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Existence and Nonexistence of Entire Positive Solutions of Semilinear Elliptic Systems

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We show that entire positive solutions exist for the semilinear elliptic system $\Delta u = p(x)v^\alpha$, $\Delta v = q(x)u^\beta$ on $\mathbf{R}^N$, $N \geq 3$, for positive α and β, provided that the nonnegative functions p and q are continuous and satisfy appropriate decay conditions at infinity. We also show that entire solutions fail to exist if the functions p and q are of slow decay.

Key Words: entire solution; semilinear elliptic system.

1. INTRODUCTION

Existence and nonexistence of solutions of the semilinear elliptic system

$$\begin{aligned} \Delta u + f(u, v) = 0, \\ \Delta v + g(u, v) = 0, \end{aligned} \qquad x \in R^N, \tag{1}$$

have received much attention recently. See, for example, [1, 3–5, 8–12] and the references therein. Most of these results have to do with the *nonexistence* of positive solutions, the existence of *radial* solutions, or the asymptotic behavior of the solutions. In particular, in [1], the *nonexistence* of

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positive *radial* solutions was shown for the system where

$$f(u,v) \geq au^k v^p \quad \text{and} \quad g(u,v) \geq bu^q v^t,$$

with $k, t > 1$, $p, q \geq 0$, and a, b both positive constants. In [5], the authors consider the system

$$\begin{aligned} \Delta u &= p(x)v^{\alpha}, \\ \Delta v &= q(x)u^{\beta}, \end{aligned} \qquad u > 0,\ v > 0,\ x \in \mathbf{R}^N,\ N \geq 3,\ \alpha \geq \beta > 0, \tag{2}$$

where p and q are nonnegative continuous functions defined on $\mathbf{R}^N$. Under the assumption that p and q are radial, they show that entire positive radial solutions exist in each of the following cases: (i) the sublinear case $1 > \alpha \geq \beta > 0$; (ii) the case where both p and q have fast decay rates at infinity,

$$\int_0^{\infty} tp(t)\,dt < \infty, \qquad \int_0^{\infty} tq(t)\,dt < \infty. \tag{3}$$

Moreover, *large* positive radial solutions ("large" means that both u and v tend to infinity at infinity) exist if either (i) $\alpha \geq \beta > 1$ (the superlinear case) and both p and q have fast decay rates (3) or (ii) $1 > \alpha \geq \beta > 0$ (the sublinear case) and the functions p and q have slow decay rates,

$$\int_0^{\infty} tp(t)\,dt = \infty, \qquad \int_0^{\infty} tq(t)\,dt = \infty. \tag{4}$$

It appears to us that little is known about entire positive *nonradial* solutions of semilinear elliptic systems in R^N. In this paper we intend to fill this gap by studying (2). In particular, we show:

- For $\alpha \geq \beta > 0$, the system has entire bounded solutions if p and q decay fast at least at the rate of $|x|^{-2-\delta}$ for $\delta > 0$ at infinity; moreover, any bounded entire solution has a limit at infinity.
- For $\alpha \geq \beta > 1$, entire solutions of (2) do not exist if the functions p and q decay no faster than $|x|^{-2}$.
- For $1 > \alpha \geq \beta > 0$, any entire solution is bounded (and therefore has a limit at infinity) if p and q decay fast at infinity in the sense that

$$\int_0^{\infty} t \max_{|x|=t} p(x)\,dt < \infty, \qquad \int_0^{\infty} t \max_{|x|=t} q(x)\,dt < \infty. \tag{5}$$

We are unable to prove the existence of *large* entire solutions as suggested by the results in [5], whose proofs rely on the radial symmetry assumption in the system. For the single equation

$$\Delta u = p(x)u^{\alpha}, \tag{6}$$

where p has a fast decay and $\alpha > 1$, Cheng and Ni [2] were able to classify all nonradial entire solutions in a clear-cut fashion. They used the comparison principle, which is not available for our system.

2. STATEMENTS AND PROOFS OF THE THEOREMS

THEOREM 1. *Suppose that* $\alpha \geq \beta > 0$ *and that* p *and* q *decay to zero at infinity faster than the quadratic ones*,

$$p(x) + q(x) \leq C|x|^{-(2+\delta)}, \tag{7}$$

for some positive constant δ.

(i) *For any pair of constants* $a, b > 0$ *satisfying*

$$b - a^{\alpha} p_0 > 0, \qquad a - b^{\beta} q_0 > 0,$$

where

$$p_0 = \max_{x \in R^N}(-\Gamma(x) * p(x)), \qquad q_0 = \max_{x \in R^N}(-\Gamma(x) * q(x)),$$

Γ *being the fundamental solution of the Laplacian*, *there exists an entire solution* $u(x), v(x)$ *of* (2) *with*

$$\lim_{|x| \to \infty} u(x) = b, \qquad \lim_{|x| \to \infty} v(x) = a.$$

(ii) *Any bounded entire solution* (u, v) *has a limit at infinity.*

Proof of Theorem 1. Let $\Gamma(x)$ be the fundamental solution of the Laplacian. We define the sequences $\{u_k\}$ and $\{v_k\}$ as

$$\begin{aligned} v_0 &= a > 0, \\ u_{k+1} &= b + \Gamma * (p v_k^{\alpha}), \\ v_k &= a + \Gamma * (q u_{k+1}^{\beta}). \end{aligned}$$

By Lemma 2.3 of [6] and the decay condition (7), both $\Gamma * p(x)$ and $\Gamma * q(x)$ are well-defined and have decay rate $O(|x|^{-\delta})$ at infinity. Thus u_1 is well-defined and if $b - a^{\alpha} p_0 > 0$ it is positive and less than b. This, in turn, implies that v_1 is positive and less than a if $a - b^{\beta} q_0 > 0$. Repeating this process, we have that for each positive integer k,

$$0 < v_k(x) < a, \; 0 < u_k(x) < b, \qquad x \in R^N,$$

and that u_k is a monotone increasing sequence, while v_k is a decreasing one.

Let $u(x), v(x)$ be the pointwise limit functions of the sequences $\{u_k\}$ and $\{v_k\}$. Then we have

$$0 \leq v(x) \leq a, \; 0 \leq u(x) \leq b, \qquad x \in R^N.$$

Furthermore, by Lebesgue's dominated convergence theorem, (u, v) satisfies

$$\begin{aligned} u &= b + \Gamma * (pv^{\alpha}), \\ v &= a + \Gamma * (qu^{\beta}). \end{aligned} \tag{8}$$

Thus (u, v) is a nonnegative bounded entire solution of the system (2). By the strong maximum principle, both u and v are positive. Moreover, by Lemma 2.3 of [6] again, the convolution term in each of the equations in (8) decays at least at the order of $|x|^{-\delta}$ at infinity. This completes the proof of Part (i) of Theorem 1.

To prove Part (ii), we apply Lemma 2.8 of [6] to the u and v equations, respectively. (By the maximum principle, the limits of u and v are positive.) ∎

THEOREM 2. *Suppose that $\alpha \geq \beta > 1$ and that p and q decay no faster than $|x|^{-2}$:*

$$p(x), q(x) \geq \frac{C}{1 + |x|^2}.$$

Then (2) *has no entire solutions.*

Proof of Theorem 2. We follow [7] where the nonexistence of (6) was studied. Suppose (2) has an entire solution (u, v). This will lead to a contradiction. The first step to reach this is to show that both

$$\int_{B_R(0)} u(x)\, dx, \qquad \int_{B_R(0)} v(x)\, dx$$

have exponential growth as $R \to \infty$. This can be achieved by showing that both

$$M(R) := \max_{|x| \leq R} u(x), \qquad N(R) := \max_{|x| \leq R} v(x)$$

have such growth, because by the mean value property of subharmonic functions (as u and v are), we have

$$\begin{aligned} M(R/2) = u(\bar{x}) &\leq \frac{1}{|B_{R/2}(\bar{x})|} \int_{B_{R/2}(\bar{x})} u(x)\, dx, \\ &\leq \frac{2^N}{|B_R(0)|} \int_{B_R(0)} u(x)\, dx, \end{aligned}$$

where $\bar{x}$ is a maximum point of u in $B_{R/2}(0)$.

Using the arguments on p. 223 of [7], we have

$$\begin{aligned} M(2R) &\geq M(R/2) + C_1 N^{\alpha}(R/2), \\ N(2R) &\geq N(R/2) + C_1 M^{\beta}(R/2). \end{aligned}$$

Thus

$$\begin{aligned} M(2R) &\geq M(R/2) + C_1^{1+\alpha} M^{\alpha\beta}(R/8) \\ &\geq M(R/8) + C_1^{1+\alpha} M^{\alpha\beta}(R/8) \\ &\geq (1 + C_1^{1+\alpha}) M(R/8). \end{aligned}$$

(Without loss of generality, assume that $u(0) = 1$ and then $M(R) \geq 1$.) From this it follows that $M(R)$ grows exponentially fast as $R \to \infty$. Similarly, we can show that the same is true for $N(R)$.

Now let $w = u + v$. Then on R^N,

$$\Delta w \geq K(x)(u^\alpha + v^\beta),$$

where $K(x) = C/(1 + |x|^2)$. Multiplying this by w and integrating on $B_R(0)$ by parts, we have

$$\int_{\partial B_r(0)} w|\nabla w| \geq \int_{B_r(0)} \left[|\nabla w|^2 + K(x)(u^\alpha + v^\beta) w \right] dx,$$

and hence

$$\int_{B_R(0)} w|\nabla w| \geq \int_0^R \int_{B_r(0)} \left[|\nabla w|^2 + K(x)(u^\alpha + v^\beta) w \right] dx\, dr. \tag{9}$$

Observe that by Hölder's inequality, for r large,

$$\begin{aligned} &\int_{B_r(0)} K(x) u^{\beta+1}\, dx \\ &\quad \leq \left(\int_{B_r(0)} K(x) u^{\alpha+1}\, dx \right)^{(\beta+1)/(\alpha+1)} \left(\int_{B_r(0)} K(x)\, dx \right)^{(\alpha-\beta)/(\alpha+1)} \\ &\quad \leq \int_{B_r(0)} K(x) u^\alpha w\, dx, \end{aligned}$$

because $\int_{B_r(0)} u\, dx$ and hence $\int_{B_r(0)} K(x) u^{\alpha+1}\, dx$ have exponential growth as $r \to \infty$. Combining this with (9), we see that there exist constants $0 < C_2 < C_3 < 1$ such that for all large R,

$$\begin{aligned} \int_{B_R(0)} w|\nabla w|\, dx &\geq C_3 \int_0^R \int_{B_r(0)} \left[|\nabla w|^2 + K(x)(u^{\beta+1} + v^{\beta+1}) \right] dx\, dr \\ &\geq C_2 \int_0^R \int_{B_r(0)} \left[|\nabla w|^2 + K(x) w^{\beta+1} \right] dx\, dr. \end{aligned}$$

Now the proof of Theorem 3.4 in [7] leads to a contradiction. ∎

The next result deals with the case when the decay rates of p and q lie between the ones required in Theorems 1 and 2.

THEOREM 3. *For $1 < \alpha \leq \beta$, all entire positive solutions (if any) of (2) are unbounded if the functions p and q satisfy*

$$\int_0^\infty t \min_{|x|=t} p(x)\, dt = \infty, \qquad \int_0^\infty t \min_{|x|=t} q(x)\, dt = \infty. \tag{10}$$

Proof of Theorem 3. For convenience we denote

$$\underline{p}(r) = \min_{|x|=r} p(x),$$
$$\underline{q}(r) = \min_{|x|=r} q(x).$$

For any function $f(x)$, define its spherical mean by

$$\bar{f}(r) \equiv \frac{1}{|\partial B_r(0)|} \int_{\partial B_r(0)} f(x)\, d\sigma.$$

We have

$$\Delta \bar{u} = \bar{u}'' + \frac{n-1}{r} \bar{u}' = \overline{\Delta u}$$
$$\geq \underline{p}(r) \overline{v^\alpha} \geq \underline{p}(r) \bar{v}^\alpha.$$

Similarly, we have

$$\Delta \bar{v} \geq \underline{q}(r) \bar{u}^\beta(r).$$

Thus we have

$$\bar{u}(r) \geq u(0) + \int_0^r t^{1-N} \int_0^t s^{N-1} \underline{p}(s) v^\alpha(s)\, ds\, dt, \tag{11}$$

$$\bar{v}(r) \geq v(0) + \int_0^r t^{1-N} \int_0^t s^{N-1} \underline{q}(s) u^\beta(s)\, ds\, dt. \tag{12}$$

Clearly,

$$\bar{u}(r) \geq u(0) + (v(0))^\alpha \int_0^r t^{1-N} \int_0^t s^{N-1} \underline{p}(s)\, ds\, dt \equiv a + b^\alpha f(r),$$

and similarly $\bar{v}(r) \geq v(0) + (u(0))^\beta g(r)$. However, $\lim_{r\to\infty} f(r) = \lim_{r\to\infty} g(r) = \infty$ as a consequence of (10). Thus (u, v) cannot be bounded. ∎

THEOREM 4. *Suppose $1 > \alpha \geq \beta > 0$ and that p and q have a fast decay in the sense of (5). Then every bounded entire solution (u, v) of system (2) is bounded in R^N.*

Proof of Theorem 4. Let

$$\bar{p}(r) = \max_{|x|=r} p(x), \qquad \bar{q}(r) = \max_{|x|=r} q(x).$$

We first show that the spherical means of u and v (as defined in the proof of Theorem 3) are bounded. Observe that

$$\bar{u}'' + \frac{n-1}{r}\bar{u}' \leq \bar{p}(r)\bar{v}^{\alpha}(r)$$

and

$$\bar{v}'' + \frac{n-1}{r}\bar{v}' \leq \bar{q}(r)\bar{u}^{\beta}(r).$$

Integrating the above inequalities and using the facts that $\bar{u}' \geq 0$ and $\bar{v}' \geq 0$ yield that for $r \geq r_0$ (r_0 to be chosen),

$$\begin{aligned}\bar{u}(r) &\leq \bar{u}(r_0) + \int_{r_0}^{r} t^{1-n} \int_0^t s^{n-1}\bar{p}(s)\bar{v}^{\alpha}(s)\,ds\,dt \\ &\leq \bar{u}(r_0) + \bar{v}^{\alpha}(r)\int_{r_0}^{r} t^{1-n}\int_0^t s^{n-1}\bar{p}(s)\,ds\,dt\end{aligned}$$

and

$$\bar{v}(r) \leq \bar{v}(r_0) + \bar{u}^{\beta}(r)\int_{r_0}^{r} t^{1-n}\int_0^t s^{n-1}\bar{q}(s)\,ds\,dt.$$

By the assumption (5), we can choose r_0 large so that

$$\int_{r_0}^{\infty} r^{1-n}\int_0^r s^{n-1}(p(s)+q(s))\,ds\,dt := M < 1.$$

Therefore

$$\begin{aligned}\bar{u}(r) &\leq \bar{u}(r_0) + M\bar{v}^{\alpha}(r) \leq \bar{u}(r_0) + M(1+\bar{v}(r)),\\ \bar{v}(r) &\leq \bar{v}(r_0) + M\bar{u}^{\beta}(r) \leq \bar{v}(r_0) + M(1+\bar{u}(r)).\end{aligned}$$

Hence $\bar{u}(r)$ and $\bar{v}(r)$ are bounded. Consequently, $\bar{u}(r)+\bar{v}(r)$ is bounded, say, by a constant L.

This implies that for $R > 0$,

$$\frac{1}{|B_R(0)|}\int_{B_R(0)} w\,dx \leq L, \qquad w := u + v.$$

This and the fact that w is subharmonic yield, for every $z \in R^N$,

$$\begin{aligned}w(z) &\leq \frac{1}{|B_{|z|}(z)|}\int_{B_{|z|}(z)} w(x)\,dx \\ &\leq \frac{1}{|B_{|z|}(z)|}\int_{B_{2|z|}(0)} w(x)\,dx \\ &\leq 2^N L.\end{aligned}$$

Thus, $u(x)$ and $v(x)$ are bounded. ∎

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REFERENCES

1. S. Chen and G. Lu, Existence and nonexistence of positive solutions for a class of semilinear elliptic system, *Nonlinear Anal.* **38**, No. 7 (1999), 919–932.
2. K.-S. Cheng and W.-M. Ni, On the structure of the conformal scalar curvature equation on R^n, *Indiana Univ. Math. J.* **41** (1992), 261–278.
3. F. David, Radial solutions of an elliptic system, *Houston J. Math.* **15** (1989), 425–458.
4. D. G. De Figueiredo and Y. Jianfu, Decay, Symmetry and existence of solutions of semilinear elliptic systems, *Nonlinear Anal.* **33**, No. 3 (1998), 211–234.
5. A. V. Lair and A. W. Wood, Existence of entire large solutions of elliptic systems, *J. Differential Equations* **164**, No. 2 (2000), 380–394.
6. Yi Li and W.-M. Ni, On conformal scalar curvature equations in R^n, *Duke Math. J.* **57** (1988), 895–924.
7. F.-H. Lin, On the elliptic equation $D_i[a^{ij}(x)D_jU] - k(x)U + K(x)U^p = 0$, *Proc. Amer. Math. Soc.* **95** (1985), 219–226.
8. E. Mitidieri, Nonexistence of positive solutions of semilinear elliptic systems in R^N, *Differential Integral Equations* **9**, No. 3 (May 1996), 465–479.
9. E. Noussair and C. A. Swanson, Asymptotics for semilinear elliptic systems, *Can. Math. Bull.* **34**, No. 4 (1991), 514–519.
10. J. Serrin and H. Zou, Non-existence of positive solutions of Lane–Emden systems, *Differential Integral Equations* **9**, No. 4 (July 1996), 635–653.
11. J. Serrin and H. Zou, Existence of positive entire solutions of elliptic Hamiltonian systems, *Comm. Partial Differential Equations* **23**, Nos. 3 & 4 (1998), 577–599.
12. C. Yarur, Existence of continuous and singular ground states for semilinear elliptic systems, *Electron. J. Differential Equations* **1998**, No. 01 (1998), 1–27.