the case of (1.2) can be found, e.g., in [20, 22, 25].

This paper is motivated by the papers [3, 23], where the Riccati technique is used to establish oscillation criteria for Eqs. (1.3) and (1.2), respectively. The aim of this paper is to connect some ideas from the oscillation theory of Eqs. (1.2) and (1.3) and give an analogous oscillation criteria for Eq. (1.1). Remark that criteria of this type were only recently published even in the case of the linear Sturm–Liouville ordinary differential equation [2].

**Notation.** Denote

$$C_p(t) = \frac{p - 1}{t^{p-1}} \int_1^t s^{p-2} \int_{\Omega(t, s)} \|x\|^{1-n} c(x) \, dx \, ds.$$

In the paper we will suppose that there exists a finite limit

$$\lim_{t \to \infty} C_p(t) = C_0. \quad (1.5)$$

We will use the method introduced by Lomtatidze et al. in [2, 3]. Denote

$$Q(t) = t^{p-1} \left( C_0 - \int_{\Omega(t, 1)} \|x\|^{1-n} c(x) \, dx \right);$$

$$H(t) = \frac{1}{t} \int_{\Omega(t, 1)} \|x\|^{p-n+1} c(x) \, dx;$$

$$Q_* = \liminf_{t \to \infty} Q(t), \quad Q^* = \limsup_{t \to \infty} Q(t);$$

$$H_* = \liminf_{t \to \infty} H(t), \quad H^* = \limsup_{t \to \infty} H(t).$$

Further \( \omega_n \) is the measure of the \( n \)-dimensional unit sphere in \( \mathbb{R}^n \) and \( q \) is the conjugate number to \( p \), i.e., \( q = \frac{p}{p-1} \).

If \( Q_* \leq (p-n)/p(\omega_n/(p-1)) \), then the equation

$$(p-1)\omega_n^{-q/p} |x|^q + (n-p)x + (p-1)Q_* = 0 \quad (1.6)$$
has two zeros (counting multiplicity). Denote by $A$ the smaller of them. Similarly, if $H_* \leq \left| \frac{p-1}{p} \right| \omega_n$, then $B$ denotes the larger of the zeros of the equation

\[
(p - 1) \omega_n^{-q/p} |x|^q + (n - p)x + H_* = 0. \tag{1.7}
\]

The oscillation criteria are formulated in terms of the functions $C_p(t)$, $Q(t)$, $H(t)$, and numbers $Q_*$, $Q^*$, $H_*$, $H^*$, $A$, and $B$.

Remark that if the condition (1.5) fails to hold and $\lim \inf_{t \to \infty} C_p(t) > -\infty$, then (1.1) is oscillatory [16, Theorem 2.2].

The main results of this paper are formulated in the next section. Section 3 is devoted to some remarks and comments. In Section 4 we introduce some properties of the nonoscillatory equation which are used in the last section in proofs of the main results.

2. MAIN RESULTS

**Theorem 2.1.** Let (1.5) hold and

\[
\lim \sup_{t \to \infty} \frac{(p - 1)}{\ln t} \left[ C_0 - C_p(t) \right] > \frac{p - n}{p} \left| \omega_n \right|. \tag{2.1}
\]

Then Eq. (1.1) is oscillatory.

Suitable modifications of the left hand side of (2.1) lead to the following

**Corollary 2.1.** Let (1.5) hold. Each of the following conditions is sufficient for Eq. (1.1) to be oscillatory:

(i) $Q_* > -\infty$ and

\[
\lim \sup_{t \to \infty} \frac{1}{\ln t} \int_{\Omega(t)} \|x\|^{p-n} c(x) \, dx > \left| \frac{n - p}{p} \right| \omega_n; \tag{2.2}
\]

(ii) let

\[
\lim \inf_{t \to \infty} \left[ Q(t) + H(t) \right] > \frac{p}{p - 1} \left| \frac{n - p}{p} \right| \omega_n; \tag{2.3}
\]

(iii) let

\[
Q_* > \frac{1}{p - 1} \left| \frac{n - p}{p} \right| \omega_n; \tag{2.4}
\]

(iv) let

\[
H_* > \left| \frac{n - p}{p} \right| \omega_n. \tag{2.5}
\]
If \( p = n \), then the oscillation constant in Theorem 2.1 and Corollary 2.1 equals zero. In this case the criteria including \( \lim \sup \), i.e., criteria (2.1) and (2.2), can be formulated in a more precise way.

**Theorem 2.2.** Let \( p = n \) and (1.5) hold. Each of the following conditions guarantees oscillation of Eq. (1.1):

(i) \( \lim_{t \to \infty} t^{n-1}[C_0 - C(t)] = \infty; \)

(ii) \( Q_* > -\infty \) and \( \lim_{t \to \infty} \int_{(1,t]} C(x) \, dx = \infty. \)

The following theorem completes criterion (2.3) in some sense.

**Theorem 2.3.** Let (1.5) and

\[
\lim_{t \to \infty} \left[ Q(t) + H(t) \right] > \left| \frac{1-n}{p} \right| \omega_n + \left| \frac{p-n+1}{p} \right| \omega_n, \quad (2.6)
\]

Then Eq. (1.1) is oscillatory.

The following theorems continue with the cases, when (2.4) and (or) (2.5) do not hold. Suppose

\[
\frac{(n-1) - p(p-1)}{p(p-1)} \Phi \left( \frac{n-1}{p} \right) \omega_n \leq Q_* \leq \left| \frac{n-p}{p} \right| \omega_n, \quad (2.7)
\]

and (or)

\[
\frac{1-n}{p} \Phi \left( \frac{p-n+1}{p} \right) \omega_n \leq H_* \leq \left| \frac{n-p}{p} \right| \omega_n. \quad (2.8)
\]

In the following theorem we suppose that only one of these inequalities holds.

**Theorem 2.4.** Let (1.5) holds. Each of the following conditions implies oscillation of Eq. (1.1):

(i) inequality (2.7) and \( H^* > \left| \frac{n-p+1}{p} \right| \omega_n - A \) hold; \quad (2.9)

(ii) inequality (2.8) and \( Q^* > \left| \frac{1-n}{p} \right| \omega_n + B \) hold. \quad (2.10)

If both (2.7) and (2.8) hold, the constants in (2.6), (2.9), and (2.10) can be decreased, as the following theorem shows.
THEOREM 2.5. Let (1.5), (2.7), and (2.8) hold. Each of the following conditions implies oscillation of Eq. (1.1):

(i) \( Q^* > Q_* - A + B \);  
(ii) \( H^* > H_* - A + B \);  
(iii) \( \limsup_{t \to \infty} [Q(t) + H(t)] > Q_* + H_* - A + B \).  

3. REMARKS AND COMMENTS

Remark 3.1. If the limit
\[
\lim_{t \to \infty} \int_{\Omega(1,t)} \|x\|^{1-n} c(x) \, dx
\]
exists, then the limit (1.5) exists too and both limits are equal. If the limit (3.1) is finite, then \( Q(t) \) takes the form
\[
Q(t) = t^{p-1} \int_{\Omega(t)} \|x\|^{1-n} c(x) \, dx.
\]
Remark that the existence of (3.1) is not necessary for existence of the limit (1.5). For \( p = 2, n = 1 \), \( c(x) \geq 0 \) is the criterion (2.4) of the well-known Hille’s criterion.

Example 3.1. For the function
\[
c(x) = \frac{\gamma}{\|x\|^p}
\]
we obtain
\[
C(t) = \gamma \omega_n \left( \frac{1}{p-1} - \frac{1}{(p-1)t^{p-1}} - \frac{\ln t}{t^{p-1}} \right),
\]
\[
C_0 = \frac{\gamma \omega_n}{p-1}, \quad Q(t) = \frac{\gamma \omega_n}{p-1}, \quad H(t) = \gamma \omega_n \frac{t - 1}{t}.
\]
Hence each of (2.3), (2.4), (2.5) implies that
\[
\text{div}(\|\nabla u\|^{p-2} \nabla u) + \frac{\gamma}{\|x\|^p} \Phi(u) = 0
\]
is oscillatory for \( \gamma > \left| \frac{p-n}{p} \right|^p \).
Remark 3.2. Criteria (2.1)–(2.5), (2.11)–(2.13) are optimal and the constants cannot be improved downwards. This follows from the fact that Eq. (3.2) with \( \gamma = \|x\|^p \) is nonoscillatory, since \( \|x\|^{(p-\alpha)/p} \) is a positive solution, and the function \( c(x) \) produces equality in the above mentioned criteria.

Remark 3.3. A closer examination of the inequalities in Theorem 2.5 shows that the right-hand side of (2.11) is less than the right-hand side of (2.10). Consequently, if (2.7) holds, then the criterion (2.11) is sharper than the criterion (2.10). The same situation also applies in the case of inequalities (2.12), (2.9) and (2.13), (2.6).

Remark 3.4. Substitution \( x = \frac{\xi}{1-x} \) in the inequality

\[
1 - px \leq |1 - x|^p
\]  

and multiplication by \( (n - 1)/(p - 1)\omega_n/(p - 1) \) yields

\[
\frac{(n - 1) - p(p - 1)}{p(p - 1)} \Phi\left(\frac{n - 1}{p}\right) \omega_n \leq \left|\frac{n - p}{p}\right| \omega_n
\]  

In the case \( n = 1 \) this inequality is trivial. Similarly we can obtain

\[
\frac{1 - n}{p} \Phi\left(\frac{p - n + 1}{p}\right) \omega_n \leq \left|\frac{n - p}{p}\right| \omega_n
\]  

from (3.3) substituting \( x = \frac{1}{p - n + 1} \) and multiplying by \( |\frac{n - p}{p}| \omega_n \). Hence both (2.7) and (2.8) are meaningful.

Substituting \( x = \frac{1}{p - n + 1} \) into the inequality

\[
|1 - px|^p + p - 1 \geq p|1 - x|^p
\]  

and multiplying by \( (p - n + 1)/p\omega_n/(p - 1) \) yields

\[
\left|\frac{n - p}{p}\right| \omega_n \geq \left|\frac{n - p}{p}\right| \omega_n
\]  

and for \( p = n - 1 \) this inequality is trivial. From here it follows that the oscillation constant in the criterion (2.3), \( \liminf_{t \to a} [Q(t) + H(t)] \), is less than that one in the criterion (2.6), \( \limsup_{t \to a} [Q(t) + H(t)] \).

Example 3.2. Let us consider the equation

\[
\text{div}(\|u\|^{p-2} \nabla u) + \alpha \left[ \frac{(p - 1)[\sin|\cdot| + 1]}{|\cdot|^{p-1}} - \frac{\cos|\cdot|}{|\cdot|} \right] \Phi(u) = 0,
\]  

\[
\alpha > 0.
\]  

(3.4)
Direct calculation shows

\[ Q(t) = \alpha (1 + \sin t) \omega_n, \]

\[ H(t) = \alpha \left[ (p - 1) - \sin t + \frac{p \cos 1 + \sin 1 - p \cos t - (p - 1)}{t} \right] \omega_n \]

\[ Q_* = 0, \quad Q^* = \alpha \omega_n, \quad H_* = \alpha (p - 2) \omega_n, \quad H^* = \alpha p \omega_n \]

\[ \lim_{t \to \infty} \left[ Q(t) + H(t) \right] = \alpha p \omega_n. \]

Hence by (2.3), Eq. (3.4) is oscillatory for \( \alpha > \frac{|z^n - z|}{p - 1} \). (Criterion 2.5 produces a poorer bound for \( \alpha \).

Suppose that, in addition, \( p(p - n) \geq n - 1 \) and \( p \geq n \). Then (2.7) holds and Eq. (1.6) has roots 0 and \( \Phi(p - n - 1) \alpha \). Hence \( A = 0 \) and (2.12) implies that (3.4) is oscillatory for every \( \alpha > 0 \).

4. NONOSCILLATORY EQUATION

The first lemma ensures that Eqs. (1.6) and (1.7) are solvable and introduces some useful properties of the function defined by the left hand side of these equations.

**LEMMA 4.1.** Let \( \alpha \in \mathbb{R} \) be an arbitrary number. The function

\[ y(x) = (p - 1) \omega_n^{-q/p} |x|^q + \alpha x \]

has the following properties:

(i) \( y(x) \) has its global minimum in the point \( \hat{x} = -\omega_n \Phi \left( \frac{x}{p} \right) \) and \( y(\hat{x}) = -\omega_n |\hat{x}|^{p/\gamma} \).

(ii) \( y(x) \) is decreasing on \( (-\infty, \hat{x}] \) and increasing on \( [\hat{x}, \infty) \);

(iii) \( \lim_{x \to \pm \infty} y(x) = \infty \).

**Proof.** This follows immediately from computation

\[ y' = (p - 1) q \omega_n^{-q/p} |x|^{q-2} x + \alpha. \]

Now suppose that Eq. (1.1) is nonoscillatory. There exists a number \( a \in \mathbb{R}^+ \) and a solution \( u \) of (1.1) which is positive on \( \Omega(a) \). The vector function \( w = \|\nabla u\|^p - 2 \nabla u / \Phi(u) \) is defined on \( \Omega(a) \) and solves here the *Riccati type equation*

\[ \text{div} w + c(x) + (p - 1) \|w\|^q = 0, \quad (4.1) \]

where \( q \) is the conjugate number of \( p \) (i.e., \( \frac{1}{p} + \frac{1}{q} = 1 \) holds).
Lemma 4.2. Suppose that there exists a finite limit (1.5) and let \( w \) be the solution of (4.1) defined on \( \Omega(a) \) for some \( a > 1 \). Then

\[
\int_{\Omega(a)} \|x\|^{1-n} \|w\|^q \, dx < \infty;
\]

and (4.1) can be written in the integral form

\[
C_a - \int_{\Omega(1,t)} \|x\|^{1-n} c(x) \, dx
= \int_{\|x\|=t} \|x\|^{1-n} \langle w, j \rangle \, d\sigma
= -(p-1) \int_{\Omega(t)} \|x\|^{1-n} \|w\|^q \, dx + (1-n) \int_{\Omega(t)} \|x\|^{-n} \langle w, j \rangle \, dx.
\]

Here \( \langle \cdot, \cdot \rangle \) is the usual scalar product in \( \mathbb{R}^n \), \( j \cdot d\sigma \) denotes the surface integral, and \( j \) is the unit outside normal vector to the sphere in \( \mathbb{R}^n \).

Proof. This follows from [16, Theorem 2.1].

If (1.1) is nonoscillatory and \( w \) is the solution of (4.1) defined on \( \Omega(a) \), then denote

\[
\rho(t) = t^{p-1} \int_{\|x\|=t} \|x\|^{1-n} \langle w, j \rangle \, d\sigma \quad \text{for } t \geq a;
\]

\[
r = \liminf_{t \to \infty} \rho(t), \quad R = \limsup_{t \to \infty} \rho(t);
\]

\[
\hat{A} = -\Phi\left( \frac{n-1}{p} \right) \omega_n, \quad \hat{B} = -\Phi\left( \frac{n-p-1}{p} \right) \omega_n.
\]

Below we prove a priori bounds for the numbers \( r, R \). First one technical lemma.

Lemma 4.3. Let (1.5) hold, (1.1) be nonoscillatory, \( w \) be the solution of (4.1) defined on \( \Omega(a) \) for some \( a > 1 \), and \( \rho(t) \) the function defined by (4.4).
The following estimations are true for every $t \geq \tau \geq a$,

$$\left| \rho(t) \right|^p \leq \omega_n^{q/p} t^p \int_{\|x\| = t} \|x||^{1-n}\|\|w\|^q \, d\sigma; \quad (4.5)$$

$$Q(t) \leq \rho(t) - t^{p-1} \int_{t}^{\infty} \left[ (p-1) \omega_n^{-q/p} |\rho(s)|^p - (1-n) \rho(s) \right] s^{-p} \, ds; \quad (4.6)$$

$$H(t) \leq -\rho(t) - \frac{1}{t} \int_{\tau}^{t} \left[ (p-1) \omega_n^{-q/p} |\rho(s)|^p - (p-n+1) \rho(s) \right] \frac{\omega_n}{p-1} \, ds$$

$$+ \frac{\tau}{t} \left[ \rho(t) + H(\tau) \right]; \quad (4.7)$$

$$Q(t) \leq \rho(t) + \left( \frac{1-n}{p} \right) \omega_n^{p-1} \frac{\omega_n}{p-1}; \quad (4.8)$$

$$H(t) \leq -\rho(t) + \left( \frac{p-n+1}{p} \right) \omega_n + \frac{\tau}{t} \left[ \rho(t) + H(\tau) \right]. \quad (4.9)$$

**Proof.** Inequality (4.5) follows from the definition of the function $\rho(t)$ and from the Schwarz and Hölder inequalities. Multiplying Eq. (4.3) by $t^{p-1}$ we obtain

$$Q(t) = \rho(t) - t^{p-1}$$

$$\times \int_{t}^{\infty} \left[ (p-1) \int_{\|x\| = s} \|x||^{1-n}\|\|w\|^q \, ds - (1-n) \rho(s)s^{-p} \right] \, ds.$$  

Now (4.5) implies (4.6). Using derivation of (4.3) we obtain

$$- \int_{\|x\| = t} \|x||^{1-n} c(x) \, d\sigma$$

$$= \frac{d}{dt} \frac{\rho(t)}{t^{p-1}}$$

$$+ (p-1) \int_{\|x\| = t} \|x||^{1-n}\|w\|^q \, d\sigma - (1-n) \int_{\|x\| = t} \|x||^{1-n} \langle w, j \rangle \, d\sigma.$$
Multiplying this equality by $t^p$ and integrating from $\tau \geq a$ to $t$ we obtain
\[
-\int_{\Omega(\tau,t)} ||x||^{p-n+1} c(x) \, dx
= t \rho(t) - \tau \rho(\tau) - p \int_{\tau}^{t} \rho(s) \, ds
+ \int_{\tau}^{t} \left( (p - 1) s^p \int_{||x||=s} ||x||^{1-n} ||w\|^{q} \, d\sigma - (1 - n) \rho(s) \right) \, ds.
\]
Now the fact that
\[
t H(t) - \tau H(\tau) = \int_{\Omega(\tau,t)} ||x||^{p-n+1} c(x) \, dx
\]
and inequality (4.5) imply (4.7). The terms in brackets in inequalities (4.6) and (4.7) can be estimated using Lemma 4.1 and after integration we get (4.8) and (4.9). Note that $\hat{A}$ and $\hat{B}$ are the points, which, substituted to $\rho(s)$ in (4.6) and (4.7), realize the minimal value of the function in brackets.

**Lemma 4.4.** Let (1.1) be nonoscillatory and (1.5) hold. If
\[
\frac{(n - 1) - p(p - 1)}{p(p - 1)} \Phi \left( \frac{n - 1}{p} \right) \omega_n \leq Q_{*} \leq \left| \frac{n - p}{p} \right| \frac{\omega_n}{p - 1},
\]
then
\[
r \geq A \geq \hat{A},
\]
where $A$ denotes the smaller of zeros of Eq. (1.6).

**Proof.** Let $w$ be the solution of the Riccati equation (4.1) defined on $\Omega(a)$, $a > 1$. If $r = \infty$, there is nothing to prove. Suppose that $r < \infty$. If $Q_{*} = \frac{(n - 1) - p(p - 1)}{p(p - 1)} \Phi \left( \frac{n - 1}{p} \right) \omega_n$, then $\hat{A}$ solves Eq. (1.6) and lies on the left hand side from the global minimum of the function defined by the left hand side of this equation. Then Lemma 4.1 implies that $A = \hat{A}$. Now inequality (4.8) implies
\[
r \geq Q_{*} - \left| \frac{1 - n}{p} \right| \frac{\omega_n}{p - 1} = \hat{A} = A
\]
and (4.10) holds.

Suppose that $\varepsilon_{0} := Q_{*} - \frac{(n - 1) - p(p - 1)}{p(p - 1)} \Phi \left( \frac{n - 1}{p} \right) \omega_n > 0$. Then clearly $A > \hat{A}$. For every $\varepsilon \in (0, \varepsilon_{0})$ there exists $t_{\varepsilon} > a$ such that for every $t \geq t_{\varepsilon}$
\[
\rho(t) \geq r - \varepsilon > r - Q_{*} + \frac{(n - 1) - p(p - 1)}{p(p - 1)} \Phi \left( \frac{n - 1}{p} \right) \omega_n.
\]
From inequality (4.8) we have

\[ Q_* \leq r + \frac{1 - n}{p} \frac{\omega_n}{p - 1} \]

and combining the last two inequalities we get

\[ \rho(t) > r - \epsilon > -\Phi \left( \frac{n - 1}{p} \right) \omega_n \]

for \( t \geq t_\epsilon \). From here and from Lemma 4.1 it follows that the right hand side of (4.6) can be increased substituting \( \rho(s) \) by \( r - \epsilon \), if \( t \geq t_\epsilon \). Hence, after integration, we have

\[
(p - 1)Q(t) < (p - 1)\rho(t) - (p - 1)\omega_n^{-q/p}|r - \epsilon|^q + (1 - n)(r - \epsilon)
\]

for every \( t \leq t_\epsilon \). The limit process \( \lim_{\epsilon \to 0^+} \liminf_{t \to \infty} \) gives

\[
(p - 1)\omega_n^{-q/p}|r|^q + (n - p)r + (p - 1)Q_* \leq 0
\]

which implies \( r \geq A \). The lemma is proved.

**Lemma 4.5.** Let (1.5) hold and let (1.1) be nonoscillatory. If

\[
\frac{1 - n}{p} \Phi \left( \frac{p - n + 1}{p} \right) \omega_n \leq H_* \leq \left| \frac{n - p}{p} \right| \omega_n,
\]

then

\[
R \leq B \leq \hat{B},
\]

where the number \( B \) denotes the larger zero of Eq. (1.7).

**Proof.** The proof is almost the same as the proof of Lemma 4.4. Let \( w \) be the solution of the Riccati equation (4.1) defined on \( \Omega(a) \), \( a > 1 \). If \( R = -\infty \), there is nothing to prove. Suppose that \( R > -\infty \). If \( H_* = \frac{1 - n}{p} \Phi \left( \frac{p - n + 1}{p} \right) \omega_n \), then \( \hat{B} \) solves Eq. (1.7) and lies on the right hand side from the global minimum of the function defined by the left hand side of this equation. Then Lemma 4.1 implies that \( B = \hat{B} \). Now inequality (4.9) implies

\[
R \leq -H_* + \left| \frac{p - n + 1}{p} \right|^p \omega_n = \hat{B} = B
\]

and (4.11) holds.
Suppose that \( \varepsilon_0 := H_* - \frac{1 - n}{p} \Phi\left(\frac{p - n + 1}{p}\right) \omega_n > 0 \). Then clearly \( B < \hat{B} \).

For every \( \varepsilon \in (0, \varepsilon_0) \) there exists \( t_\varepsilon \) such that for every \( t \geq t_\varepsilon \)

\[
\rho(t) \leq R + \varepsilon < R + H_* - \frac{1 - n}{p} \Phi\left(\frac{p - n + 1}{p}\right) \omega_n.
\]

From inequality (4.9) we have

\[
H_* \leq -R + \left| \frac{p - n + 1}{p} \right| \omega_n.
\]

Combining the last two inequalities we get

\[
\rho(t) < R + \varepsilon < -\Phi\left(\frac{p - n + 1}{p}\right) \omega_n
\]

for \( t \geq t_\varepsilon \). From here and from Lemma 4.1 it follows that the right hand side of (4.7) can be increased substituting \( \rho(s) \) by \( R + \varepsilon \), if \( \tau \geq t_\varepsilon \). Hence, after integration, we have

\[
H(t) < -\rho(t) - (p - 1) \omega_n^{-q/p} |R + \varepsilon|^q
\]

\[
+ (p - n + 1)(R + \varepsilon)\left(1 - \frac{\tau}{t}\right) + \frac{\tau}{t} \left[ \rho(\tau) + H(\tau) \right]
\]

for every \( t \leq t_\varepsilon \). The limit process \( \lim_{\varepsilon \to 0^+} \liminf_{t \to \infty} \) gives

\[
(p - 1) \omega_n^{-q/p} |R|^q + (n - p) R + H_* \leq 0
\]

which implies \( R \leq B \). The lemma is proved. \( \blacksquare \)

5. PROOFS OF THE MAIN RESULTS

Proof of Theorem 2.1. Suppose, by contradiction, that (1.5) holds, there exists a solution \( w(x) \) of the Riccati equation defined on \( \Omega(a) \) for some \( a > 1 \), and (2.1) does not hold. We start with equality (4.3). Multiplying it
by \(( p - 1)t^{p-2}\), integrating over \([a, t]\), and using (4.4) we obtain

\[
( p - 1) \int_a^t \frac{\rho(s)}{s} \, ds
\]

\[
= ( p - 1) \int_a^t \frac{\rho(s)}{s} \, ds
\]

\[
- ( p - 1) \int_{\Omega(t)} \|x\|^{|1-n|w|^q} \, dx
\]

\[
+ ( p - 1) \int_{\Omega(\omega)} \|x\|^{|1-n|w|^q} \, dx
\]

\[
- ( p - 1) \int_a^t s^{p-1} \int_{\|x\|=s} \|x\|^{|1-n|w|^q} \, d\sigma \, ds
\]

\[
+ (1-n) t^{p-1} \int_{\Omega(t)} \|x\|^{|1-n|w,j} \, dx
\]

\[
- (1-n) \int_{\Omega(\omega)} \|x\|^{|1-n|w,j} \, dx
\]

\[
+ (1-n) \int_a^t \frac{\rho(s)}{s} \, ds,
\]

where on the right hand side the last six terms arise from integration by
parts from the last two terms in (4.3). From here and from inequality (4.5)
we conclude

\[
( p - 1) \int_a^t \left[ \left( p - 1 \right) \omega_n^{-\frac{q}{p}} \rho(s) \right]^q - (1-n) \rho(s) \right] s^{-p} \, ds
\]

\[
- \int_a^t \left[ \left( p - 1 \right) \omega_n^{-\frac{q}{p}} \rho(s) \right]^q - \left( p - n \right) \rho(s) \right] \frac{1}{s} \, ds,
\]

where the constant terms are joint in one constant \(K\). The terms in the
integrals can be estimated by Lemma 4.1 and after integration we obtain

\[
( p - 1) \int_a^t \left[ \left( p - 1 \right) \omega_n^{-\frac{q}{p}} \rho(s) \right]^q - \left( p - n \right) \rho(s) \right] \frac{1}{s} \, ds
\]

\[
+ \int_a^t \frac{1-n}{p} \left( \frac{\omega_n}{p-1} \right)^{\frac{p}{p-1}} \omega_n \ln \frac{t}{a}
\]

(5.1)

for \(t \geq a\), which contradicts (2.1). The theorem is proved.
Proof of Corollary 2.1. Suppose that (1.5) holds. From the definition of the function \( Q(t) \) it follows

\[
t^{p-1}(C_0 - C(t)) = Q(t) + t^{p-1} \int_{\Omega(t, t)} \|x\|^{1-n} c(x) \, dx
\]

\[ - (p - 1) \int_1^t s^{p-2} \int_{\Omega(s, t)} \|x\|^{1-n} c(x) \, dx \, ds.
\]

Integration by parts in the second integral gives

\[
t^{p-1}(C_0 - C(t)) = Q(t) - \int_{\Omega(t, t)} \|x\|^{p-n} c(x) \, dx. \tag{5.2}
\]

Now the statement (i) follows from Theorem 2.1. Similarly from the definition of the function \( Q(t) \) it follows

\[
t^{p-1}(C_0 - C(t)) = (p - 1) \int_1^t \frac{Q(s)}{s} \, ds + C_0.
\]

From here and from Theorem 2.1 it follows that (1.1) is oscillatory if (iii) holds. Integrating by parts in the last equality we have

\[
t^{p-1}(C_0 - C(t))
\]

\[ = (p - 1) \left[ \frac{1}{t} \int_1^t Q(s) \, ds + \int_1^t \frac{1}{s^2} \int_1^s Q(\xi) \, d\xi \, ds \right] + C_0. \tag{5.3}
\]

Further

\[
\int_1^t Q(s) \, ds = tQ(t) - Q(1) - \int_1^t sQ'(s) \, ds
\]

\[ = tQ(t) - C_0 - C_0 \frac{p - 1}{p} (t^p - 1)
\]

\[ + (p - 1) \int_1^t s^{p-1} \int_{\Omega(s, t)} \|x\|^{1-n} c(x) \, dx \, ds
\]

\[ + \int_1^t s^p \int_{\|x\| = s} \|x\|^{1-n} c(x) \, d\sigma \, ds.
\]

Integration by parts in the first integral on the right hand side and the definitions of the functions \( Q(t) \) and \( H(t) \) give

\[
\int_1^t Q(s) \, ds = \frac{1}{p} (tQ(t) - tH(t) - C_0).
\]
Hence

\[ Q(t) + H(t) = \frac{p}{t} \int_{1}^{t} Q(s) \, ds + \frac{C_0}{t} \quad (5.4) \]

holds for every \( t \geq 1 \). If (ii) holds, then

\[ \liminf_{t \to \infty} \frac{1}{t} \int_{1}^{t} Q(s) \, ds > \frac{1}{p - 1} \left| \frac{n - p}{p} \right| \omega_n \]

and equality (5.3) with the Theorem 2.1 implies oscillation of Eq. (1.1). The last statement follows from the equalities

\[
\begin{aligned}
C(t) - C(\tau) &= (p - 1) \int_{\tau}^{1} \frac{\ln s}{s^p} \int_{\Omega(1,s)} \|x\|^{n-p} c(x) \, dx \, ds \\
\frac{1}{\ln t} \int_{\Omega(1,t)} \|x\|^{n-p} c(x) \, dx &= \frac{H(t)}{\ln t} + \frac{1}{\ln t} \int_{1}^{t} \frac{H(s)}{s} \, ds,
\end{aligned}
\]

which can be derived in a similar way. \( \square \)

**Proof of Theorem 2.2.** The proof follows immediately from the proof of Theorem 2.1 and Corollary 2.1, where we substitute \( p = n \) in (5.1) and (5.2). \( \square \)

**Proof of Theorem 2.3.** Suppose, by contradiction, that Eq. (1.1) is nonoscillatory and (1.5) holds. Then inequalities (4.8) and (4.9) hold. The sum of these two inequalities contradicts (2.6). This contradiction ends the proof. \( \square \)

**Proof of Theorem 2.4.** Let us prove (i). Suppose, by contradiction, that (1.1) is nonoscillatory and (2.7) holds. Inequality (4.9) implies

\[
H^* \leq -r + \left| \frac{p - n + 1}{p} \right| \omega_n.
\]

From Lemma 4.4 it follows

\[
H^* \leq \left| \frac{p - n + 1}{p} \right| \omega_n - A,
\]
a contradiction. The statement (ii) can be proved similarly using inequality (4.9) and Lemma 4.4, which implies

\[ Q^* \leq B + \frac{1}{p - 1} \left| \frac{n - 1}{p} \right| \omega_n, \quad (5.5) \]

a contradiction to (2.10).

Proof of Theorem 2.5. Suppose that (1.1) is nonoscillatory and (2.7) and (2.8) hold. Suppose \( Q_* > \frac{(n - 1)(p - 1)}{p(n - 1)} \Phi(\frac{n - 1}{p}) \omega_n \) and \( H_* > \frac{1 - n}{p} \Phi(\frac{p - n + 1}{p}) \omega_n \). Then \( A > \hat{A} \) and \( B < \hat{B} \). By Lemmas 4.4 and 4.5 for every \( \varepsilon \in (0, \min(A - \hat{A}, -B + \hat{B})) \) there exists \( t_\varepsilon \) such that

\[ \hat{A} < A - \varepsilon < \rho(t) < B + \varepsilon < \hat{B} \]

for every \( t \geq t_\varepsilon \). Lemma 4.1 implies that the right hand sides of inequalities (4.6) and (4.7) can be for \( t \geq \tau \geq t_\varepsilon \) increased substituting \( \rho(t) \) by \( A - \varepsilon, B + \varepsilon \), respectively. Hence

\[ Q(t) \leq \rho(t) - \omega_n^{-q/p} |A - \varepsilon|^q + \frac{1 - n}{p - 1} (A - \varepsilon) \quad (5.6) \]

\[ H(t) \leq -\rho(t) \]

\[ - \left[ (p - 1) \omega_n^{-q/p} |B + \varepsilon|^q - (p - n + 1)(B + \varepsilon) \right] \left[ 1 - \frac{\tau}{t} \right] \]

\[ + \frac{\tau}{t} [\rho(\tau) + H(\tau)] \quad (5.7) \]

for large \( t \) and \( \tau \). From (5.6) using the limit process \( \lim_{\varepsilon \to 0^+} \lim_{t \to \infty} \sup \) and Lemma 4.5 we obtain

\[ Q_* \leq B - \omega_n^{-q/p} |A|^q + \frac{1 - n}{p - 1} A. \quad (5.8) \]

Combining this inequality and equality (1.6) we obtain

\[ Q_* \leq Q_* - A + B \]

which contradicts (2.11). The condition (i) is proved. The condition (ii) follows in a similar way from (5.7), Lemma 4.4, and equality (1.7). The sum
of equalities (5.6), (5.7) gives
\[
Q(t) + H(t) \leq -\omega_n^{-q/p}|A - \varepsilon|^q + \frac{1 - n}{p - 1}(A - \varepsilon)
\]
\[
-\left[(p - 1)\omega_n^{-q/p}|B + \varepsilon|^q - (p - n + 1)(B + \varepsilon)\right]
\]
\[
\times \left[1 - \frac{\tau}{t}\right] + \frac{\tau}{t}\left[\rho(\tau) + H(\tau)\right].
\]  \hspace{1cm} (5.9)

The limit process \(\lim_{t \to a^+} \limsup_{t \to a^+}\) and Eqs. (1.6) and (1.7) imply
\[
\limsup_{t \to a^+} [Q(t) + H(t)] \leq Q_* + H_* - A + B,
\]
which contradicts (2.13).

If \(Q_* = \frac{(n - 1)}{p(p - 1)} \Phi^{(n - 1)/p} \omega_n, (H_* = \frac{1 - n}{p} \Phi^{(n - 1)/p} \omega_n)\) then \(A = \hat{A}, (B = \hat{B})\) and equality (5.6) (5.7) with \(\varepsilon = 0\) follows from the statement (ii) of Lemma 4.1 for every \(t \geq a\). The rest of the proof is identical with those given above. \(\square\)

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REFERENCES

7. O. Došlý and A. Lomtatidze, Oscillation and nonoscillation criteria for half-linear second order differential equations, submitted for publication.


