Finite groups whose subgroups of equal order are conjugate

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INTRODUCTION

It is our purpose to clarify the structure of the solvable finite groups whose subgroups of equal order are conjugate. In [1], A. Bensaïd and this author succeeded in determining the structure of the non-solvable finite groups satisfying that property up to that of some solvable finite groups sharing that property.

All groups in this paper will be finite. Notations and conventions follow the same scheme as in [1]. The group of semilinear maps \( x \mapsto ax^\sigma \) with \( x \in \mathbb{F}_p^* \), \( p \) prime, \( \sigma \in \text{Gal}(\mathbb{F}_p^*/\mathbb{F}_p) \), \( a \in \mathbb{F}_p^* \), will be denoted by \( \mathcal{L}(1, p^n) \).

§1. GENERAL PROPERTIES OF B-GROUPS

To be self-contained we recall the definition of B-group as formulated in [1]. The reader is referred also to [1] for a host of basic results.

Definition 1. Let \( G \) be a group.

a) \( G \) is called a B-group if any two subgroups of the same order are conjugate.

b) \( G \) is called an iso-group if any two subgroups of the same order are isomorphic.

Now we elucidate the structure of a Sylow \( p \)-subgroup \( \mathcal{P} \) of a B-group. If \( p = 2 \), then it was shown in ([1], Theorem 4) that \( \mathcal{P} \) is a cyclic or an elementary abelian 2-group or that \( \mathcal{P} \) is a quaternion group of order 8.
As to the structure of a Sylow $p$-subgroup of a $B$-group we will present a proof of the statement to Theorem 2 using results of [1].

**Theorem 2.** Let $P \in \text{Syl}_p(G)$, $p$ odd prime, $G$ a $B$-group. Then $P$ is cyclic or elementary abelian.

**Proof.** Of course, we will assume that $P \neq \{1\}$. Thus the center $Z(P)$ of $P$ is not trivial.

Therefore, as $P$ is an iso-group, it follows that $P$ contains only one subgroup of order $p$ (whence $P$ is cyclic by ([4], Satz III.8.2) or else $\exp(P) = p$.

In order to proceed we argue by means of an inductive argument. Let $U \neq \{1\}$ be a minimal normal subgroup of $G$. Then, due to Theorem 5 of [1], the following holds.

Either 1) $U$ is an elementary abelian $t$-group, $t$ prime, but $U$ is not cyclic, or 2) $U$ is cyclic of prime order, or 3) $U \cong A_5$ or $U \cong PSL(2,8)$ or $U \cong PSL(2,32)$.

**Re 1)** Due to ([1], Theorem 2.e), $U \in \text{Syl}_t(G)$ holds. So, as a quotient of a $B$-group is a $B$-group ([1], Theorem 2.c), $G/U$ is a $B$-group with $|G/U| > t$. Hence the statement of the Theorem follows immediately by induction.

**Re 2)** If $U \cong C_t$, $t$ odd prime, then the $B$-group property of $G$ reveals that $\Omega_1(S_u) = U$ for some $S_u \in \text{Syl}_t(G)$. Hence $S_u$ is cyclic by ([4], Satz III.8.2). So the Theorem holds in this case by induction. Next suppose $U \cong C_2$. Let $t$ be an odd prime dividing $|G/U|$. Then, by induction, any $S_u \in \text{Syl}_t(G)$ is cyclic or elementary abelian. So the Theorem holds too.

**Re 3)** Assume $U \cong A_5$, so that $|U| = 60$. Let $T_u \in \text{Syl}_u(G)$ for some odd prime $u$. So, as $G$ is a $B$-group, $|\Omega_1(T_u)| = u$ in case $u$ divides 15. Therefore, just as above, $T_3$ and $T_5$ are both cyclic. Hence assume $u | 15$. Then $T_u$ is isomorphic to a Sylow $u$-subgroup of $G/U$, and so $T_u$ is cyclic or elementary abelian as $G/U$ is a $B$-group of smaller order than the $B$-group $G$.

Next assume $U \cong PSL(2,8)$ (whence $|U| = 2^3 \cdot 7 \cdot 3^2$) or $U \cong PSL(2,32)$ (whence $|U| = 2^5 \cdot 3 \cdot 11 \cdot 31$). Now note that $PSL(2,8)$ has cyclic Sylow 3-subgroups of order 9. Therefore $G$ has cyclic Sylow 3-subgroups in case $U \cong PSL(2,8)$. Again, if $T_u \in \text{Syl}_u(G)$ for $u = 7$ in case $U \cong PSL(2,8)$ or for $u$ dividing 1023 in case $U \cong PSL(2,32)$, then we see that $|\Omega_1(T_u)| = u$ whence that $T_u$ is cyclic. Hence assume $u | 21$ if $U \cong PSL(2,8)$ or $u | 1023$ if $U \cong PSL(2,32)$, but with $u$ odd in both cases. Then $T_u$ is isomorphic to a Sylow $u$-subgroup of the $B$-group $G/U$, so that by induction $T_u$ is cyclic or elementary abelian.

The proof of the Theorem is complete. □

**Remark.** It turns out that Theorem 2 can be sharpened. Namely, it holds that, if $P$ is a noncyclic Sylow $p$-subgroup of the $B$-group $G$ with $p$ an odd prime, the group $P$ is elementary abelian of order $p^3$ or $p^3$. This is a consequence of
the following Theorem 3. Indeed, suppose $N$ is a minimal normal subgroup of $G$. Then by ([1], Theorem 5), either $N \cong A_5$ or $N \cong PSL(2, 8)$ or $N \cong PSL(2, 32)$ or $N$ is cyclic of prime order or $N$ is elementary abelian, but not cyclic. Suppose $P$ is elementary abelian of order $p'$, $p$ odd prime, $t \geq 4$. Then our assertion holds by an inductive argument in case $N$ is not elementary abelian. For the $B$-group property of $G$ and $G/N$ shows that $P$ is isomorphic to a Sylow $p$-subgroup of $G/N$ in that case. Thus there remains to investigate the case $N$ elementary abelian, but not cyclic. Again by induction we may assume that $N$ is a $p$-group, whereas the $B$-group property of $G$ shows that here $N \cong P$. Our statement follows now from Theorem 3.

**Theorem 3.** Let $G$ be a $B$-group. Suppose $M$ is a minimal normal subgroup of order $r'$, $r$ prime, $t \geq 3$. Then $t = 3$ or $t = 5$. If $t = 5$, then $r = 2$ and $G/C_G(M)$ is isomorphic to the group $\Gamma L(1, 32)$ of semilinear transformations over $F_{32}$ of order $5(2^5 - 1)$.

**Proof.** Since $M$ is solvable and characteristic simple, we see that $M$ is elementary abelian. From ([1], Theorem 2.e) we know that $M$ is the unique Sylow $r$-subgroup of $G$. Suppose $T$ is another nontrivial minimal normal subgroup of $G$. Then $G/T$ is a $B$-group, $MT/T \cong M$ and by induction, $t \leq 3$ or $t = 5$ and $r = 2$ and $G/C_G(M) \cong \Gamma L(1, 32)$, where $G = G/T$ and $M = MT/T$ by definition. Since $\bar{M}$ is a Sylow $r$-subgroup of $\bar{G}$ and $C_G(\bar{M})$ it holds that $\bar{M}$ is a direct factor of $C_G(\bar{M})$, due to the Schur–Zassenhaus theorem. Put $C/T = C_G(\bar{M})$. As $M$ is characteristic in $MT$ by $MT = M \times T$, we see that also $MT \leq C$, whence that $M \leq C$. Therefore $M$ is direct factor of $C$. Hence $C_G(M)/T = C_G(\bar{M})$. So $G/C_G(M) \cong \bar{G}/C_G(\bar{M})$. Thus the theorem holds in this case.

So we may assume that $M$ is the unique minimal normal subgroup of $G$. Then the Theorems 9 and 11 of [1] show that the $B$-group $G$ is solvable. So we have $C_G(M) = M$, for instance; see ([4], Satz 111.4.2). In the sequel of the proof of this Theorem we maintain that $t \geq 4$. All the subgroups of $G$ of order $r$ are conjugate in $G$ and all are contained in $M$. So, if $M_1 < G$ with $|M_1| = r$, it follows that $|G: N_G(M_1)| = (r'-1)/(r-1)$. Hence $(r^t-1)/(r-1)$ divides $|GL(t, r)|$, as $G/M$ is isomorphic to a subgroup of $Aut(M)$ (note $N_G(M_1) \supseteq M = C_G(M)$ and $Aut(M) \cong GL(t, r)$). Then it holds by applying Theorem 5.7 of [2] (here the solvability of $G$ plays a rôle) that $G/M$ can be viewed as subgroup of the group $\Gamma L(1, r')$ of semilinear transformations over $F_{p'}$ of order $t(r^t - 1)$, unless $r = 2$, $t = 6$ and $r = 3$, $t = 4$. Each of these two exceptional cases does not lead to a $B$-group structure for $G$, to be shown in the Theorems 4 and 5. So $G/M \cong \Gamma L(1, r')$ holds. Define $\alpha_t = (r^t-1)(r^{t-1}-1)(r^{t-2}-1)(r^3-1)^{-1}(r^2-1)^{-1}(r-1)^{-1}$. Then $G$, whence also $M$, contains $\alpha_t$ elementary abelian subgroups of order $r^3$. So, as $G$ is a $B$-group, we see that $\alpha_t = |G: N_G(L)|$ for any subgroup $L$ of $G$ of order $r^3$. Hence $\alpha_t$ divides $|\Gamma L(1, r')| = t(r^t - 1)$.

Suppose $t \geq 7$. Then $|\alpha_t| | (t^t - 1)$ implies that $(r^{t-1}-1)(r^{t-2}-1)$ divides $t(r^3-1)(r^2-1)(r-1)$. From this inequality

$$r^{2t-5} = r^{t-2}, r^{t-3} < t r^6$$
emerges, and so \(2^{2t-11} \leq r^{2t-11} < t\) should hold. However, there is no integer \(t \geq 7\) satisfying \(2^{2t-11} < t\).

Suppose \(t = 6\). Then \(\alpha_s \mid 6(r^6 - 1)\) which implies \((r^5 - 1)(r^4 - 1)\) divides \(6(r^6 - 1)(r^2 - 1)(r - 1)\). So \((r^5 - 1)(r^4 + 1) \mid 6(r^3 - 1)(r - 1)\). Hence \(r^6 = r^4 \cdot r^2 < 6r^4\), so that \(r^2 < 6\). But \(r = 2\) would lead to \(((2^5 - 1)(2^2 + 1)\) divides \(6(2^3 - 1)(2 - 1)\)'', which is a contradiction.

Suppose \(t = 4\). Let \(U\) be an elementary abelian subgroup of \(G\) of order \(r^2\). Then there are \(|G: N_G(U)| = \frac{(r^6 - 1)}{(r^2 - 1)}\) conjugate subgroups in \(G\) to \(U\). Therefore, as \(|G: N_G(U)|\) divides \(|I\mathcal{L}(1, r^4)| = 4(r^4 - 1)\), we see that \(1 + r + r^2\) divides \(4(r^2 - 1)\). Now \(1 + r + r^2\) is odd. So \(1 + r + r^2\) should divide \(r^2 - 1\), which is never possible.

Finally, assume \(t = 5\). Then \(\alpha_s \mid 5(r^5 - 1)\). This implies that \((r^4 - 1)(r^3 - 1)\) divides \(5(r^5 - 1)(r^2 - 1)(r - 1)\), whence \(r^2 + 1 \mid 5(r - 1)\). Hence \(r = 2\) or \(r = 3\). We have \(\alpha_s = 5(2^5 - 1)\) or \(\alpha_s = 5(3^5 - 1)\), respectively, simply by the definition of \(\alpha_s\). Suppose \(r = 3\). Then \(5(3^5 - 1) = |I\mathcal{L}(1, 3^3)| \geq |G/M| \geq |G : N_G(L)| = \alpha_s = 5(3^5 - 1)\), i.e. \(|G/M| = |G : N_G(L)| = 5(3^5 - 1)\). Since \(G\) is a \(B\)-group and since \(C_G(M) = M\), we see that an element \(\tau \in G\) satisfying \(|\tau M| = 2\), must act on \(M\) by conjugation by inverting all of its elements. Therefore, \(|G : N_G(L)| \leq \frac{1}{2} |G/M|\) which contradicts the equality \(|G/M| = |G : N_G(L)|\) just found. So there remains \(r = 2\), in which case \(G/M \cong I\mathcal{L}(1, 32)\) is of order \(5(2^5 - 1)\).

The proof of the Theorem is complete. \(\square\)

Next suppose that some group \(G\) has a chief section \(L/K\) for which \(|L/K| = 3^4\) or \(|L/K| = 2^6\). Then \(G\) is not a \(B\)-group. This will be shown in Theorem 4 and in Theorem 5, respectively. The group \(G/K\) has a minimal normal subgroup of order \(81\) or of order \(64\) respectively; in both possibilities \(G/K\) is not a \(B\)-group by the very same Theorems. Therefore the exceptional cases \("p = 2, t = 6"\) and "\(p = 3, t = 4\)" brought up in the proof of Theorem 3 do not lead to a \(B\)-group structure for \(G\) in Theorem 3.

Thus the reader is now aware of the fact that the Theorems 4 and 5 do not follow from the statement of Theorem 3, but that each of them has to be proved in a separate way!

**Theorem 4.** Let \(G\) be a \(B\)-group. Then there does not exist a chief section of \(G\) of order \(81\).

**Proof.** Assume the contrary. As the \(B\)-group property is hereditary on factor groups, we also have that there exists a \(B\)-group containing a minimal normal subgroup of order \(81\). Furthermore, by an inductive argument we also see that there exists a \(B\)-group \(G\) containing a unique minimal normal subgroup \(M\) where in addition the order of \(M\) is \(81\). Note that \(M\) is the unique elementary abelian Sylow 3-subgroup of \(G\). In particular, it follows from ([1], Theorems 9 and 11) that \(G\) is solvable. In addition \(C_G(M) = M\). Therefore \(M\) can be regarded as a 4-dimensional faithful irreducible \(\mathbb{F}_3[G/M]\)-module \(\tilde{M}\). Write \(\tilde{G} = G/M\). Assume \(O_{13}(\tilde{G}) > \{1\}\). This assumption is based on the fact that \(\tilde{G}\)
injects in $\text{Aut}(M)$ so that $|\bar{G}| \mid 2^9.5.13$ due to $3^j | \bar{G}|$. Then Clifford’s theorem ([4], Satz V.17.3) reveals that the cyclic group $O_{13}(\bar{G})$ acts irreducible on $\bar{M}$, implying that $X \cong C_3 \times C_3 \times C_3 \times C_3$ where $X/\bar{M} := O_{13}(\bar{G})$. This contradicts $C_G(M) = M$. So $O_{13}(\bar{G}) = \{1\}$. Since $G$ is a $B$-group we have $G > N_G(U) \simeq M$ with $|G : N_G(U)| = (3^3 - 1)(3^2 - 1)(3^2 - 1)(3 - 1)^{-1} = 130$ for any subgroup $U$ of $M$ of order 9. Hence $65 \mid |\bar{G}|$ and remember $|\bar{G}| \mid 2^9.65$. Assume $O_5(\bar{G})$ is (cyclic) of order 5. So $O_3(\bar{G})$ is a direct factor of $C_G(O_3(\bar{G}))$ (due to the Schur–Zassenhaus theorem) and observe that $|\bar{G}/C_G(O_3(\bar{G}))|$ divides 4. Now it follows from applying Sylow’s theorem on $C_G(O_3(\bar{G}))/O_3(\bar{G})$ that $O_{13}(C_G(O_3(\bar{G})))$ has order 13. As $O_{13}(C_G(O_3(\bar{G}))) \triangleleft \bar{G}$, this is in conflict to $O_{13}(\bar{G}) = \{1\}$.

Therefore the Fitting subgroup $F(\bar{G})$ of the $B$-group $\bar{G}$ is a nontrivial 2-group. Note that $F(\bar{G}) \neq \bar{G}$ as $65 \mid |\bar{G}|$. As $C_G(M) = M$ and as $G$ is a $B$-group it follows that $F(\bar{G})$ has precisely one element of order 2, and this element acts on $\bar{M}$ by conjugation by inverting all of its elements. So, by ([1], Theorem 4) it holds that $F(\bar{G})$ is quaternion of order 8 or $F(\bar{G})$ is a cyclic 2-group. Again by ([1], Theorem 4) $F(\bar{G}) \subseteq \text{Syl}_2(\bar{G})$ if $F(\bar{G})$ is quaternion of order 8. Hence as $C_G(F(\bar{G})) \triangleleft F(\bar{G})$, $|\bar{G}/F(\bar{G})|$ divides then $|\text{Aut}(Q)| = 3$ in that case. So as $3^j \mid |\bar{G}|$ (note $|\bar{G}|$ divides $2^9.65$) we have here $\bar{G} = F(\bar{G}) \cong Q$. However, $Q$ is not a $B$-group as it should be. So, finally, we see that $F(\bar{G})$ is a cyclic 2-group. Recall that $C_G(F(\bar{G})) \triangleleft F(\bar{G})$ as $\bar{G}$ is solvable. Hence as $\bar{G}/C_G(F(\bar{G})) \cong \text{Aut}(F(\bar{G}))$ and as $\text{Aut}(F(\bar{G}))$ is a 2-group whenever $F(\bar{G})$ is a cyclic 2-group, it holds that $\bar{G}$ is a 2-group. But $65 \mid |\bar{G}|$. This is the final contradiction, thereby proving the Theorem.

**Theorem 5.** Let $G$ be a $B$-group. Then there does not exist a chief section of $G$ of order 64.

**Proof.** As in the proof of Theorem 4 we may assume that $G$ is solvable, and that $M$ is the unique minimal normal subgroup of $G$ so that $F(G) = M$, where $|M| = 64$. Note $M$ is elementary abelian. We have $C_G(F(G)) = F(G) = M$. Thus $|G/M|$ divides $|\text{Aut}(M)| = 3^3.5.7.31$. Put $\bar{H} = HM/M$ whenever $H \subseteq G$.

Suppose $O_p(\bar{G}) \neq \{1\}$, where $p = 5$ or $p = 31$. As $O_p(\bar{G}) \leq F(\bar{G})$, Clifford’s theorem ([4], Satz V.17.3) applied on $C_G(M) = M$ with $M \cong \mathbb{F}_{2^i}$ yields $[O_p(\bar{G}), M] = \{1\}$. This contradicts $C_G(M) = M$!

Hence $|F(\bar{G})| = 3^i7^j$ with $i \leq 4, j \leq 2$, must hold. Note $ij \geq 1$ has to be the case and observe that $F(\bar{G})$ is abelian, due to Theorem 2. Now $M$ contains $(2^6 - 1)(2^5 - 1)(2^2 - 1)^{-1} = 3.7.31$ subgroups of order 4. So, as $G$ is a $B$-group, $31 \mid |G : N_G(V)|$ where $V \triangleleft G$ is of order 4. Thus $31 \parallel |\bar{G}|$. This means that $F(\bar{G})$ is not cyclic. [Indeed, suppose $F(\bar{G})$ were cyclic. Since $\bar{G}/F(\bar{G})$ is isomorphic to a quotient group of the 3-group $(\text{Aut}(F(\bar{G})))_2$, we see that $|\bar{G}/F(\bar{G})| \mid |F(\bar{G})|$ is not divisible by 31. This is a contradiction.] So, as $G$ is a $B$-group, either $9 \mid |O_3(F(\bar{G}))|$ and $O_3(F(\bar{G}))$ is elementary abelian, or $49 \mid |O_7(F(\bar{G}))|$ and $O_7(F(\bar{G}))$ is elementary abelian.

Suppose $O_3(F(\bar{G}))$ is elementary abelian of order at least 9 (and at most 81 as we saw above). As $\bar{G}$ is a $B$-group, $O_3(F(\bar{G})) \in \text{Syl}_3(\bar{G})$. Assume that $\bar{G}$ has
$t$ subgroups of order 3. Hence $t = |G : N_G(R)|$ divides $3^4.5.7.2^2.31$, where $R < G$ with $|R| = 3$, as $|G| / |\text{Aut}(M)| = 2 = 3^4.5.7.2^2.31$. On the other hand $t = (3^i - 1).2^{-1}$ with $2 \leq i \leq 4$, as $G$ is a $B$-group. So as $t$ must be odd, we have $t = 13$. This contradicts $13 / 3^4.5.7.2^2.31$.

There remains $O_2(F(\bar{G})) = F(\bar{G})$, where $F(\bar{G})$ is elementary abelian of order 49. Hence the $B$-group $\bar{G}$ contains eight subgroups of order 7, as here $F(\bar{G}) \in \text{Syl}_7(\bar{G})$. Therefore 8 divides $|\bar{G}|$, which is not true.

The proof of the Theorem is complete.

\section{Structure of $B$-Groups}

In this section we deepen the structure of the $B$-groups. We have to introduce some notation. Namely, if $M$ is an elementary abelian $p$-group, then $P \text{Aut}(M)$ will be the set of the so-called power automorphisms of $M$. We see that $P \text{Aut}(M)$ is nothing else but the centre of $\text{Aut}(M)$; whence given $\alpha \in Z(\text{Aut}(M))$ it holds that $m^\alpha = m^i$ for some specific integer $i$ independent from the choice of the element $m \in M$. Furthermore, if $p$ is prime dividing an integer $s \geq 1$, then $p^\alpha | s$ stands for $p^\alpha | s$ together with $p^\alpha + 1 | s$.

\textbf{Theorem 6.} Suppose $M$ is a minimal normal subgroup of the $B$-group $G$, where $M \cong C_p \times C_p \times C_p$, $p$ prime. Put $M_1 < M$, $|M_1| = p$. Then either one of the following three possibilities occurs.

1. $3^\delta | p - 1$ with $\delta \geq 1$; $G/C_G(M) = L \times K$, where $K$ is cyclic of order dividing $(p - 1).3^{-\delta}$ and $|L| = 3^\delta (p^2 + p + 1)$ with $L = \langle a, \lambda \rangle$, where $a$ has order $(p^2 + p + 1).3^{-\delta}, \lambda$ has order $3^\delta + 1, \lambda^{-1} a \lambda = a^{p^2};$

$N_G(M_1)/C_G(M) = \langle K, \lambda^3 \rangle \cong P \text{Aut}(M);$ note $N_G(M_1) \not\subseteq G$ in this case.

2. $3 | p - 1$; $G/C_G(M) = L \times K$, where $K$ is cyclic and whose order divides $p - 1$, $L = \langle a, \lambda \rangle$ has order $3(p^2 + p + 1)$ with $|a| = p^2 + p + 1$, $|\lambda| = 3, \lambda^{-1} a \lambda = a^{p^2};$

$\langle \lambda \rangle = C_G(M_1)/C_G(M);$

$N_G(M_1)/C_G(M_1) \cong K \hookrightarrow P \text{Aut}(M);$ note $N_G(M_1) \not\subseteq G$ in this case.

3. $G/C_G(M)$ is cyclic of order $t(p^2 + p + 1)$ with $t | p - 1;

N_G(M_1)/C_G(M) \hookrightarrow P \text{Aut}(M)$ and $N_G(M_1)/C_G(M)$ has order $t$; it holds that $3 \nmid t$ if and only if $3 | p - 1$.

In each of these cases $G/C_G(M)$ acts like a group of semilinear transformations on $M$, where $M$ is regarded as a 3-dimensional vector space over $\mathbb{F}_p$.

\textbf{Proof.} It holds that $G/C_G(M)$ is solvable; see ([11], Theorems 9 and 11). From ([2], Theorem 5.7) we know that $H := G/C_G(M)$ can be viewed as subgroup of $\Gamma L(1, p^5)$, where $\Gamma L(1, p^5) = \langle \alpha, \beta | |\alpha| = p^3 - 1, |\beta| = 3, \beta^{-1} \alpha \beta = \alpha^{p^2} \rangle$, so that $|\Gamma L(1, p^5)| = 3(p^5 - 1)$. Put $A = \langle \alpha \rangle$. As $G$ is a $B$-group, we have $|G : N_G(M_1)| = \# \{\text{conjugates of } M_1 \text{ in } G\} = p^2 + p + 1$. Put $U = N_G(M_1)/C_G(M)$. One of the following three possibilities does occur.

1. $H \neq H \cap A$ and $U \subseteq H \cap A$;

2. $H \neq H \cap A$ and $U \not\subseteq H \cap A$;

3. $H \leq A$.

Note that due the structure of $\Gamma L(1, p^5)$, $U \cap H \cap A \not\subseteq H$ in each case and also
$H \cap A \leq H$. As $M \in \text{Syl}_p(G)$ by the $B$-group property, we may assume without loss of generality that $C_G(M) = M$.

Re 1) Suppose $H \neq H \cap A$, $U \leq H \cap A$. Hence $|H/(H \cap A)| = 3$, $|H/U| = p^2 + p + 1$. So $3 | p - 1$, but $3 | (H \cap A)/U$. View $M$ as faithful irreducible $\mathbb{F}_p H$-module. [In the sequel, $M_D$ will be the $\mathbb{F}_p D$-module $M$, with $D \leq H$.] It follows here that $M_U$ must be homogeneous in such a way that $M_U \cong W + W + W$, where $W$ is a 1-dimensional $\mathbb{F}_p U$-submodule of $M_U$. [For otherwise, the inertia group in $H$ of some 1-dimensional $\mathbb{F}_p U$-submodule of $M_U$, would be equal to a subgroup $X$ of $H$ in such a way that $|H:X| = 3$; but $U \geq X$ by definition of $U$, violating $3 < p^2 + p + 1$.] So, as no element of $U \setminus \{1\}$ operates trivially on $M$, it holds that $U \simeq \mathbb{P} \text{Aut}(M)$. Hence $|U| | p - 1$. Consider $S < H$ with $S/U \leq \text{Syl}_3(H/U)$. Then $|S/U| = 3$ as we saw above. Apparently, $U \leq \langle \alpha^{p^2 + p + 1} \rangle$, so $U$ is cyclic. Therefore $S$ is abelian as is easy to check. It follows from the $B$-structure of $H$ that $S$ has cyclic Sylow 3-subgroups whence that $S$ itself is cyclic. As $p | |H|$, it holds that $M_S$ is a completely reducible $\mathbb{F}_p S$-module. If $M_S$ is not irreducible, $M_S$ contains a 1-dimensional $\mathbb{F}_p S$-submodule. This, however, violates the $B$-property of $G$ and $H$, just as here $U = U \cap H \cap A \leq H$, whence $N_G(M_U) \leq G$. Hence $M_S$ is an irreducible $\mathbb{F}_p S$-module. Of course, $M_S$ is also a faithful $\mathbb{F}_p S$-module. Put $3^\delta | |p - 1$. So $\delta \geq 1$ as we saw above. If $3^\delta$ is divisible by the 3-part of $|S|$, then we see that $p \equiv 1 \mod |S|$, just as $S = S \cap A$. In that case our remarks on $M_S$ yield a contradiction to ([5], Exercise V.9.20), concerning the $\mathbb{F}_p$-dimension of faithful irreducible $\mathbb{F}_p S$-modules. Hence $3^{\delta+1} \parallel |S|$, i.e. $3^{\delta} \parallel |U|$. The other statements in 1) of the Theorem follow easily.

Re 2) Suppose $H \neq H \cap A$, $U \nleq H \cap A$. Put $K = U \cap H \cap A$. As $|A| = p^3 - 1$, $|H/H \cap A| = 3$ and $|(H \cap A)/U \cap (H \cap A)| = |H/U| = p^2 + p + 1$, we see that $|K| | p - 1$. We will show that $3 \parallel |K|$. For assume that $3 \nmid |K|$. Then 3 divides $3 + 3(p - 1) + (p - 1)^2 = p^2 + p + 1$. Now $|U/K| = 3$. As $H$ is a $B$-group, however, the assumption $3 \nmid |K|$ yields that either $H$ has cyclic Sylow 3-subgroups or $K$ contains elementary abelian Sylow 3-subgroups of $H$, whereas $3 \mid \gcd(|U/K|, |K|, |(H \cap A)/K|)$ produces a contradiction to that structure. Thus we have indeed unconditionally that $3 \parallel |K|$. Now $H/K$ is a $B$-group and $|U/K| = 3$ with $U \leq H \cap A$, $H \cap A \leq H$, $|H/(H \cap A)| = 3$, $U(H \cap A) = H$. Hence $|H/(H \cap A)/K|$ would violate the Sylow 3-structure of the $B$ group $H/K$. Therefore, $3 \parallel |(H \cap A)/K| = p^2 + p + 1$, so that $3 \parallel |H \cap A|$. Next observe that $(|K|, |H/K|) = 1$ as here $(p - 1, 3(p^2 + p + 1)) = 1$. Just as we have shown earlier it holds that each element of $K$ acts like a power automorphism on $M$. But we can say more. The $\mathbb{F}_p U$-module $M$ is a direct sum of a 1-dimensional $\mathbb{F}_p U$-module $M_1$ and a 2-dimensional $\mathbb{F}_p U$-module $L$, say. Regard $M_1$ and $L$ as subgroups of $M$. Hence, as $G$ is a $B$-group, $U = N_G(L)/M$ note that $p^2 + p + 1 = |G:N_G(M_1)| = |G:N_G(L)|$. 245
Consider $C_G(L)/M$. Suppose $a \in C_G(L) \setminus M$. By Maschke’s theorem, $a \in N_G(R)$ for some 1-dimensional $\mathbb{F}_p C_G(L)$-module $R$ complementing $L$ in $M$. The $B$-group property reveals that the trivial action of $a$ on some $L_1 \leq L$ with $|L_1| = p$, implies that $a$ acts also trivially on $R$. Hence $a \in C_G(R \times L) = C_G(M) = M$. Therefore $C_G(L) = M$. Since $3 \nmid p - 1$, we see that, as $3 \mid |N_G(M_1)/M|$, $C_G(M_1) \neq M$ holds.

Now $(C_G(M_1)/M) \cap K \leq H$ as $H \cap A$ is a cyclic normal subgroup of $H$. Observe that $(C_G(M_1)/M) \cap K$ acts via power automorphisms on $M$, whence, as $C_G(M) = M$, it follows that $(C_G(M_1)/M) \cap K = \{1\}$. Hence $C_G(M_1)/M \not\leq K$ and so $|C_G(M_1)/M| = 3$. Our previous remark $C_G(L) = M$ yields now $N_G(M_1) \not\leq G$. This proves the statements in part 2) of the Theorem.

Re 3) Finally assume $H \leq A$. So $H$ is cyclic. We have $U \leq H$ with $|H/U| = p^2 + p + 1$. It follows from Clifford’s theorem combined with the $B$-group property of $H$ and with $|U| \mid p - 1$, that each element of $U$ acts like a power automorphism on $M$. By definition of $U$, $M \subseteq M$ is an irreducible $\mathbb{F}_p X$-module for any $U \leq H$. Hence Exercise V.9.20 of [5] yields that if $3 = |X/U|$ (whence $3 \mid p - 1$ also, as now $3 \mid |H/U| = p^2 + p + 1$) it also holds that $3 \mid |U|$. Conversely, if $3 \nmid |X/U|$ for any $U \leq H$, then $3 \nmid |H/U| = p^2 + p + 1$, whence $3 \nmid p - 1$. In that case $3 \nmid |U|$, as $|U| \mid p - 1$.

Herewith the proof of the Theorem is complete. \qed

The following topic of investigation will be the situation $M \leq G$, where $M \cong C_p \times C_p$, $p$ prime, is a minimal normal subgroup of the $B$-group $G$. In Theorem 7 we treat the particular situation in which $P$ is a Mersenne prime. We mention that it is a separate possibility in Theorem 5.7 of [2].

**Theorem 7.** Suppose $p = 2^j - 1$ is a Mersenne prime. Let $G$ be a $B$-group containing a minimal normal subgroup $M \cong C_p \times C_p$. Let $M \leq M$, $|M| = p$. Then
the following holds.
1) $G/C_G(M)$ is cyclic of order $2^{i+1}t$, where $t$ is odd and $t | p - 1$;
2) if $K/C_G(M)$ is of order $t$, then $K/C_G(M)$ is characterized by
   
   \[ K/C_G(M) \leq P \text{Aut}(M) \]
   
   (i.e. considering $G/C_G(M) \leq \text{Aut}(M)$, it holds that
   \[ P \text{Aut}(M) \cap \frac{G}{C_G(M)} = \frac{K}{C_G(M)}. \]
3) $N_G(M_1) = K$.

**Proof.** Without loss of generality we may assume that $C_G(M) = M$. [Again because of the Schur-Zassenhaus theorem combined with $M \in \text{Syl}_p(G)$.] As $G$ is a $B$-group, $|G : N_G(M_1)| = p + 1 = 2^i > 1$ holds. So suppose there are at least two involutions in $G$. Since $G$ is a $B$-group at least one of them must act trivially on $M$ by conjugation, thereby contradicting $C_G(M) = M$. So by (II, Theorem 4) the Sylow 2-subgroups of the $B$-group $G/M$ are either cyclic or isomorphic to $Q$, the quaternion group of order 8. Observe that $G/M \leq \text{Aut}(M)$ and so, as $p | |G/M|$, it follows that $|G/M|$ divides $(p^2 - 1)(p - 1)$. Suppose $Q \in \text{Syl}_2(G/M)$, i.e. $|G : N_G(M_1)| = p + 1 = 4$ or 8. Suppose $p = 3$. Then, however, $|G/M|$ divides $8 \cdot 2 = 16$. So the 2-group $G/M,$ being isomorphic to $Q$, has to be a $B$-group. This does not hold. Suppose $p = 7$ in case $Q \in \text{Syl}_2(G/M)$. So, as $|G/M| > 2^5 \cdot 3^2$, $|G : N_G(M_1)| = p + 1 = 8$ and $N_G(M_1)/M_1$ divides 9. However, as $G$ is a $B$-group, it is clear that, as $C_G(M) = M$, the unique involution of $G$ inverts each element of $M$ under the conjugation action. So $2 | |N_G(M_1)/M_1|$ should hold; a contradiction.

Therefore, we have proved that $G/M$ has cyclic Sylow 2-subgroups. Observe that $|N_G(M_1)/M_1|$ divides $p - 1$. [Namely, look at the action of $C_G(M_1)$ on $M$, by applying Maschke’s theorem on the 2-dimensional $\mathbb{F}_p$-vector space $M$; remember that $G$ is a $B$-group so that now $C_G(M_1) = M$ has to be the case!] Thus as $2^i = |G : N_G(M_1)|$, we see that $2^{i+1} | |G/M|$, for it is clear that, as $C_G(M) = M$, an involution of $G$ must act on $M$ under conjugation, by inverting its elements. Burnside’s theorem ([4], Satz IV.2.8) yields that $G/M$ is 2-nilpotent now, so there exists $M \leq U \leq G$ with $G/U \cong C_{2^{i+1}}$, $|U|$ odd, $|U| | q - 1$. Note that $U/M \leq N_G(M_1)/M = N_G(M_1)/C_G(M_1) \leq \text{Aut}(M_1) = \text{Aut}(C_p) \cong C_{p - 1}$. The group $M$ can be regarded as an irreducible faithful $\mathbb{F}_p(G/M)$-module (as $C_G(M) = M$). Since $|G : N_G(M_1)| = p + 1 = 2^i > 2$, it follows immediately that $M$ is a homogeneous $\mathbb{F}_p(U/M)$-module. So, as $C_G(M) = M$, any element of $U$ acts like a power automorphism on $M$, which means that $U/M \leq Z(G/M)$. We also see that $U\langle \tau \rangle / M$, where $|\tau| = 2$, $\tau \in G$, acts like a group of power automorphisms on $M$. Since $N_G(M_1) = U\langle \tau \rangle$, it follows that all statements of the Theorem have been proved now.  

In the next theorem $M < G$, $M \cong C_p \times C_p$, $p$ prime, $G$ a $B$-group will be treated in its most generality. It holds that the statements of Theorem 7 are covered by the first part of Theorem 8, although the proof of Theorem 8 has to rely in essence on the result of Theorem 7. Observe that in Theorem 8 $G/C_G(M)$ is assumed to be solvable if $p = 11, 19, 29$ or 59. It is obvious that these primes
must be treated in a particular way, as the Theorems 9 and 11 of [1] show. The structure of $G/C_G(M)$ can be read off of those theorems for $p = 11, 19, 29$ and 59 when $G/C_G(M)$ is supposed to be non-solvable.

**Theorem 8.** Suppose $M$ is a minimal normal subgroup of the B-group $G$, with $M \cong C_p \times C_p$, $p$ prime. Assume in addition that $G/C_G(M)$ is solvable in the cases $p = 11, 19, 29$ and 59. Let $M_1 < M$ be of order $p$.

Then one of the following possibilities does occur.

1. $G/C_G(M) \cong C_{(p+1)}$ with $2^t \mid p - 1$, $2^t \mid t$, $p - 1$, $\delta \geq 0$. Regard $G/C_G(M)$ as subgroup of $\text{Aut}(M)$. Then $P \text{Aut}(M) \cap G/C_G(M)$ is cyclic of order $t$ and so $N_G(M_1)/C_G(M)$ is of order $t$.

2. $G/C_G(M) \cong \langle S, \lambda \rangle \times T$, where $|\lambda| = 2^{(p+1)}$, $|\sigma| = (p + 1)/2 \neq 0 \pmod{2}$ with $S = \langle \sigma \rangle$, $\lambda^{-1} \sigma \lambda = \sigma^{-1}$, $T$ is cyclic of odd order, $T\langle \lambda^2 \rangle$ acts like $P \text{Aut}(M) \cap G/C_G(M)$ on $M$ and $T\langle \lambda^2 \rangle \cong N_G(M_1)/C_G(M)$; note that $|T|$ divides $p - 1$ and that also $2^t \mid p - 1$ with $\delta \geq 1$ in this case.

3. either $p = 5$ and $G/C_G(M) \cong SL(2,3)$, or $p = 11$ and $G/C_G(M) \cong SL(2,3) \times C_u$ with $u = 1$ or $u = 5$.

In the cases 1) and 2) $G/C_G(M)$ acts like a group of semilinear transformations on the 2-dimensional $\mathbb{F}_p$-vector space $M$. As $G$ is a B-group, it holds that $p \mid |G/C_G(M)|$ in all three cases.

**Proof.** In case $p$ is a Mersenne prime, then part 1 of the Theorem holds as has been shown in Theorem 7. So assume $p$ is not a Mersenne prime and assume in addition that $G/C_G(M)$ is solvable in case $p = 11, 19, 29$ and 59. Therefore, applying Theorems 9 and 11 of [1], it holds that under the before mentioned conditions, $G/C_G(M)$ is solvable. Now it follows by the proof of Theorem 5.7 of [2] that $G/C_G(M)$ can be regarded as subgroup of $GL(1, p^2)$, unless $p \in \{5, 11, 23\}$ in which cases an extra possibility might occur.

We divide up. Note that is no loss of generality to assume that $C_G(M) = M$.

A) Let us here consider the situation in which $G/C_G(M) \leq GL(1, p^2)$. Put $H = G/M$, $GL(1, p^2) = A\langle \xi \rangle$, with $|\xi| = 2$, $A = \langle \alpha \rangle$, $|\alpha| = p^2 - 1$, $\xi^{-1} \alpha \xi = \alpha^p$. Define $U = N_G(M_1)/M$. At first sight one of the following situations comes into the picture.

1. $H \cap A < H$ and $U \not\leq H \cap A$;
2. $H \cap A < H$ and $U \leq H \cap A$;
3. $H = A$.

In each of these cases the B-group $H$ is 2-nilpotent. So, by Theorem 4 of [1], $H$ has cyclic Sylow 2-subgroups.

A,1) It holds that $|H \cap A|/(U \cap H \cap A)| = |G : N_G(M_1)| = p + 1$ by the B-group property of $G$. Put $K = U \cap H \cap A$. We have $K \leq H$, so $H/K$ is a B-group and also it is true that $(H \cap A)/K \leq H/K$. If $p$ is odd, then we see that any Sylow 2-subgroup $S$ of $H/K$ contains now (at least) three involutions (so that $S$ is neither cyclic nor quaternion) and that not all of the involutions are contained in the cyclic group $(H \cap A)/K$. This violates the cyclicity of the Sylow 2-subgroups of $H$. So let $p = 2$. 248
Then $H/K$ is of order 6, whereas $M$ is elementary abelian of order 4. As $G$ is a $B$-group, this contradicts Theorem 2.2 of [1]. Hence the case $A,1$) does not occur.

A,2) Now suppose $|H/(H \cap A)|=2$ and $|(H \cap A)/U|=\frac{1}{2}(p+1)$. Here $p$ is odd. Let $H \cap A=\langle \beta \rangle$; whence $|\beta||p^2-1$. We see that $H=(H \cap A)\langle \lambda \rangle$, where $\langle \lambda \rangle$ is a cyclic Sylow $2$-subgroup of $H$. If $R \leq H \cap A$, $|R|$ odd, $|R|\mid p+1$, then $\lambda^{-1}a^R=a^p=a^{-1}$ for each $a \in R$ if $T \leq H \cap A$, $|T|$ odd, $|T|\mid p-1$, then $\lambda^{-1}c^T=c^p=c$ for each $c \in T$. Let $2^\delta$ be the 2-part of $|U|$. Now $m= m^{-1}$ for any $m \in M$, where $|\tau|=2$, $\tau \in H$. Then $\delta \geq 1$ as $G$ is a $B$-group with $C_G(M)=M$. It is clear that $T \leq P \text{Aut}(M)$, as $p+1=|G:N_G(M)|$. Now observe that $\Gamma L(1,p^2)$ has noncyclic Sylow $2$-subgroups whenever $p$ is an odd prime. Thus that $2$-nilpotent group $\Gamma L(1,p^2)$ is not a $B$-group. Therefore the $B$-group $H$ can be regarded as a proper subgroup of $\Gamma L(1,p^2)$. The 2-element $\lambda$ is of the form $\xi \beta$, where $\beta \in A$. Hence $(\xi \beta)^2=\beta^{p+1}$, so that the order of $(\xi \beta)^2$ divides $p-1$. Then $\lambda^2$ acts like a power automorphism on $M$. So $\lambda^2$ normalizes certainly $M_1$. This means all together that the order of $(H \cap A)/U$ is an odd number. That is, $\frac{1}{2}(p+1)$ is odd. Note also that $\lambda^2 \neq 1$, as the $B$-group property for $G$ yields the existence of an involution in $G$ that acts on $M$ by conjugation by inverting all of its elements. Hence $2 \mid |U|$. Let $2^\delta \mid |U|$. Now consider $U\langle \lambda \rangle$, whence $|U\langle \lambda \rangle|=2|U|$. Then $M$, considered as $F_2(U\langle \lambda \rangle)$-module, is irreducible and faithful. The group $U\langle \lambda \rangle$ is cyclic. So according to Exercise 9.20 of [5], it holds that $p^2=1(\text{mod }2^{\delta+1})$, but $p \neq 1(\text{mod }2^{\delta+1})$. We have seen above that $p+1$ equals twice an odd number. So anyway $2^\delta \mid p-1$. This finishes the proof of 2).

A,3) Assume $H \subseteq A$. We have $|H/U|=|G:N_G(M_1)|=p+1$. Assume $p$ is odd. Now $H$ contains a unique involution inverting any element of $M$ under conjugation. Suppose $2^\delta$ equals the 2-part of $|U|$, whence $\delta \geq 1$. Thus $2^\delta \mid p-1$. By ([5], Exercise 9.20) it holds that $p^2=1(\text{mod }2^{\delta+1})$, but $p \neq 1(\text{mod }2^{\delta+1})$. It is clear now that $U \leq P \text{Aut}(M)$. Suppose $p=2$. As $G$ is a $B$-group, this leads to $|H/U|=3$ and $|U|\subseteq \{1\}$. Hence in both cases we are in statement 1) of the Theorem.

B) Next assume $p \in \{5,11,23\}$. Recall $C_G(M)=M$ might be assumed. We also maintain throughout that $G/M \cong \Gamma L(1,p^2)$. Suppose $p=23$. It follows from the proof of ([2], Theorem 5.7) that the $B$-group $G/M$ is isomorphic in its conjugation action on $M$ to a subgroup of $SL(2,3) \times C_{11}$, in which the latter group acts transitively on the elements of $(C_{23} \times C_{23}) \setminus \{1\}$. The action of $G/M$ is such that $Z(SL(2,3)) \times C_{11} \leq P \text{Aut}(C_{23} \times C_{23})$. Hence $G/M$ cannot act on $M$ by conjugation, thereby transitively permuting the twenty-four subgroups of $G$ of order 23. So $p \neq 23$. The very same theorem in combination with $C_G(M)=M$, the $B$-group property of $G$, and the proof of Theorem 5.7 of [2] yields immediately $G/M=SL(2,3)$ in case $p=5$, or $G/M \cong SL(2,3) \times C_i$ ($i=1$ or $i=5$) in case $p=11$. We omit the details.
This concludes the proof of the Theorem.

To close section 2 we state a theorem that gives more information on the exceptional possibility occurring in Theorem 3.

**Theorem 9.** Let $G$ be a solvable group of order 4960 whose Sylow 2-subgroups are elementary abelian. Then $G$ is a $B$-group if and only if the Fitting subgroup of $G$ has order 32.

**Proof.** a) Let $G$ be a solvable group of order $5 \cdot 31 \cdot 2^5$ whose Fitting subgroup $F(G)$ is elementary abelian of order 32. We will show that $G$ is a $B$-group.

It follows from Sylow's theorem that there exists $N/F(G) \trianglelefteq G/F(G)$ with $|G/N| = 5$. Now note that $G$ acts irreducibly on the $\mathbb{F}_2$-"vector space" $F(G)$. Indeed, otherwise by Sylow's theorem, it would easily follow that $N$ would contain a characteristic subgroup of order 31, in contradiction to $|F(G)| = 32$. We have that $|G/F(G)/F(G/F(G))|$ equals one or five. If $G/F(G)$ would be nilpotent then Clifford's theorem applied on $G/F(G)$ under its action on $F(G)$, would give that some 5-element would centralize the whole of $F(G)$, contrary to the fact that the Fitting subgroup of $G$ is supposed to have thirty-two elements. Hence $F(G/F(G)) = N/F(G)$ after all. An ample observation shows that the thirty-one subgroups of $G$ of order 2 are conjugate within $N$ and that the same holds for all the thirty-one subgroups of order 16 of $G$. Now observe that there are $155 = 5(2^5 - 1)$ subgroups of order 4 in $G$ and also 155 subgroups of order 8 in $G$. Assume that some 5-element $b \in G$ would normalize such a group $T$ of order 4. Then Maschke's theorem reveals that $F(G) = T \times S$, where $S$, being of order 8, is also normalized by $b$. Then the element $b$ centralizes in fact $F(G)$, which was not allowed as we saw. An analogous argumentation holds with respect to the subgroups of order 8. The same statement holds for any 31-element instead of a 5-element. This means that $N_G(T) = F(G)$ for any subgroup $T$ of $G$ with $|T| = 4$ or with $|T| = 8$. So all subgroups of order 4 of $G$ are conjugate in $G$ and also all subgroups of order 8 of $G$ are conjugate in $G$. Now, in order to speed up the argumentation, we conclude that $G$ satisfies the conditions of ([3], Theorem 3) from which it directly follows that $G$ is a $B$-group.

b) Assume that $G$ is a solvable $B$-group whose Sylow 2-subgroups are elementary abelian of order 32, where $|G| = 5 \cdot 31 \cdot 2^5$. We are going to prove that the order of $F(G)$ is thirty-two.

Now, as $G/O_2(G)$ is also a $B$-group we see that $O_2(G/O_2(G))$ is a minimal normal subgroup of $G/O_2(G)$, in such a way that $|O_2(G/O_2(G))| = 32$. This means that the implication "$2 \mid |F(G)| \Rightarrow 32 \mid |F(G)|" holds here.

So assume $32 \mid |F(G)|$. If $32 = |F(G)|$, then we are done. Hence we may assume that $O_p(F(G)) \neq \{1\}$, for some $p$ dividing 155. Thus $G/O_p(G)$ is a $B$-group of order $5 \cdot 31 \cdot 2^5 \cdot p^{-1}$. In such a group $O_2(F(G/O_p(G)))$ must be a minimal normal subgroup of order 32. It is forced then that $p = 5$. Note, however, that a group $G/O_5(G)$ or order $31 \cdot 2^5$ cannot be a $B$-group in case $G/O_3(G)$ has elementary abelian Sylow 2-subgroups.
Therefore there remains to investigate $F(G) \equiv C_t$ with $t$ dividing 155. As $G$ is solvable, it holds that $F(G) = C_G(F(G))$ and $G/C_G(F(G))$ is isomorphic to a subgroup of Aut($C_t$). Hence $|G/C_G(F(G))|$ divides the least common multiple of the integers 4, 30 and 120, i.e. $|G/C_G(F(G))|$ divides 120. This contradicts the actual value 32 of the 2-part of $|G/F(G)|$ in case $F(G) \equiv C_t$, $t \mid 155$.

The proof of the Theorem is complete.  

§3. PROPERTIES OF B-GROUPS

Let $G$ be a solvable $B$-group. Suppose that $L/K$ is an elementary abelian chief section of order $p'$, $p$ prime, $t \geq 2$, $p' \neq 4$. We will show that there exists an elementary abelian minimal normal subgroup $M$ of $G$ of order $p'$. Moreover, we know that in that case $M$ is the unique Sylow $p$-subgroup of the $B$-group $G$ and that, of course, $M \leq F(G)$. Namely, let $K \neq \{I\}$. Then there exists a minimal normal subgroup $U$ of $G$, contained in $K$. As $G/U$ is a $B$-group, we see that by induction there exists $V \not\leq G$ with $U \leq V$ such that $V/U \equiv L/K$. As $G/U$ is a $B$-group, it holds that $V/U$ is a chief section of $G$. Consider $V \cong C_G(U) \cap V \geq U$. Then $V/(C_G(U) \cap V) \equiv V C_G(U)/C_G(U)$. Now $U$ can be cyclic of prime order, or $U$ satisfies the hypothesis of one of the Theorems 3, 6, 7 and 8. If $U \neq C_2$, then it follows from these theorems that $V C_G(U)/C_G(U)$ must be a cyclic $p$-group, whence that $V = C_G(U) \cap V$ and so $V \leq C_G(U)$. If $U \equiv C_2$, then we see that not only $V \leq C_G(U)$ but that even $V - A \times U$, where $A \equiv L/K$ with $p$ odd, due to $p' \neq 4$ and to Theorem 5 of [1]; in this case $A$ is characteristic in $V$, so that $A$ is the subgroup $M$ as required. In the former case $U \neq C_2$ and $V \leq C_G(U)$, Theorem 2 of this paper combined with the Theorems 2.d) and 2.e) of [1] and the Schur-Zassenhaus theorem yields now that $V = A \times U$, where $(|A|, |U|) = 1$ and $A \equiv L/K$. So here $A$ is the required subgroup $M$. When we also take the Theorems 9 and 11 of [1] into account, then it turns out that the following theorem holds.

**Theorem 10.** Let $G$ be a $B$-group. Suppose that $G$ has an elementary abelian chief section of order $p'$, $p$ prime, $t \geq 2$.

Assume $p' \neq 4$. Then $G$ has a unique Sylow $p$-subgroup $S$ and so $S$ is an elementary abelian minimal normal subgroup of $G$ of order $p'$, whence also $p = 2$, $t = 3$ or $p = 2$, $t = 5$ or $p$ odd, $t = 2$ or $p$ odd, $t = 3$.

If $p' = 4$, then we have two possibilities. Either $G$ has an elementary abelian minimal normal subgroup of order 4 (and so $4 \mid |G|$) or $G$ has Sylow 2-subgroups $Q$ isomorphic to the quaternion group of order 8. In both of these cases with $p' = 4$, it holds that $G$ is solvable. In the latter case where $Q \in \text{Syl}_2(G)$, we get that either $Q \leq G$ or otherwise there exists $L \leq K \leq G$ with $L \leq G$ such that $K/L \cong Q$, $L \cong C_5 \times C_5$ or $L = C_{11} \times C_{11}$ such that $G/C_G(L) \equiv \text{SL}(2, 3)$ in case $L \cong C_5 \times C_5$ or $G/C_G(L) \equiv \text{SL}(2, 3) \times C_a$ with $a = 1$ or $a = 5$ in case $L \cong C_{11} \times C_{11}$.

**Proof.** The case $p' \neq 4$ has been considered earlier. So let $p' = 4$. Let $V/U$ be a chief section of $G$ of order 4. Then the $B$-group $G/U$ is solvable by Theorem 2.e) of [1], as $G/V$ is apparently of odd order. Further it follows from Theorem

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2.d) of [1] and from Theorem 4, that $2 \parallel |U|$ or that $2 \not\parallel |U|$. In both cases $U$ is 2-nilpotent (by [4], Satz IV.2.8) and so $U$ is solvable. Therefore $G$ is solvable. Now suppose that $4 \parallel |G|$. Let $R \neq \{1\}$ be a minimal normal subgroup of $G$, contained in $U$. As $R$ is of odd prime power order, it follows from Theorems 3, 6, 7 and 8 that $V \leq C_G(U)$ after the inductive argumentation that $U$ may be chosen equal to $R$ (note $G/R$ is a $B$-group). Hence the Schur–Zassenhaus theorem implies that there exists $M \triangleleft G$, $M \cong C_2 \times C_2$, $M \in \text{Syl}_2(G)$. Next consider the case where $Q \in \text{Syl}_2(G)$. Let $V/U$ be an elementary abelian chief section of order 4. Again $4 \parallel |G/U|$, and also $U$ is 2-nilpotent. There exists $S < U$, $S \triangleleft G$, $|U/S| = 2$, $S$ of odd order. We may assume that $S \neq \{1\}$. Let $M$ be a minimal normal subgroup of $G$ contained in $S$. Now $G/M$ is a $B$-group, so that by induction there are two situations to consider.

1) Firstly, there exists (by lack of symbols) $V \leq G$, $U \leq G$, $S \triangleleft G$, $S = M$, $V/M \cong Q$, $M < U < V$, $V/U \cong C_2 \times C_2$. If $M \cong C_5 \times C_3$ or $M \cong C_{11} \times C_{11}$, then the theorem holds by elaborating the contents of Theorem 8. So assume $M \cong C_5 \times C_3$ and $M \cong C_{11} \times C_{11}$. Then $M$ is either cyclic of prime order or $M$ satisfies either hypothesis of the Theorems 3, 6, 7 or 8. Note that $M$ is of odd order, anyway. Because of the Theorems 3, 6, 7 and 8 we see that the (meta)cyclic structure of $G/C_G(M)$ yields that $M$ is central in $V$. Therefore the Schur–Zassenhaus theorem implies that $V = A \times M$, where $A \cong Q$. Hence, as $A$ is characteristic in $V$, it holds that $A \triangleleft G$, as was to be proved.

2) Secondly, there exists (by lack of symbols) $V \leq G$, $U \leq G$, $S \triangleleft G$, $M \triangleleft S$ with $M < S < U < V$ and $V/S \cong Q$, $V/U \cong C_2 \times C_2$, $S/M \cong C_3 \times C_5$ or $S/M \cong C_5 \times C_{11}$, but with $Q$ not isomorphic to some normal subgroup of $G/M$. Here the order of $M$ is odd too. So the $B$-group property reveals that either $M$ is cyclic of odd prime order, or $M$ satisfies either hypothesis of the Theorems 3, 6, 7 or 8. So we see that therefore $G/C_G(M)$ is (meta)cyclic, but there can be an exception, namely $M \cong C_{11} \times C_{11}$ in case $S/M \cong C_5 \times C_5$, or $M \cong C_3 \times C_5$ in case $S/M \cong C_{11} \times C_{11}$. Whenever $G/C_G(M)$ is (meta)cyclic it holds that $M$ is central in $V$ and it is not difficult to see that $V/M$ splits over $M$, but here is indeed a little point. Anyway, we see that if $G/C_G(M)$ is (meta)cyclic, the conclusion $V = A \times M$, where $A \cong Q \times (C_3 \times C_3)$ with $r = 5$ or $r = 11$ and $M \cong C_3 \times C_3$ with $rs = 55$ holds, so that $A \triangleleft G$ as wanted. Thus there remains the case $S/M \cong C_r \times C_r$, $M \cong C_3 \times C_3$, $rs > 1$, $rs = 55$. Look at $G/C_G(M)$ and consider $C_G(M) \cap S$: view Theorem 8. As $S/M$ is a chief section of $G$ (remember $G/M$ is a $B$-group!), it holds then that $G/C_G(M) \cong SC_G(M)/C_G(M) \cong S/(S \cap C_G(M))$. Hence we have, by Theorem 8, that $C_G(M) \supsetneq S$, i.e. $S \cong C_2 \times C_3 \times C_{11} \times C_{11}$, by the Schur–Zassenhaus theorem. Put $X \leq S$, $X \cong C_2 \times C_3$, $Y \cong C_{11} \times C_{11}$. Then $C_G(S) = C_G(X) \cap C_G(Y)$. The group $S$ contains seventy-two subgroups of order 55. Let $D < S$, $|D| = 55$. These subgroups have to be permuted inter alia, i.e. $|G:N_G(D)| = 72$, where $D \triangleleft G$, $|D| = 55$ (note that $S$ is a Hall $\{5, 11\}$-subgroup of $G$). It holds here that there exists $\tau \in U$, $|\tau| = 2$ inverting each element of $S$. As $8 \mid |G|$, the conclusion is that the seventy-two subgroups of $G$ of order 55 will not be permuted transitively under conjugation by the elements.
of $G$ (note $\tau \in N_G(D)$, so that $8 \not| (G : N_G(D))$). Hence the exceptional case in which $V$ acts nontrivially on $M$ does not exist.

The Theorem has been proved. \hfill $\square$

§4. SUFFICIENT CONDITIONS FOR A SOLVABLE GROUP TO BE A $B$-GROUP

We have seen that the structure of the Fitting subgroup of a $B$-group is of a rather restricted nature. Now we will describe a converse situation suggested by the outcome of the Theorems 6, 8 and 10.

**Theorem 11.** Let $G$ be a solvable group. Assume the following.

(a) Any non-cyclic Sylow subgroup of $G$ is normal in $G$, and  
(b) Any two subgroups of $G$ of equal order that are contained in the Fitting subgroup of $G$, are conjugate in $G$.

Then $G$ is a $B$-group.

**Proof.** Suppose $H$ and $K$ are subgroups of $G$ of equal order. We will show that $H$ and $K$ are conjugate in $G$.

(i) Suppose $G$ contains at least one non-cyclic Sylow subgroup. Then $X \not= \{1\}$, where by definition $X$ stands for the product of all the non-cyclic Sylow subgroups of $G$; note that $X \leq F(G)$ by (a) and that $X \unlhd G$. Hence $X \cap H$ is a Hall subgroup of $H$ and $X \cap K$ is a Hall subgroup of $K$. Observe $|X \cap H| = |X \cap K|$. Thus by (b), $X \cap H$ and $X \cap K$ are conjugate in $G$ and so we can and we will assume from not on that (by replacing $H$ by one of its conjugates if necessary) that $X = X \cap H = X \cap K$.

Now it is true that $G/X$ and all of its subgroups are $B$-groups. Indeed, $G/X$ does not contain non-cyclic Sylow subgroups. This means that every subgroup of $G/X$ is itself a $B$-group, as shown in ([3], Corollary to Theorem 3).

Next we unravel the structure of $X$ somewhat more. It is not difficult to see (by (b) and ([4], Sätze 1.6.9 and III.8.2) that a (normal) Sylow $p$-subgroup $P$ of $X$ is elementary abelian when $p$ is odd, and that $P$ is either an elementary abelian 2-group or a generalized quaternion group $Q$ with $|Q| = 8$; note that $Q$ is not an iso-group when $|Q| \geq 16$. Therefore $X$ is a Hamilton group, i.e. each subgroup of $X$ is a normal subgroup of $X$. In particular $H \cap X \leq X$.

Put $T = N_G(H \cap X)$, whence $H$, $K$ and $X$ are all contained in $T$. Since $X$ satisfies $|X| = |G/X|$ = 1, the Schur-Zassenhaus theorem yields that there exists $L \leq T$ with $T = LX$ and $L \cap X = \{1\}$ and also that, as $(|H \cap X|, |H/(H \cap X)|) = (1, (|K \cap X|, |K/(K \cap X)|))$, there exists $I \leq H$ with $H = I(H \cap X)$ and $I \cap H \cap X = \{1\}$ and $J \leq K$ with $K = J(H \cap X)$ and $J \cap H \cap X = \{1\}$. So $|I| = |J|$ and $|I|$ divides $|L|$. Since $L$ is a Hall subgroup of $T$ we see that, by Hall’s theorem ([4], Hauptsatz VI.1.8), we are able to replace $H$ and $K$ by $T$-conjugates if necessary, in order to achieve that $L$ shall contain both $I$ and $J$. Above it was shown that $L$, being isomorphic to a subgroup of $G/X$, is itself a $B$-group. Therefore $I$ and $J$ are $L$-conjugate, i.e. there exists $x \in L$ with $J = I^x$. So $H^x = I^x(H \cap X)^x = I^x(H \cap X) = J(H \cap X) = J(K \cap X) = K$, and we are done.

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(2) Finally assume that all Sylow subgroups of $G$ are cyclic. Then $G$ is a $B$-group; see again ([3], Corollary to Theorem 3).

The proof of the Theorem is complete.

Remark. In Theorem 10 it is stated that a solvable $B$-group might have non-normal Sylow 2-subgroups isomorphic to the quaternion group of order eight. Presumably a theorem can be exhibited in the spirit of Theorem 11, dealing with the just mentioned situation.

REFERENCES