Europ. J. Combinatorics (1990) **11**, 21–34

On the Finding of Final Polynomials

JÜRGEN BOKOWSKI AND JÜRGEN RICHTER

Final polynomials have been used to prove non-representability for oriented matroids, i.e. to decide whether geometric embeddings of combinatorial structures exist. They received more attention when Dress and Sturmfels, independently, pointed out that non-representable oriented matroids always possess a final polynomial as a consequence of an appropriate real version of Hilbert's Nullstellensatz. We discuss the more difficult problem of determining such final polynomials algorithmically. We introduce the notion of bi-quadratic final polynomials, and we show that finding them is equivalent to solving an LP-Problem. We apply a new theorem about symmetric oriented matroids to a series of cases of geometrical interest.

1. INTRODUCTION

The method of using *final polynomials,* as a tool for proving non-representability of oriented matroids or for deciding geometric embeddings of combinatorial structures, was introduced when the first author studied 3-spheres and their polytopality. Sturmfels and Dress have independently pointed out that, through the use of a real version of Hilbert's Nullstellensatz, a final polynomial exists for every non-representable oriented matroid. This gave new impetus to the more difficult task of finding such polynomials algorithmically. For a comprehensive introduction the reader is referred to [14, 15, 16]. Any further progress in this direction would be a great advance, and this paper is devoted precisely to this question.

Oriented matroids (and matroids) are general discrete structures which can perhaps be compared with graphs or lattices. Their general structure is an appropriate tool for tackling mathematical problems when linear dependencies and independencies play a substantial role. According to these very general properties, the rather young theory of oriented matroids has been used and applied in many different fields of mathematics.

A fundamental problem in the theory of oriented matroids is deciding representability. Final polynomials were suggested as a general tool for proving non-representability. We emphasize the advantage of this method: once a final polynomial is known, the calculations for finding it are no longer needed in the proof. So far, only a rather small number of final polynomials have been found. In every case, these polynomials were determined by using rather long computer-algebra-like calculations, and sometimes even greater effort was needed in solving a specific problem, e.g. in [3]. This more difficult problem of finding a particular final polynomial for specific applications will be discussed in Section 2. We provide a polynomial algorithmic method, using essentially a suitable variant of linear programming for finding classes of final polynomials. It is very likely that there are final polynomials which we cannot find in this way. On the other hand, all final polynomials known to us so far can be found in this way. Thus our method, deciding first a suitable LP-problem, can at least be applied as a good pre-processing.

Section 3 is devoted to symmetrical chirotopes. For symmetric d-chirotopes it is natural to decide whether there is a symmetrical realization with respect to a given finite subgroup of the orthogonal group $O(d)$. Our main result in Section 3 asserts that, once the non-realizability of our chirotope along with (geometric) symmetry assumptions has been achieved by our method, we can solve the general case as well, i.e. the

0195-6698/90/010021 + 14 $02.00/0

symmetry assumption can be dropped. This is a substantial shortcut in attaining the final polynomial, algorithmically.

In Section 4 we apply our method of finding new non-embeddability proofs to a series of examples: for instance, we apply it to an example in the class of *star-chirotopes*, cf. [7], which were key examples in establishing the Non-Steinitz-theorem by Sturmfels [13], and in which it seemed to be difficult to find corresponding final polynomials. Yet another application of our method will be given in a forthcoming paper [4], in which we determine all (simplicial) embeddings of a certain class of matroid manifolds.

In conclusion, we would like to mention that our approach to the problem has been coordinate-free. This new development has also been presented in the collection of papers written by Crapo, Havel, Sturmfels, White and Whiteley in Minnesota in 1987 [8].

2. Finding Final Polynomials Through Linear Programming

It is generally considered that the algorithmic search for final polynomials is very difficult. Sometimes the effort seems to be beyond our reach when the number of points increase. In this section, we provide an algorithmic method for finding final polynomials in a reasonable length of time. All final polynomials known to us so far can be achieved by our method. In describing the details, we restrict ourselves to the simplicial case, thus simplifying the main idea.

Let χ be a given simplicial d-chirotope of n points $\chi: \Lambda(n, d) \to \{-1, 1\}$. Here $\Lambda(n, d)$ denotes the set of all ordered d-tuples of n elements $\lambda = (\lambda_1, \ldots, \lambda_d)$ with $\lambda_1 \leqslant \cdots \leqslant \lambda_d$. If necessary, we refer to the tuples λ of $\Lambda(n, d)$ as

$$\lambda^1 = (1, 2, \ldots, d), \quad \lambda^2 = (1, 2, \ldots, d-1, d+1), \quad \ldots, \quad l^{\binom{n}{d}} = (n-d-1, \ldots, n).$$

The map χ is considered to be extended in an alternating fashion onto all d-tuples $\lambda \in \{1, \ldots, n\}^d$. To every $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d) \in \{1, \ldots, n\}^d$, we assign a formal variable $[\lambda_1, \lambda_2, \ldots, \lambda_d]$, called a *bracket*, which in case of a realizable chirotope with points $x_1, \ldots, x_n \in R^d$, $x_i = (x_i^1, x_i^2, \ldots, x_i^d)$ will be equal to the determinant

$$[\lambda_1, \lambda_2, \ldots, \lambda_d] = \det\begin{pmatrix} x_{\lambda_1}^1 & \cdots & x_{\lambda_1}^d \\ \vdots & & \vdots \\ x_{\lambda_d}^1 & \cdots & x_{\lambda_d}^d \end{pmatrix}.$$

But, in general, we consider these *brackets* as only being formal variables. We abbreviate $\tau := \tau_1, \ldots, \tau_{d-2}$. The following term will be called the *Graßmann–Plücker polynomial*:

$$\{\tau \mid \lambda_1, \lambda_2, \lambda_3, \lambda_4\} := [\tau, \lambda_1, \lambda_2][\tau, \lambda_3, \lambda_4] - [\tau, \lambda_1, \lambda_3][\tau, \lambda_2, \lambda_4] + [\tau, \lambda_1, \lambda_4][\tau, \lambda_2, \lambda_3].$$

Again, in the case of the brackets being determinants, this Graßmann–Plücker polynomial is equal to zero, since in this case we have precisely a Graßmann–Plücker syzygy.

The definition of a chirotope requires one of the following relations:

$$\{-1, +1\} \subset P \qquad \text{or} \qquad \{0\} = P$$

with

$$P := \{\chi((\tau, \lambda_1, \lambda_2))\chi((\tau, \lambda_3, \lambda_4)), \quad -\chi((\tau, \lambda_1, \lambda_3))\chi((\tau, \lambda_2, \lambda_4)), \quad \chi((\tau, \lambda_1, \lambda_4))\chi((\tau, \lambda_2, \lambda_3))\}.$$

This is a necessary condition for χ being realizable; otherwise, no choice of coordinates for the points $x_1, \ldots, x_n$ would fulfill both the requirement

$$\chi(\lambda) = \chi((\lambda_1, \ldots, \lambda_d)) = \operatorname{sign}[\lambda_1, \ldots, \lambda_d]$$

and the above single Graßmann–Plücker syzygy.

The set of vectors $\alpha = (\alpha_1, \ldots, \alpha_{\binom{n}{d}}) \in R^{\binom{n}{d}}$ with

$$\alpha_1 = [1, 2, \ldots, d], \qquad \alpha_2 = [1, 2, \ldots, d-1, d+1], \qquad \ldots, \qquad \alpha_{\binom{n}{d}} = [n-d-1, \ldots, n],$$

such that all Graßmann–Plücker syzygies are fulfilled, defines the Grassmannian $G(n, d) \subset R^{\binom{n}{d}}$. The chirotope χ is realizable iff there is a point

$$\alpha = (\alpha_1, \ldots, \alpha_{\binom{d}{n}}) \in G(n, d)$$

in the Grassmannian $G(n, d)$, which also lies in the cone $C(\chi) \subset R^{\binom{n}{d}}$, defined as

$$C(\chi) := \left\{ \alpha = (\alpha_1, \ldots, \alpha_{\binom{d}{n}}) \mid \operatorname{sign}(\alpha_j) = \chi(\lambda^j),\ 1 \leqslant j \leqslant \binom{n}{d} \right\}.$$

The change in the above Graßmann–Plücker polynomial, caused by a permutation of the points on the right-hand side of the vertical line in

$$\{\tau \mid \lambda_1, \lambda_2, \lambda_3, \lambda_4\},$$

keeps the relation property (the chirotope condition) mentioned above invariant. The notation was accordingly chosen in order to avoid multiple checkings of this condition. The polynomial remains equal up to choosing the actual brackets

$$[\lambda_1, \ldots, \lambda_d] \qquad \text{or} \qquad \operatorname{sign} \pi[\lambda_{\pi(1)}, \ldots, \lambda_{\pi(d)}]$$

for a permutation $\pi \in S_d$, where S_d denotes the symmetric group, or it is equal up to ordering the bracket products. By a suitable permutation of the points $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, we can attain that all three bracket products are positive:

$$\underbrace{[\tau, \lambda_1, \lambda_2][\tau, \lambda_3, \lambda_4]}_{>0} - \underbrace{[\tau, \lambda_1, \lambda_3][\tau, \lambda_2, \lambda_4]}_{>0} + \underbrace{[\tau, \lambda_1, \lambda_4][\tau, \lambda_2, \lambda_3]}_{>0}.$$

In this case, we say, we have *normalized* our Graßmann–Plücker polynomial. In case χ is realizable and in the normalized case, we can deduce that the inequality

$$[\tau, \lambda_1, \lambda_2][\tau, \lambda_3, \lambda_4] < [\tau, \lambda_1, \lambda_3][\tau, \lambda_2, \lambda_4]$$

must hold. In what follows, it will be useful to have a special notation for these *formal inequalities*, arising from normalized Graßmann–Plücker polynomials.

Definition 2.1. Let χ be a simplicial chirotope and let

$$\{\tau \mid \lambda_1, \lambda_2, \lambda_3, \lambda_4\} := [\tau, \lambda_1, \lambda_2][\tau, \lambda_3, \lambda_4] - [\tau, \lambda_1, \lambda_3][\tau, \lambda_2, \lambda_4] + [\tau, \lambda_1, \lambda_4][\tau, \lambda_2, \lambda_3]$$

be a Graßmann–Plücker polynomial such that

$$\begin{aligned} &\chi((\tau, \lambda_1, \lambda_2)) \cdot \chi((\tau, \lambda_3, \lambda_4)) \\ &\qquad = \chi((\tau, \lambda_1, \lambda_3)) \cdot \chi((\tau, \lambda_2, \lambda_4)) = \chi((\tau, \lambda_1, \lambda_4)) \cdot \chi((\tau, \lambda_2, \lambda_3)) = 1; \end{aligned}$$

then we call the 4-tuple B of formal variables,

$$B = \langle [\tau, \lambda_1, \lambda_2], [\tau, \lambda_3, \lambda_4] \mid [\tau, \lambda_1, \lambda_3], [\tau, \lambda_2, \lambda_4] \rangle,$$

a *bi-quadratic inequality of* χ. The set of all bi-quadratic inequalities of χ will be denoted by $\mathcal{B}_\chi$.

REMARK 2.1. Whenever χ is realizable, we have

$$A \cdot B < C \cdot D \qquad \text{for all } \langle A, B \mid C, D \rangle \in \mathcal{B}_\chi.$$

In order to prepare a definition for *bi-quadratic final polynomials,* we must identify those brackets, which in the realizable case would change according to the alternating determinant rules; i.e. for any permutation $\pi \in S_d$ we set

$$[\lambda_1, \lambda_2, \ldots, \lambda_d] - \operatorname{sign} \pi \cdot [\lambda_{\pi(1)}, \lambda_{\pi(2)}, \ldots, \lambda_{\pi(d)}] = 0.$$

In other words, if R is the integer polynomial ring generated by the formal brackets $\{[\lambda] \mid \lambda \in \{1, \ldots, n\}^d\}$ and if I is the ideal generated by polynomials of the form

$$[\lambda_1, \lambda_2, \ldots, \lambda_d] - \operatorname{sign} \pi \cdot [\lambda_{\pi(1)}, \lambda_{\pi(2)}, \ldots, \lambda_{\pi(d)}],$$

we calculate within the quotient ring R/I. Note that R/I is similar, but not identical, to the bracket ring introduced by White, see [8], where I contains also the Graßmann–Plücker polynomials. Now we are ready to define *bi-quadratic final polynomials* as follows:

DEFINITION 2.2. A simplical chirotope χ admits a *bi-quadratic final polynomial,* whenever there is a collection of bi-quadratic inequalities

$$\langle A_i, B_i \mid C_i, D_i \rangle \in \mathcal{B}_\chi; \qquad 1 \leq i \leq k$$

such that the following equality holds within the ring R/I:

$$\prod_{i=1}^{k} A_i \cdot B_i \stackrel{I}{=} \prod_{i=1}^{k} C_i \cdot D_i.$$

Here $a \stackrel{J}{=} b$ expresses that the ring variables a and b are equal modulo the set J.

The above Definition 2.2 allows the following claim.

LEMMA 2.1. *If χ admits a bi-quadratic final polynomial, then χ is not realizable.*

PROOF. χ admits a bi-quadratic final polynomial, i.e. there is a collection of bi-quadratic inequalities

$$\langle A_i, B_i \mid C_i, D_i \rangle \in \mathcal{B}_\chi; \qquad 1 \leq i \leq k$$

such that

$$\prod_{i=1}^{k} A_i \cdot B_i \stackrel{I}{=} \prod_{i=1}^{k} C_i \cdot D_i.$$

In assuming that χ is realizable we have by Remark 1, $A_i \cdot B_i < C_i \cdot D_i$ for all $i \in \{1, \ldots, k\}$, and since both sides in the above equation are by definition positive, we also have

$$\prod_{i=1}^{k} A_i \cdot B_i < \prod_{i=1}^{k} C_i \cdot D_i.$$

In using

$$[\lambda_1, \lambda_2, \ldots, \lambda_d] - \operatorname{sign} \pi \cdot [\lambda_{\pi(1)}, \lambda_{\pi(2)}, \ldots, \lambda_{\pi(d)}] = 0,$$

whenever necessary, we arrive at a contradiction. □

REMARK 2.2. Now, it is a straightforward task to translate a bi-quadratic final polynomial into an ordinary final polynomial as introduced in [5].

Of course, it seems as if one looses quite a bit of information in searching for bi-quadratic final polynomials rather then looking for final polynomials in general. But there are two reasons for doing this. (1) The solvability of the remaining system of inequalities can now be seen to be a LP-problem, yielding a polynomial algorithm by the ellipsoid method (cf. [10]). (2) Again, all known examples of non-representable oriented matroids, where non-representability was proven by means of final polynomials, can be treated automatically and more easily than before.

And, above all, this might be the only accessible method of solving a more difficult problem under consideration. In any case, we suggest solving this LP-problem as a pre-processing method. How do we obtain the LP-problem?

We consider the set of all formal inequalities $\mathcal{B}_\chi$. We replace each variable $X=[\lambda_1, \ldots, \lambda_d]$ by its formal absolute value $X \cdot \chi((\lambda_1, \ldots, \lambda_d))$. It is clear how one obtains new formal inequalities $\langle A, B \mid C, D\rangle$ for the formal absolute values. Assuming realizability, we can take the logarithm on both sides, and we obtain a linear problem (*) with integer coefficients and strict inequalities for the new variables $Y := \log(X \cdot \chi((\lambda_1, \ldots, \lambda_d)))$ chosen appropriately.

In principle, it is now possible to use the special structure of this inequality system when solving this system of inequalities. Consider the inequalities with positive coefficients

$$\sum \cdots + \alpha_t Y + \cdots < \sum \cdots + \cdots$$

or

$$\sum \cdots + \cdots < \sum \cdots + \beta_r Y + \ldots,$$

and multiply those inequalities, which contain the variable Y, with suitable factors in order to obtain mY, where m denotes the smallest common multiple of all (positive) factors α_l, β_r of Y. We rewrite all inequalities in the form

$$\sum \cdots < mY < \sum \cdots.$$

The decisive inductive step of our process for solving all variables is to replace this system of inequalities by all pairs of inequalities, and to compare all possible parts, left and right, that one has solved for Y together with the remaining system of inequalities. Now, either one obtains a solution this way, or one finally arrives at a contradiction, $0<0$, which then can be traced back in order to find the bi-quadratic final polynomial. We formulate this result as follows.

THEOREM 1. *A chirotope χ admits a bi-quadratic final polynomial iff the dual of the above LP-problem (*) is admissible (LP-Phase I). Moreover, the solution of the LP-problem can be used for constructing the bi-quadratic final polynomial.*

REMARK 3. We do not recommend the procedure used above for algorithmic purposes in general. For practical solutions we applied the MINOS software system. When judging the above theorem it is the algorithmic point of view which one has to keep in mind. We were surprised ourselves about the practical significance, as will be shown in concrete applications in Section 4.

3. FINAL POLYNOMIALS WITH SYMMETRIES

This section is devoted to symmetric chirotopes and corresponding final polynomials with respect to a given symmetry. For symmetric d-chirotopes it is natural not only to look for a realization at all, but also to decide whether there is a symmetrical realization with respect to a given finite subgroup of the orthogonal group $O(d)$.

In a straightforward way, this leads to the tool of *bi-quadratic final polynomials with respect to a symmetry group* for disproving symmetrical embeddings. Our main result in this section asserts that once the non-realizability of our chirotope according to (geometric) symmetry assumptions has been established, by using this tool, we can then derive a final polynomial even for the general case, by dropping the (geometric) symmetry assumption. The result is that our chirotope is not realizable at all. The advantage of our theorem is immediate in these cases, as it is an essential shortcut in finding a final polynomial, algorithmically.

In order to formulate this theorem, we first have to fix our notation. We abbreviate for any $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d) \in E^d$ and any $\sigma \in S_n$:

$$\sigma\lambda = (\sigma(\lambda_1), \sigma(\lambda_2), \ldots, \sigma(\lambda_d)).$$

Let χ be a simplicial d-chirotope. An element $\sigma \in S_n$ of the permutation group S_n is called a *rotation of* χ if

$$\chi(\lambda) = \chi(\sigma\lambda) \qquad \text{for all } \lambda = (\lambda_1, \ldots, \lambda_d) \in \{1, \ldots, n\}^d,$$

denoted by $\sigma\chi = \chi$. Similarly, an element $\sigma \in S_n$ is called a *reflection of* χ if

$$\chi(\lambda) = -\chi(\sigma\lambda) \qquad \text{for all } \lambda = (\lambda_1, \ldots, \lambda_d) \in \{1, \ldots, n\}^d.$$

In this case, we write $\sigma\lambda = -\chi$. We call

$$R_\chi = \{\sigma \in S_n \mid \sigma\chi = \chi\}$$

the *set of rotations of* χ and

$$M_\chi = \{\sigma \in S_n \mid \sigma\chi = -\chi\}$$

the *set of reflections of* χ. The union of these sets forms the group of automorphisms $\mathrm{Aut}(\chi)$ of χ denoted by

$$G_\chi := R_\chi \cup M_\chi = \mathrm{Aut}(\chi).$$

The group G_χ(resp. any subgroup of G_χ) acts on the polynomial ring R as follows. For any bracket and any $\sigma \in S_n$, we define

$$\sigma * [\lambda] = \begin{cases} [\sigma\lambda] & \text{if } \sigma \in R_\chi, \\ -[\sigma\lambda] & \text{if } \sigma \in M_\chi. \end{cases}$$

For any polynomial $P([\lambda], [\mu], \ldots)$ in the bracket variables $[\lambda]$, $[\mu], \ldots,$ we define

$$\sigma * P([\lambda], [\mu], \ldots) = P(\sigma * [\lambda], \sigma * [\mu], \ldots).$$

The action of the group can be extended to any quotient R/J, whenever the ideal J remains fixed under the group. In particular, we define for the above polynomial ring R/I, for any $P + I \in R/I$; $P \in R$ and any $\sigma \in G_\chi$:

$$\sigma * (P + I) := (\sigma * P) + I.$$

Our next claim is that according to

$$\sigma * \langle A, B \mid C, D \rangle = \langle \sigma * A, \sigma * B \mid \sigma * C, \sigma * D \rangle,$$

G_χ also acts on the set of bi-quadratic inequalities. The only non-trivial fact we have to prove is the following:

LEMMA 3.1. *The property of $\langle A, B \mid C, D \rangle$ being a bi-quadratic inequality remains fixed under the action of G_χ, i.e. for any $\sigma \in G_\chi$, $\sigma * \langle A, B \mid C, D \rangle$ is again a bi-quadratic inequality.*

PROOF. If $\langle A, B \mid C, D \rangle$ is a bi-quadratic inequality, then there are brackets E and F such that $AB - CD + EF$ is a Graßmann–Plücker polynomial, and $AB > 0$, $CD > 0$, $EF > 0$. Since Graßmann–Plücker polynomials in R/I do not change by renumbering the vertices,

$$(\sigma * A)(\sigma * B) - (\sigma * C)(\sigma * D) + (\sigma * E)(\sigma * F)$$

is a Graßmann–Plücker polynomial as well. For any rotation or reflection σ, we have:

$$(\sigma * A)(\sigma * B) > 0, \qquad (\sigma * C)(\sigma * D) > 0, \qquad (\sigma * E)(\sigma * F) > 0.$$

Thus, by Definition 2.1, $\langle \sigma * A, \sigma * B \mid \sigma * C, \sigma * D \rangle$ yields a bi-quadratic inequality again; in other words, G acts on the set of bi-quadratic inequalities. □

Now, we introduce the notion of a *symmetric realization of χ with respect to G_R*, where $G_n < G_\chi$ and $G_R < O(d)$, is any subgroup of G_χ and $O(d)$ respectively. By that, we mean a realization R of χ with a geometric automorphism group isomorphic to G_R.

Given such a symmetric realization, we have for any $\lambda \in E^d$ and any $\sigma \in G_R$

$$[\lambda] - \sigma * [\lambda] = 0.$$

Now, we want to factorize our ring R/I, such that elements together with their symmetric images are identified. Therefore, we define the ideal I_{G_R} in R which is generated by polynomials of the following type: $[\lambda] - \sigma * [\lambda]$ for any $\sigma \in G_R$ and any $\lambda \in E^d$. We also consider the polynomial ring $R/\overline{I_{G_R}}$, where $\overline{I_{G_R}}$ denotes the ideal generated by $I \cup I_{G_R}$.

Here we have factored out the symmetry of χ. A *symmetric bi-quadratic final polynomial with respect to G_R* will now be defined, as follows.

DEFINITION 3.1. A simplicial chirotope χ admits a *symmetric bi-quadratic final polynomial with respect to G_R* whenever there is a collection of bi-quadratic inequalities

$$\langle A_i, B_i \mid C_i, D_i \rangle \in \mathscr{B}_\chi; \qquad 1 \leq i \leq k,$$

such that

$$\prod_{i=1}^{k} A_i \cdot B_i \overset{\overline{I_{G_R}}}{=} \prod_{i=1}^{k} C_i \cdot D_i.$$

With this definition we obtain the following result:

THEOREM 2. *For any subgroup $G_R < G_\chi$ we have that a (symmetric) chirotope admits a bi-quadratic final polynomial, iff it admits a symmetric bi-quadratic final polynomial with respect to G_R.*

REMARK. The result of Theorem 2 heavily and positively influences the computational part of our applications. Whenever we want to prove that a symmetric chirotope is not realizable and we are looking for a bi-quadratic final polynomial, we can reduce the computational time tremendously. In particular, if the inequality system being considered generally has N inequalities, we only have to solve a system of approximately $N/|G_\chi|$ inequalities. Our proof will show how to construct the bi-quadratic final polynomial whenever the symmetric one is given.

PROOF OF THEOREM 2. One part of this theorem is very easily proven. Since for any $P, Q \in R$,

$$P \overset{I}{=} Q \text{ also implies } P \overset{\overline{I_{G_R}}}{=} Q,$$

the existence of a bi-quadratic final polynomial implies the existence of a symmetric one. To prove the reverse, we first provide the following two lemmas.

LEMMA 3.1. *For any $\lambda_1, \lambda_2 \in E^d$, we have*

$$[\lambda_1] \overset{\overline{I_{G_R}}}{=} [\lambda_2] \quad \Rightarrow \quad [\lambda_1] \in G_R * ([\lambda_2] + I).$$

PROOF. $[\lambda_1] =^{\overline{I_{G_R}}} [\lambda_2]$ implies that there exists a $\sigma \in G_R$ such that $[\lambda_1] =^I \sigma * [\lambda_2]$ holds, and this in turn can be written as

$$[\lambda_1] \in \sigma * [\lambda_2] + I = \sigma * ([\lambda_2] + I) \in G_R([\lambda_2] + I). \quad \square$$

LEMMA 3.2. *For any $\lambda_1, \lambda_2 \in E^d$, we have*

$$[\lambda_1] \overset{\overline{I_{G_R}}}{=} [\lambda_2] \quad \Rightarrow \quad \prod_{\sigma \in G_R} (\sigma * [\lambda_1]) \overset{I}{=} \sum_{\sigma \in G_R} (\sigma * [\lambda_2]).$$

PROOF. For $\lambda_1, \lambda_2 \in E^d$, we assume that

$$[\lambda_1] \overset{\overline{I_{G_R}}}{=} [\lambda_2].$$

We consider the orbits $\Omega([\lambda_i]) := \prod_{\sigma \in G_R} (\sigma * [\lambda_i])$, $i = 1, 2$, of G_R within R generated by $[\lambda_1]$, and $[\lambda_2]$, respectively. They induce corresponding orbits

$$\Omega([\lambda_i] + I) = \prod_{\sigma \in G_R} \sigma * \{[\lambda_i] + I\} = \prod_{\sigma \in G_R} (\sigma * [\lambda_i]) + I, \qquad i = 1, 2. \quad \square$$

To pursue the proof of Theorem 2 let us assume that χ admits a symmetric bi-quadratic final polynomial with respect to G_R. Then there is a collection of bi-quadratic inequalities

$$\langle A_i, B_i \mid C_i, D_i \rangle \in \mathscr{B}_\chi; \qquad 1 \leq i \leq k,$$

such that

(**)
$$\prod_{i=1}^{k} A_i \cdot B_i \overset{\overline{I_{G_R}}}{=} \prod_{i=1}^{k} C_i \cdot D_i.$$

We will show that, in replacing any bi-quadratic inequality by its orbit under G_R, we obtain a bi-quadratic final polynomial in R/I. In other words, the collection

$$\sigma * \langle A_i, B_i \mid C_i, D_i \rangle \in \beta_\chi; \qquad 1 \leq i \leq k; \qquad \sigma \in G_R$$

is a bi-quadratic final polynomial. Finally, we have to show that

$$\prod_{\substack{1 \leq i \leq k \\ \sigma \in G_R}} (\sigma * A_i) \cdot (\sigma * B_i) \overset{I}{=} \prod_{\substack{1 \leq i \leq k \\ \sigma \in G_R}} (\sigma * C_i) \cdot (\sigma * D_i).$$

This is equivalent to

$$\prod_{1 \leq i \leq k} \Big(\prod_{\sigma \in G_R} \sigma * A_i \Big) \cdot \Big(\prod_{\sigma \in G_R} \sigma * B_i \Big) \overset{I}{=} \prod_{1 \leq i \leq k} \Big(\prod_{\sigma \in G_R} \sigma * C_i \Big) \cdot \Big(\prod_{\sigma \in G_R} \sigma * D_i \Big).$$

Since (**) is only valid if for any bracket $[\lambda]$ on the left of (**) there is a bracket $[\lambda']$ on the right of (**) with $[\lambda] =^{\overline{I_{G_R}}} [\lambda']$, the result follows according to Lemma 3.2. $\square$

4. Applications

In this section we provide a collection of non-representable oriented matroids together with their bi-quadratic final polynomials. We first give some examples of known non-representable proofs and even known final polynomials. By using our theorems as discussed in Section 3, we can provide extremely short non-realizability proofs.

Example 1: the Vamos matroid. As a first example, we give a non-realizability proof for an oriented version of the well known Vamos matroid of 8 points in rank 4. To give a combinatorial description of the oriented matroid, we start with the chirotope χ which corresponds to the following rank 4 configuration:

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

Notice that this configuration has a symmetry group isomorphic to $Z_2 \times S_4$ since the points $1, 2, 3$ and 4 form a regular tetrahedron in the subspace $\{[w, x, y, z) \mid w = 1\}$ and the points $5, 6, 7$ and 8 form a similar one in the subspace $\{(w, x, y, z) \mid w = -1\}$. Notice furthermore that we have

$$[1, 2, 5, 6] = [1, 3, 5, 7] = [1, 4, 5, 8] = [2, 3, 6, 7] = [2, 4, 6, 8] = [3, 4, 7, 8] = 0.$$

We modify the chirotope χ in order to obtain a new map χ_V by replacing exactly these zero-orientations by 1, i.e. we require

$$\begin{aligned} \chi_V(1, 2, 5, 6) &= \chi_V(1, 3, 5, 7) = \chi_V(1, 4, 5, 8) \\ &= \chi_V(2, 3, 6, 7) = \chi_V(2, 4, 6, 8) = \chi_V(3, 4, 7, 8) = +1 \end{aligned}$$

and $\chi = \chi_V$ for all other 4-tuples. We will show that χ_V is again a chirotope. We consider any Graßmann–Plücker polynomial $\{a, b \mid \cdots\}$, where a, b are vertices of the same tetrahedron, as mentioned above. In this polynomial $\{a, b \mid \cdots\}$, there occurs no other basis λ with $\chi(\lambda) = 0$, since all 4-tuples contain a and b. But if a and b lie in different tetrahedra, and if there would be another λ in the polynomial with $\chi(\lambda) = 0$, we see from the structure of the polynomial $\{a, b \mid \cdots\}$ and from the six 4-tuples above that this zero-valued $\chi(\lambda)$ is the second factor in the same summand of the polynomial which is already equal to zero. Two other summands of opposite signs must exist, showing that the chirotope condition for χ_V is fulfilled.

Notice that χ_V has a combinatorial symmetry isomorphic to the alternating group A_4 which can be generated by the permutations

$$(123)(567) \qquad \text{and} \qquad (12)(34)(56)(78).$$

So far, we have only given a description of our (simplicial) chirotope χ_V. Now we claim that χ_V is not realizable.

LEMMA 4.1. *The Vamos matroid χ_V is not realizable, and χ_V admits a bi-quadratic final polynomial.*

PROOF. Consider the Graßmann–Plücker polynomial

$$\{1,2\,|\,3,6,5,4\} = \underbrace{[1,2,3,6]}_{+}\underbrace{[1,2,5,4]}_{+} - \underbrace{[1,2,3,5]}_{+}\underbrace{[1,2,6,4]}_{+} + \underbrace{[1,2,3,4]}_{-}\underbrace{[1,2,6,5]}_{-} = 0.$$

$\langle[1,2,3,6],\ [1,2,5,4]\,|\,[1,2,3,5],\ [1,2,6,4]\rangle$ is a bi-quadratic inequality of χ_V. We assume that χ_V is symmetrically (A_4) realizable, therefore, we have permutations $\sigma_1 := (123)(567)$, $\sigma_2 := (124)(568) \in G_{\chi_V}$, such that

$$\sigma_1^{-1} * [1,2,3,6] = [1,2,3,5] \qquad \text{and} \qquad \sigma_2 * [1,2,5,4] = [1,2,6,4].$$

Thus, we have $[1,2,3,6][1,2,5,4] =^{\bar{I}_{G_{\chi_V}}} [1,2,3,5][1,2,6,4]$. In conclusion, Theorem 2 proves the lemma; i.e. χ_V is not realizable at all. □

Example 2: Desargues Theorem. Consider the configuration D_{10}^3 given in Figure 1. This configuration is an oriented version of Desargues Theorem. Notice that in the corresponding chirotope χ_D, we have the mutations

$[1,0,6], [2,0,4], [3,0,5], [1,3,7], [2,1,8], [3,2,9], [7,5,6], [8,6,4], [9,4,5], [7,8,9]$.

When we require $\chi(\lambda) = 0$ for the corresponding 3-tuples λ, we obtain exactly a chirotope corresponding to Desargues configuration. χ_D has a symmetry isomorphic to Z_3 generated by $\sigma = (123)(456)(789)$.

LEMMA 4.2. *The Desargue chirotope χ_D is not realizable, and χ_D admits a bi-quadratic final polynomial.*

PROOF. Consider the Graßmann–Plücker relations

$$\{1\,|\,2,0,6,3\} = [1,2,0][1,6,3] - [1,2,6][1,0,3] + [1,2,3][1,0,6],$$
$$\{1\,|\,7,6,5,3\} = [1,7,6][1,5,3] - [1,7,5][1,6,3] + [1,7,3][1,6,5],$$
$$\{7\,|\,5,1,8,6\} = [7,5,1][7,8,6] - \underline{[7,5,8][7,1,6]} + \underline{[7,5,6][7,1,8]},$$
$$\{7\,|\,8,5,0,9\} = [7,8,5][7,0,9] - [7,8,0][7,5,9] + [7,8,9][7,5,0].$$

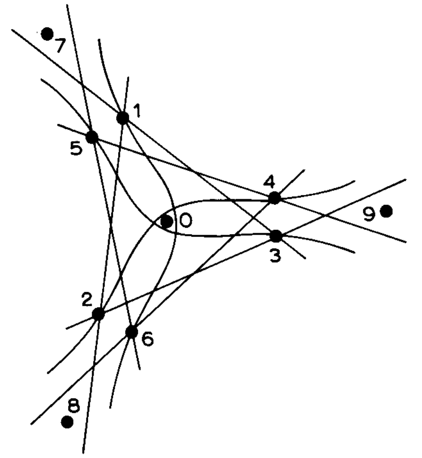

FIGURE 1.

The underlined brackets have to be negative in any realization of χ_D. This yields the bi-quadratic inequalities

$$\begin{gathered}\langle [1,2,0],[1,6,3] \mid [1,2,6],[1,0,3] \rangle,\\ \langle [1,7,6],[1,5,3] \mid [1,7,5],[1,6,3] \rangle,\\ \langle [7,5,1],[7,8,6] \mid [7,5,8],[7,1,6] \rangle,\\ \langle [7,8,5],[7,0,9] \mid [7,8,0],[7,5,9] \rangle.\end{gathered}$$

Now the symmetry of χ_D yields the following identities:

$$\begin{aligned} &\sigma^2 * [1,2,0] = [3,1,0], && \sigma * [1,5,3] = [2,6,1],\\ &\sigma^2 * [7,8,6] = [9,7,5], && \sigma * [7,0,9] = [8,0,7],\end{aligned}$$

and this in turn leads to the identity

$$\begin{gathered}[1,2,0][1,6,3][1,7,6][1,5,3][7,5,1][7,8,6][7,5,8][7,0,9] \overset{I_{G\chi_D}}{=}\\ [1,2,6][1,0,3][1,7,5][1,6,3][7,5,8][7,1,6][7,8,0][7,5,9].\end{gathered}$$

Now Theorem 2 gives the desired result. □

Example 3: a star-chirotope. In [7] an example of an infinite class of non-realizable chirotopes was given, such that any proper minor of each of those chirotopes is realizable. The non-realizability proof was given by standard geometric arguments, and finding corresponding final polynomials turned out to be difficult, when specific coordinates were chosen. Here, we give a bi-quadratic final polynomial for the smallest example of this class (see Figure 2).

LEMMA 4.3. *The star-chirotope χ_5 corresponding to the configuration in Figure 2 is not realizable, and χ_5 admits a bi-quadratic final polynomial.*

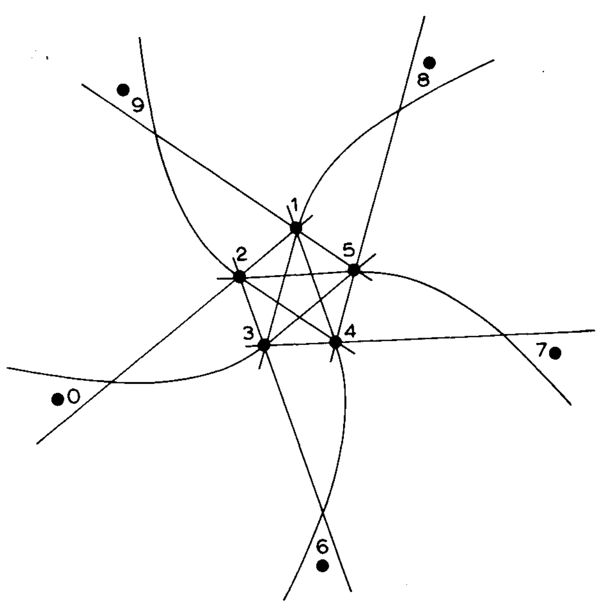

FIGURE 2.

PROOF. We give a list of the Graßmann–Plücker relations used and the corresponding bi-quadratic inequalities:

$$\{6 \mid 0, 7, 2, 1\} = 0 \;\Rightarrow\; \langle [6, 0, 2], [6, 7, 1] \mid [6, 0, 1], [6, 7, 2] \rangle \in \beta_{\chi_5}$$
$$\{7 \mid 6, 8, 3, 1\} = 0 \;\Rightarrow\; \langle [7, 6, 3], [7, 8, 1] \mid [7, 6, 1], [7, 8, 3] \rangle \in \beta_{\chi_5}$$
$$\{8 \mid 7, 9, 4, 1\} = 0 \;\Rightarrow\; \langle [8, 7, 4], [8, 9, 1] \mid [8, 7, 1], [8, 9, 4] \rangle \in \beta_{\chi_5}$$
$$\{9 \mid 8, 0, 5, 1\} = 0 \;\Rightarrow\; \langle [9, 8, 5], [9, 0, 1] \mid [9, 8, 1], [9, 0, 5] \rangle \in \beta_{\chi_5}$$
$$\{6 \mid 2, 3, 7, 1\} = 0 \;\Rightarrow\; \langle [6, 2, 7], [6, 3, 1] \mid [6, 2, 1], [6, 7, 1] \rangle \in \beta_{\chi_5}$$
$$\{7 \mid 3, 4, 8, 2\} = 0 \;\Rightarrow\; \langle [7, 3, 8], [7, 4, 2] \mid [7, 3, 2], [7, 4, 8] \rangle \in \beta_{\chi_5}$$
$$\{8 \mid 4, 5, 9, 3\} = 0 \;\Rightarrow\; \langle [8, 4, 9], [8, 5, 3] \mid [8, 4, 3], [8, 5, 9] \rangle \in \beta_{\chi_5}$$
$$\{9 \mid 5, 1, 0, 4\} = 0 \;\Rightarrow\; \langle [9, 5, 0], [9, 1, 4] \mid [9, 5, 4], [9, 1, 0] \rangle \in \beta_{\chi_5}$$
$$\{0 \mid 1, 2, 6, 5\} = 0 \;\Rightarrow\; \langle [0, 1, 6], [0, 2, 5] \mid [0, 1, 5], [0, 2, 6] \rangle \in \beta_{\chi_5}$$
$$\{1 \mid 4, 6, 3, 2\} = 0 \;\Rightarrow\; \langle [1, 4, 3], [1, 6, 2] \mid [1, 4, 2], [1, 6, 3] \rangle \in \beta_{\chi_5}$$
$$\{2 \mid 5, 7, 4, 3\} = 0 \;\Rightarrow\; \langle [2, 5, 4], [2, 7, 3] \mid [2, 5, 3], [2, 4, 7] \rangle \in \beta_{\chi_5}$$
$$\{3 \mid 1, 8, 5, 4\} = 0 \;\Rightarrow\; \langle [3, 1, 5], [3, 8, 4] \mid [3, 1, 4], [3, 8, 5] \rangle \in \beta_{\chi_5}$$
$$\{4 \mid 2, 9, 1, 5\} = 0 \;\Rightarrow\; \langle [4, 2, 1], [4, 9, 5] \mid [4, 2, 5], [4, 9, 1] \rangle \in \beta_{\chi_5}$$
$$\{5 \mid 3, 0, 2, 1\} = 0 \;\Rightarrow\; \langle [5, 3, 2], [5, 0, 1] \mid [5, 3, 1], [5, 0, 2] \rangle \in \beta_{\chi_5}$$

Any involved bracket appears exactly once, both on the right and on the left. In conclusion, the product of all brackets on the right equals in the ring R/I the product of all brackets on the left, proving our lemma. □

Example 4: the non-polytopal Altshuler sphere M^{10}_{425}. We give the combinatorial neighborly description of Altshuler's 3-sphere M^{10}_{425} with 10 vertices in terms of its facets:

1045	1048	4579	8275
3267	1062	4531	0497
5489	0135	7604	2619
7901	0197	7628	4831
9823	3260	6791	6053
0132	3284	6753	
2354	2357	9826	
4576	2319	9840	
6798	5482	8913	
8910	5406	8975	

The cyclic symmetry-group G generated by $\sigma = (13579)(24680)$ is a subgroup of the automorphism group of this sphere. Since the sphere M^{10}_{425} is neighborly, it is also rigid by Shemer's result, [12], i.e. there are only two chirotopes χ and $-\chi$ which are compatible in the sense of [6] to the polyhedral structure of the polytope. M^{10}_{425} is realizable, iff these chirotopes are realizable. Since the chirotopes are unique up to a multiplication by -1, they must have the same automorphism group as in the sphere M^{10}_{425}. The facial structure of M^{10}_{425} forces the following identities (for details see [3]):

$$\chi((2, 6, 0, 1, 8)) = \chi((2, 6, 0, 1, 4)) = \chi((2, 6, 0, 1, 3))$$
$$= \chi((2, 6, 0, 4, 3)) = \chi((2, 6, 0, 8, 3)) = \chi((8, 6, 0, 1, 7)).$$

On the other hand, the symmetry requires:

$$\chi((2,6,0,1,4)) = \chi(\sigma^3(2,6,0,1,4)) = \chi((8,2,6,7,0)),$$
$$\chi((2,6,0,4,3)) = \chi(\sigma^2(2,6,0,4,3)) = \chi((6,0,4,8,7)),$$
$$\chi((2,6,0,8,3)) = \chi(\sigma^4(2,6,0,8,3)) = \chi((0,4,8,6,1)).$$

We claim that M^{10}_{425} is not realizable. To prove this, we have simply to consider two Graßmann–Plücker polynomials $\{6,8,0 \mid 1,7,4,2\}$ and $\{2,6,0 \mid 1,3,8,4\}$. With these preparations, we leave it to the reader to work out the conclusion by using Theorem 2.

Example 5: a projective theorem. Our examples chosen above were oriented versions of projective theorems of the form

whenever the brackets $[a]$, $[b]$, $[c]$, ... *are zero, then* $[x]$ *has to be zero, too.*

Another type of projective theorem is of the form

whenever the brackets $[a]$, $[b]$, $[c]$, ... *are zero and* $[r]$, $[s]$, $[t]$, ... *are non-zero, then* $[x]$ *has to be non-zero, too.*

The following theorem of this type has been proven in [8] by using a computer algebra system.

THEOREM. *For any* 10 *points given in* R^3, *we have, whenever*

$$[1,2,3] = [1,6,0] = [1,7,9] = [2,4,0] = [2,5,6]$$
$$= [3,4,9] = [3,5,7] = [4,5,9] = [6,7,8] = 0,$$

and whenever all other brackets except $[8,9,0]$ *are non-zero, then* $[8,9,0]$ *is non-zero. Especially, we have* $[8,9,0] = [2,3,4] \cdot [1,6,7]^2 \cdot [3,4,7]$.

One oriented version of this theorem is that the configuration in Figure 3 is not realizable. This fact can again be proven by using a bi-quadratic final polynomial. The smallest final polynomial we know of was found on a computer algebra program that solves LP-problems with integer coefficients. This final polynomial is horribly large and consists of 135 bi-quadratic inequalities. A symmetric version with respect to the symmetry-group generated by (176)(253)(908) with 45 bi-quadratic inequalities can be sent by the first author on request.

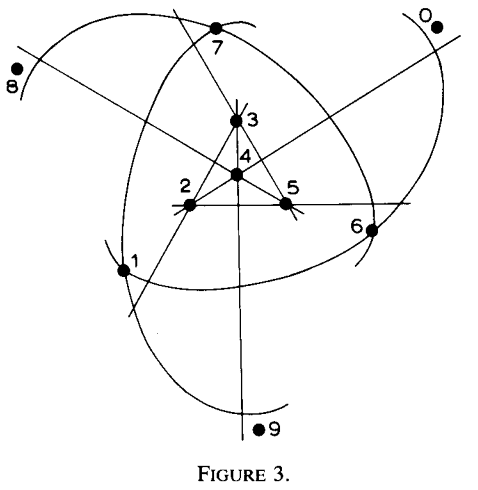

FIGURE 3.

Example 6: classification of matroid manifolds. What actually stimulated the investigations which led to our results in this paper was a classification problem for 31 matroid manifolds. This was recently completed but will be described in a forthcoming paper [4].

Acknowledgment

We wish to express our gratitude to Alfred Wanka from the University of Cologne, who helped us in using the MINOS optimization program which was not available at our university.

References

1. A. Altshuler, Neighborly 4-polytopes and neighborly combinatorial 3-manifolds with ten vertices, *Can. J. Math.*, **XXIX** (2) (1977), 400–420.
2. A. Altshuler, J. Bokowski and L. Steinberg, The classification of simplicial 3-spheres with nine vertices into polytopes and nonpolytopes, *Discr. Math.*, **31** (1980), 115–124.
3. J. Bokowski and K. Garms, Altshuler's sphere M^{10}_{425} is not polytopal, *Europ. J. Combin.*, **8** (1987), 227–229.
4. J. Bokowski and J. Richter, Bi-quadratic final polynomials and embedded symmetric polyhedra, in preparation.
5. J. Bokowski, J. Richter and B. Sturmfels, Nonrealizability proofs in computational geometry, *Discr. Comput. Geom.*, to appear.
6. J. Bokowski and B. Sturmfels, Polytopal and non-polytopal spheres: an algorithmic approach, *Israel J. Math.*, **57** (1987), 257–271.
7. J. Bokowski and B. Sturmfels, An infinite family of minorminimal nonrealizable simplicial 3-chirotopes, *Math. Z.*, **200** (1989), 583–589.
8. H. Crapo, T. F. Havel, B. Sturmfels, W. Whiteley and N. L. White, Symbolic computations in geometry, *IMA Preprint Series*, **389** (January 1988).
9. A. Dress, Chirotops and oriented matroids, *Bayreuther Math. Schr.*, **21** (1986), 14–68.
10. L. G. Khachian, A polynomial algorithm in linear programming, *Sov. Math. Dokl.*, **20** (1979) 191–194.
11. M. Las Vergnas, Matroides orientables, Preprint, April 1974; announced in *C. R. Acad. Sci. Paris*, **280** (1975), 61–64.
12. I. Shemer, Neighborly polytopes, *Israel J. Math.*, **43** (1982), 291–314.
13. B. Sturmfels, Boundary complexes of convex polytopes cannot be characterized locally, *J. London Math. Soc.* **35** (1987), 257–269.
14. B. Sturmfels, Computational synthetic geometry, Ph.D. dissertation, University of Washington, Seattle, 1987.
15. B. Sturmfels, Computing final polynomials and final syzygies using Buchberger's Gröbner bases method. *Results Math.*, **15** (1989), 551–360.
16. J. Bokowski and B. Sturmfels, *Computational synthetic geometry*, Lecture Notes in Mathematics, 1355 Berlin, Springer-Verlag, 1989.

Received 30 June 1988 and accepted 1 June 1989

Jürgen Bokowski and Jürgen Richter
Fachbereich Mathematik,
Technische Hochschule Darmstadt,
D-6100 Darmstadt,
Federal Republic of Germany