J. Math. Anal. Appl. 349 (2009) 245–258

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Diffraction of sound waves by a finite barrier in a moving fluid

M. Ayub *, R. Nawaz, A. Naeem

Department of Mathematics, Quaid-i-Azam University, 45320, Islamabad 44000, Pakistan

ARTICLE INFO

Article history:
Received 2 February 2008
Available online 2 September 2008
Submitted by P. Broadbridge

Keywords:
Diffraction
Absorbing finite barrier
Wiener–Hopf technique
Far field

ABSTRACT

The diffraction of a line source by an absorbing finite barrier, satisfying Myers' impedance condition [M.K. Myers, On the acoustic boundary condition in the presence of flow, J. Sound Vibration 71 (1980) 429–434] in the presence of a subsonic flow is studied. The problem is solved analytically by using Integral transforms, Wiener–Hopf technique and the asymptotic methods. The expression for the diffracted field is shown to be the sum of the fields produced by the two edges of the strip and a field due to the interaction of the two edges. The diffracted field in the far zone is determined by the method of steepest decent.

1. Introduction

Noise reduction by barriers has become a common measure of environmental protection. A practical method for the reduction of noise radiated from the aero-engines and inside of wind tunnels is to use absorbing linings. It is therefore desirable to have theoretical work which attempts to determine how effectively the sound radiation is reduced by absorbing lining in the presence of fluid flow. Clearly, the radiated sound will be a complicated function of the Mach number and the absorptive properties of the surface scattering the sound waves and an analysis of this dependence is a problem of some significance. It is also worth mentioning that the use of absorbing barrier particularly for the reduction of traffic noise has received much attention in recent years (see list of references in Butler [1]). The present work arose in connection with noise reduction by means of barriers. It is hoped that the present work would have possible applications in noise abatement.

Rawlins [2] discussed the sound scattered by semi-infinite absorbing plane due to a cylindrical acoustic wave, satisfying Ingard's condition [3] in a moving fluid. Later on Asghar et al. [4] extended Rawlin's idea to calculate the diffraction of spherical acoustic wave from an absorbing plane. But there has been a considerable discussion in the literature regarding the proper impedance boundary conditions on acoustic field in the presence of fluid flow. Effects of moving medium was first correctly given by Miles [5] and Ribner [6] for a plane interface of relative motion. Ingard [3] discussed the effect of flow on boundary conditions at a plane impedance surface. Later on Myers [7] discussed the diffraction of cylindrical acoustic waves, by a semi-infinite absorbing plane, which was in fact generalization of the Ingard's condition [3]. Now a days, Myers' condition [7] is the accepted form of the boundary condition for impedance walls with flow.

Diffraction from a strip is a well-studied phenomenon. Many scientists worked on diffraction problems related to strip geometry [8–11]. In this paper, we have investigated the diffraction of an acoustic wave from a finite absorbing barrier by using Myers' impedance boundary condition. The integral transforms, Wiener–Hopf technique [9,12] and asymptotic methods [13] are used to calculate the diffracted field. It is found that the two edges of the finite barrier give rise to two diffracted fields (one from each edge) and to an interaction field. The results for the half plane [14] can be recovered by taking an appropriate limit.

* Corresponding author.
E-mail address: mayub59@yahoo.com (M. Ayub).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.08.044

2. Formulation of the problem

We consider the diffraction of an acoustic wave incident on the finite absorbing plane occupying a space $y = 0$, $-l \leqslant x \leqslant 0$. The line source is located at (x_0, y_0). The system is placed in a fluid moving with subsonic velocity U parallel to the x-axis. The time dependence is taken to be of harmonic type $e^{-i\omega t}$ (ω is the angular frequency) and the plane is assumed to be satisfying the Myers' impedance condition [7]

$$u_n = \frac{-p}{Z_a} + \frac{U}{i\omega Z_a}\frac{\partial p}{\partial x},$$

where u_n is the normal derivative of the perturbation velocity, p is the surface pressure, Z_a is the acoustic impedance of the surface and $-n$ a normal pointing from the fluid into the surface. The perturbation velocity $\mathbf{u}$ of the irrotational sound wave can be written in terms of the velocity potential Φ as $\mathbf{u} = \nabla\Phi$. Then the resulting pressure p of the sound field is

$$p = -\rho_0\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\Phi,$$

where ρ_0 is the density of the undisturbed stream.

Thus, we have to solve the following boundary value problem

$$\left[\left(1-M^2\right)\frac{\partial^2}{\partial x^2} + 2ikM\frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2} + k^2\right]\Phi(x,y) = \delta(x-x_0)\delta(y-y_0), \tag{1}$$

subject to the following boundary conditions

$$\left[\frac{\partial}{\partial y} \mp 2\beta M\frac{\partial}{\partial x} \pm ik\beta \mp \frac{i\beta M^2}{k}\frac{\partial^2}{\partial x^2}\right]\Phi\left(x,0^{\pm}\right) = 0, \quad -l < x < 0, \tag{2}$$

$$\frac{\partial}{\partial y}\Phi\left(x,0^+\right) = \frac{\partial}{\partial y}\Phi\left(x,0^-\right), \quad -\infty < x < -l,\ x > 0,$$

$$\Phi\left(x,0^+\right) = \Phi\left(x,0^-\right), \quad -\infty < x < -l,\ x > 0. \tag{3}$$

In above equations $k = \frac{\omega}{c}$, which is the wave number, $\beta = \frac{\rho_0 c}{Z_a}$ is the specific complex admittance and $M = \frac{U}{c}$ where M is the Mach number and c is the velocity of sound. It is assumed that the flow is subsonic, i.e., $|M| < 1$, and $\operatorname{Re}\beta > 0$, a necessary condition for an absorbing surface.

3. Wiener–Hopf equations

Since we are dealing with subsonic flow, so we can make the following real substitutions

$$x = \sqrt{1-M^2}X, \qquad x_0 = \sqrt{1-M^2}X_0, \qquad y = Y, \qquad y_0 = Y_0, \qquad \beta = \sqrt{1-M^2}B, \qquad k = \sqrt{1-M^2}K, \tag{4}$$

writing

$$\Phi(x,y) = \psi^{(t)}(X,Y)e^{-iKMX}, \tag{5}$$

and using the relation (4), Eqs. (1) to (3) can be written as

$$\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + K^2\right)\psi^{(t)}(X,Y) = \frac{\delta(X-X_0)\delta(Y-Y_0)}{\sqrt{1-M^2}}e^{-iKMX_0}, \tag{6}$$

$$\left[\frac{\partial}{\partial Y} \mp 2BM\frac{\partial}{\partial X} \pm iKB\left(1+M^2\right) \mp \frac{iBM^2}{(1-M^2)K}\frac{\partial^2}{\partial x^2}\right]\psi^{(t)}\left(X,0^{\pm}\right) = 0, \quad -l < X < 0, \tag{7}$$

$$\frac{\partial}{\partial Y}\psi^{(t)}\left(X,0^+\right) = \frac{\partial}{\partial Y}\psi^{(t)}\left(X,0^-\right), \quad -\infty < X < -l,\ X > 0,$$

$$\psi^{(t)}\left(X,0^+\right) = \psi^{(t)}\left(X,0^-\right), \quad -\infty < X < -l,\ X > 0. \tag{8}$$

The total velocity potential may be expressed as

$$\psi^{(t)}(X,Y) = \Psi(X,Y) + \Psi_0(X,Y), \tag{9}$$

where $\Psi_0(X,Y)$ is the incident field (corresponding to the inhomogeneous equation), $\Psi(X,Y)$ is the diffracted field (corresponding to the homogeneous equation). Thus $\Psi_0(X,Y)$ and $\Psi(X,Y)$ satisfy the following equations

$$\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + K^2\right)\Psi_0(X,Y) = \frac{\delta(X-X_0)\delta(Y-Y_0)}{\sqrt{1-M^2}}e^{-iKMX_0}, \tag{10}$$

$$\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + K^2\right)\Psi(X,Y) = 0. \tag{11}$$

The solution of (10) can be obtained by Green's function method

$$\Psi_0(X,Y)=\frac{a}{4i}H_0^1(KR)=\frac{a}{4\pi i}\int_{-\infty}^{\infty}\frac{1}{\kappa}e^{i[\alpha(X-X_0)+\kappa|Y-Y_0|]}\,d\alpha, \tag{12}$$

where $a=\frac{e^{iKMX_0}}{\sqrt{1-M^2}}$, $R=\sqrt{(X-X_0)^2+(Y-Y_0)^2}$, $\kappa=\sqrt{K^2-\alpha^2}$, K is the wave number, and α is the Fourier transform variable. Now we introduce the spatial Fourier transform over the variable X by

$$\bar\Psi(\alpha,Y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi(X,Y)e^{-i\alpha X}\,dX=\bar\Psi_+(\alpha,Y)+e^{-i\alpha l}\bar\Psi_-(\alpha,Y)+\bar\Psi_1(\alpha,Y), \tag{13}$$

where

$$\bar\Psi_+(\alpha,Y)=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\Psi(X,Y)e^{i\alpha X}\,dX,$$

$$\bar\Psi_-(\alpha,Y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-l}\Psi(X,Y)e^{i\alpha(X+l)}\,dX,$$

$$\bar\Psi_1(\alpha,Y)=\frac{1}{\sqrt{2\pi}}\int_{-l}^{0}\Psi(X,Y)e^{i\alpha X}\,dX, \tag{14}$$

where $\bar\Psi_-(\alpha,Y)$ is regular for $\operatorname{Im}\alpha<\operatorname{Im}K$, and $\bar\Psi_+(\alpha,Y)$ is regular for $\operatorname{Im}\alpha>-\operatorname{Im}K$ while $\bar\Psi_1(\alpha,Y)$ is an integral function and therefore $\bar\Psi_1(\alpha,Y)$ is analytic in the common region $-\operatorname{Im}K<\alpha<\operatorname{Im}K$.

On taking the Fourier transform of Eq. (11), we obtain

$$\left(\frac{d^2}{dY^2}+\kappa^2\right)\bar\Psi(\alpha,Y)=0, \tag{15}$$

where $\kappa=\sqrt{K^2-\alpha^2}$. Thus, the Fourier transform of Eqs. (7) and (8) gives

$$\begin{aligned}
&\bar\Psi_-(\alpha,0^+)=\bar\Psi_-(\alpha,0^-)=\bar\Psi_-(\alpha,0),\\
&\bar\Psi_+(\alpha,0^+)=\bar\Psi_+(\alpha,0^-)=\bar\Psi_+(\alpha,0),\\
&\bar\Psi'_-(\alpha,0^+)=\bar\Psi'_-(\alpha,0^-)=\bar\Psi'_-(\alpha,0),\\
&\bar\Psi'_+(\alpha,0^+)=\bar\Psi'_+(\alpha,0^-)=\bar\Psi'_+(\alpha,0),
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
\bar\Psi_1'(\alpha,0^+)={}&-2i\alpha MB\left[\bar\Psi_1(\alpha,0^+)+\bar\Psi_0(\alpha,0)\right]-iKB(1+M^2)\left[\bar\Psi_1(\alpha,0^+)+\bar\Psi_0(\alpha,0)\right]\\
&-\frac{iBM^2\alpha^2}{(1-M^2)K}\left[\bar\Psi_1(\alpha,0^+)+\bar\Psi_0(\alpha,0)\right]-\bar\Psi_0'(\alpha,0),
\end{aligned} \tag{17}$$

$$\begin{aligned}
\bar\Psi_1'(\alpha,0^-)={}&2i\alpha MB\left[\bar\Psi_1(\alpha,0)+\bar\Psi_0(\alpha,0)\right]+iKB(1+M^2)\left[\bar\Psi_1(\alpha,0^-)+\bar\Psi_0(\alpha,0)\right]\\
&+\frac{iBM^2\alpha^2}{(1-M^2)K}\left[\bar\Psi_1(\alpha,0^-)+\bar\Psi_0(\alpha,0)\right]-\bar\Psi_0'(\alpha,0).
\end{aligned} \tag{18}$$

The solution of Eq. (15) satisfying radiation condition is given by

$$\bar\Psi(\alpha,Y)=\begin{cases}A_1(\alpha)e^{i\kappa Y} & \text{if } Y\geqslant 0,\\ A_2(\alpha)e^{-i\kappa Y} & \text{if } Y<0.\end{cases} \tag{19}$$

Using Eqs. (13) and (16) in Eq. (19), we get

$$\bar\Psi_+(\alpha,0)+e^{-i\alpha l}\bar\Psi_-(\alpha,0)+\bar\Psi_1(\alpha,0^+)=A_1(\alpha), \tag{20a}$$

$$\bar\Psi_+(\alpha,0)+e^{-i\alpha l}\bar\Psi_-(\alpha,0)+\bar\Psi_1(\alpha,0^-)=A_2(\alpha), \tag{20b}$$

$$\bar\Psi'_+(\alpha,0)+e^{-i\alpha l}\bar\Psi'_-(\alpha,0)+\bar\Psi'_1(\alpha,0^+)=i\kappa A_1(\alpha), \tag{20c}$$

$$\bar\Psi'_+(\alpha,0)+e^{-i\alpha l}\bar\Psi'_-(\alpha,0)+\bar\Psi'_1(\alpha,0^-)=-i\kappa A_2(\alpha). \tag{20d}$$

Subtracting (20b) from (20a) and (20d) from (20c), we obtain

$$A_1(\alpha) - A_2(\alpha) = \bar{\Psi}_1(\alpha, 0^+) - \bar{\Psi}_1(\alpha, 0^-) = 2J_1(\alpha, 0), \tag{21}$$

and

$$A_1(\alpha) + A_2(\alpha) = \frac{1}{i\kappa}\left[\bar{\Psi}_1'(\alpha, 0^+) - \bar{\Psi}_1'(\alpha, 0^-)\right] = \frac{2J_1'(\alpha, 0)}{i\kappa}, \tag{22}$$

where

$$J_1(\alpha, 0) = \frac{1}{2}\left[\bar{\Psi}_1(\alpha, 0^+) - \bar{\Psi}_1(\alpha, 0^-)\right], \tag{23}$$

and

$$J_1'(\alpha, 0) = \frac{1}{2}\left[\bar{\Psi}_1'(\alpha, 0^+) - \bar{\Psi}_1'(\alpha, 0^-)\right]. \tag{24}$$

By adding and subtracting Eqs. (21) and (22), we get

$$A_1(\alpha) = J_1(\alpha, 0) + \frac{J_1'(\alpha, 0)}{i\kappa}, \tag{25}$$

and

$$A_2(\alpha) = -J_1(\alpha, 0) + \frac{J_1'(\alpha, 0)}{i\kappa}. \tag{26}$$

Making use of (20a) in (20c) and (20b) in (20d), we can write

$$\bar{\Psi}_+'(\alpha, 0) + e^{-i\alpha l}\bar{\Psi}_-'(\alpha, 0) + \bar{\Psi}_1'(\alpha, 0^+) = i\kappa\left[\bar{\Psi}_+(\alpha, 0) + e^{-i\alpha l}\bar{\Psi}_-(\alpha, 0) + \bar{\Psi}_1(\alpha, 0^+)\right], \tag{27a}$$

$$\bar{\Psi}_+'(\alpha, 0) + e^{-i\alpha l}\bar{\Psi}_-'(\alpha, 0) + \bar{\Psi}_1'(\alpha, 0^-) = -i\kappa\left[\bar{\Psi}_+(\alpha, 0) + e^{-i\alpha l}\bar{\Psi}_-(\alpha, 0) + \bar{\Psi}_1(\alpha, 0^-)\right]. \tag{27b}$$

By eliminating $\bar{\Psi}_1'(\alpha, 0^+)$ from (27*a*) and (17) and $\bar{\Psi}_1'(\alpha, 0^-)$ from (27b) and (18), we get

$$\begin{aligned}
&\bar{\Psi}_+'(\alpha, 0) + e^{-i\alpha l}\bar{\Psi}_-'(\alpha, 0) - 2i\alpha MB\left[\bar{\Psi}_1(\alpha, 0^+) + \bar{\Psi}_0(\alpha, 0)\right] - iKB\left(1 + M^2\right)\left[\bar{\Psi}_1(\alpha, 0^+) + \bar{\Psi}_0(\alpha, 0)\right] \\
&\quad - \frac{iBM^2\alpha^2}{(1 - M^2)K}\left[\bar{\Psi}_1(\alpha, 0^+) + \bar{\Psi}_0(\alpha, 0)\right] - \bar{\Psi}_0'(\alpha, 0) \\
&\quad = i\kappa\left[\bar{\Psi}_+(\alpha, 0) + e^{-i\alpha l}\bar{\Psi}_-(\alpha, 0) + \bar{\Psi}_1(\alpha, 0^+)\right],
\end{aligned} \tag{28a}$$

and

$$\begin{aligned}
&\bar{\Psi}_+'(\alpha, 0) + e^{-i\alpha l}\bar{\Psi}_-'(\alpha, 0) + 2i\alpha MB\left[\bar{\Psi}_1(\alpha, 0) + \bar{\Psi}_0(\alpha, 0)\right] + iKB\left(1 + M^2\right)\left[\bar{\Psi}_1(\alpha, 0^-) + \bar{\Psi}_0(\alpha, 0)\right] \\
&\quad + \frac{iBM^2\alpha^2}{(1 - M^2)K}\left[\bar{\Psi}_1(\alpha, 0^-) + \bar{\Psi}_0(\alpha, 0)\right] - \bar{\Psi}_0'(\alpha, 0) \\
&\quad = -i\kappa\left[\bar{\Psi}_+(\alpha, 0) + e^{-i\alpha l}\bar{\Psi}_-(\alpha, 0) + \bar{\Psi}_1(\alpha, 0^-)\right].
\end{aligned} \tag{28b}$$

By adding Eqs. (28a) and (28b), we get

$$\bar{\Psi}_+'(\alpha, 0) + e^{-i\alpha l}\bar{\Psi}_-'(\alpha, 0) - i\kappa L(\alpha)J_1(\alpha) = \bar{\Psi}_0'(\alpha, 0). \tag{29}$$

Similarly, by eliminating $\bar{\Psi}_1(\alpha, 0^+)$ from (27a) and (17) and $\bar{\Psi}_1(\alpha, 0^-)$ from (27b) and (18), and then subtracting the resulting equations, we get

$$\bar{\Psi}_+(\alpha, 0) + e^{-i\alpha l}\bar{\Psi}_-(\alpha, 0) + \frac{iL(\alpha)J_1'(\alpha, 0)}{[(2\alpha MB + KB(1 + M^2) + \frac{BM^2\alpha^2}{(1-M^2)K})]} = \bar{\Psi}_0(\alpha, 0), \tag{30}$$

where

$$L(\alpha) = 1 + \frac{B}{\kappa}\left[\left(2\alpha M + K\left(1 + M^2\right) + \frac{M^2\alpha^2}{(1 - M^2)K}\right)\right]. \tag{31}$$

Eqs. (29) and (30) are the Wiener–Hopf equations. We proceed to find the solutions for these equations.

4. Solution of the Wiener–Hopf equations

For the solution of the Wiener–Hopf equations (29) and (30), one can use the following factorizations [14]

$$L(\alpha)=1+\frac{B}{\kappa}\left[\left(2\alpha M+K\left(1+M^2\right)+\frac{M^2\alpha^2}{(1-M^2)K}\right)\right]=L_+(\alpha)L_-(\alpha) \tag{32a}$$

and

$$\kappa(\alpha)=\kappa_+(\alpha)\kappa_-(\alpha), \tag{32b}$$

where $L_+(\alpha)$ and $\kappa_+(\alpha)$ are regular for $\operatorname{Im}\alpha > -\operatorname{Im}K$, i.e., for upper half plane and $L_-(\alpha)$ and $\kappa_-(\alpha)$ are regular for $\operatorname{Im}\alpha < \operatorname{Im}K$, i.e., lower half plane. Putting the value of $J_1(\alpha,0)$ and $J_1'(\alpha,0)$ from Eqs. (29) and (30) into (25) and (26), we get

$$\begin{aligned}A_1(\alpha)={}&\frac{1}{i\kappa L(\alpha)}\left(\bar\Psi_+'(\alpha,0)+e^{-i\alpha l}\bar\Psi_-'(\alpha,0)-\bar\Psi_0'(\alpha,0)\right)\\&+\frac{B[(2\alpha M+K(1+M^2)+\frac{M^2\alpha^2}{(1-M^2)K})]}{\kappa L(\alpha)}\left\{\bar\Psi_+(\alpha,0)+e^{-i\alpha l}\bar\Psi_-(\alpha,0)-\bar\Psi_0(\alpha,0)\right\},\end{aligned} \tag{33}$$

and

$$\begin{aligned}A_2(\alpha)={}&-\frac{1}{i\kappa L(\alpha)}\left(\bar\Psi_+'(\alpha,0)+e^{-i\alpha l}\bar\Psi_-'(\alpha,0)-\bar\Psi_0'(\alpha,0)\right)\\&+\frac{B[(2\alpha M+K(1+M^2)+\frac{M^2\alpha^2}{(1-M^2)K})]}{\kappa L(\alpha)}\left\{\bar\Psi_+(\alpha,0)+e^{-i\alpha l}\bar\Psi_-(\alpha,0)-\bar\Psi_0(\alpha,0)\right\}.\end{aligned} \tag{34}$$

For a plane wave incidence, one can assume that

$$\Psi_0(X,Y)=e^{-iKX\cos\theta_0-iKY\sin\theta_0}.$$

Then after taking the Fourier transform, we get

$$\bar\Psi_0(\alpha,0)=\frac{1}{\sqrt{2\pi}}\left[\frac{1-e^{-il(\alpha-K\cos\theta_0)}}{i(\alpha-K\cos\theta_0)}\right], \tag{35}$$

and

$$\bar\Psi_0'(\alpha,0)=\frac{b}{\sqrt{2\pi}}\left[\frac{1-e^{-il(\alpha-K\cos\theta_0)}}{(\alpha-K\cos\theta_0)}\right], \tag{36}$$

where $b=-iK\sin\theta_0$. Making use of Eqs. (32a), (32b), (35) and (36) in Eqs. (29) and (30), we obtain

$$\bar\Psi_+'(\alpha,0)+e^{-i\alpha l}\bar\Psi_-'(\alpha,0)+S_+(\alpha)S_-(\alpha)J_1(\alpha)=\frac{b}{\sqrt{2\pi}}\left[\frac{1-e^{-il(\alpha-K\cos\theta_0)}}{(\alpha-K\cos\theta_0)}\right], \tag{37}$$

and

$$\bar\Psi_+(\alpha,0)+e^{-i\alpha l}\bar\Psi_-(\alpha,0)+\frac{iL_+(\alpha)L_-(\alpha)J_1'(\alpha,0)}{[\{2\alpha MB+KB(1+M^2)+\frac{BM^2\alpha^2}{(1-M^2)K}\}]}=\frac{1}{\sqrt{2\pi}}\left[\frac{1-e^{-il(\alpha-K\cos\theta_0)}}{i(\alpha-K\cos\theta_0)}\right], \tag{38}$$

where

$$S(\alpha)=-i\kappa(\alpha)L(\alpha)=S_+(\alpha)S_-(\alpha), \tag{39}$$

and $S_+(\alpha)$ and $S_-(\alpha)$ are regular in upper and lower half planes, respectively. Equations of type (37) and (38) have been considered by Noble [9] and a similar analysis may be employed to obtain an approximate solution for large K. Thus, we follow the procedure given in [9, Section 5.5, p. 196].

Dividing Eq. (37) by $S_+(\alpha)$ on both the sides, we get

$$\frac{\bar\Psi_+'(\alpha,0)}{S_+(\alpha)}+e^{-i\alpha l}\frac{\bar\Psi_-'(\alpha,0)}{S_+(\alpha)}+S_-(\alpha)J_1(\alpha)=\frac{b}{\sqrt{2\pi}}\left[\frac{1-e^{-il(\alpha-K\cos\theta_0)}}{S_+(\alpha)(\alpha-K\cos\theta_0)}\right]. \tag{40}$$

The first term on the left-hand side of above equation is regular in the upper half plane. Hence, we can write

$$\frac{e^{-i\alpha l}\bar\Psi_-'(\alpha,0)}{S_+(\alpha)}=U_+(\alpha)+U_-(\alpha), \tag{41}$$

and

$$\frac{b}{\sqrt{2\pi}S_+(\alpha)}\left[\frac{e^{-il(\alpha-K\cos\theta_0)}}{(\alpha-K\cos\theta_0)}\right]=V_+(\alpha)+V_-(\alpha). \tag{42}$$

Using Eqs. (41) and (42) in Eq. (37) and also adding the pole contribution on both sides of the resulting equation, we get

$$\begin{aligned}&\frac{\bar{\Psi}'_+(\alpha,0)}{S_+(\alpha)}+U_+(\alpha)+V_+(\alpha)-\frac{b}{\sqrt{2\pi}S_+(\alpha)(\alpha-K\cos\theta_0)}\left[\frac{1}{S_+(\alpha)}-\frac{1}{S_+(K\cos\theta_0)}\right]\\&\quad=\frac{b}{\sqrt{2\pi}S_+(K\cos\theta_0)(\alpha-K\cos\theta_0)}-iS_-(\alpha)J_1(\alpha)-U_-(\alpha)-V_-(\alpha).\end{aligned} \tag{43}$$

Similarly, dividing Eq. (37) by $S_-(\alpha)e^{-i\alpha l}$ on both sides, we have

$$\frac{e^{i\alpha l}\bar{\Psi}'_+(\alpha,0)}{S_-(\alpha)}+\frac{\bar{\Psi}'_-(\alpha,0)}{S_-(\alpha)}+ie^{i\alpha l}S_+(\alpha)J_1(\alpha)=\frac{be^{i\alpha l}}{\sqrt{2\pi}S_-(\alpha)}\left[\frac{1-e^{-il(\alpha-K\cos\theta_0)}}{(\alpha-K\cos\theta_0)}\right]. \tag{44}$$

The second term on the left-hand side of above equation is regular in lower half plane. Therefore, we can write

$$\frac{e^{i\alpha l}\bar{\Psi}'_+(\alpha,0)}{S_-(\alpha)}=R_+(\alpha)+R_-(\alpha), \tag{45}$$

and

$$\frac{be^{i\alpha l}}{\sqrt{2\pi}S_-(\alpha)(\alpha-K\cos\theta_0)}=N_+(\alpha)+N_-(\alpha). \tag{46}$$

Using Eqs. (45) and (46) in Eq. (44), we obtain

$$\frac{\bar{\Psi}'_-(\alpha,0)}{S_-(\alpha)}+\frac{be^{ilK\cos\theta_0}}{\sqrt{2\pi}S_-(\alpha)(\alpha-K\cos\theta_0)}+R_-(\alpha)-N_-(\alpha)=ie^{i\alpha l}S_+(\alpha)J_1(\alpha)-R_+(\alpha)+N_+(\alpha). \tag{47}$$

The LHS of Eq. (43) and RHS of Eq. (47) are regular in $\tau>-K_2$ and the RHS of Eq. (43) and the LHS of Eq. (47) are regular in $\tau<K_2\cos\theta_0$. Hence using the extended form of Liouville's theorem, each side of Eqs. (43) and (47) is equal to zero, i.e.,

$$\frac{\bar{\Psi}'_+(\alpha,0)}{S_+(\alpha)}+U_+(\alpha)+V_+(\alpha)-\frac{b}{\sqrt{2\pi}S_+(\alpha)(\alpha-K\cos\theta_0)}\left[\frac{1}{S_+(\alpha)}-\frac{1}{S_+(K\cos\theta_0)}\right]=0, \tag{48}$$

and

$$\frac{\bar{\Psi}'_-(\alpha,0)}{S_-(\alpha)}+\frac{be^{ilK\cos\theta_0}}{\sqrt{2\pi}S_-(\alpha)(\alpha-K\cos\theta_0)}+R_-(\alpha)-N_-(\alpha)=0. \tag{49}$$

Using the general decomposition theorem [9], explicit expressions for $U_+(\alpha)$, $V_+(\alpha)$, $R_-(\alpha)$ and $N_-(\alpha)$ are given by

$$U_+(\alpha)=\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{e^{-i\zeta l}\bar{\Psi}'_-(\zeta,0)}{S_+(\zeta)(\zeta-\alpha)}\,d\zeta, \tag{50}$$

$$V_+(\alpha)=\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{be^{-il(\zeta-K\cos\theta_0)}}{\sqrt{2\pi}(\zeta-K\cos\theta_0)S_+(\zeta)(\zeta-\alpha)}\,d\zeta, \tag{51}$$

$$R_-(\alpha)=-\frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id}\frac{e^{i\zeta l}\bar{\Psi}'_+(\zeta,0)}{S_-(\zeta)(\zeta-\alpha)}\,d\zeta, \tag{52}$$

and

$$N_-(\alpha)=-\frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id}\frac{be^{i\zeta l}}{\sqrt{2\pi}(\zeta-K\cos\theta_0)S_-(\zeta)(\zeta-\alpha)}\,d\zeta, \tag{53}$$

where $-K_2<c<K_2\cos\theta_0$ and $-K_2<d<K_2\cos\theta_0$, also $\tau>c$ and $\tau<d$.

Using Eqs. (50) and (51) in Eq. (48), we get

$$\frac{1}{S_+(\alpha)}\left[\bar{\Psi}'_+(\alpha,0)-\frac{b}{\sqrt{2\pi}(\alpha-K\cos\theta_0)}\right]+\frac{b}{\sqrt{2\pi}S_+(K\cos\theta_0)(\alpha-K\cos\theta_0)}$$
$$+\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{e^{-i\zeta l}}{S_+(\zeta)(\zeta-\alpha)}\left[\bar{\Psi}'_-(\zeta,0)+\frac{be^{ilK\cos\theta_0}}{\sqrt{2\pi}(\zeta-K\cos\theta_0)}\right]d\zeta=0. \tag{54}$$

Using Eqs. (52) and (53) in Eq. (49), we get

$$\frac{1}{S_-(\alpha)}\left[\bar{\Psi}'_-(\alpha,0)+\frac{be^{i\alpha K\cos\theta_0}}{\sqrt{2\pi}(\alpha-K\cos\theta_0)}\right]-\frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id}\frac{e^{-i\zeta l}}{S_-(\zeta)(\zeta-\alpha)}\left[\bar{\Psi}'_+(\zeta,0)-\frac{b}{\sqrt{2\pi}(\zeta-K\cos\theta_0)}\right]d\zeta=0. \tag{55}$$

Let us introduce the following notations

$$\bar{\Psi}'_+(\alpha,0)-\frac{b}{\sqrt{2\pi}(\alpha-K\cos\theta_0)}=\bar{\Psi}^*_+(\alpha,0), \tag{56}$$

and

$$\bar{\Psi}'_-(\alpha,0)+\frac{be^{i\alpha K\cos\theta_0}}{\sqrt{2\pi}(\alpha-K\cos\theta_0)}=\bar{\Psi}^*_-(\alpha,0). \tag{57}$$

Using Eqs. (56) and (57) in Eqs. (54) and (55), we get

$$\frac{\bar{\Psi}^*_+(\alpha,0)}{S_+(\alpha)}+\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{e^{-i\zeta l}\bar{\Psi}^*_-(\zeta,0)}{S_+(\zeta)(\zeta-\alpha)}d\zeta+\frac{b}{\sqrt{2\pi}S_+(K\cos\theta_0)(\alpha-K\cos\theta_0)}=0, \tag{58}$$

and

$$\frac{\bar{\Psi}^*_-(\alpha,0)}{S_-(\alpha)}-\frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id}\frac{e^{-i\zeta l}\bar{\Psi}^*_-(\zeta,0)}{S_-(\zeta)(\zeta-\alpha)}d\zeta=0. \tag{59}$$

From the assumption that $0<\theta_0<\pi$, we can choose a so that $-K_2\cos\theta_0<a<K_2\cos\theta_0$ and take $c=d=a$.

In Eq. (58) replacing ζ by $-\zeta$ and in Eq. (59) replacing α by $-\alpha$ and noting that $S_+(-\alpha)=S_-(+\alpha)$, we obtain

$$\frac{\bar{\Psi}^*_+(\alpha,0)}{S_+(\alpha)}-\frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\frac{e^{i\zeta l}\bar{\Psi}^*_-(-\zeta,0)}{S_-(\zeta)(\zeta+\alpha)}d\zeta+\frac{b}{\sqrt{2\pi}S_+(K\cos\theta_0)(\alpha-K\cos\theta_0)}=0, \tag{60}$$

and

$$\frac{\bar{\Psi}^*_-(-\alpha,0)}{S_+(\alpha)}-\frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\frac{e^{-i\zeta l}\bar{\Psi}^*_-(\zeta,0)}{S_-(\zeta)(\zeta+\alpha)}d\zeta=0. \tag{61}$$

Adding and subtracting Eqs. (60) and (61), we get

$$\frac{S^*_+(\alpha,0)}{S_+(\alpha)}-\frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\frac{e^{i\zeta l}S^*_+(\zeta,0)}{S_-(\zeta)(\zeta+\alpha)}d\zeta+\frac{b}{\sqrt{2\pi}S_+(K\cos\theta_0)(\alpha-K\cos\theta_0)}=0, \tag{62}$$

and

$$\frac{D^*_+(\alpha,0)}{S_+(\alpha)}+\frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\frac{e^{i\zeta l}D^*_+(\zeta,0)}{S_-(\zeta)(\zeta+\alpha)}d\zeta+\frac{b}{\sqrt{2\pi}S_+(K\cos\theta_0)(\alpha-K\cos\theta_0)}=0, \tag{63}$$

where

$$\left[\bar{\Psi}^*_+(\alpha,0)+\bar{\Psi}^*_-(-\alpha,0)\right]=S^*_+(\alpha,0), \tag{64}$$

and

$$\left[\bar{\Psi}_+^*(\alpha,0)-\bar{\Psi}_-^*(-\alpha,0)\right]=D_+^*(\alpha,0). \tag{65}$$

Now Eqs. (62) and (63) are of the same type and we obtain an approximate solution by a method due to Jones [12]. Setting

$$S_+^*(\alpha,0)=D_+^*(\alpha,0)=F_+^*(\alpha,0), \tag{66}$$

Eqs. (62) and (63) take the form

$$\frac{F_+^*(\alpha,0)}{S_+(\alpha)}+\frac{\lambda}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\frac{e^{i\zeta l}F_+^*(\zeta,0)}{S_-(\zeta)(\zeta+\alpha)}\,d\zeta+\frac{b}{\sqrt{2\pi}S_+(K\cos\theta_0)(\alpha-K\cos\theta_0)}=0, \tag{67}$$

where $\lambda=\pm 1$. Calculating $F_+^*(\alpha,0)$ with the help of Eqs. (56), (57), (64) and (65) and combining the resulting equations, we get

$$F_+^*(\alpha,0)=F_+(\alpha,0)-\frac{b}{\sqrt{2\pi}(\alpha-K\cos\theta_0)}+\lambda\frac{be^{ilK\cos\theta_0}}{\sqrt{2\pi}(\alpha+K\cos\theta_0)}, \tag{68}$$

where

$$F_+(\alpha,0)=\bar{\Psi}'_+(\alpha,0)-\lambda\bar{\Psi}'_-(-\alpha,0). \tag{69}$$

Using Eq. (68) in Eq. (67), we obtain

$$\frac{1}{S_+(\alpha)}\left[F_+(\alpha,0)-\frac{b}{\sqrt{2\pi}(\alpha-K\cos\theta_0)}+\lambda\frac{be^{i\alpha K\cos\theta_0}}{\sqrt{2\pi}(\alpha+K\cos\theta_0)}\right]+\frac{b}{\sqrt{2\pi}S_+(K\cos\theta_0)(\alpha-K\cos\theta_0)}+\frac{\lambda}{2\pi i}I, \tag{70}$$

where

$$I=\int_{-\infty+ia}^{\infty+ia}\frac{e^{i\zeta l}}{S_-(\zeta)(\zeta+\alpha)}\left[F_+(\zeta,0)-\frac{b}{\sqrt{2\pi}(\zeta-K\cos\theta_0)}+\lambda\frac{be^{ilK\cos\theta_0}}{\sqrt{2\pi}(\zeta+K\cos\theta_0)}\right]d\zeta,$$

or

$$I=I_1-\frac{b}{\sqrt{2\pi}}I_2+\lambda\frac{be^{ilK\cos\theta_0}}{\sqrt{2\pi}}I_3, \tag{71}$$

where

$$I_1=\int_{-\infty+ia}^{\infty+ia}\frac{e^{i\zeta l}F_+(\zeta,0)}{S_-(\zeta)(\zeta+\alpha)}\,d\zeta,$$

$$I_2=\int_{-\infty+ia}^{\infty+ia}\frac{e^{i\zeta l}}{S_-(\zeta)(\zeta+\alpha)(\zeta-K\cos\theta_0)}\,d\zeta,$$

$$I_3=\int_{-\infty+ia}^{\infty+ia}\frac{e^{i\zeta l}}{S_-(\zeta)(\zeta+\alpha)(\zeta+K\cos\theta_0)}\,d\zeta.$$

We note that $F_+(\alpha,0)$ is regular in $\tau>-K_2$. Thus, we should expect that $F_+(\alpha,0)$ will have a branch point at $\alpha=-K$ but for large l this is sufficiently far from the point $\alpha=K$ to enable us to evaluate the above integrals in the asymptotic expansion as

$$I_1=2\pi iT(\alpha)F_+(K), \tag{72}$$

where

$$T(\alpha)=\frac{1}{2\pi i}E_{-1}W_{-1}\left\{-i(K+\alpha)l\right\}, \tag{73}$$

$$E_{-1}=2e^{i\frac{\pi}{4}}e^{iKl}(l)^{-1}(i)^{\frac{-1}{2}}h_{-1}, \tag{74}$$

where $h_{-1}=e^{i\frac{\pi}{4}}$,

$$W_{n-\frac{1}{2}}(z)=\int_0^{\infty}\frac{u^ne^{-u}}{u+z}\,du=\Gamma(n+1)e^{\frac{z}{2}}z^{\frac{n}{2}-\frac{1}{2}}W_{-\frac{1}{2}(n+1),\frac{n}{2}}(z), \tag{75}$$

where $z = -i(K+\alpha)l$, $W_{m,n}$ is known as a Whittaker function. Similarly, the other integrals may be evaluated easily by giving

$$I_2 = 2\pi i\left[\frac{e^{ilK\cos\theta_0}}{\sqrt{2\pi}S_-(K\cos)(\alpha+K\cos\theta_0)} + R_2(\alpha)\right], \tag{76}$$

$$I_3 = 2\pi i R_1(\alpha), \tag{77}$$

where

$$R_{1,2}(\alpha) = \frac{E_{-1}[W_{-1}\{-i(K\pm K\cos\theta_0)l\} - W_{-1}\{-i(K+\alpha)l\}]}{2\pi i(\alpha\mp K\cos\theta_0)}. \tag{78}$$

Substituting Eqs. (72), (76) and (77) in (71), we obtain

$$I = 2\pi i T(\alpha)F_+(K) - \frac{b}{\sqrt{2\pi}}2\pi i\left[\frac{e^{ilK\cos\theta_0}}{\sqrt{2\pi}S_-(K\cos\theta_0)(\alpha+K\cos\theta_0)} + R_2(\alpha)\right] + \lambda\frac{be^{ilK\cos\theta_0}}{\sqrt{2\pi}}2\pi i R_1(\alpha).$$

Using above equation in Eq. (70), we get

$$\frac{F_+(\alpha,0)}{S_+(\alpha)} = \frac{b}{\sqrt{2\pi}}\left[G_1(\alpha) - \lambda G_2(\alpha)\right] - i\lambda bT(\alpha)F_+(K), \tag{79}$$

where

$$P_1(\alpha) - e^{ilK\cos\theta_0}R_1(\alpha) = G_1(\alpha), \tag{80a}$$

$$e^{ilK\cos\theta_0}P_2(\alpha) - R_2(\alpha) = G_2(\alpha), \tag{80b}$$

and

$$\frac{1}{(\alpha-K\cos\theta_0)}\left[\frac{1}{S_+(\alpha)} - \frac{1}{S_+(K\cos\theta_0)}\right] = P_1(\alpha), \tag{81a}$$

$$\frac{1}{(\alpha+K\cos\theta_0)}\left[\frac{1}{S_+(\alpha)} - \frac{1}{S_-(K\cos\theta_0)}\right] = P_2(\alpha). \tag{81b}$$

Now, we can find out $F_+(K)$ by setting $\alpha = K$ in Eq. (79), we get

$$F_+(K) = \frac{b}{\sqrt{2\pi}}\left[\frac{G_1(K) - \lambda G_2(K)}{\frac{1}{S_+(K)} + \lambda T(K)}\right]. \tag{82}$$

Using Eq. (82) in Eq. (79), we get

$$F_+(\alpha,0) = \frac{bS_+(\alpha)}{\sqrt{2\pi}}\left[G_1(\alpha) - \lambda G_2(\alpha)\right] - \frac{\lambda bT(\alpha)S_+(\alpha)}{\sqrt{2\pi}}\left[\frac{G_1(K) - \lambda G_2(K)}{\frac{1}{S_+(K)} + \lambda T(K)}\right]. \tag{83}$$

For $\lambda = -1$, Eq. (83) becomes

$$\bar{\Psi}'_+(\alpha,0) + \bar{\Psi}'_-(-\alpha,0) = \frac{bS_+(\alpha)}{\sqrt{2\pi}}\left[G_1(\alpha) + G_2(\alpha)\right] + \frac{bT(\alpha)S_+(\alpha)}{\sqrt{2\pi}}\left[\frac{G_1(K) + G_2(K)}{\frac{1}{S_+(K)} - T(K)}\right]. \tag{84}$$

For $\lambda = 1$, Eq. (83) becomes

$$\bar{\Psi}'_+(\alpha,0) - \bar{\Psi}'_-(-\alpha,0) = \frac{bS_+(\alpha)}{\sqrt{2\pi}}\left[G_1(\alpha) - G_2(\alpha)\right] - \frac{bT(\alpha)S_+(\alpha)}{\sqrt{2\pi}}\left[\frac{G_1(K) - G_2(K)}{\frac{1}{S_+(K)} + T(K)}\right]. \tag{85}$$

By adding and subtracting Eqs. (84) and (85), we get

$$\bar{\Psi}'_+(\alpha,0) = \frac{bS_+(\alpha)G_1(\alpha)}{\sqrt{2\pi}} + \frac{bT(\alpha)S_+(\alpha)C_1}{\sqrt{2\pi}}, \tag{86}$$

and

$$\bar{\Psi}'_-(\alpha,0) = \frac{bS_-(\alpha)G_2(-\alpha)}{\sqrt{2\pi}} + \frac{bT(-\alpha)S_-(\alpha)C_2}{\sqrt{2\pi}}, \tag{87}$$

where

$$S_+(\alpha) = L_+(\alpha)\kappa_+(\alpha),$$

and

$$S_-(\alpha) = L_-(\alpha)\kappa_-(\alpha),$$

while $\kappa_\pm(\alpha) = \sqrt{K \pm \alpha}$,

$$C_1 = S_+(K)\left[\frac{G_2(K) + S_+(K)G_1(K)T(K)}{1 - S_+^2(K)T^2(K)}\right], \tag{88}$$

and

$$C_2 = S_+(K)\left[\frac{G_1(K) + S_+(K)G_2(K)T(K)}{1 - S_+^2(K)T^2(K)}\right]. \tag{89}$$

In a similar way, the solution for the second Wiener–Hopf equation, namely (38) is presented as

$$\bar{\Psi}_+(\alpha,0) = \frac{-iL_+(\alpha)G_1'(\alpha)}{\sqrt{2\pi}[\{2\alpha MB + KB(1+M^2) + \frac{BM^2\alpha^2}{(1-M^2)K}\}]} + \frac{-iT(\alpha)L_+(\alpha)C_1'}{\sqrt{2\pi}[\{2\alpha MB + KB(1+M^2) + \frac{BM^2\alpha^2}{(1-M^2)K}\}]}, \tag{90}$$

and

$$\bar{\Psi}_-(\alpha,0) = \frac{-iL_-(\alpha)G_2'(-\alpha)}{\sqrt{2\pi}[\{-2\alpha MB + KB(1+M^2) + \frac{BM^2\alpha^2}{(1-M^2)K}\}]} - \frac{-iT(-\alpha)L_-(\alpha)C_2'}{\sqrt{2\pi}[(-2\alpha MB + KB(1+M^2) + \frac{BM^2\alpha^2}{(1-M^2)K})]}, \tag{91}$$

where

$$G_1'(\alpha) = \frac{[\{2\alpha MB + KB(1+M^2) + \frac{BM^2\alpha^2}{(1-M^2)K}\}]}{(\alpha - K\cos\theta_0)}\left[\frac{1}{L_+(\alpha)} - \frac{1}{L_+(K\cos\theta_0)}\right] - e^{ilK\cos\theta_0}R_1(\alpha), \tag{92}$$

$$G_2'(\alpha) = \frac{e^{ilK\cos\theta_0}}{(\alpha + K\cos\theta_0)}\left[\frac{[\{-2\alpha MB + KB(1+M^2) + \frac{BM^2\alpha^2}{(1-M^2)K}\}]}{L_+(\alpha)} + \frac{B[(2MK\cos\theta_0) + K(1+M^2) + \frac{M^2K^2\cos^2\theta_0}{(1-M^2)K}]}{L_-(K\cos\theta_0)}\right] - R_2(\alpha), \tag{93}$$

$$C_1' = L_+(K)\left[\frac{G_2'(K) + L_+(K)G_1'(K)T(K)}{1 - L_+^2(K)T^2(K)}\right], \tag{94}$$

$$C_2' = L_+(K)\left[\frac{G_1'(K) + L_+(K)G_2'(K)T(K)}{1 - L_+^2(K)T^2(K)}\right]. \tag{95}$$

Substituting Eqs. (35), (36), (86), (87), (90) and (91) in Eqs. (33) and (34), we get

$$\left.\begin{matrix} A_1(\alpha) \\ A_2(\alpha) \end{matrix}\right\} = \frac{b\,\mathrm{sgn}(Y)}{i\kappa L(\alpha)}\Bigg[\left(\frac{S_+(\alpha)G_1(\alpha)}{\sqrt{2\pi}} + \frac{T(\alpha)S_+(\alpha)C_1}{\sqrt{2\pi}}\right) + e^{-i\alpha l}\left(\frac{S_-(\alpha)G_2(-\alpha)}{\sqrt{2\pi}} + \frac{T(-\alpha)S_-(\alpha)C_2}{\sqrt{2\pi}}\right) - \frac{1}{\sqrt{2\pi}}\left[\frac{1 - e^{-il(\alpha - K\cos\theta_0)}}{(\alpha - K\cos\theta_0)}\right]\Bigg]$$
$$- \frac{B[(2\alpha M + K(1+M^2) + \frac{M^2\alpha^2}{(1-M^2)K})]}{i\kappa L(\alpha)}\Bigg[\frac{L_+(\alpha)G_1'(\alpha)}{\sqrt{2\pi}[(2\alpha MB + KB(1+M^2) + \frac{BM^2\alpha^2}{(1-M^2)K})]} + \frac{T(\alpha)L_+(\alpha)C_1'}{\sqrt{2\pi}[(2\alpha MB + KB(1+M^2) + \frac{BM^2\alpha^2}{(1-M^2)K})]}$$
$$+ e^{-i\alpha l}\left[\frac{L_-(\alpha)G_2'(-\alpha)}{\sqrt{2\pi}[(-2\alpha MB + KB(1+M^2) + \frac{BM^2\alpha^2}{(1-M^2)K})]} + \frac{T(-\alpha)L_-(\alpha)C_2'}{\sqrt{2\pi}[(-2\alpha MB + KB(1+M^2) + \frac{BM^2\alpha^2}{(1-M^2)K})]}\right] + \frac{1}{\sqrt{2\pi}}\left[\frac{1 - e^{-il(\alpha - K\cos\theta_0)}}{(\alpha - K\cos\theta_0)}\right]\Bigg], \tag{96}$$

where

$$\mathrm{sgn}(Y) = \begin{cases} 1 & \text{if } Y \geqslant 0, \\ -1 & \text{if } Y < 0, \end{cases}$$

and $A_1(\alpha)$ corresponds to $Y \geqslant 0$ and $A_2(\alpha)$ corresponds to $Y < 0$. Now $\Psi(X,Y)$ can be obtained by taking the inverse Fourier transform of Eq. (19), thus

$$\Psi(X,Y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left.\begin{matrix}A_1(\alpha)\\A_2(\alpha)\end{matrix}\right\}e^{i\kappa|Y|-i\alpha X}\,d\alpha, \tag{97}$$

where $A_1(\alpha)$ and $A_2(\alpha)$ are given by Eq. (96).

Substituting the values of $A_1(\alpha)$ and $A_2(\alpha)$ from Eq. (96) into Eqs. (97) and using the approximations (80a), (80b), (92) and (93), we can break up the field $\Psi(X,Y)$ into two parts

$$\Psi(X,Y)=\Psi^{\mathrm{sep}}(X,Y)+\Psi^{\mathrm{int}}(X,Y), \tag{98}$$

where

$$\begin{aligned}\Psi^{\mathrm{sep}}(X,Y)={}&\frac{b}{2\pi i}\int_{-\infty}^{\infty}\frac{S_+(\alpha)e^{i\kappa|Y|-i\alpha X}}{\kappa L(\alpha)S_+(K\cos\theta_0)(\alpha-K\cos\theta_0)}\,d\alpha+\frac{b}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{-il(\alpha-K\cos\theta_0)}S_-(\alpha)e^{i\kappa|Y|-i\alpha X}}{\kappa L(\alpha)S_-(K\cos\theta_0)(\alpha-K\cos\theta_0)}\,d\alpha\\&-\frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{[2\alpha MB+KB(1+M^2)+\frac{BM^2\alpha^2}{(1-M^2)K}]L_+(\alpha)e^{-il(\alpha-K\cos\theta_0)}e^{i\kappa|Y|-i\alpha X}}{\kappa L(\alpha)(\alpha-K\cos\theta_0)L_+(K\cos\theta_0)}\,d\alpha\\&-\frac{i}{2\pi}\int_{-\infty}^{\infty}\left[\frac{(2\alpha M+K(1+M^2)+\frac{M^2\alpha^2}{(1-M^2)K})(2BMK\cos\theta_0+BK(1+M^2)+\frac{BM^2K^2\cos^2\theta_0}{(1-M^2)K})}{(-2\alpha M+K(1+M^2)+\frac{M^2\alpha^2}{(1-M^2)K})}\right]\\&\times\frac{L_-(\alpha)e^{i\kappa|Y|-i\alpha X}}{\kappa L(\alpha)(\alpha-K\cos\theta_0)L_-(K\cos\theta_0)}\,d\alpha+\frac{A}{2\pi}\int_{-\infty}^{\infty}\left[\frac{\{4\alpha MB(2\alpha M+K(1+M^2)+\frac{M^2\alpha^2}{(1-M^2)K}\}}{\{-2\alpha M+K(1+M^2)+\frac{M^2\alpha^2}{(1-M^2)K}\}}\right]\\&\times\frac{e^{-il(\alpha-K\cos\theta_0)}e^{i\kappa|Y|-i\alpha X}}{\kappa L(\alpha)(\alpha-K\cos\theta_0)}\,d\alpha,\end{aligned} \tag{99}$$

and

$$\begin{aligned}\Psi^{\mathrm{int}}(X,Y)={}&\frac{b}{2\pi i}\int_{-\infty}^{\infty}\frac{1}{\kappa L(\alpha)}\big[T(\alpha)S_+(\alpha)C_1+e^{-i\alpha l}T(-\alpha)S_-(\alpha)C_2\\&-S_+(\alpha)e^{ilK\cos\theta_0}R_1(\alpha)-S_-(\alpha)e^{-il\alpha}R_2(-\alpha)\big]e^{i\kappa|Y|-i\alpha X}\,d\alpha\\&+\frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{\{2\alpha MB+KB(1+M^2)+\frac{BM^2\alpha^2}{(1-M^2)K}\}}{\kappa L(\alpha)}\left[\frac{T(\alpha)L_+(\alpha)C_1'}{\{2\alpha MB+KB(1+M^2)+\frac{BM^2\alpha^2}{(1-M^2)K}\}}\right.\\&+\frac{e^{-i\alpha l}T(-\alpha)L_-(\alpha)C_2'}{(-2\alpha MB+KB(1+M^2)+\frac{BM^2\alpha^2}{(1-M^2)K})}-\frac{e^{ilK\cos\theta_0}L_+(\alpha)R_1(\alpha)}{(2\alpha MB+KB(1+M^2)+\frac{BM^2\alpha^2}{(1-M^2)K})}\\&\left.-\frac{e^{-i\alpha l}L_-(\alpha)R_2(-\alpha)}{(-2\alpha MB+KB(1+M^2)+\frac{BM^2\alpha^2}{(1-M^2)K})}\right]e^{i\kappa|Y|-i\alpha X}\,d\alpha.\end{aligned} \tag{100}$$

Here $\Psi^{\mathrm{sep}}(X,Y)$ consists of two parts each representing the diffracted field produced by the edges at $x=0$ and $x=-l$, respectively, as though the other edge was absent while $\Psi^{\mathrm{int}}(X,Y)$ gives the interaction of one edge upon the other.

5. Far field approximation

The far field may now be calculated by evaluating the integral in Eq. (97) asymptotically [13]. For that we substitute $X=R\cos\theta$, $|Y|=R\sin\theta$ and deform the contour by the transformation $\alpha=-K\cos(\theta+iq_3)$ $(0<\theta\leqslant\pi,\ -\infty<q_3<\infty)$.

Hence for large KR, Eqs. (97), (99) and (100), respectively, become

$$\Psi(X,Y)=\frac{i}{\sqrt{2\pi}}\left(\frac{\pi}{2KR}\right)^{\frac{1}{2}}\left\{\begin{matrix}A_1(-K\cos\theta)\\A_2(-K\cos\theta)\end{matrix}\right\}e^{iKR+i\frac{\pi}{4}}, \tag{101}$$

$$\Psi^{\mathrm{sep}}=\big[b\,\mathrm{sgn}(Y)f_1(-K\cos\theta)+g_1(-K\cos\theta)\big]\frac{i}{4\pi K}\left(\frac{1}{KR}\right)^{\frac{1}{2}}\exp\left(iKR+i\frac{\pi}{4}\right), \tag{102}$$

and

$$\Psi^{\mathrm{int}}=-\big[ib\,\mathrm{sgn}(Y)f_2(-K\cos\theta)+g_2(-K\cos\theta)\big]\frac{i}{4\pi K}\left(\frac{1}{KR}\right)^{\frac{1}{2}}\exp\left(iKR+i\frac{\pi}{4}\right), \tag{103}$$

where $A_1(-K\cos\theta)$ and $A_2(-K\cos\theta)$ are given by Eq. (96), and

$$\begin{aligned} f_1(-K\cos\theta) = {} & \frac{S_+(-K\cos\theta)}{L(-K\cos\theta)S_+(K\cos\theta_0)(-K\cos\theta - K\cos\theta_0)} \\ & - \frac{e^{-il(-K\cos\theta - K\cos\theta_0)}S_+(K\cos\theta)}{L(-K\cos\theta)S_+(-K\cos\theta_0)(-K\cos\theta - K\cos\theta_0)}, \end{aligned} \tag{104}$$

$$\begin{aligned} g_1(-K\cos\theta) = {} & \frac{1}{K(\cos\theta + \cos\theta_0)}\Bigg[\frac{B[-2MK\cos\theta + K(1+M^2) + \frac{M^2K\cos^2\theta}{(1-M^2)}]}{L(-K\cos\theta)}\frac{L_+(-K\cos\theta)e^{-il(-K\cos\theta - K\cos\theta_0)}}{L_+(K\cos\theta_0)} \\ & - \frac{[-2MK\cos\theta + K(1+M^2) + \frac{M^2K\cos^2\theta}{(1-M^2)}]L_-(-K\cos\theta)}{[2MK\cos\theta + K(1+M^2) + \frac{M^2K\cos^2\theta}{(1-M^2)}]}\frac{[2BMK\cos\theta_0 + BK(1+M^2) + \frac{BM^2K^2\cos^2\theta_0}{(1-M^2)K}]}{L(-K\cos\theta)} \\ & + \frac{4BMK\cos\theta[-2MK\cos\theta + K(1+M^2) + \frac{M^2K\cos^2\theta}{(1-M^2)}]e^{-il(-K\cos\theta - K\cos\theta_0)}}{L(-K\cos\theta)[2MK\cos\theta + K(1+M^2) + \frac{M^2K\cos^2\theta}{(1-M^2)}]}\Bigg], \end{aligned} \tag{105}$$

$$\begin{aligned} f_2(-K\cos\theta) = {} & \frac{1}{L(-K\cos\theta)}\big[T(-K\cos\theta)S_+(-K\cos\theta)C_1 + e^{ilK\cos\theta}T(K\cos\theta)S_-(-K\cos\theta)C_2 \\ & + S_+(-K\cos\theta)e^{ilK\cos\theta_0}R_1(-K\cos\theta) + S_-(-K\cos\theta)e^{ilK\cos\theta}R_2(K\cos\theta)\big], \end{aligned} \tag{106}$$

and

$$\begin{aligned} g_2(-K\cos\theta) = {} & \frac{B[-2MK\cos\theta + K(1+M^2) + \frac{M^2K\cos^2\theta}{(1-M^2)}]}{L(-K\cos\theta)}\Bigg[\frac{T(-K\cos\theta)L_+(-K\cos\theta)C_1'}{B[-2MK\cos\theta + K(1+M^2) + \frac{M^2K\cos^2\theta}{(1-M^2)}]} \\ & + \frac{e^{ilK\cos\theta}T(K\cos\theta)S_-(-K\cos\theta)C_2'}{B[2MK\cos\theta + K(1+M^2) + \frac{M^2K\cos^2\theta}{(1-M^2)}]} - \frac{e^{ilK\cos\theta_0}L_+(-K\cos\theta)R_1(-K\cos\theta)}{B[-2MK\cos\theta + K(1+M^2) + \frac{M^2K\cos^2\theta}{(1-M^2)}]} \\ & - \frac{e^{ilK\cos\theta}L_-(-K\cos\theta)R_2(K\cos\theta)}{B[-2MK\cos\theta + K(1+M^2) + \frac{M^2K\cos^2\theta}{(1-M^2)}]}\Bigg]. \end{aligned} \tag{107}$$

Thus from Eq. (9), we obtain

$$\psi_t(X,Y) = \frac{a}{4\pi i}H_0^1(KR) + \frac{i}{\sqrt{2\pi}}\left(\frac{\pi}{2KR}\right)^{\frac{1}{2}}\begin{Bmatrix} A_1(-K\cos\theta) \\ A_2(-K\cos\theta) \end{Bmatrix} e^{iKR + i\frac{\pi}{4}}. \tag{108}$$

Using Eq. (108) in Eq. (5), the total far field is given by

$$\Phi(x,y) \sim \frac{e^{iKMX_0}}{4\pi i\sqrt{1-M^2}}H_0^1(KR)e^{-iKMX} + \frac{i}{\sqrt{2\pi}}\left(\frac{\pi}{2KR}\right)^{\frac{1}{2}} e^{iKR + i\frac{\pi}{4}}e^{-iKMX}\begin{Bmatrix} A_1(-K\cos\theta) \\ A_2(-K\cos\theta) \end{Bmatrix}. \tag{109}$$

6. Graphical results

In this section, we will present some graphs showing the effects of sundry parameters on the separated field produced by the two edges of the barrier.

A computer program MATHEMATICA has been used for the numerical evaluation and graphical plotting of the separated field given by the expression (102). The absorbing parameter B is to be taken from 0.1 to 0.9 and the Mach number is allowed to take the values from -1 to 1. Positive Mach number indicates that the stream flow is from left to right and negative Mach number indicates that the stream flow is from right to left. The following situations are considered.

(i) Where the source is fixed in one position (for all Mach numbers) relative to the finite barrier ($\theta_0 = 45°$, M and θ are allowed to vary).

(ii) Where the source is fixed in one position, relative to the finite barrier ($\theta_0 = 45°$, B and θ are allowed to vary).

For all the situations, $\theta_0 = 45°$, Figs. 1–4 show that the field, in the region $0 < \theta \leqslant \pi$, is most affected by the changes in M, B and K. The main features of the graphical results, some of which can be seen in Figs. 1–4, are as follows:

(a) In Fig. 1, as the value of M increases, the amplitude of the separated field decreases, i.e., the sound intensity reduces, by fixing all other parameters. For the same values of M, it can be seen from Figs. 1 and 2 that by increasing the wave number K, the number of oscillations increases.

(b) In Fig. 3, by increasing the absorbing parameter B, again the amplitude of the separated field decreases, i.e., the sound intensity reduces. For the same values of B, it can be seen from Figs. 3 and 4 that by increasing the wave number K, the number of oscillations increases.

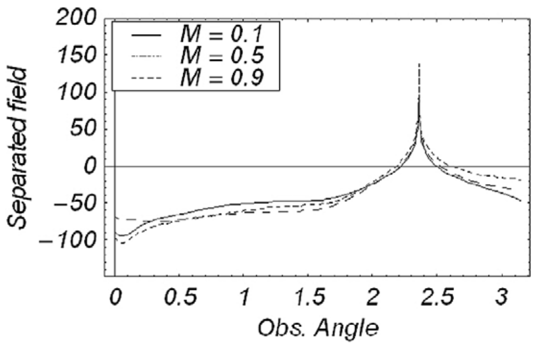

Fig. 1. Variation of separated field with observation angle θ, for different values of M at $\theta_0 = \frac{\pi}{4}$, $K = 1$, $B = 0.5$.

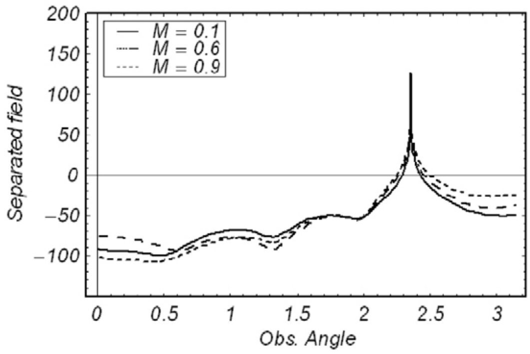

Fig. 2. Variation of separated field with observation angle θ, for different values of M at $\theta_0 = \frac{\pi}{4}$, $K = 2$, $B = 0.5$.

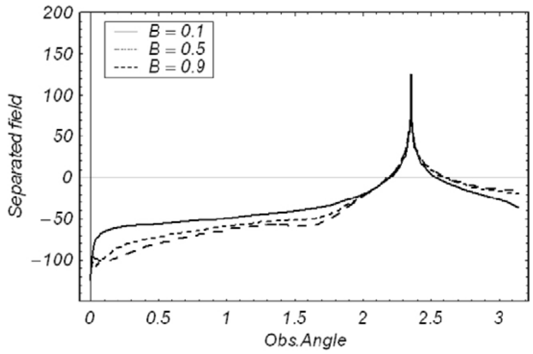

Fig. 3. Variation of separated field with observation angle θ, for different values of B at $\theta_0 = \frac{\pi}{4}$, $K = 1$, $M = 0.5$.

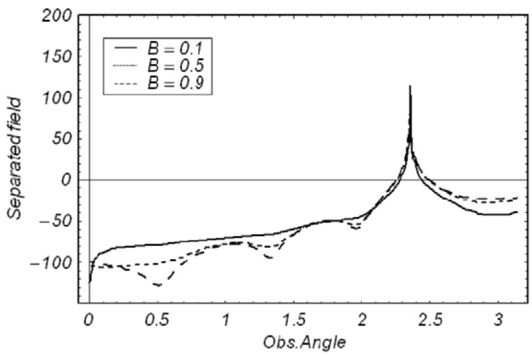

Fig. 4. Variation of separated field with observation angle θ, for different values of B at $\theta_0 = \frac{\pi}{4}$, $K = 2$, $M = 0.5$.

7. Conclusion

The diffracted field due to an absorbing finite barrier satisfying Myers' condition [7] in a moving fluid is obtained. The total field is shown to be the sum of the field produced by the two edges of the finite barrier and a field due to the interaction of the two edges. Further the consideration of finite strip will help to understand the acoustic diffraction and will go a step further to complete the discussion for the half plane by taking limit $l \to \infty$ in the expression (109). Also the results for the still fluid can be found by putting $M = 0$.

Acknowledgments

The authors are grateful to the referee for his valuable suggestions. These suggestions were found useful in enhancing the quality of the paper. One of the authors, A. Naeem, gratefully acknowledges the financial support provided by the Higher Education Commission (HEC) of Pakistan.

References

[1] D.S. Jones, J.D. Morgan, The instability of a vortex sheet on a subsonic stream under acoustic radiation, Proc. Camb. Philos. Soc. 72 (1972) 465–488.
[2] A.D. Rawlins, Acoustic diffraction by an absorbing semi-infinite half plane in a moving fluid, Proc. Roy. Soc. Edinb. Ser. A 72 (1975) 337–357.
[3] P.M. Morse, K.U. Ingard, Encyclopedia of Physics Acoustics, vol. 1, Springer, Berlin, 1961.
[4] S. Asghar, B. Ahmed, M. Ayub, Point source diffraction by an absorbing half plane, IMA J. Appl. Math. 46 (1991) 217–224.
[5] J.W. Miles, On the reflection of sound at an interface of relative motion, J. Acoust. Soc. Am. 29 (1957) 226–228.
[6] H.S. Ribner, Reflection, transmission and amplification of sound by a moving medium, J. Acoust. Soc. Am. 29 (1957) 435–441.
[7] M.K. Myers, On the acoustic boundary condition in the presence of flow, J. Sound Vibration 71 (1980) 429–434.
[8] S. Asghar, Acoustic diffraction by an absorbing finite strip in a moving fluid, J. Acoust. Soc. Am. 83 (2) (1988) 812–816.
[9] B. Noble, Methods Based on the Wiener–Hopf Technique, Pergamon Press, London, 1958.
[10] S. Asghar, T. Hayat, Cylindrical wave diffraction by an absorbing strip, Arch. Acoust. 233 (1998) 391–401.
[11] S. Asghar, T. Hayat, B. Asghar, Cylindrical wave diffraction by a perfectly conducting strip in a homogeneous bi-isotropic medium, J. Modern Opt. 45 (3) (1998) 515–528.
[12] D.S. Jones, Acoustic and Electromagnetic Waves, Clarendon Press, Oxford, 1986.
[13] E.T. Copson, Asymptotic Expansions, Cambridge University Press, 1967.
[14] Bashir Ahmad, An improved model for noise barriers in a moving fluid, J. Math. Anal. Appl. 321 (2) (2006) 609–620.