Nonlinear Fredholm Operators with Noncompact Fibers and Applications to Elliptic Problems on $\mathbb{R}^N$

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Under suitable conditions, an equation $F(x) = y$ between Banach spaces involving a nonlinear Fredholm mapping $F$ of nonnegative index is shown to have a noncompact and hence infinite set of solutions for almost every $y$ for which the equation is solvable. The proof of this nonuniqueness (but not existence) result relies on an entirely new line of arguments in which the concept of generalized critical value plays a central role. When $F: W^{2,p}(\mathbb{R}^N) \rightarrow L^p(\mathbb{R}^N)$ is associated with a quasilinear elliptic PDE on $\mathbb{R}^N$ with “constant coefficient,” it often happens that a crucial denseness hypothesis in the abstract theorem is equivalent to the existence of a nontrivial solution to the equation $F(x) = F(0)$ and hence can be verified in practice. Generalizations exist for some classes of problems with nonconstant coefficients and for problems on exterior domains.

1. INTRODUCTION

Let $X$ and $Y$ be real Banach spaces and let $F: X \rightarrow Y$ be a nonlinear Fredholm mapping of index $k \geq 0$. The main abstract result of this paper, Theorem 5.4, asserts that under appropriate additional conditions the set of solutions of the equation $F(x) = y$ is noncompact for almost every $y \in Y$ for which the equation is solvable, that is, almost every $y \in F(X)$. Here, “almost every” must be understood in the sense of the Baire category relative to $F(X)$.

The literature contains numerous statements ensuring the compactness of the set $F^{-1}(y)$ under various assumptions about $F$. In particular, all the uniqueness theorems are of this type. There are much fewer results that provide the opposite noncompactness property, especially in a nonvariational setting. When $F$ is a gradient, this may sometimes be obtained via Lyusternik–Schnirelmann theory or variants thereof, but usually only for special values of $y$ (e.g., $y = 0$) and very specific choices of $F$. Aside from...
the holomorphic case, the only other example that comes to mind is when $F$ is a locally trivial fiber bundle and $F$ is Fredholm of strictly positive index. If so, $F^{-1}(y)$ is noncompact for every $y \in Y$. This is explained in Remark 1.1 below. Unfortunately, in practice, it is hopeless to expect $F$ to be a fiber bundle without assuming that $DF(x)$ is onto $Y$ for every $x \in X$ (and even this is not enough; see for instance [12]), a very restrictive assumption of limited value in many applications. In contrast, Theorem 5.4 requires only a very mild assumption about the surjectivity of $DF(x)$ and it is valid in the fundamental index 0 case.

Remark 1.1. The foregoing comments use the fact that if $F: X \to Y$ is a fiber bundle, the fibers $F^{-1}(y)$ are contractible because $X$ and $Y$ are contractible. This (classical) property follows from Whitehead’s theorem and the homotopy sequence of a fiber bundle. For more details, see Earle and Eells [5]. Also, since the all fibers are homeomorphic, they are all nonempty and then, by the Sard–Smale theorem (if $F$ is smooth enough; see [1] or [16]) one of them at least is a $k$-dimensional $C^1$ submanifold of $X$, $k = \text{index } F$. It is well known that there is no compact contractible manifold of positive dimension [1, p. 559], so that $F^{-1}(y)$ is noncompact for at least one $y \in Y$ and hence for every $y \in Y$ since the fibers are homeomorphic.

The last section of this paper is devoted to applications of Theorem 5.4 to quasilinear second order elliptic equations on $\mathbb{R}^N$ viewed as functional equations in Sobolev spaces. The discussion of applications of Theorem 5.4 is especially important because it contains a perhaps perplexing assumption, namely, the denseness in $F(X)$ of some subset $A(F)$ of “asymptotic values.” In a general functional setting, it is by no means clear that the verification of this assumption is any easier than the very conclusion of the theorem. In fact, it is apparently mostly for problems with noncompact nonlinearities that the verification of the denseness of $A(F)$ in $F(X)$ may become a much simpler matter. A result in this direction is given in Lemma 5.7, which hints that the single equation $F(x) = F(0)$ may contain the key to the answer. The hypotheses of Lemma 5.7 are satisfied by some quasilinear elliptic equations on $\mathbb{R}^N$, one of the simplest examples being

$$-u'' + u - u^3 = f \quad \text{in } \mathbb{R}.$$  

As we shall see in Section 6, here the key ingredients are the translation invariance of the domain and the fact that the homogeneous problem ($f = 0$) has a nonzero solution. More generally, for a broad class of quasilinear second order elliptic PDEs on $\mathbb{R}^N$ beyond the scope of Lemma 5.7 and posed over $X := W^{2,p}(\mathbb{R}^N)$ for suitable $p$, the denseness of $A(F)$ in $F(W^{2,p}(\mathbb{R}^N))$ continues to amount to the existence of a nonzero solution
(in $W^2, r(\mathbb{R}^N)$) of an associated PDE, possibly different from the original one. Coincidentally, the existence of nonzero solutions to homogeneous PDEs on $\mathbb{R}^N$ has been studied for its own sake in a number of special cases and a substantial amount of literature already exists about this issue. However, the surprising connection with some denseness property is made here for the first time.

A second unusual, but this time not perplexing, hypothesis made in Theorem 5.4 is that the set of critical points of $F$ is required to have empty interior. We emphasize that it is indeed the set of critical points, not critical values, which is involved there and hence the Sard–Smale theorem has little to do with the verification of this condition. In the finite dimensional case, it is plain that the critical points of $F$ have empty interior when $F$ is real-analytic and $DF(x)$ is onto $Y$ for at least one $x \in X$. It turns out that the same criterion is valid when $X$ and $Y$ are infinite dimensional and $F$ is Fredholm of nonnegative index. We prove this property in Corollary 4.4 since it seems to have been omitted in the literature devoted to nonlinear Fredholm operators. By way of a counterexample, we also show that it breaks down in the non-Fredholm case (Remark 4.3).

As a nonuniqueness result not based upon existence, Theorem 5.4 is of a new type in functional analysis and indeed its proof does not follow from classical arguments. It relies primarily upon the concept of generalized critical value, introduced by the author in [13], and related results. Section 2 is devoted to a review of the material in [13] relevant to the present work along with a few complements. Theorem 3.3 in Section 3 is the decisive new tool needed to obtain Theorem 5.4: At a first sight, the proof of Theorem 5.4 seems to require knowing that the set of generalized critical values of $F$ is of first category in $Y$. Unfortunately, simple finite dimensional examples show that this is not true even for smooth $F$ and no criterion for such a property is currently available when $X$ and $Y$ are infinite dimensional (for the finite dimensional case, see Remark 3.3). Theorem 3.3 highlights a somewhat surprising feature making it unnecessary to have any estimate about the size of the set of generalized critical values of $F$, provided that the set of critical points of $F$ has empty interior in $X$. This motivates investigating the properties of real-analytic Fredholm mappings in Section 4. Naturally, Section 5 centers around the main result and corollaries.

As previously recorded, the applications to PDEs on $\mathbb{R}^N$ are discussed in Section 6. Only simple examples for which the hypotheses of Theorem 5.4 can be checked without recourse to sophisticated arguments are fully discussed. The consideration of more general examples requires developing additional background material, notably but not only about the real-analyticity of Nemytskii operators and also reviewing some relevant material from Rabier and Stuart [15]. Those issues have little to do with
the central theme of this paper and, accordingly, they are discussed in detail elsewhere [14]. Nevertheless, some salient results from [14] along with sample examples, including nonanalytic ones, are briefly mentioned for completeness.

2. GENERALIZED CRITICAL VALUES

Recall that a continuous linear mapping \( L \in \mathcal{L}(X, Y) \) between real Banach spaces \( X \) and \( Y \) is surjective if and only if \( v(L) > 0 \), where

\[
v(L) := \inf \{\|L^*y^*\| : \|y^*\| = 1\}, \tag{2.1}
\]

and \( L^* \in \mathcal{L}(Y^*, X^*) \) is the adjoint of \( L \) (see Brézis [4, p. 29]). As a result, given a \( C^1 \) mapping \( F: X \to Y \), we have

\[
\text{rge } DF(x) = Y \iff v(DF(x)) > 0, \tag{2.2}
\]

and the set of critical values of \( F \), henceforth denoted by \( K_0(F) \), is characterized by

\[
K_0(F) := \{y \in Y : \exists x \in X, F(x) = y, v(DF(x)) = 0\}. \tag{2.3}
\]

The broader concept of generalized critical value, introduced in [13], is defined via membership to the set

\[
K(F) := \{y \in Y : \exists x_n \in X, \lim_{n \to \infty} F(x_n) = y, \lim_{n \to \infty} v(DF(x_n)) = 0\}. \tag{2.4}
\]

That \( K_0(F) \subseteq K(F) \) is obvious, as is the fact that the converse need not be true (example: \( X = Y = \mathbb{R} \) or \( \mathbb{C} \) and \( F(x) := e^x \), for which \( K_0(F) = \emptyset \) but \( K(F) = \{0\} \)).

It is easily seen [13, Lemma 2.1] that the mapping \( v \) in (2.1) is continuous, so that the mapping \( v \circ DF \) is continuous. From the definition of \( K(F) \) it follows at once that:

**Theorem 2.1.** The set \( K(F) \) of generalized critical values of \( F \) is closed in \( Y \).

From (2.4), generalized critical values of \( F \) need not be values of \( F \). Actually, by [13, Proposition 3.1 and Corollary 6.1], \( K(F) \subseteq F(X) \) if and only if \( F(X) \) is closed in \( Y \) provided that \( F \) is of class \( C^2 \) and \( F \) is Fredholm of index \( k \geq 0 \) (if \( k = 0 \), this remains true when \( F \) is \( C^1 \)). Such a
result is informative but will not be used here. We now introduce a subset of $K(F) \cap F(X)$ of “almost” critical values of $F$.

$$\Sigma(F) := \{ y \in Y : \exists (x_n) \subset X, F(x_n) = y, \lim_{n \to \infty} y(DF(x_n)) = 0 \}.$$  (2.5)

Clearly, points of $\Sigma(F)$ are values of $F$ and

$$K_0(F) \subset \Sigma(F) \subset K(F) \cap F(X).$$  (2.6)

More precisely, if $y \in \Sigma(F)$ and one of the subsequences $(x_n)$ from the definition of $\Sigma(F)$ in (2.5) has a convergent subsequence, then $y \in K_0(F)$. Hence, if we define

$$\Sigma_\infty(F) := \{ y \in Y : \exists (x_n) \subset X \text{ with no convergent subsequence, } F(x_n) = y, \lim_{n \to \infty} y(DF(x_n)) = 0 \},$$  (2.7)

we obtain that $\Sigma(F)$ is the (nondisjoint) union

$$\Sigma(F) = K_0(F) \cup \Sigma_\infty(F).$$  (2.8)

The subscript “$\infty$” in $\Sigma_\infty(F)$ is motivated by the remark that, when $X$ is finite dimensional, a sequence $(x_n)$ as in the definition of $\Sigma_\infty(F)$ in (2.7) must necessarily satisfy $\lim_{n \to \infty} |x_n| = \infty$. Of course, the situation is no longer as simple when $\dim X = \infty$ but, at any rate, it is obvious that regardless of $\dim X$, $F^{-1}(y)$ is noncompact whenever $y \in \Sigma_\infty(F)$. Our subsequent results depend in part upon the smallness of the set $K(F) \setminus \Sigma_\infty(F)$. Conditions ensuring that this set and even the larger set $K(F) \setminus (\Sigma_\infty(F) \setminus K_0(F))$ are small in the sense of the Baire category relative to $Y$ are given in Section 3. Neither set is conveniently described in plain words but it is clear that they both contain all the points $y \in K(F)$ such that $F^{-1}(y)$ is compact. Their smallness thus implies that if $F^{-1}(y)$ is compact and nonempty for every $y$ in some “large” subset $S \subset Y$, then $S$ must be mostly contained in $Y \setminus K(F)$. However, if it can also be shown that $F^{-1}(y)$ is either empty or noncompact whenever $y \in Y \setminus K(F)$, then such a large subset $S$ cannot exist. This is the basic idea in the proof of Theorem 5.4. Now, the reason why a precise information about $F^{-1}(y)$ can be obtained when $y \in Y \setminus K(F)$ is due to the following result from [13] showing that $F$ is very well behaved in the (open) set $F^{-1}(Y \setminus K(F))$.

**Theorem 2.2.** Suppose that $F : X \to Y$ is Fredholm of index $k \geq 0$ and of class $C^1$ if $k = 0$, $C^{2-}$ if $k \geq 1$. Let $V \subset Y$ be a connected component of

$^1$That is, $C^1$ with a locally Lipschitz continuous derivative.
\[ Y \setminus K(F), \text{ so that } V \text{ is open in } Y \text{ by Theorem 2.1. Then, either } F^{-1}(V) = \emptyset \text{ or } F: F^{-1}(V) \to V \text{ is a locally trivial } C^0 \text{ fiber bundle (in particular, } F \text{ is onto } V). \]

Theorem 2.2 is a special case of [13, Theorem 6.1] where the result is proved for more general, i.e., not necessarily Fredholm, \( F \) under the assumption that \( F \) has “uniformly split kernels” in \( X \setminus F^{-1}(K(F)) \). This terminology means that there should be a constant \( C > 0 \) such that for every \( x \in X \setminus F^{-1}(K(F)) \) there is a projection \( P_x \in \mathcal{L}(X) \) with \( \ker P_x = \ker DF(x) \) and \( \|P_x\| \leq C \). Such a condition always holds when \( F \) is Fredholm of index \( k \neq 0 \). Indeed, \( DF(x) \) is onto \( Y \) for \( x \in X \setminus F^{-1}(K(F)) \) because \( K_0(F) \subset K(F) \), whence \( \dim \ker DF(x) = k \). It is known (see Beaupre [2] or Jameson [9]) that there is a constant \( C(k) > 0 \) such that for every subspace \( Z \) of \( X \) with \( \dim Z = k \) there is a projection \( P \in \mathcal{L}(X) \) with \( \ker P = Z \) and \( \|P\| \leq C(k) \). Thus, the “uniformly split kernel” assumption holds for Fredholm mappings of index \( k \neq 0 \). We also point out that Theorem 6.1 of [13] is given for \( F \) of class \( C^{2-} \). That it remains true for \( F \) of class \( C^1 \) when \( k = 0 \) (so that \( DF(x) \in GL(X,Y) \) for \( x \in X \setminus F^{-1}(K(F)) \)) follows from [13, Remark 4.2].

\textbf{Corollary 2.3.} Let \( F \) and \( V \) be as in Theorem 2.2. If \( F^{-1}(y) \) is compact for some \( y \in V \), then \( F: F^{-1}(V) \to V \) is proper.

\textbf{Proof.} With no loss of generality, assume \( F^{-1}(V) \neq \emptyset \). By Theorem 2.2, \( F^{-1}(y) \) is homeomorphic to the fiber \( \Xi \) of the bundle \( F: F^{-1}(V) \to V \), whence \( \Xi \) is compact since \( F^{-1}(y) \) is compact. Let then \( z \in V \) be arbitrary. The local triviality property of Theorem 2.2 means that there is an \( r > 0 \) such that \( F^{-1}(B(z,r)) \) is homeomorphic to \( B(z,r) \times \Xi \) where \( B(z,r) \subset V \) is the open ball with center \( z \) and radius \( r \) and the homeomorphism is fiber-preserving. Thus, if \( C \subset B(z,r) \) is a compact subset of \( V \), we have that \( F^{-1}(C) \) is homeomorphic to \( C \times \Xi \) and therefore compact. If now \( C \) is an arbitrary compact subset of \( V \), cover \( C \) with a finite number of balls \( B(z,r/2) \) with \( B(z,r) \) as above. Then, \( C \cap \bar{B}(z,r/2) \) is a compact subset of \( V \) contained in \( B(z,r) \), so that \( F^{-1}(C \cap \bar{B}(z,r/2)) \) is compact. It follows that \( F^{-1}(C) \) is compact, being a finite union of such subsets. \( \square \)

In Section 5 we shall need a variant of Corollary 2.3, given in Corollary 2.4 below, that involves asymptotic values (not necessarily critical) defined as follows.

\( C(k) \) is even independent of \( X \); for instance, \( C(k) = 1 + 2 \sqrt{k} \) works (not optimal).
Definition 2.1. We shall say that \( y \in Y \) is an asymptotic value of \( F \) if there is a sequence \((x_n)\) from \( X \) without convergent subsequence such that \( \lim_{n \to \infty} F(x_n) = y \). If also the sequence \((x_n)\) is bounded, we shall say that \( y \) is a boundedly asymptotic value of \( F \). The set of asymptotic values (resp. boundedly asymptotic values) of \( F \) will be denoted by \( A(F) \) (resp. \( A_b(F) \)).

(\text{Note } A_b(F) \subset A(F) \subset F(X).)

The relevance of (boundedly) asymptotic values can be hinted from the elementary remark that \( A(F) = \emptyset \) if and only if \( F \) is proper while \( A_b(F) = \emptyset \) if and only if \( F \) is proper on the closed bounded subsets of \( X \).

Corollary 2.4. Let \( F \) be as in Theorem 2.2. Suppose that the set \( A(F) \) of asymptotic values of \( F \) is dense in \( F(X) \). Then, given any (open) connected component \( V \) of \( Y \setminus K(F) \), we have either \( V \cap F(X) = \emptyset \) or \( F^{-1}(y) \) is non-compact for every \( y \in V \).

Proof. Suppose \( V \cap F(X) \neq \emptyset \), whence \( V \cap \overline{F(X)} \neq \emptyset \). Since \( V \) is open in \( Y \) and \( A(F) \) is dense in \( F(X) \), this implies that \( V \cap A(F) \neq \emptyset \). Let then \( z \in V \cap A(F) \) be given and let \((x_n)\) be a sequence from \( X \) with no convergent subsequence such that \( \lim_{n \to \infty} F(x_n) = z \). Once again by the openness of \( V \), we have \( F(x_n) \in V \) for \( n \) large enough, thus for every \( n \in \mathbb{N} \) with no loss of generality. The set \( C := (F(x_n)) \cup \{z\} \) is a compact subset of \( V \) whose inverse image contains the sequence \((x_n)\) and hence is non-compact. By Corollary 2.3, it follows that \( F^{-1}(y) \) cannot be compact for any \( y \in V \).

As we shall see later, a simple condition about \( F \), which is also relevant in semilinear elliptic problems on \( \mathbb{R}^N \), ensures that the set \( A_b(F) \), and hence also \( A(F) \), is dense in \( F(X) \). However, observe that such a condition cannot have value when \( F \) is proper on the closed bounded subsets of \( X \) (in particular, when \( \dim X < \infty \)) since \( A_b(F) = \emptyset \) in this case.

Another result from [13] needed later is the “generalized Ekeland principle,” so labelled for reasons duly explained in that reference.

Theorem 2.5 (Generalized Ekeland principle). Suppose that \( F : X \to Y \) is Fredholm of index \( k \geq 0 \) and of class \( C^1 \) if \( k = 0 \), \( C^{2-} \) if \( k \geq 1 \). Then \( \partial(F(X)) \subset K(F) \), where \( \partial(F(X)) \) is the boundary of \( F(X) \) in \( Y \).

Theorem 2.5 is a special case of [13, Corollary 6.1] in which the “uniformly split kernels” assumption holds, as explained earlier. Once again, the validity of Theorem 2.5 in the \( C^1 \) case when \( k = 0 \) follows from [13, Remark 4.2]. If \( y \in \partial(F(X)) \), there is a sequence \((x_n) \subset X \) such that \( \lim_{n \to \infty} F(x_n) = y \). The nonobvious fact stressed by Theorem 2.5 is that, in addition, the sequence \((x_n)\) may be chosen so that \( \lim_{n \to \infty} \nu(DF(x_n)) = 0 \).
Remark 2.1. Another, more refined, definition of $K(F)$ is also used in [13], where the condition $\lim_{n \to \infty} \nu(DF(x_n)) = 0$ in (2.4) is replaced by $\lim_{n \to \infty} \nu(DF(x_n)) = 0$. All the results previously quoted from [13] carry over to this case. If this definition is used, then the sets $\Sigma(F)$ and $\Sigma_0(F)$ in (2.5) and (2.7) must be defined consistently. Although such a refined concept is not needed here, it is of importance in some other issues (see Remark 3.3).

3. THE SIZE OF $K(F) \setminus (\Sigma_0(F) \setminus K_0(F))$

Let $F: X \to Y$ be a $C^1$ mapping. For $m \in \mathbb{N}$ and with $\nu$ from (2.1) we set

$$S_m := \{ x \in X : \nu(DF(x)) < 1/m \} \subset X \quad (3.1)$$

and

$$S_m^* := \{ x \in X : 0 < \nu(DF(x)) < 1/m \} \subset S_m. \quad (3.2)$$

**Lemma 3.1.** We have

(i) $\Sigma(F) = \bigcap_{m \in \mathbb{N}} F(S_m)$

and

(ii) $K(F) = \bigcap_{m \in \mathbb{N}} F(S_m^*)$.

**Proof.** A routine verification. 

**Lemma 3.2.** Suppose that $F: X \to Y$ is of class $C^1$ and that the set of singular points of $F$, i.e., the set $\{ x \in X : \nu(DF(x)) = 0 \} \subset X$ has empty interior in $X$. Then, the set $K(F) \setminus \Sigma(F)$ is of first category in $Y$.

**Proof.** From the continuity of $\nu \circ DF$ and the hypothesis that the set of critical points of $F$ has empty interior in $X$, it follows that $S_m \subset S_m^*$ for every $m \in \mathbb{N}$. Hence, $F(S_m) \subset F(S_m^*) \subset F(S_m^*)$, the latter by the continuity of $F$. This implies $F(S_m) \subset F(S_m^*)$, whence $F(S_m) = F(S_m^*)$ since the other inclusion is obvious. Together with Lemma 3.1(ii) we infer that

$$K(F) = \bigcap_{m \in \mathbb{N}} F(S_m^*). \quad (3.3)$$

The set $S_m^*$ is open in $X$ (continuity of $\nu \circ DF$) and $F$ is a submersion on $S_m^*$. Since submersions are open maps (even without split kernels; see [1] for a proof), it follows that $F(S_m^*)$ is open in $Y$. As a result, $\Gamma_m := \partial F(S_m^*)$ (boundary relative to $Y$) is a closed subset with empty interior in $Y$ and, by (3.3),

$$K(F) = \bigcap_{m \in \mathbb{N}} (F(S_m^*) \cup \Gamma_m). \quad (3.4)$$
Let then \( y \in K(F) \setminus \Sigma(F) \). From Lemma 3.1(i), there is some \( m_0 \in \mathbb{N} \) such that \( y \notin F(S^*_{m_0}) \). On the other hand, \( y \in F(S^*_{m_0}) \cup \Gamma_{m_0} \) by (3.4). Thus \( y \in \Gamma_{m_0} \), since \( F(S^*_{m_0}) \subset F(S_{m_0}) \). This shows that \( K(F) \setminus \Sigma(F) \subset \bigcup_{m \in \mathbb{N}} \Gamma_m \) and hence that \( K(F) \setminus \Sigma(F) \) is of first category in \( Y \).

**Remark 3.1.** Lemma 3.2 does not say, and it is generally not true, that \( K(F) \setminus \Sigma(F) \) is of first category in \( K(F) \). For instance, the “exponential” examples given in the previous section have \( K(F) \neq \emptyset \) but \( \Sigma(F) = \emptyset \).

Also, when \( X \) and \( Y \) are finite dimensional, the question whether \( K(F) \setminus \Sigma(F) \) has Lebesgue measure 0 is open, unless of course \( K(F) \) has measure 0 (see Remark 3.3).

There are alternatives, but apparently not very satisfactory ones, to the assumption in Lemma 3.2 that the set of critical points of \( F \) is empty interior. However, it is worth pointing out that this assumption is not needed if \( Y = \mathbb{R} \). Indeed, let \( C \) be a connected component of the interior of the set of critical points of \( F \). Since \( Y = \mathbb{R} \), \( F \) is constant on \( C \). If \( C = X \), then \( F \) is constant on \( X \) and \( \Sigma(F) = K(F) \) in this case, so that Lemma 3.2 is true. Otherwise, \( F(C) = F(\partial C) \) and every point \( x \in \partial C \) can be approximated by a sequence \( (x_n) \) such that \( v(DF(x_n)) > 0 \). Since \( \lim_{n \to \infty} v(DF(x_n)) = v(DF(x)) = 0 \), we have \( x_n \in S^*_m \) for every \( m \in \mathbb{N} \) if \( n \) is large enough. Thus, the critical values of \( F \) are all in \( F(S^*_m) \) for every \( m \in \mathbb{N} \). For \( x \in S_m \) we have either \( F(x) \in F(S^*_m) \) or \( F(x) \) is a critical value of \( F \), so that \( F(S_m) \subset F(S^*_m) \) holds. This suffices to repeat the proof of Lemma 3.2. When \( Y \neq \mathbb{R} \), Lemma 3.2 may badly fail if the set of critical points of \( F \) has nonempty interior, as we now show.

**Remark 3.2.** It is easy to find a \( C^\infty \) curve \( \gamma \) from \([0, \infty)\) to \( \mathbb{R}^2 \) containing all the points of \( \mathbb{Q}^2 \). Let then \( F: \mathbb{R}^2 \to \mathbb{R}^2 \) be defined by \( F(x) := \gamma(|x|^2) \). Then, \( F \) is of class \( C^\infty \), Fredholm of index 0 and \( F(\mathbb{R}^2) \supset \mathbb{Q}^2 \). Evidently, every point \( x \in \mathbb{R}^2 \) is a critical point of \( F \), hence every point of \( F(\mathbb{R}^2) \) is a critical value of \( F \), i.e., \( K_0(F) = F(\mathbb{R}^2) \). Since \( K(F) \) is closed in \( \mathbb{R}^2 \) and \( \mathbb{Q}^2 \subset F(\mathbb{R}^2) = K_0(F) \subset K(F) \) it follows that \( K(F) = \mathbb{R}^2 \). Also, \( \Sigma(F) \subset F(\mathbb{R}^2) = K_0(F) \) so that \( \Sigma(F) \) is of first category in \( \mathbb{R}^2 \). As a result, \( K(F) \setminus \Sigma(F) = \mathbb{R}^2 \setminus \Sigma(F) \) is residual in \( \mathbb{R}^2 \).

**Theorem 3.3.** Suppose that \( X \) and \( Y \) are separable and that \( F: X \to Y \) is Fredholm of index \( k \geq 0 \) and of class \( C^{k+1} \). Suppose also that the set of critical points of \( F \) has empty interior in \( X \). Then, the set \( K(F) \setminus (\Sigma_\infty(F) \setminus \mathcal{K}_0(F)) \) is of first category in \( Y \).

**Proof.** By Lemma 3.2 and the Sard–Smale theorem [16], both the sets \( K(F) \setminus \Sigma(F) \) and \( \mathcal{K}_0(F) \) are of first category, whence \( (K(F) \setminus \Sigma(F)) \cup \mathcal{K}_0(F) = K(F) \setminus (\Sigma(F) \setminus \mathcal{K}_0(F)) \) is of first category. The conclusion follows from the relation \( \Sigma(F) \setminus \mathcal{K}_0(F) = \Sigma_\infty(F) \setminus \mathcal{K}_0(F) \) (see (2.8)).
Remark 3.3. Theorem 3.3 is obvious when $K(F)$ itself is of first category in $Y$. However, the problem of identifying classes of maps with this property is highly nontrivial, even in the finite dimensional case. In fact, a polynomial example when $K(F)$ is not of first category in $Y$ is given by $F: \mathbb{R}^3 \to \mathbb{R}$, $F(x_1, x_2, x_3) := x_2x_3(x_1x_2 - 1)$. For $a \in \mathbb{R}$ and $n \in \mathbb{N}$, observe that $F(n, 1/2n, -4an) = a$ and $DF(n, 1/2n, -4an) = (-a/n, 0, -1/4n) \to 0$ as $n \to \infty$. Thus, $K(F) = \mathbb{R}$ is not of first category in $\mathbb{R}$. The value of Theorem 3.3 is precisely to bypass the difficult question about the size of $K(F)^3$ for the applications we have in mind. A recent breakthrough in these issues can be found in the work of Kurdyka et al. [8] who show that $K(F)$ has Lebesgue measure 0 (hence, being closed, is of first category) when $X = \mathbb{R}^p$, $Y = \mathbb{R}^q$, $F$ is semialgebraic (a generalization of polynomial maps) and $K(F)$ is defined as in Remark 2.1. The polynomial example given above, for which $K(F)$ is of second category, shows that the definition chosen for $K(F)$ does matter. To date, there is no adequate generalization of semialgebraic maps to the infinite dimensional setting and hence no corresponding foreseeable generalization of the result in [8].

More generally, Theorem 3.3 is true when $X$ and $Y$ are not separable but $F$ is $\sigma$-proper; i.e., the inverse image of every compact subset of $Y$ is a countable union of compact subsets of $X$. Indeed, as shown by Quinn and Sard [11], the Sard–Smale theorem is true in that case as well. When $X$ and $Y$ are separable, every Fredholm mapping is locally proper [16] and hence $\sigma$-proper. In practice, $\sigma$-properness is true in the nonseparable case for mappings which are proper on closed bounded subsets. In our PDE applications, it will be essential that the mapping of interest is not proper on some bounded subset and hence this generalization to nonseparable spaces has no value for our purposes.

The usefulness of Theorem 3.3 depends upon what it takes to show that the set of critical points of $F$ has empty interior in $X$. As we shall see in the next section (Corollary 4.4) this turns out to be quite simple when $F$ is Fredholm and real-analytic. We shall also see (Remark 4.3) that real-analyticity does not help if $F$ is not Fredholm.

4. SOME PROPERTIES OF REAL-ANALYTIC FREDHOLM MAPPINGS

Recall that given an open subset $U \subset X$, a mapping $F: U \to Y$ is said to be real-analytic if $F$ is of class $C^\infty$ and if the identity

\[ \sum_{|\alpha| \leq n} \frac{\partial^n F}{\partial x_\alpha} (a) = f(a) \]

This issue arises only for nonproper maps since $K(F) = K_0(f)$ when $F$ is proper.
\[ F(x) = \sum_{j=0}^{\infty} \frac{D^j F(x_0)}{j!} (x-x_0)^j \]

holds for every \( x_0 \in U \) and every \( x \) in some open ball \( B(x_0, \rho) \subset U \) centered at \( x_0 \). It is shown in Fučík et al. [6, Theorem 3.11, p. 144] that \( F \) above has a complex-analytic extension, still denoted by \( F \), to some open subset of \( \mathbb{C} \otimes X \) containing \( U \). Furthermore, after shrinking \( \rho > 0 \) above if necessary ([6, p. 135]) the inequality

\[ ||DF(x_0)h|| \leq M j! \left( \frac{||h||}{\rho} \right)^j, \]

holds for every \( j \in \mathbb{N} \) and every \( h \in \mathbb{C} \otimes X \) with \( ||h|| < \rho \), where \( M > 0 \) is a constant independent of \( j \). This implies \( ||DF(x_0)|| \leq M j! \rho^{-j} \) for every \( j \in \mathbb{N} \) and hence that

\[ \sum_{j=0}^{\infty} \frac{||D^j F(x_0)||}{j!} ||x-x_0||^j < \infty, \quad \forall x \in B(x_0, \rho). \]  \hspace{1cm} (4.2)

Among other things, it follows from (4.2) and by the same arguments as in the finite dimensional case that \( DF:U \to \mathcal{L}(X,Y) \) is real-analytic with

\[ DF(x) = \sum_{j=0}^{\infty} \frac{D^{j+1} F(x_0)}{j!} (x-x_0)^j, \quad \forall x \in B(x_0, \rho). \]

The proof of Lemma 4.1 below is similar to the standard one when \( \dim X < \infty \) and is omitted.

**Lemma 4.1.** Let \( U \subset X \) be a nonempty open connected subset and let \( g: U \to \mathbb{R} \) be real-analytic. If \( g \) vanishes on some nonempty open subset of \( U \), then \( g = 0 \) in \( U \).

The mapping \( v: \mathcal{L}(X,Y) \to [0, \infty) \) from (2.1) is not even of class \( C^1 \). However, when attention is confined to Fredholm operators, the condition \( v(L) = 0 \) turns out to be locally equivalent to one of the form \( \delta(L) = 0 \) where \( \delta \) is real-analytic. Obviously, the case of negative index has no interest (\( \delta = 0 \) works). For nonnegative index, the precise form of this result is given in

**Lemma 4.2.** Let \( L_0 \in \mathcal{L}(X,Y) \) be Fredholm of index \( k \geq 0 \) and suppose that \( v(L_0) = 0 \). There is an open neighborhood \( \mathcal{V}_0 \) of \( L_0 \) in \( \mathcal{L}(X,Y) \) and a real-analytic mapping\(^4\) \( \delta: \mathcal{V}_0 \to \mathbb{R} \) such that for every \( L \in \mathcal{V}_0 \), \( v(L) = 0 \) if and only if \( \delta(L) = 0 \).\(^5\)

\(^4\)The proof reveals that just like \( v \), \( \delta \) takes values in \([0, \infty)\), but this is here a minor point.

\(^5\)In general, \( \delta \) is not defined on all of \( \mathcal{L}(X,Y) \) or even on the whole set of Fredholm operators of index \( k \), unless \( X \) and \( Y \) are finite dimensional. In the latter case, \( \delta(L) := \det LL^* \) works.
Proof. In this proof, we shall repeatedly use the fact that the sum, product and composition of real-analytic maps between Banach spaces is real-analytic whenever these operations are defined. For instance, the “product” of a mapping with values in \( \mathcal{L}(X, Y) \) and a mapping with values in \( X \) is defined. The analyticity of products follows easily from the absolute summability property (4.2) together with the well known fact that absolutely convergent series in a Banach space are convergent. For the analyticity of composites, see [6, Lemma 3.3].

Call \( Y_0 \) some closed complement of \( \text{rge} \ L_0 \) in \( Y \) and let \( P_0 \) and \( Q_0 \) denote the projections onto \( \text{rge} \ L_0 \) and \( Y_0 \), respectively. These projections exist and are continuous because \( L_0 \) is Fredholm and \( 1 \leq r_0 := \dim Y_0 < \infty \) since \( L_0 \) is singular. Also, \( k_0 := \dim \ker L_0 = k + r_0 \) since \( L_0 \) has index \( k \).

Let \( \{e_{01}, \ldots, e_{0k_0}\} \) be a basis of \( \ker L_0 \) chosen once and for all and let \( X_0 \) be a closed complement of \( \ker L_0 \) in \( X \). Then, \( P_0 L_0 = L_0 \) is a linear isomorphism of \( X_0 \) onto \( \text{rge} L_0 \), whence \( P_0 L \) remains an isomorphism of \( X_0 \) onto \( \text{rge} L_0 \) for \( L \) in some open neighborhood \( \mathcal{U}_0 \) of \( L_0 \) in \( \mathcal{L}(X, Y) \). For \( L \in \mathcal{U}_0 \), set

\[
\xi_i(L) := -[P_0 L_{|X_0}]^{-1} P_0 L e_{0i} \in X_0, \quad 1 \leq i \leq k_0, \tag{4.3}
\]

and

\[
e_i(L) := e_{0i} + \xi_i(L), \quad 1 \leq i \leq k_0. \tag{4.4}
\]

Clearly, \( \xi_i(L_0) = 0 \), so that \( e_i(L_0) = e_{0i} \) and \( e_i \) is continuous on \( \mathcal{U}_0 \), \( 1 \leq i \leq k_0 \). Therefore, the vectors \( \{e_1(L), \ldots, e_{k_0}(L)\} \) remain linearly independent after shrinking \( \mathcal{U}_0 \) if necessary. From (4.3) and (4.4), we infer that \( P_0 L e_i(L) = 0 \), \( 1 \leq i \leq k_0 \), i.e., \( \{e_1(L), \ldots, e_{k_0}(L)\} \subset \ker P_0 L \) for every \( L \in \mathcal{U}_0 \).

The operator \( P_0 L_0 \) from \( X \) to \( \text{rge} L_0 \) is Fredholm of index \( k_0 = \dim \ker L_0 \). By the openness of the set of Fredholm operators of index \( k_0 \) in \( \mathcal{L}(X, \text{rge} L_0) \), it follows that \( P_0 L \in \mathcal{L}(X, \text{rge} L_0) \) is Fredholm of index \( k_0 \) for every \( L \in \mathcal{U}_0 \) after shrinking \( \mathcal{U}_0 \) once again if necessary. Since \( P_0 L \) is also onto \( \text{rge} L_0 \) (see above) it follows that \( \dim \ker P_0 L = k_0 \) for every \( L \in \mathcal{U}_0 \). This shows that \( \{e_1(L), \ldots, e_{k_0}(L)\} \) is a basis of \( \ker P_0 L \) for every \( L \in \mathcal{U}_0 \).

By the well-known real-analyticity of the mapping \( T \to T^{-1} \) in the general linear group of two Banach spaces and the linearity (hence analyticity) of the mappings \( L \in \mathcal{L}(X, Y) \to P_0 L_{|X_0} \in \mathcal{L}(X_0, \text{rge} L_0) \) and \( L \in \mathcal{L}(X, Y) \to P_0 L e_{0i} \in \text{rge} L_0 \), we find that the mappings \( \xi_i \) in (4.3) and \( e_i \) in (4.4) are real-analytic in \( \mathcal{U}_0 \).
For \( L \in \mathcal{L}_0 \), write \( L = P_0 L + Q_0 L \). Since \( P_0 L \) is onto \( \text{rge} \ L_0 \), it follows that \( L \) is onto \( Y \) if and only if \( Q_0 L_{\text{ker} \ L_0} \) is onto \( Y_0 \). In turn, given any basis \( \{y_1, \ldots, y_n\} \) of \( Y_0 \) with dual basis \( \{y_1^*, \ldots, y_n^*\} \subset Y_0^* \), this amounts to saying that the \( r_0 \times k_0 \) matrix \( M(L) := (m_{ij}(L)) \) where \( m_{ij}(L) := \langle y_j^*, Q_0 L e_i(L) \rangle \), has rank \( r_0 \). This happens if and only if \( \delta(L) := \det(M(L))M(L)^T \) is nonzero.

By the analyticity of \( e_i \) for \( 1 \leq j \leq k_0 \), the coefficients \( m_{ij} \) and hence also \( \delta \) are real-analytic functions of \( L \). As was just seen, \( L \in \mathcal{L}_0 \) is singular if and only if \( \delta(L) = 0 \). This completes the proof. \( \Box \)

**Remark 4.1.** If \( k = 0 \), Lemma 4.2 remains true in the complex case; the entire proof carries over with “real-analytic” replaced by “complex-analytic,” except that the zero set of \( \delta(L) := \det(M(L))M(L)^T \) does not characterize the singular operators \( L \in \mathcal{L}_0 \). However, since \( k = 0 \), the matrix \( M(L) \) is square and \( \delta(L) := \det(M(L)) \) works. If \( k > 0 \), then the single mapping \( \delta \) must be replaced by the collection of all the principal minors of \( M(L) \) to preserve complex-analyticity.

**Theorem 4.3.** Let \( Z \) be a third Banach space and let \( U \subset X \) be a nonempty open connected subset. Let \( A : U \rightarrow \mathcal{L}(Z, Y) \) be a real-analytic mapping such that \( A(x) \) is Fredholm of index \( k \geq 0 \) for every \( x \in U \).

(i) If \( A(x) \) is singular (i.e., \( v(A(x)) = 0 \)) for \( x \) in some nonempty open subset of \( U \), then \( A(x) \) is singular for every \( x \in U \). Equivalently, if there is an \( x_0 \in U \) such that \( A(x_0) \) is onto \( Y \), then the set \( \{ x \in U : v(A(x)) = 0 \} \) is a closed subset of \( U \) with empty interior.

(ii) If \( \dim X < \infty \) and there is an \( x_0 \in U \) such that \( A(x_0) \) is onto \( Y \), the set \( \{ x \in U : v(A(x)) = 0 \} \) is either \( \emptyset \) or a real-analytic subvariety of \( U \) with dimension at most \( \dim X - 1 \) (and in particular consists of isolated points when \( \dim X = 1 \)).

**Proof.** (i) Denote by \( S \) the interior of the set of points \( x \in U \) such that \( v(A(x)) = 0 \), so that \( S \neq \emptyset \) is open in \( U \). To prove that \( S = U \) it suffices to show that \( S \) is closed in \( U \). Let then \( x_0 \in \text{cl}_U(S) \), the closure of \( S \) in \( U \). Since \( v \circ A \) is continuous on \( U \) and vanishes on \( S \), we have \( v(A(x_0)) = 0 \). From Lemma 4.2, there is an open neighborhood \( \mathcal{L}_0 \) of \( A(x_0) \) in \( \mathcal{L}(Z, Y) \) and a real-analytic function \( \delta : \mathcal{L}_0 \rightarrow \mathbb{R} \) such that \( v(L) = 0 \) if and only if \( \delta(L) = 0 \) whenever \( L \in \mathcal{L}_0 \). By the continuity of \( A \), there is thus an \( r > 0 \) such that \( v(A(x)) = 0 \) if and only if \( \delta(A(x)) = 0 \) whenever \( x \in B(x_0, r) \subset U \). Now, \( \delta \circ A \) is real-analytic and since \( x_0 \in \text{cl}_U(S) \), the set \( B(x_0, r) \cap S \) is a nonempty open subset of \( B(x_0, r) \). From the above, \( \delta \circ A = 0 \) on \( B(x_0, r) \cap S \), whence \( \delta \circ A = 0 \) on \( B(x_0, r) \) by Lemma 4.1. Thus, \( v(A(x)) = 0 \) for every \( x \in B(x_0, r) \), which shows that \( x_0 \in S \).
(ii) It suffices to show that if the set \( \{ x \in U : v(A(x)) = 0 \} \) is not empty, it coincides locally with the zero set of a nonzero real-analytic function. Let \( x_0 \in U \) be such that \( v(A(x_0)) = 0 \). By Lemma 4.2, there is an \( r > 0 \) such that the points \( x \in B(x_0, r) \subset U \) such that \( v(A(x)) = 0 \) are characterized by the condition \( \delta \circ A(x) = 0 \) with \( \delta \) real-analytic in the vicinity of \( A(x_0) \). Thus, \( \delta \circ A \) is real-analytic, and \( \delta \circ A \neq 0 \) by (i).

**Remark 4.2.** Part (i) of Theorem 4.3 is of course true in the complex case since passing to the underlying real structure merely doubles the index \( k \). In part (ii), the set \( \{ x \in U : v(A(x)) = 0 \} \) is either empty or a complex-analytic subvariety of \( U \) with dimension at most \( \dim X - 1 \). That may be seen by modifying the choice of \( \delta \) (in general vector-valued) as indicated in Remark 4.1. However, if \( k = 0 \), \( \delta \) remains scalar (also from Remark 4.1) and hence the set \( \{ x \in U : v(A(x)) = 0 \} \) is either empty or a complex-analytic subvariety of \( U \) with dimension exactly \( \dim X - 1 \).

**Corollary 4.4.** Suppose that \( F : X \to Y \) is Fredholm of index \( k \geq 0 \) and real-analytic. The set of critical points of \( F \) has empty interior in \( X \) if and only if there is an \( x_0 \in X \) such that \( DF(x_0) \) is onto \( Y \).

**Proof.** The necessity is obvious. At the beginning of this section, we observed that the real-analyticity of \( F \) entails the real-analyticity of \( DF \). Thus, the sufficiency follows from Theorem 4.3 with \( U = X = Z \) and \( A = DF \).

**Remark 4.3.** Corollary 4.4 is false for general real-analytic mappings, even polynomial ones, between infinite-dimensional Banach spaces. Therefore, the Fredholm assumption is needed. A quadratic counterexample is as follows. Let \( Y := C^0([−1, 1], \mathbb{R}) \) and let \( L \in \mathcal{L}(Y) \) be defined by \( (Ly)(t) := t^2y(t) \) for every \( y \in Y \) and every \( t \in [−1, 1] \). It is straightforward to check that \( L−\lambda I \) is onto \( Y \) for \( \lambda < 0 \) or \( \lambda > 1 \). Now, let \( X := \mathbb{R} \times Y \) and let \( F : X \to Y \) be defined by \( F(\lambda, y) := Ly−\lambda y \). This mapping is quadratic in \((\lambda, y)\). For \((\lambda, y)\) and \((\mu, v)\) in \( X \), we have \( DF(\lambda, y)(\mu, v) = Lv−\lambda v−\mu y \). Since \( L−\lambda I \) is onto \( Y \) for \( \lambda < 0 \) or \( \lambda > 1 \), it is plain that \( DF(\lambda, y) \) is onto \( Y \) for every pair \((\lambda, y)\) in \((−\infty, 0) \times Y \cup (1, \infty) \times Y \). We claim that \( DF(\lambda, y) \) is singular for every \((\lambda, y)\) in \((0, 1) \times Y \), a nonempty open subset of \( X \). To see this, fix \( 0 < \lambda < 1 \) and \( y \in Y \). By contradiction, suppose that \( DF(\lambda, y) \) is onto \( Y \). Then, given any \( z \in Y \), the equation \( Lv−\lambda v−\mu y = z \) has a solution \((\mu, v)\) in \( X \). Since \( (Lv)(t) = t^2v(t) \), the choices \( t = −\sqrt{\lambda} \) and \( t = \sqrt{\lambda} \) in \( (−1, 1) \) show that both the conditions \( −\mu y(−\sqrt{\lambda}) = z(−\sqrt{\lambda}) \) and \( −\mu y(\sqrt{\lambda}) = z(\sqrt{\lambda}) \) must hold for some \( \mu \in \mathbb{R} \). But it is always possible to find \( z \in Y \) such that one at least
among these two relations is violated for every \( \mu \in \mathbb{R} \). If \( y(-\sqrt{\lambda}) = y(\sqrt{\lambda}) = 0 \), the conditions reduce to \( z(-\sqrt{\lambda}) = z(\sqrt{\lambda}) = 0 \), and of course \( z \in Y \) can be found such that this does not hold. If \( y(-\sqrt{\lambda}) \neq 0 \) or \( y(\sqrt{\lambda}) \neq 0 \), \( z \in Y \) can be found such that \( y(-\sqrt{\lambda}) z(\sqrt{\lambda}) - z(-\sqrt{\lambda}) y(\sqrt{\lambda}) \neq 0 \) and for this choice there can be no \( \mu \in \mathbb{R} \) such that \( -\mu y(-\sqrt{\lambda}) = z(-\sqrt{\lambda}) \) and \( -\mu y(\sqrt{\lambda}) = z(\sqrt{\lambda}) \). Thus, \( DF(\lambda, y) \) is not onto \( Y \). A counterexample in Hilbert space is obtained by simply replacing \( Y \) above by \( H^1((-1,1)) \subset C^0([-1,1], \mathbb{R}) \).

5. GENERIC NONCOMPACTNESS OF THE FIBERS

We shall now combine the results from Sections 2 and 3 to derive the main abstract result of this paper (Theorem 5.4). An important special case (Corollary 5.5) will be obtained via Corollary 4.4. We need three rather simple technical lemmas.

**Lemma 5.1.** Suppose that \( F : X \to Y \) is of class \( C^1 \) and that the set of critical points of \( F \) has empty interior in \( X \). Then, the interior \( \text{int} F(X) \) of \( F(X) \) in \( Y \) is dense in \( F(X) \).

**Proof.** Let \( y \in F(X) \) be fixed and let \( x \in X \) be such that \( F(x) = y \). The hypotheses of the lemma ensure the existence of a sequence \((x_n) \subset X \) with \( \lim_{n \to \infty} x_n = x \) and \( DF(x_n) \) onto \( Y \). Since submersions are open maps, we have \( F(x_n) \in \text{int} F(X) \) and, of course, \( \lim_{n \to \infty} F(x_n) = F(x) = y \). Thus, \( y \) can be approximated by a sequence of points from \( \text{int} F(X) \). □

Recall that a Baire space is a topological space whose nonempty open subsets are of second category.

**Lemma 5.2.** Let \( W \subset Y \) be an open subset. The following properties hold:

(i) If \( E \subset \bar{W} \) is a subset of first category in \( Y \), then \( E \) is of first category in \( \bar{W} \).

(ii) If \( S \subset Y \) is any subset such that \( W \subset S \subset \bar{W} \), then \( S \) is a Baire space. (Equivalently: If \( \text{int} S \) is dense in \( S \), then \( S \) is a Baire space.)

**Proof.** The lemma is trivial if \( W = \emptyset \). From now on, we assume \( W \neq \emptyset \).

(i) Since \( E \) is of first category in \( Y \), we have \( E \subset \bigcup_{n \in \mathbb{N}} E_n \) where \( E_n \) is a closed subset of \( Y \) with empty interior in \( Y \). Thus, \( E \subset \bigcup_{n \in \mathbb{N}} (E_n \cap \bar{W}) \) and \( E_n \cap \bar{W} \) is closed in \( \bar{W} \) for every \( n \in \mathbb{N} \). Also, \( E_n \cap \bar{W} \) has empty interior
in \( \bar{W} \), for otherwise there is a \( y \in E_n \cap \bar{W} \) and an open ball \( B(y, r) \) in \( Y \) such that \( B(y, r) \cap \bar{W} \subset E_n \cap \bar{W} \). Since \( y \in E_n \cap \bar{W} \subset \bar{W} \), the (open) subset \( B(y, r) \cap W \) of \( Y \) is nonempty. But \( B(y, r) \cap W \subset B(y, r) \cap \bar{W} \subset E_n \cap W \subset E_n \) and hence \( E_n \) has nonempty interior in \( Y \), which is a contradiction.

(ii) By contradiction, if \( S \) contains a nonempty open subset of first category in \( S \), then there is a \( y \in S \) and an open ball \( B(y, r) \) in \( Y \) such that \( B(y, r) \cap W \) is of first category in \( S \). Since \( y \in S \subset \bar{W} \), the open subset \( B(y, r) \cap W \) of \( Y \) is nonempty and, since \( W \subset S \), we have \( B(y, r) \cap W \subset B(y, r) \cap S \). It follows that \( B(y, r) \cap W \) is of first category in \( S \) and therefore in \( Y \) (see Remark 5.1 below), which contradicts the Baire category theorem.

**Remark 5.1.** The above proof uses the following remark: If \( E \subset S \) is closed in \( S \) with empty interior in \( S \), then \( E \) is contained in a closed subset of \( Y \) with empty interior in \( Y \). To see this, write \( E = C \cap S \) with \( C \) closed in \( Y \). Note that \( (\text{int} C) \cap S = \emptyset \) since \( E \) has empty interior in \( S \), whence \( E = C \cap S = (\partial C) \cap S \subset \partial C \). Since \( C \) is closed in \( Y \), \( \partial C \) has empty interior in \( Y \).

The last preliminary result below is intuitively obvious.

**Lemma 5.3.** Let \( S \subset Y \) be a Baire space. If \( \mathcal{R} \subset S \) is a subset of \( S \) which is residual in \( S \), then \( \mathcal{R} \) is also residual in \( S \).

**Proof.** By hypothesis, there is a sequence of open and dense subsets \( \mathcal{O}_n \) of \( S \) such that \( \mathcal{R} \supset \bigcap_{n \in \mathbb{N}} \mathcal{O}_n \supset \bigcap_{n \in \mathbb{N}} (\mathcal{O}_n \cap S) \). For each \( n \in \mathbb{N} \), \( (\mathcal{O}_n \cap S) \) is an open subset of \( S \) and it suffices to show that if \( \mathcal{O} \subset S \) is an open and dense subset of \( S \), then \( \mathcal{O} \cap S \) is dense in \( S \). Let then \( y_0 \in S \) and \( \varepsilon > 0 \) be given. Since \( y_0 \in S \), there is a \( y \in \mathcal{O} \) such that \( \|y - y_0\| < \varepsilon/2 \) and, by the openness of \( \mathcal{O} \), there is a \( \delta < \varepsilon/2 \) such that \( z \in S \) and \( \|y - z\| < \delta \) implies \( z \in \mathcal{O} \). Since \( y \in \mathcal{O} \subset S \), we may find a \( z \in S \) such that \( \|y - z\| < \delta \). Thus, \( z \in \mathcal{O} \cap S \) and \( \|y_0 - z\| < \varepsilon \).

**Theorem 5.4.** Suppose that \( X \) and \( Y \) are separable real Banach spaces and that \( F: X \to Y \) is Fredholm of index \( k \geq 0 \) and of class \( C^{k+1} \). Suppose also that the set of critical points of \( F \) has empty interior in \( X \) and that the set \( A(F) \) of asymptotic values of \( F \) (see Definition 2.1) is dense in \( F(X) \). Then, \( F(X) \) is a Baire space and the set \( \mathcal{R} \) of the regular values of \( F \) such that \( F^{-1}(y) \) is noncompact is residual in \( F(X) \) and in \( F(X) \).
Proof. In a first step, we show that \( \overline{F(X)} \setminus \mathcal{R} \) is of first category in \( Y \). Let \( \tilde{V} \) denote the union of the connected components of \( Y \setminus K(F) \) that intersect \( F(X) \), so that \( \tilde{V} \subset \mathcal{R} \) by Corollary 2.4 (if \( y \in \tilde{V} \), then \( y \) is a regular value of \( F \) since \( K_0(F) \subset K(F) \)). Also, \( \Sigma_\alpha(F) \setminus K_0(F) \subset \mathcal{R} \) (see (2.7) for the definition of \( \Sigma_\alpha(F) \)) and hence
\[
\tilde{V} \cup \Sigma_\alpha(F) \setminus K_0(F) \subset \mathcal{R}.
\] (5.1)

On the other hand, we have \( F(X) \setminus K(F) = \tilde{V} \) by Theorem 2.2, whence \( F(X) \subset \tilde{V} \cup K(F) \). By Theorem 2.5, \( \partial F(X) \subset K(F) \), so that \( F(X) \subset \tilde{V} \cup K(F) \). Conversely, \( \tilde{V} \subset F(X) \subset F(X) \) and \( K(F) \subset K(F) \) (trivial) imply that \( \tilde{V} \cup K(F) \subset F(X) \). Therefore, \( F(X) = \tilde{V} \cup K(F) \) and now, by (5.1), \( F(X) \setminus \mathcal{R} \subset (\tilde{V} \cup K(F)) \setminus (\tilde{V} \cup \Sigma_\alpha(F) \setminus K_0(F)) \subset K(F) \setminus (\Sigma_\alpha(F) \setminus K_0(F)) \) (in fact, the latter inclusion is an equality). It thus follows from Theorem 3.3 that \( F(X) \setminus \mathcal{R} \) is of first category in \( Y \).

From Lemma 5.1, we have \( \text{int} F(X) = \overline{F(X)} \). Accordingly, we may choose \( W = \text{int} F(X) \) and \( E = F(X) \setminus \mathcal{R} \) in Lemma 5.2(i). This yields that \( F(X) \setminus \mathcal{R} \) is of first category in \( F(X) \), i.e., that \( \mathcal{R} \) is residual in \( F(X) \). Next, with \( S = F(X) \) in Lemma 5.2(ii), we obtain that \( F(X) \) is a Baire space. That \( \mathcal{R} \) is residual in \( F(X) \) follows from Lemma 5.3 with \( S = F(X) \).

Recall that it is only in Baire spaces that residual subsets may be called “large.” This is the reason why this property is underscored in Theorem 5.4. Of course, \( F(X) \) is also a Baire space, being a closed subset of \( Y \). Thus, in spirit, Theorem 5.4 states that almost every \( y \) in \( F(X) \) or in \( \overline{F(X)} \) is a regular value of \( F \) with \( F^{-1}(y) \) noncompact. We emphasize that “almost every” is here understood relative to \( F(X) \) or \( \overline{F(X)} \) and not merely relative to \( Y \). In particular, the result that almost every point of \( F(X) \) is a regular value of \( F \) does not follow from the Sard–Smale theorem, which does not even ensure that any point of \( F(X) \) is a regular value. We also stress that Theorem 5.4 does not say much about the solvability of the equation \( F(x) = y \): Just like the uniqueness results, but in the other direction, it gives a qualitative property (generic noncompactness) of the set of solutions when that set of solutions is nonempty. The only contribution of Theorem 5.4 to the existence question is the implicit result that \( F(X) \) is residual in \( F(X) \). In particular, if \( F(X) \) is dense in \( Y \), it must be residual in \( Y \), which is stronger.

Remark 5.2. When \( k = 0 \) in Theorem 5.4, the separability of \( X \) implies that \( F^{-1}(y) \) consists of an infinite sequence of distinct isolated points for every \( y \in \mathcal{R} \).
In Theorem 5.4, it is not true that $F^{-1}(y)$ is noncompact for every regular value $y \in F(X)$. A counterexample is given by $X = Y = \mathbb{C}$ and $F(x) := xe^x$ for which the hypotheses of the theorem hold, $y = 0$ is a regular value of $F$ and $F^{-1}(0) = \{0\}$ contains only one point.

There are two rather obvious technical difficulties in using Theorem 5.4 in practice. The first one is to check that the critical points of $F$ form a subset with empty interior in $X$. In light of Corollary 4.4, an important special case of Theorem 5.4 when this condition holds is as follows.

**Corollary 5.5.** Suppose that $X$ and $Y$ are separable real Banach spaces and that $F: X \to Y$ is real-analytic and Fredholm of index $k \geq 0$. Suppose also that $DF(x_0)$ is onto $Y$ for some $x_0 \in X$ and that the set $A(F)$ of asymptotic values of $F$ (see Definition 2.1) is dense in $F(X)$. Then, $F(X)$ is a Baire space and the set $\mathcal{R}$ of the regular values of $F$ such that $F^{-1}(y)$ is noncompact is residual in $F(X)$ and in $F(X)$.

Corollary 5.5 is not the only useful version of Theorem 5.4. In other words, the set of critical points of $F$ may sometimes be shown to have empty interior even when $F$ is not real-analytic. Examples will be given in the next section, with full details in [14].

The second technical difficulty in using Theorem 5.4 is of course the verification that the set $A(F)$ of asymptotic values of $F$ is dense in $F(X)$. This condition is necessary for the validity of Theorem 5.4 because $\mathcal{R} \subset A(F)$. In many problems, especially finite dimensional ones, the denseness issue is not easier to check than the very conclusion of the theorem, but the following result seems to be worth mentioning.

**Theorem 5.6.** Let $\mathcal{A}$ denote the space of real-analytic mappings from $\mathbb{R}^p$ to $\mathbb{R}^q$, $p \geq q \geq 1$ and let $\mathcal{S} \subset \mathcal{A}$ be the subset of those $F$ such that $\mathcal{R}_F := \{ y \in \mathbb{R}^q \text{ is a regular value of } F, F^{-1}(y) \text{ is unbounded} \}$ is residual in $\mathbb{R}^q$. Then, $\mathcal{S}$ is open in $\mathcal{A}$ equipped with the Whitney $C^1$ topology.

**Proof.** Let $F \in \mathcal{S}$ be given. Since $\mathcal{R}_F \subset F(X)$ is residual in $\mathbb{R}^q$ there is an $x_0 \in \mathbb{R}^p$ such that $DF(x_0)$ is onto $\mathbb{R}^q$ and hence there is an $\varepsilon_0 > 0$ such that every $L \in L(\mathbb{R}^p, \mathbb{R}^q)$ with $\|L - DF(x_0)\| < \varepsilon_0$ is onto $\mathbb{R}^q$. Let $\varepsilon: \mathbb{R}^p \to (0, \infty)$ be a continuous function such that $\lim_{|x| \to \infty} \varepsilon(x) = 0$ and $\varepsilon(x) < \varepsilon_0$ for every $x \in \mathbb{R}^p$. The set $\mathcal{U} := \{ G \in \mathcal{A} : \sup_{x \in \mathbb{R}^p} |G(x) - F(x)| + \sup_{x \in \mathbb{R}^p} \|DG(x) - DF(x)\| < \varepsilon(x) \}$ is an open neighborhood of $F$ in $\mathcal{A}$ for the Whitney $C^1$ topology. We now show that $\mathcal{U}$ is contained in $\mathcal{S}$.

Let $G \in \mathcal{S}$, so that $\|DG(x_0) - DF(x_0)\| < \varepsilon_0$ and hence $DG(x_0)$ is onto $\mathbb{R}^q$ from the above. Next, let $y \in \mathcal{R}_F$ and let $(x_n) \subset \mathbb{R}^p$ be a sequence tending to infinity in norm such that $F(x_n) = y$. Clearly, $\lim_{n \to \infty} G(x_n) = y$, whence
$y \in A(G)$. This shows that $\mathcal{R}_F \subset A(G)$. Since $\mathcal{R}_F$ is residual in $\mathbb{R}^n$, it follows that $A(G) \subset \overline{G(X)}$ is dense in $\mathbb{R}^q$. This implies that $\overline{G(X)} = \mathbb{R}^q$ and that $A(G)$ is dense in $\overline{G(X)}$. By Corollary 5.5, the set $\mathcal{R}_G$ is residual in $\overline{G(X)} = \mathbb{R}^q$, so that $G \in \mathcal{S}$.

Naturally, when $p = q = 2$, the set $\mathcal{S}$ of Theorem 5.6 contains all the nonpolynomial holomorphic functions. Observe that when $q = 1$, a much stronger variant of Theorem 5.6 can be given a straightforward proof based on the intermediate value theorem.

In the infinite dimensional setting, a considerable simplification of the problem of the denseness of $A(F)$ in $F(X)$ arises when $F$ has further properties with respect to the weak topology of $X$. Specifically:

**Lemma 5.7.** Suppose that $F: X \to Y$ satisfies the following two conditions:

(i) For every $x \in X$, the mapping $z \in X \to F(x + z) - F(z) \in Y$ is completely continuous, i.e., $F(x + z_n) - F(z_n) \to F(x) - F(z)$ in $Y$ whenever $(z_n) \subset X$ is a sequence converging weakly to $z$ in $X$.

(ii) There are a sequence $(x_n) \subset X$ and a constant $c > 0$ such that $\|x_n\| \geq c$ for every $n \in \mathbb{N}$, $x_n \to 0$ in $X$ and $F(x_n) \to F(0)$ in $Y$.

Then, $A(F)$ is dense in $F(X)$.

**Proof.** Neither the assumptions nor the conclusion of the lemma are affected by changing $F$ into $F - F(0)$, so that we may assume $F(0) = 0$ in the first place. Let $x \in X$ be fixed and let $(x_n)$ be the sequence given in condition (ii) of the lemma. Obviously, $(x_n)$ has no subsequence converging (strongly) in $X$ and hence the sequence $(x + x_n)$ has no subsequence converging in $X$ either. Write $F(x + x_n) = (F(x + x_n) - F(x_n)) + F(x_n)$. Since $F(x_n) \to F(0) = 0$, it follows from condition (i) of the lemma that $F(x + x_n) \to F(x)$. From Definition 2.1 and the above remark that $(x + x_n)$ has no convergent subsequence, we infer that $F(x) \in A(F)$ (even $A_b(F)$). This shows that $F(X) \subset A(F)(\subset F(X))$ and hence that $A(F)$ is dense in $F(X)$.

If $F$ is proper on the closed bounded subsets of $X$, then $A_b(F) = \emptyset$ and condition (ii) of Lemma 5.7 cannot hold. Nevertheless, nonlinear operators satisfying the conditions of Lemma 5.7 do arise in semilinear elliptic problems on $\mathbb{R}^n$ as we shall see in Section 6. From Theorem 5.4 and Lemma 5.7 it follows at once that:

**Corollary 5.8.** Suppose that $X$ and $Y$ are separable real Banach spaces and that $F: X \to Y$ is Fredholm of index $k \geq 0$ and of class $C^{k+1}$. Suppose also that the set of critical points of $F$ has empty interior in $X$ and that $F: X \to Y$ satisfies the following two conditions:
(i) For every \( x \in X \), the mapping \( z \in X \to F(x + z) - F(z) \in Y \) is completely continuous.

(ii) There is a sequence \( (x_n) \subset X \) such that \( \|x_n\| \geq c > 0 \) for every \( n \in \mathbb{N} \), \( x_n \to 0 \) in \( X \) and \( F(x_n) \to F(0) \) in \( Y \).

Then \( F(X) \) is a Baire space and the set \( \mathcal{R} \) of the regular values of \( F \) such that \( F^{-1}(y) \) is noncompact is residual in \( F(X) \) and in \( F(X) \).

By using Corollary 5.5 instead of Theorem 5.4 we obtain:

**Corollary 5.9.** Suppose that \( X \) and \( Y \) are separable real Banach spaces and that \( F: X \to Y \) is real-analytic and Fredholm of index \( k \neq 0 \). Suppose also that \( DF(x_0) \) is onto \( Y \) for some \( x_0 \in X \) and that \( F: X \to Y \) satisfies the following two conditions:

(i) For every \( x \in X \), the mapping \( z \in X \to F(x + z) - F(z) \in Y \) is completely continuous.

(ii) There is a sequence \( (x_n) \subset X \) such that \( \|x_n\| \geq c > 0 \) for every \( n \in \mathbb{N} \), \( x_n \to 0 \) in \( X \) and \( F(x_n) \to F(0) \) in \( Y \).

Then \( F(X) \) is a Baire space and the set \( \mathcal{R} \) of the regular values of \( F \) such that \( F^{-1}(y) \) is noncompact is residual in \( F(X) \) and in \( F(X) \).

Note that a special case of condition (ii) of Corollaries 5.8 and 5.9 arises when \( F(x_n) = F(0) \). In this form, it expresses a property of the solutions of the single equation \( F(x) = F(0) \), which therefore has an impact on the structure of the solutions of the equation \( F(x) = y \) for general \( y \). The following (academic) simple example illustrates the results of this section and shows that \( F \) need not be surjective or even have a dense image. Let \( X = Y = \ell^2 \) and with \( x : = (x_i) \), let \( F \) be defined by \( F(x) : = (x_i - x_i^2) \).

Clearly, \( F(x) \) is polynomial in \( x \), hence analytic and the equation \( F(x) = 0 = F(0) \) has the sequence \( (x^{(n)}) \) of solutions defined by \( x^{(n)}_i : = \delta_{ni} \) (Kronecker delta) with norm 1 and tending weakly to 0 in \( \ell^2 \). Also, for fixed \( x \) the mapping \( A_x : h \in \ell^2 \to 2(x_i h_i) \in \ell^2 \) is linear and compact, so that \( DF(x) = I - A_x \) is a compact perturbation of the identity \( F \) is Fredholm of index 0 and \( DF(0) = I \) is invertible. Lastly, for \( x, z \in \ell^2 \), we have that \( F(x + z) - F(z) = (x_i - x_i^2 - 2x_i z_i) = F(x) - A_x z \) is a completely continuous function of \( z \). Thus, Corollary 5.9 applies. In fact, here the validity of Corollary 5.9 can be verified by a direct calculation: For \( y \in \ell^2 \), the equation \( F(x) = y \) is solvable if and only if \( y_i \leq 1/4 \) for all indices \( i \). Thus, \( F(\ell^2) \) is closed in \( \ell^2 \) and \( F(\ell^2) \neq \ell^2 \). Now, let \( y \in F(\ell^2) \). The equation \( F(x) = y \) has a solution \( x^y \in \ell^2 \) with \( x^y_i = O(|y_i|) \) for every \( i \). On the other hand, since \( \lim_{n \to \infty} y_n = 0 \), the scalar equation \( x_n^y = y_n \) also has a solution \( x_n^y = O(1) \) for \( n \) large enough. Thus, for every large enough \( n \) the equation \( F(x) = y \) has (among many others) the sequence of distinct solutions \( (x^{(n)}) \subset \ell^2 \).
defined by $x_i^{(n)} = x_i^0$ for $i \neq n$ and $x_n^{(n)} = x_n^1$, with no convergent subsequence. Therefore, in this example, $F^{-1}(y)$ is noncompact for every $y$ for which the equation $F(x) = y$ is solvable.

6. APPLICATIONS TO QUASILINEAR ELLIPTIC EQUATIONS ON $\mathbb{R}^N$

First, consider the ODE

$$-u'' + u - u^3 = f \quad \text{in } \mathbb{R},$$

(6.1)

where $f \in L^p(\mathbb{R})$ for some $1 < p < \infty$ and the solutions $u$ are sought in the Sobolev space $W^{2,p}(\mathbb{R})$. It is trivial to check that the operator $F$ given by

$$F: u \in W^{2,p}(\mathbb{R}) \mapsto F(u) := -u'' + u - u^3 \in L^p(\mathbb{R})$$

(6.2)

is well defined and real-analytic (it is polynomial in $u$). Also, $DF(0)h = -h'' + h$, so that $DF(0) \in GL(W^{2,p}(\mathbb{R}), L^p(\mathbb{R}))$. The latter assertion is a special case of the well-known result that the operator $-D^2 + 1$ is an isomorphism of $W^{1,q}(\mathbb{R}^N)$ to $L^q(\mathbb{R}^N)$ for $1 < q < \infty$. When $q = 2$, this follows from the definition of the space $H^2(\mathbb{R}^N)$ by Fourier transform.

More generally, for $u \in W^{2,p}(\mathbb{R})$, $DF(u) = DF(0) + 3u^2$ where $3u^2$ is viewed as a multiplication operator. Since $u \in C^0(\mathbb{R})$ and tends to $0$ at infinity and since the multiplication by a measurable bounded function tending to $0$ at infinity is a compact linear operator from $W^{2,p}(\mathbb{R})$ (even $W^{1,q}(\mathbb{R})$) into $L^q(\mathbb{R})$, it follows that $DF(u)$ is a compact perturbation of the isomorphism $DF(0)$ and hence a Fredholm operator of index $0$. This shows that $F$ is Fredholm of index $0$.

By the same argument that multiplication by a measurable bounded function tending to $0$ at infinity is compact, the linear mappings $v \in W^{2,p}(\mathbb{R}) \to v\cdot v \in L^p(\mathbb{R})$ and $v \in W^{2,p}(\mathbb{R}) \to (u \cdot v)^{1/2} \in L^{2p}(\mathbb{R})$ are compact, hence completely continuous. This implies that the mappings $v \in W^{2,p}(\mathbb{R}) \to u \cdot v \in L^p(\mathbb{R})$ and hence also $v \in W^{2,p}(\mathbb{R}) \to uv^2 \in L^p(\mathbb{R})$ are completely continuous. As a result, the mapping $v \in W^{2,p}(\mathbb{R}) \to F(u + v) - F(v) = -u'' + u - 3uv^2 - 3uv^2 \in L^p(\mathbb{R})$ is completely continuous, i.e., condition (i) of Corollary 5.9 holds.

Remark 6.1. Above, the whole point is that the difference $F(u + v) - F(v)$ eliminates $v''$, $v$, and $v^3$, that is, the three terms that are not completely continuous as functions of $v \in W^{2,p}(\mathbb{R})$.

That condition (ii) of Corollary 5.9 also holds can be seen as follows: Let $v_0(x) = \sqrt{2}/\cosh x$. By a direct verification, $F(v_0) = 0 = F(0)$ and $v_0 \in W^{2,p}(\mathbb{R})$ because $v_0$, $v_0'$, and $v_0''$ tend exponentially to $0$ at infinity. For
\( n \in \mathbb{N} \), set \( v_n(x) := v_0(x+n) \). Then \( F(v_n) = 0 \) and \( v_n \rightharpoonup 0 \) in \( W^{2, p}(\mathbb{R}) \). For the latter property, note that \( v_n \rightharpoonup 0 \) in \( L^p(\mathbb{R}) \) since \( |v_0|_{0, p, \mathbb{R}} = \lim_{n \to \infty} |v_n|_{0, p, \mathbb{R}} = 0 \) for every \( \phi \in \mathcal{D}(\mathbb{R}) \). By the same argument, \( v_n' \rightharpoonup 0 \) in \( L^p(\mathbb{R}) \) and \( v_n'' \rightharpoonup 0 \) in \( L^p(\mathbb{R}) \) so that, altogether, \( v_n \rightharpoonup 0 \) in \( W^{2, p}(\mathbb{R}) \). Thus, by Corollary 5.9 we obtain:

**Theorem 6.1.** Let \( 1 < p < \infty \). The subset \( \mathcal{R} \) of those \( f \in L^p(\mathbb{R}) \) such that \( f \) is a regular value of the operator \( F: W^{2, p}(\mathbb{R}) \to L^p(\mathbb{R}) \) in (6.2) and \( F^{-1}(f) \subset W^{2, p}(\mathbb{R}) \) is noncompact, is residual in \( F(W^{2, p}(\mathbb{R})) \) and in \( F(W^{2, p}(\mathbb{R})) \).

From Remark 5.2, it follows that for \( f \in \mathcal{R} \), \( F^{-1}(f) \) in Theorem 6.1 is an infinite sequence of isolated points. Observe that despite the fact that translation invariance plays a key role in the proof of Theorem 6.1, it cannot be used in an obvious way to prove that \( F^{-1}(f) \) is noncompact, unless \( f = 0 \) (which is not even a regular value of \( F \)). Apparently, it is still unknown whether \( F(W^{2, p}(\mathbb{R})) = L^p(\mathbb{R}) \) or even dense in \( L^p(\mathbb{R}) \). In fact, the literature does not seem to contain much nontrivial information about the Eq. (6.1) when \( f \neq 0 \). Since \( DF(0) \) is an isomorphism, one such trivial piece of information is that \( F(W^{2, p}(\mathbb{R})) \) contains an open ball about 0 in \( L^p(\mathbb{R}) \).

**Remark 6.2.** The arguments used in the proof of Theorem 6.1 do not carry over to the case when the Eq. (6.1) is considered on a bounded open interval \( I \) and \( W^{2, p}(\mathbb{R}) \) and \( L^p(\mathbb{R}) \) are replaced (say) by \( W^{2, p}(I) \cap W^{1, p}_0(I) \) and \( L^p(I) \), respectively. In that case, the corresponding operator \( F \) is a (nonlinear) compact perturbation of the linear isomorphism \( DF(0) \) and hence proper on the closed bounded subsets of \( W^{2, p}(I) \cap W^{1, p}_0(I) \). As pointed out in the previous section, Corollary 5.9 is not available and the validity of Theorem 6.1 is an open question.

Theorem 6.1 can be generalized in many ways and in particular to higher dimensional problems

\[-Au + |u|^s u = f \quad \text{in} \quad \mathbb{R}^N, \tag{6.3}\]

with \( s > 0 \) a real number and \( 0 < s < 4/(N-2) \) if \( N \geq 3 \). Theorem 6.1 remains valid as stated, except for the restriction \( p > N/2 \) if \( N \geq 3 \) and, of course, \( \mathbb{R} \) replaced by \( \mathbb{R}^N \). As before, the existence of a nonzero solution in \( W^{2, p}(\mathbb{R}^N) \) of the homogeneous equation

\[-Au + |u|^s u = 0 \quad \text{in} \quad \mathbb{R}^N \tag{6.4}\]

plays a key role. That such a (radially symmetric) solution exists in \( H^1(\mathbb{R}^N) \) was found by Strauss \[17\] when \( N \geq 2 \). The case \( N = 1 \) is in Berestycki and Lions \[3\]. By a bootstrapping argument, the solutions of (6.4) in
$H^1(\mathbb{R}^N)$ can be shown to be in $W^{2,q}(\mathbb{R}^N)$ for every $1 < q < \infty$. If $s$ is an even integer, which cannot happen if $N > 4$ due to the restriction $s < 4/(N-2)$, the operator $F$ associated with (6.3), namely (with $N/2 < p < \infty$)

$$F: u \mapsto F(u) := -Du + u - |u|^s u \in L^p(\mathbb{R}),$$

remains real-analytic and Corollary 5.9 is still available.

**Remark 6.3.** To use Corollary 5.9 it must be checked that $DF(u)$ is Fredholm of index $k \geq 0$. By a result of Rabier and Stuart [15, Theorem 3.8] (or, here, by a direct argument) this follows from the fact that $DF(0) = -A + 1$ is an isomorphism from $W^{2,p}(\mathbb{R}^N)$ onto $L^p(\mathbb{R}^N)$.

The case when $s$ in (6.3) is not an even integer can be handled by Corollary 5.8, but this requires proving that the set of critical points of $F$ in (6.5) has empty interior in $W^{2,p}(\mathbb{R}^N)$. See [14] for details.

**Remark 6.4.** If $N \geq 3$ and $s \geq 4/(N-2)$, it is proved in Willem [18, p. 64] that (6.4) has no nonzero solution in $H^1(\mathbb{R}^N) \cap L^{s+2}(\mathbb{R}^N)$, and this easily implies that it has no nonzero solution in $W^{2,p}(\mathbb{R}^N)$ when $p > N/2$. As a result, by [15, Theorem 6.5], $F$ in (6.5) is proper on the closed bounded subsets of $W^{2,p}(\mathbb{R}^N)$ and hence Corollary 5.8 (or 5.9) cannot be used. (Once again, Theorem 6.5 in [15] is given for $p > N$ but remains valid when $p > N/2$ due to the independence of $F$ upon $Vu$.) The generic structure of $F^{-1}(f)$ (if any) for $N \geq 3$ and $s \geq 4/(N-2)$ remains unknown.

Theorem 5.4, but neither Corollary 5.8 nor Corollary 5.9, can still be used to extend Theorem 6.1 to genuinely quasilinear problems. For such problems, condition (i) of Lemma 5.7 fails to hold. One of the simplest examples is given by

$$-(1+u^2) u'' + u - u^3 - 3u^5 + 2u^7 = f$$

in $\mathbb{R}$, (6.6)

where, as before, $f \in L^p(\mathbb{R})$ for some $1 < p < \infty$ and the solutions $u$ are sought in $W^{2,p}(\mathbb{R})$. It is readily seen that condition (i) of Lemma 5.7 does not hold for this example, which, however, retains the basic property that the homogeneous equation

$$-(1+u^2) u'' + u - u^3 - 3u^5 + 2u^7 = 0$$

in $\mathbb{R}$ (6.7)

has a nontrivial solution in $W^{2,p}(\mathbb{R})$. In fact, the coefficients were chosen that $v_0(x) := 1/\cosh x$ is a solution of (6.7). The corresponding operator $F$ is real-analytic (obvious) and Fredholm of index 0 by [15, Theorem 3.8]
and the fact that $DF(0)$ is an isomorphism. Although Lemma 5.7 is not available, it can still be proved that the subset $A_b(F)$; hence also $A(F)$, is dense in $F(W^{2,p}(\mathbb{R}))$. This is mostly obtained as a corollary of Theorem 6.5 in [15] with some extra technicalities fully explained in [14]. This property makes Theorem 5.4 available. Higher dimensional and/or nonanalytic variants of (6.7) can be considered as well. In all cases, the existence of a nontrivial solution of the corresponding homogeneous equation is essential.

All the examples discussed above have “constant coefficients.” Generalizations exist for problems with $N$-periodic or even “asymptotically $N$-periodic” coefficients. See once more [14] for details. Here, we give only a simple example: A variant of (6.1) with asymptotically constant coefficients (a special case) is given by

$$-u'' + u - u^3 + \varphi(x) u^k = f \quad \text{in } \mathbb{R},$$  

(6.8)

where $k \geq 2$ is a positive integer and $\varphi: \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\lim_{|x| \to \infty} \varphi(x) = 0$. As a mapping from $W^{2,p}(\mathbb{R})$ to $L^p(\mathbb{R})$ (with $1 < p < \infty$), $F(u) := -u'' + u - u^3 + \varphi(x) u^k$ is still real-analytic and Fredholm of index 0. As explained in [14], an interesting feature is that the homogeneous equation $F(u) = 0$ has no longer any particular relevance for the validity of Theorem 6.1. What is important is the nonemptiness of the set of solutions of $F(u) = 0$, where $F^\infty$ is obtained from $F$ by letting $|x|$ tend to infinity. From the assumption that $\lim_{|x| \to \infty} \varphi(x) = 0$, the operator $F^\infty$ is thus $F^\infty(u) = -u'' + u - u^3$ and the equation $F^\infty(u) = 0$ is just Eq. (6.2). The existence of a nontrivial solution to (6.2) ensures the denseness of $A(F)$ in $F(W^{2,p}(\mathbb{R}))$ and Theorem 6.1 for the problem (6.8) follows from Corollary 5.9.

For more general problems with asymptotically $N$-periodic coefficients, the limiting operator $F^\infty$ has $N$-periodic coefficients. The existence of nontrivial solutions of $F^\infty(u) = 0$ for this case is more delicate than when the coefficients are constant but results of this type have been proved for instance by Kryszewski and Szulkin [10].

Generalizations of Theorem 6.1 exist in at least three other reactions: To systems, to higher order elliptic problems, and to exterior domains. Indeed, there is nothing in the proof of Theorem 6.1 that specifically relies on the fact that the equation is scalar or second order. That a generalization to an exterior domain $\Omega$ is possible uses the remark that the limit operator $F^\infty$ can be extended to $W^{2,p}(\mathbb{R}^N)$ because it depends only upon the coefficients of $F$ “at infinity.” By a cut-off argument similar to that used in Galdi and Rabier [7] for the Navier–Stokes problem, the denseness of $A(F)$ in

\footnote{But for the opposite purpose to prove properness on closed bounded subsets.}
$F(\mathbb{W}^{2, p}(\Omega) \cap \mathbb{W}^{1, p}_0(\Omega))$ can be derived from the existence of a nonzero solution to the equation $F''(u) = 0$ in $\mathbb{W}^{2, p}(\mathbb{R}^N)$ (not $\mathbb{W}^{2, p}(\Omega) \cap \mathbb{W}^{1, p}_0(\Omega)$), that is, to the exact same equation as when $\Omega = \mathbb{R}^N$.

REFERENCES

15. P. J. Rabier and C. A. Stuart, Fredholm and properness properties of quasilinear elliptic operators on $\mathbb{R}^N$, in press.