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Ring ideals and the Stone-Čech compactification in pointfree topology

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ABSTRACT

This paper shows that the compact completely regular coreflection in the category of frames is given by the frame of Jacobson radical ideals of the ring $\mathfrak{R}L$ of real-valued continuous functions on L, as an alternative to its familiar representations in terms of (i) the l-ideals of $\mathfrak{R}L$ as lattice-ordered ring or (ii) the ideals of the bounded part of $\mathfrak{R}L$ which are closed in the usual uniform topology. Further, in analogy with this, the compact zero-dimensional coreflection will also be described in terms of ring ideals, this time of the ring $\mathfrak{Z}L$ of integer-valued continuous functions on L.

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Both familiar descriptions of the Stone-Čech compactification, that is, the compact completely regular coreflection, in pointfree topology which involve ideals of continuous real-valued functions do not use mere ring ideals but operate either with suitable *l-ring ideals* or with ideals of bounded functions that are *closed in the uniform topology*. Naturally, this raises the question whether there is a purely ring-theoretical notion which could serve the same purpose, given that this does indeed hold in the corresponding classical situation. It is the object of this paper to show that this is also the case here by establishing that *the frame of Jacobson radical ideals of the ring $\mathfrak{R}L$ of continuous real-valued functions on a frame L provides the compact completely regular coreflection of L.*

It should be noted that there are ways of piecing this result together from various facts involving the relation between the frames of l-ideals of f-rings and their ring ideal lattices, the existing description of the coreflection in question in terms of the bounded part of $\mathfrak{R}L$ as l-ring, and the relation between certain ideals of f-rings and of their convex subrings (Banaschewski [3,4,6]) but it seemed worthwhile to provide a succinct autonomous proof of this.

As a convenient tool for this purpose, we introduce the notion of a *regular support* on a commutative ring A with unit as a support $A \to L$, L any frame, satisfying a natural regularity condition (Section 1) and show that, in the case of Gelfand rings, the map from A to the frame $\mathrm{JRId}A$ of its Jacobson radical ideals which takes each $a \in A$ to the Jacobson radical ideal generated by it is the universal regular support on A (Lemma 1). From this we readily obtain that the familiar cozero map $\mathrm{coz} : \mathfrak{R}L \to L$ induces a frame homomorphism $\sigma_L : \mathrm{JRId}\mathfrak{R}L \to L$ (Corollary 1) which will then be identified as the coreflection map from compact completely regular frames (Lemma 2, Proposition 1). In addition, we establish an analogue of this result for the compact zero-dimensional coreflection, this time using the ring of integer-valued continuous functions on L (Proposition 2).

Regarding the background required here we refer to Johnstone [10], Pultr [11], or Vickers [12] for general facts about frames, to Banaschewski [4,7] for details concerning the function ring $\mathfrak{R}L$, and to Bigard, Keimel and Wolfenstein [9] for lattice ordered rings. As to foundations, the setting adopted here is Zermelo–Fraenkel set theory treated within classical logic. The natural question to what extent the present results are constructively valid in the familiar sense of topos theory remains to be investigated.

Further, we recall the following. The ring $\mathfrak{R}L$ is a Gelfand ring, meaning that the frame $\mathrm{RId}\mathfrak{R}L$ of its radical ideals J ($a \in J$ whenever some $a^n \in J$) is normal and consequently a *compact normal* frame (Banaschewski [5,7]). Next, for any compact

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frame L, we have its *saturation quotient* SL determined by the saturation nucleus s_L on L where

$$s_L(a) = \bigvee\{b \in L \mid b \vee c = e \text{ implies } a \vee c = e\},$$

calling the $b \in L$ involved the *a-small* elements; further, for compact *normal* L, SL is compact regular and the correspondence $L \mapsto SL$ is functorial (Banaschewski [6]). Finally, for any commutative ring A with unit, the saturation quotient S(RIdA) of its frame of radical ideals is equal to the frame JRIdA of its Jacobson radical ideals, that is, the ideals $J \subseteq A$ such that $a \in A$ belongs to J whenever all $1 + ar$, $r \in A$, are invertible modulo J.

The latter result can be derived from Banaschewski–Harting [1], but it seems convenient to include an independent proof here. Consider then any $J \in$ S(RIdA) and $a \in A$ for which all $1 + ar$, $r \in A$, are invertible modulo J. Now if $[a] \vee H = [1]$ in RIdA ($[\cdot]$ for the principal radical ideal) then $x + y = 1$ where $x \in [a]$ and $y \in H$, and using a suitable power of $x = 1 - y$ we see that $as + b = 1$ for some $s \in A$ and $b \in H$; further as $b = 1 - as$ is invertible modulo J, we have $c \in A$ for which $bc - 1 \in J$. It follows that $J \vee H$ contains $1 = (1 - bc) + bc$ so that $J \vee H = [1]$; hence, $a \in J$ as J is saturated, and this shows $J \in$ JRIdA. Conversely, given any $J \in$ JRIdA, consider a J-small $I \in$ RIdA and $a \in I$. Now trivially, for any $r \in A$, $I + [1 + ar] = [1]$ so that $J + [1 + ar] = [1]$ by the nature of I, and therefore $x + y = 1$ where $x \in J$ and $y^n = (1 + ar)b$ for some $b \in A$ and n. Hence,

$$(1 + ar)b = (1 - x)^n = 1 + c$$

where $c \in J$, showing that $1 + ar$ is invertible modulo J and hence $a \in J$.

1. Regular supports and Gelfand rings

Recall that, for a commutative ring A with unit, a *support* on A is a map $\sigma : A \to L$ into a frame L such that

$$\sigma(ab) = \sigma(a) \wedge \sigma(b), \qquad \sigma(1) = e$$
$$\sigma(a + b) \le \sigma(a) \vee \sigma(b), \qquad \sigma(0) = 0$$

where e and 0 are the unit (=top) and zero (=bottom) of L, respectively.

In particular, the map $A \to$ RIdA taking each $a \in A$ to the corresponding principal radical ideal $[a]$ is a support, and in fact the *universal support* on A, meaning that any support $\sigma : A \to L$ induces a frame homomorphism RId$A \to L$ taking each $J \in$ RIdA to $\bigvee \sigma[J]$ and hence $[a]$ to $\sigma(a)$. Further, σ will be called *regular* if

$$\sigma(a) = \bigvee\{\sigma(b) \mid [b] \prec [a] \text{ in RId}A\}$$

where $\prec$ signifies the familiar "rather below" (or "well inside") relation in frames.

In the following, s is the saturation nucleus on RIdA and $\langle a \rangle = s([a])$, the saturated radical (=Jacobson radical) ideal generated by a. Also, any ring is taken to be commutative with unit.

Lemma 1. *For any Gelfand ring A, $A \to JRIdA$, $a \mapsto \langle a \rangle$, is the universal regular support on A.*

Proof. We have to show that (1) the indicated map is a regular support, and (2) for any regular support $\sigma : A \to L$ there exists a unique frame homomorphism $h :$ JRId$A \to L$ such that $h(\langle a \rangle) = \sigma(a)$ for all $a \in A$.

Re (1): Since $a \mapsto [a]$, as is familiar, is a support, $A \to$ JRIdA, $a \mapsto \langle a \rangle$ is evidently also a support. On the other hand, by the regularity of JRIdA (=S(RIdA)!) in the present situation,

$$\langle a \rangle = \bigvee\{I \in \text{JRId}A \mid I \prec \langle a \rangle\} = \bigvee\{\langle b \rangle \mid \langle b \rangle \prec \langle a \rangle\} = \bigvee\{\langle b \rangle \mid [b] \prec [a]\},$$

the final step because $\langle b \rangle \prec \langle a \rangle$ iff $[b] \prec [a]$, by the properties of s.

Re (2): To begin with, any regular support $\sigma : A \to L$ determines a homomorphism $h :$ RId$A \to L$ such that $h(J) = \bigvee \sigma[J]$ by the properties of RIdA, and this will induce the desired homomorphism JRId$A \to L$ if it is shown that $h(s(J)) \le h(J)$. Further, the latter will follow if we prove that $h(I) \le h(J)$ for any J-small $I \in$ RIdA, meaning: $\sigma(a) \le h(J)$ for all $a \in I$. Now for any $b \in A$ such that $[b] \prec [a]$, take $H \in$ RIdA for which $[b] \cap H = [0]$ and $[a] \vee H = [1]$. Then also $J \vee H = [1]$, I being J-small and $a \in I$, hence $[b] \subseteq J$ since $[b] \cap H = [0]$, so that $\sigma(b) \le h(J)$ and the regularity of σ then shows $\sigma(a) \le h(J)$, as desired. Finally, the uniqueness is automatic because the $\langle a \rangle$ generate JRIdA. □

For the application we want to make of this lemma, recall the cozero map $\text{coz} : \mathfrak{R}L \to L$ for which

$$\text{coz}(\gamma) = \bigvee\{\gamma(p, 0) \vee \gamma(0, q) \mid p < 0 < q \text{ in } \mathbb{Q}\}.$$

This is well known to be a support, and since $[(n\gamma - \mathbf{1})^+] \prec [\gamma]$ (use $(\mathbf{1} - n\gamma)^+$) for any $\gamma \in \mathfrak{R}L$, the familiar fact that

$$\text{coz}(\gamma) = \bigvee\{\text{coz}((n\gamma - \mathbf{1})^+) \mid n = 1, 2, \ldots\}$$

shows that it is regular. Hence we conclude, by Lemma 1 and the fact that $\mathfrak{R}L$ is Gelfand:

Corollary 1. *For any frame L, there exists a homomorphism $\sigma_L : JRId\mathfrak{R}L \to L$ such that $\sigma_L(\langle \gamma \rangle) = \text{coz}(\gamma)$.*

Remark 1. Lemma 1 has the notable consequence that, as in the case of RIdA and arbitrary commutative rings A with unit, JRIdA may be described, for any Gelfand ring A, by generators and relations, namely generators x_a corresponding to the elements $a \in A$ and the relations

$$x_{ab} = x_a \wedge x_b, \qquad x_1 = e, \qquad x_{a+b} \leq x_a \vee x_b, \qquad x_0 = 0, \qquad x_a = \bigvee \{x_b \mid b \prec a\}$$

where $b \prec a$ means that

$$au + v = 1 \quad \text{and} \quad (bv)^n = 0$$

for some $u, v \in A$ and n (since $b \prec a$ iff $[b] \prec [a]$ in RIdA).

Remark 2. We note in passing that the regularity of the support $A \to$ JRIdA, $a \mapsto \langle a \rangle$, used above is actually characteristic of Gelfand rings: *a commutative ring with unit is Gelfand whenever that support is regular.*

For any $I, J \in$ RIdA such that $I \vee J = [1]$, $[a] \vee [b] = [1]$ for some $a \in I$ and $b \in J$, and hence by hypothesis also

$$\langle 1 \rangle = \bigvee \{\langle u \rangle \mid [u] \prec [a]\} \vee \bigvee \{\langle v \rangle \mid [v] \prec [b]\}$$

in JRIdA. Further, by compactness and the properties of $\prec$, suitable finite joins of the $[u]$ and $[v]$ provide H and K in RIdA such that

$$H \prec [a], \qquad K \prec [b], \qquad H \vee K = [1].$$

Now, take $\overline{H}$ and $\overline{K}$ in RIdA such that

$$H \cap \overline{H} = [0], \qquad [a] \vee \overline{H} = [1], \qquad K \cap \overline{K} = [0], \qquad [b] \vee \overline{K} = [1].$$

Then $I \vee \overline{H} = [1] = J \vee \overline{K}$ while $\overline{H} \cap \overline{K} = \overline{H} \cap \overline{K} \cap (H \vee K) = [0]$, showing that RId$A$ is normal, that is, A is Gelfand.

2. The compact completely regular coreflection

Recall that a lattice-ordered ring with unit is said to have *bounded inversion* if every element $a \geq 1$ is invertible, and note that such a ring is automatically over $\mathbb{Q}$, and hence any $a \geq r$, $r > 0$ in $\mathbb{Q}$, is invertible.

Lemma 2. *For any commutative f-ring A with unit and bounded inversion,* JRIdA *is completely regular.*

Proof. In the following s, $[\cdot]$, and $\langle \cdot \rangle$ are used as before, and $\rho(\cdot)$ stands for the radical of an ideal.

Since the present A is Gelfand, RIdA is compact normal, making JRIdA = S(RIdA) compact regular (Banaschewski [5,7]). Hence it will be enough to show that, in JRIdA, $I \prec J$ implies $I \prec\prec J$. Let then $I \cap H = \langle 0 \rangle$ and $I \vee H = \langle 1 \rangle$ in JRIdA where the latter means $s\rho(J + H) = \langle 1 \rangle$ and hence $\rho(J + H) = [1]$ so that $J + H = [1]$ by the properties of s and ρ. Thus $a + b = 1$ for $a \in J$ and $b \in H$. Now, for any $r \in \mathbb{Q}$ such that $0 < r < 1$

$$(a - r)^+ + b = (a + b - r) \vee b = (1 - r) \vee b \geq 1 - r,$$

and since $1 - r > 0$, $(a - r)^+ + b$ is invertible so that $\langle (a - r)^+ \rangle \vee \langle b \rangle = \langle 1 \rangle$. As a result, $I \subseteq \langle (a - r)^+ \rangle$ since $I \cap \langle b \rangle \subseteq I \cap H = \langle 0 \rangle$. On the other hand, $\langle a \rangle \vee \langle (r - a)^+ \rangle = \langle 1 \rangle$ because $a + (r - a)^+ = r \vee a \geq r > 0$, and

$$\langle (a - r)^+ \rangle \cap \langle (r - a)^+ \rangle = s([(a - r)^+] \cap [(r - a)^+]) = s([(a - r)^+ (r - a)^+]) = s(0) = \langle 0 \rangle$$

by the general rule $[x] \cap [y] = [xy]$ for principal radical ideals and $x^+ x^- = 0$ by the properties of f-rings. As a result, $\langle (a - r)^+ \rangle \prec \langle a \rangle$ and therefore also

$$\langle (a - q)^+ \rangle \prec \langle (a - p)^+ \rangle \quad \text{whenever } 0 < p < q < 1,$$

showing that, in fact, $\langle (a - r)^+ \rangle \prec\prec \langle a \rangle$ for any $0 < r < 1$. Finally, $a \in J$ implies $\langle a \rangle \subseteq J$, and

$$I \subseteq \langle (a - r)^+ \rangle \prec\prec \langle a \rangle \subseteq J$$

then proves $I \prec\prec J$, as claimed. □

Remark 3. It should be noted that any compact regular frame, in view of the fact that the relation $\prec$ interpolates by compactness, is completely regular if the Axiom of Countable Dependent Choice is assumed. The point here is that, in the present case, the complete regularity of JRIdA can be obtained *without any choice assumption*.

Proposition 1. *For any frame L, σ_L : JRId$\mathfrak{R}L \to L$ is the coreflection map from compact completely regular frames.*

Proof. By Lemma 2, the frames $\mathrm{JRId}\mathfrak{R}L$ are of the right kind because the $\mathfrak{R}L$ are known to have bounded inversion. On the other hand, the correspondence $L \mapsto \mathrm{JRId}\mathfrak{R}L$ is functorial since $\mathfrak{R}$ and RId are functors on all frames and all commutative rings with unit, respectively, while the passage $M \mapsto SM$ to the saturation quotient for compact frames M is functorial, as already mentioned, for compact normal M so that $\mathrm{SRId} = \mathrm{JRId}$ is at least functorial for Gelfand rings which is sufficient here. Further, for any frame homomorphism $h : L \to M$, the corresponding homomorphism $\overline{h} : \mathrm{JRId}\mathfrak{R}L \to \mathrm{JRId}\mathfrak{R}M$ takes $\langle\gamma\rangle$ to $\langle h\gamma\rangle$ and consequently

$$h\sigma_L(\langle\gamma\rangle) = h(\mathrm{coz}(\gamma)) = \mathrm{coz}(h\gamma) = \sigma_M\overline{h}(\langle\gamma\rangle)$$

by Lemma 1 and the definition of coz, showing the naturality of the σ_L. Finally, if L itself is compact completely regular then σ_L is an isomorphism: it is onto because its image contains all $\mathrm{coz}(\gamma)$, $\gamma \in \mathfrak{R}L$, and they generate L by complete regularity, and since it is clearly dense ($\gamma = \mathbf{0}$ whenever $\mathrm{coz}(\gamma) = 0$), it is one–one by compactness. This proves the non-trivial part of the required condition that L is compact completely regular iff σ_L is an isomorphism, showing that the σ_L are the desired coreflection maps. □

Proposition 1 has an interesting consequence concerning strong 0-dimensionality. For this, recall that

(1) in analogy with the terminology in classical topology, a completely regular frame is called *strongly* zero-*dimensional* if its compact completely regular coreflection is zero-dimensional;

(2) a commutative ring A with unit is called an *exchange ring* (or: *clean*) if, for each $a \in A$, there exists an idempotent u such that $a + u$ is invertible; and

(3) a Gelfand ring is an exchange ring iff $\mathrm{JRId}A$ is zero-dimensional (Banaschewski [5]).

Consequently, Proposition 1 implies the following characterization.

Corollary 2. *A completely regular frame L is strongly* zero-*dimensional iff $\mathfrak{R}L$ is an exchange ring.*

Remark 4. Given the familiar fact that

(1) for any topological space X and its frame $\mathfrak{O}X$ of open sets, $\mathfrak{R}(\mathfrak{O}X) \cong C(X)$, and

(2) for the spectrum functor Σ from frames to spaces and any commutative ring A with unit, $\Sigma(\mathrm{JRId}A) = \mathrm{Max}A$, the maximal ideal space of A,

Proposition 1 immediately exhibits the *Stone-Čech compactification* of a space X as $\mathrm{Max}C(X)$ with the familiar map $X \to \mathrm{Max}C(X)$. Note, however, that here one has to use (unavoidably) the Prime Ideal Theorem. Thus, Proposition 1 is the pointfree and choice-free antecedent of "βX as $\mathrm{Max}C(X)$".

3. The compact zero-dimensional coreflection

We now turn to the zero-dimensional counterpart of the Stone-Čech compactification, the spatial form of which is known as the Stone–Banaschewski compactification. As is familiar, this has an extremely simple description, namely as the map $\mathfrak{I}(BL) \to L$ from the ideal lattice $\mathfrak{I}(BL)$ of the Boolean algebra BL of complemented elements of L which takes each $J \in \mathfrak{I}(BL)$ to its join $\bigvee J$ in L (Banaschewski [2]).

As far as representing this coreflection in terms of continuous functions on L is concerned, it is natural to turn to the integer-valued continuous functions, that is, the maps $\alpha : \mathbb{Z} \to L$ such that

$$\alpha(k) \wedge \alpha(l) = 0 \quad \text{for } k \neq l \quad \text{and} \quad \bigvee\{\alpha(m) \mid m \in \mathbb{Z}\} = e,$$

which form the familiar l-ring $\mathfrak{Z}L$ (Banaschewski [8]) as "integer-valued" counterpart of $\mathfrak{R}L$. In particular, then, the frame $\mathfrak{L}\mathfrak{Z}L$ of l-ring ideals is compact zero-dimensional, and the coreflection map involved here is then

$$\mathfrak{L}\mathfrak{Z}L \to L, J \mapsto \bigvee \mathrm{coz}[J],$$

where the cozero map $\mathfrak{Z}L \to L$ is now given by

$$\mathrm{coz}(\alpha) = \bigvee\{\alpha(m) \mid 0 \neq m \in \mathbb{Z}\}.$$

The question arising here, given our earlier result concerning the Stone-Čech compactification, is whether this coreflection can also be represented in terms of the *ring ideals* of $\mathfrak{Z}L$, and our aim now is to show that this is indeed the case.

Note at the outset that the frame of Jacobson radical ideals will not provide what is required in this setting: if $L = \mathbf{2}$, the two-element chain, then $\mathfrak{Z}L = \mathbb{Z}$, and the compact zero-dimensional coreflection of $\mathbf{2}$ is of course $\mathbf{2}$ itself while $\mathrm{JRId}\mathbb{Z}$ is certainly not isomorphic to that. This example suggests what is required here is a quotient of the frame of radical ideals which makes the regular elements, that is, the non-zerodivisors, act like invertible ones. For an arbitrary commutative ring A with unit this may be described as follows. Let $\mathrm{Reg}A$ be the set of its regular elements and k the operator on $\mathrm{RId}A$ for which

$$k(J) = \{a \in A \mid ar \in J \text{ for some } r \in \mathrm{Reg}A\}.$$

By the obvious properties of $\mathrm{Reg}A$, one easily sees that this is indeed a radical ideal and that k is a nucleus, making $\mathrm{Fix}(k)$ a frame which will be denoted as $\mathrm{KRId}A$. Further, this frame is *compact*, by the fact that its up-directed joins are simply unions, as one readily checks.

Now, in the present case of $A = \mathfrak{Z}L$, it turns out this frame is also zero-*dimensional*. In actual fact, this holds for a considerably larger class of rings, namely the $\mathbb{Z}$-rings, meaning the commutative f-rings with unit which satisfy the $\mathbb{Z}$-identity

$$a \wedge (1 - a) \leq 0.$$

As is familiar, the $\mathfrak{Z}L$ are of this kind because they satisfy all l-ring identities valid in $\mathbb{Z}$.

In any $\mathbb{Z}$-ring A, we let $[a]$ be the radical ideal generated by $a \in A$ and $\langle a \rangle = k([a])$ for each $a \in A$. Note that $[a] = [|a|]$ for any $a \in A$, by the fact that $a^2 = |a|^2$ in f-rings, and hence also $\langle a \rangle = \langle |a| \rangle$. Next, KRId$A$ is obviously generated by these $\langle a \rangle$ so that the claimed zero-dimensionality will follow if we show that any $\langle a \rangle$ is complemented in KRIdA.

Now, for any $a \in A$, $|a| \wedge 1$ is idempotent by the $\mathbb{Z}$-identity so that $(|a| \wedge 1)(1 - (|a| \wedge 1)) = 0$. Consequently, by the properties of f-rings,

$$0 = |a|(1 - (|a| \wedge 1)) \wedge (1 - (|a| \wedge 1))$$

and hence also

$$0 = |a|(1 - (|a| \wedge 1))(1 - (|a| \wedge 1)) = |a|(1 - (|a| \wedge 1)).$$

As a result we then have

$$\begin{aligned}\langle |a| \rangle \cap \langle 1 - (|a| \wedge 1) \rangle &= k([|a|]) \cap k([1 - (|a| \wedge 1)]) \\ &= k([|a|] \cap [1 - (|a| \wedge 1)]) = k([|a|(1 - (|a| \wedge 1))]) = k([0]) = \langle 0 \rangle\end{aligned}$$

the third step by the familiar rules concerning principal radical ideals. On the other hand, $\langle |a| \rangle \vee \langle 1 - (|a| \wedge 1) \rangle$ contains $|a| + 1 - (|a| \wedge 1) \geq 1$ and is therefore $\langle 1 \rangle$, showing that $\langle 1 - (|a| \wedge 1) \rangle$ is the desired complement of $\langle a \rangle = \langle |a| \rangle$. The fact that $|a|(1 - (|a| \wedge 1)) = 0$ in any $\mathbb{Z}$-ring A also implies that Reg$A = \{a \in A \mid 1 \leq |a|\}$ for such A: for any $a \in$ Reg$A$, this condition shows $1 - (|a| \wedge 1) = 0$ so that $1 = |a| \wedge 1 \leq |a|$; on the other hand, if $1 \leq |a|$ then $ab = 0$ implies $|b| \leq |a||b| = |ab| = 0$, and therefore $b = 0$, proving that $a \in$ Reg$A$.

Next, we have to show that KRId$(\mathfrak{Z}L)$ is functorial in L, and since $L \mapsto \mathfrak{Z}L$ is a functor to the category of $\mathbb{Z}$-rings and l-ring homomorphisms this will follow if we prove that $A \mapsto$ KRIdA is functorial on the latter. Now, $A \mapsto$ RIdA is certainly functorial such that the frame homomorphism $\tilde{\varphi}$: RId$A \to$ RIdB determined by $\varphi : A \to B$ takes any radical ideal J of A to the radical ideal of B generated by its image $\varphi[J]$. Hence all that has to be checked is that this $\tilde{\varphi}$ interacts properly with the nuclei k_A on RIdA and k_B on RIdB which determine the quotient frames KRIdA and KRIdB. For this it has to be shown that $\tilde{\varphi}(k_A(J)) \subseteq k_B(\tilde{\varphi}(J))$ which follows immediately from the fact that $\varphi : A \to B$ takes regular elements to regular elements because they are the r for which $|r| \geq 1$, by the properties of $\mathbb{Z}$-rings, and φ preserves this condition, being an l-ring homomorphism.

Finally, to see that KRId$\mathfrak{Z}L$ provides the compact zero-dimensional coreflection of L, note first that the map h : RId$\mathfrak{Z}L \to L$ taking J to $\bigvee\{\text{coz}(\alpha) \mid \alpha \in J\} = \bigvee \text{coz}[J]$ is a frame homomorphism by the fact that coz : $\mathfrak{Z}L \to L$ is a support. To see that it induces a homomorphism KRId$\mathfrak{Z}L \to L$ we have to show that $h(k(J)) \leq h(J)$ for any radical ideal J in $\mathfrak{Z}L$. Now if $\alpha \in k(J)$ then $\alpha\beta \in J$ for some regular $\beta \in \mathfrak{Z}L$, meaning that $|\beta| \geq \mathbf{1}$ as noted earlier for arbitrary $\mathbb{Z}$-rings; hence, $\text{coz}(\beta) = \text{coz}(|\beta|) = e$, and then

$$\text{coz}(\alpha) = \text{coz}(\alpha) \wedge \text{coz}(\beta) = \text{coz}(\alpha\beta) \leq h(J)$$

which proves the point. In the following, κ_L : KRId$\mathfrak{Z}L \to L$ will be the homomorphism thus obtained.

Regarding the naturality of these κ_L, note first that $\alpha \in \langle \gamma \rangle$ for any $\alpha, \gamma \in \mathfrak{Z}L$ iff $(\alpha\beta)^n = \gamma\delta$ for some n and $\beta, \delta \in \mathfrak{Z}L$ with $|\beta| \geq \mathbf{1}$, and hence

$$\text{coz}(\alpha) = \text{coz}(\alpha\beta) = \text{coz}((\alpha\beta)^n) = \text{coz}(\gamma\delta) \leq \text{coz}(\gamma)$$

which shows that

$$\kappa_L(\langle \gamma \rangle) = \bigvee \text{coz}[\langle \gamma \rangle] = \text{coz}(\gamma).$$

Further, as $h(\text{coz}(\gamma)) = \text{coz}(h\gamma)$ and $\langle \gamma \rangle \mapsto \langle h\gamma \rangle$ by the homomorphism KRId$\mathfrak{Z}(L) \to$ KRId$\mathfrak{Z}(M)$ induced by $h : L \to M$, this proves the desired naturality.

In all, we therefore have

Proposition 2. *For any frame L, κ_L : KRId$\mathfrak{Z}L \to L$ is the coreflection map from compact* zero-*dimensional frames.*

Remark 5. We note that the above result could also be obtained with the help of an appropriate notion of support, in the way Proposition 1 was proved. The relevant concept here is that of an *inverting* support, meaning a support $\sigma : A \to L$ such that $\sigma(r) = e$ for all $r \in$ Reg$A$. In particular, then, the map

$$A \to \text{KRId}A, \qquad a \mapsto \langle a \rangle,$$

considered here is easily seen to be the universal inverting support for any commutative ring A with unit. On the other hand, coz : $\mathfrak{Z}L \to L$ is an inverting support because $r \in$ Reg$A$ iff $1 \leq |r|$ for any $\mathbb{Z}$-ring A while $1 \leq |\alpha|$ implies $\text{coz}(\alpha) = e$, and this determines then κ_L : KRId$\mathfrak{Z}L \to L$ as the unique frame homomorphism such that $\kappa_L(\gamma) = \text{coz}(\gamma)$.

Remark 6. As a counterpart to Remark 4, $\mathfrak{Z}(\mathfrak{O}X) \stackrel{\sim}{=} C(X, \mathbb{Z})$ for any space X. On the other hand, one readily sees for any commutative ring A with unit and its classical ring QA of quotients that $\mathrm{KRId}A \stackrel{\sim}{=} \mathrm{RId}(QA)$, and Proposition 2 then shows that the *compact* zero-*dimensional reflection of a space* X is given by $\mathrm{Max}QC(X, \mathbb{Z})$ with the obvious map $X \to \mathrm{Max}QC(X, \mathbb{Z})$.

Remark 7. As noted at the beginning of this section, the coreflection considered here is most readily given by the map $\mathfrak{I}(BL) \to L$ taking each ideal J of BL to $\bigvee J$ in L. Now, by the familiar equivalence between Boolean algebras and Boolean rings, $\mathfrak{I}(BL)$ is also the ideal lattice of the *Boolean ring BL*. Moreover, the latter can easily be identified as the ring of $\mathbb{F}_2$-valued continuous functions on the frame L ($\mathbb{F}_2$ the two-element field) whose elements are understood in the same way as the $\mathbb{Z}$-valued continuous functions on L, that is, as the maps $\gamma : \mathbb{F}_2 \to L$ with $\gamma(0)$ and $\gamma(1)$ complementary to each other. Thus $\mathfrak{I}(BL)$ is seen as the frame of ideals of *some* ring of continuous functions on L. By way of contrast, our focus on $\mathfrak{Z}L$ was prompted by the fact that it is a subring of the "usual" ring of continuous functions, namely $\mathfrak{R}L$ (in which, as is familiar, it is characterized by the condition $\gamma(m, m+1) = 0$ for all $m \in \mathbb{Z}$). Thus, we are motivated here by the circumstance that, in classical terms, $C(X, \mathbb{Z})$ has been considered as a natural counterpart of $C(X)$ which is not the case with $C(X, \mathbb{F}_2)$.

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