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The product of operators with closed range in Hilbert C^*-modules ☆

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ABSTRACT

Suppose T and S are bounded adjointable operators with close range between Hilbert C^*-modules, then TS has closed range if and only if $Ker(T)+Ran(S)$ is an orthogonal summand, if and only if $Ker(S^*)+Ran(T^*)$ is an orthogonal summand. Moreover, if the Dixmier (or minimal) angle between $Ran(S)$ and $Ker(T)\cap[Ker(T)\cap Ran(S)]^{\perp}$ is positive and $\overline{Ker(S^*)+Ran(T^*)}$ is an orthogonal summand then TS has closed range.

1. Introduction

The closeness of range of operators is an attractive and important problem which appears in operator theory, especially, in the theory of Fredholm operators and generalized inverses. In this paper we will investigate when the product of two operators with closed range again has closed range. This problem was first studied by Bouldin for bounded operators between Hilbert spaces in [3,4]. Indeed, for Hilbert space operators T, S whose ranges are closed, he proved that the range of TS is closed if and only if the Dixmier (or minimal) angle between $Ran(S)$ and $Ker(T)\cap[Ker(T)\cap Ran(S)]^{\perp}$ is positive, where

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the Dixmier angle between subspaces M and N of a certain Hilbert space is the angle $\alpha_0(M, N)$ in $[0, \pi/2]$ whose cosine is defined by $c_0(M, N) = \sup\{\|\langle x, y\rangle\| : x \in M,\ \|x\| \leqslant 1\ ,\ y \in N,\ \|y\| \leqslant 1\}$. Nikaido [24,25] also gave topological characterizations of the problem for the Banach space operators. Recently (Dixmier and Friedrichs) angles between linear subspaces have been studied systematically by Deutsch [7], he also has reconsidered the closeness of range of the product of two operators with closed range. In this note we use C*-algebras techniques to reformulate some results of Bouldin and Deutsch in the framework of Hilbert C*-modules. Some further characterizations of modular operators with closed range are obtained.

Hilbert C*-modules are essentially objects like Hilbert spaces, except that the inner product, instead of being complex-valued, takes its values in a C*-algebra. Since the geometry of these modules emerges from the C*-valued inner product, some basic properties of Hilbert spaces like Pythagoras' equality, self-duality, and decomposition into orthogonal complements do not hold. The theory of Hilbert C*-modules, together with adjointable operators forms an infrastructure for some of the most important research topics in operator algebras, in Kasparov's KK-theory and in noncommutative geometry.

A (left) *pre-Hilbert C*-module* over a C*-algebra $\mathcal{A}$ is a left $\mathcal{A}$-module E equipped with an $\mathcal{A}$-valued inner product $\langle\cdot,\cdot\rangle : E \times E \to \mathcal{A}\ ,\ (x, y) \mapsto \langle x, y\rangle$, which is $\mathcal{A}$-linear in the first variable x (and conjugate-linear in y) and has the properties:

$$\langle x, y\rangle = \langle y, x\rangle^*,\ \langle ax, y\rangle = a\langle x, y\rangle \text{ for all } a \text{ in } \mathcal{A},$$

$$\langle x, x\rangle \geqslant 0 \text{ with equality only when } x = 0.$$

A pre-Hilbert $\mathcal{A}$-module E is called a *Hilbert $\mathcal{A}$-module* if E is a Banach space with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{1/2}$. A Hilbert $\mathcal{A}$-submodule E of a Hilbert $\mathcal{A}$-module F is an orthogonal summand if $F = E \oplus E^{\perp}$, where $E^{\perp} := \{y \in F :\ \langle x, y\rangle = 0 \text{ for all } x \in E\}$ denotes the orthogonal complement of E in F. The papers [9,10] and the books [19,22] are used as standard sources of reference.

Throughout the present paper we assume $\mathcal{A}$ to be an arbitrary C*-algebra (i.e. not necessarily unital). We use the notations $Ker(\cdot)$ and $Ran(\cdot)$ for kernel and range of operators, respectively. We denote by $\mathcal{L}(E, F)$ the Banach space of all bounded adjointable operators between E and F, i.e., all bounded $\mathcal{A}$-linear maps $T : E \to F$ such that there exists $T^* : F \to E$ with the property $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in E, y \in F$. The C*-algebra $\mathcal{L}(E, E)$ is abbreviated by $\mathcal{L}(E)$.

In this paper we first briefly investigate some basic facts about Moore–Penrose inverses of bounded adjointable operators on Hilbert C*-modules and then we give some necessary and sufficient conditions for closeness of the range of the product of two orthogonal projections. These lead us to our main results. Indeed, for adjointable module maps T, S whose ranges are closed we show that the operator TS has closed range if and only if $Ker(T) + Ran(S)$ is an orthogonal summand, if and only if $Ker(S^*) + Ran(T^*)$ is an orthogonal summand. The Dixmier angle between submodules M and N of a Hilbert C*-module E is the angle $\alpha_0(M, N)$ in $[0, \pi/2]$ whose cosine is defined by

$$c_0(M, N) = \sup\{\|\langle x, y\rangle\| : x \in M,\ \|x\| \leqslant 1\ ,\ y \in N,\ \|y\| \leqslant 1\}.$$

If the Dixmier angle between $Ran(S)$ and $Ker(T) \cap [Ker(T) \cap Ran(S)]^{\perp}$ is positive and $\overline{Ker(S^*) + Ran(T^*)}$ is an orthogonal summand then TS has closed range. Since every C*-algebra is a Hilbert C*-module over itself, our results are also remarkable in the case of bounded adjointable operators on C*-algebras.

2. Preliminaries

Closed submodules of Hilbert modules need not to be orthogonally complemented at all, but Lance states in [19, Theorem 3.2] under which conditions closed submodules may be orthogonally complemented (see also [22, Theorem 2.3.3]). Let E, F be two Hilbert $\mathcal{A}$-modules and suppose that an operator T in $\mathcal{L}(E, F)$ has closed range, then one has:

- $Ker(T)$ is orthogonally complemented in E, with complement $Ran(T^*)$,
- $Ran(T)$ is orthogonally complemented in F, with complement $Ker(T^*)$,
- the map $T^* \in \mathcal{L}(F, E)$ has closed range, too.

Lemma 2.1. *Suppose $T \in \mathcal{L}(E, F)$. The operator T has closed range if and only if TT^* has closed range. In this case, $Ran(T) = Ran(TT^*)$.*

Proof. Suppose T has closed range, the proof of Theorem 3.2 of [19] indicates that $Ran(TT^*)$ is closed and $Ran(T) = Ran(TT^*)$.

Conversely, if TT^* has closed range then $F = Ran(TT^*) \oplus Ker(TT^*) = Ran(TT^*) \oplus Ker(T^*) \subset Ran(T) \oplus Ker(T^*) \subset F$ which implies T has closed range. □

Let $T \in \mathcal{L}(E, F)$, then a bounded adjointable operator $S \in \mathcal{L}(F, E)$ is called an *inner inverse* of T if $TST = T$. If $T \in \mathcal{L}(E, F)$ has an inner inverse S then the bounded adjointable operator $T^\times = STS$ in $\mathcal{L}(F, E)$ satisfies

$$TT^\times T = T \ \text{ and } \ T^\times TT^\times = T. \tag{2.1}$$

The bounded adjointable operator $T^\times$ which satisfies (2.1) is called *generalized inverse* of T. It is known that a bounded adjointable operator T has a generalized inverse if and only if $Ran(T)$ is closed, see e.g. [5,31].

Let $T \in \mathcal{L}(E, F)$, then a bounded adjointable operator $T^\dagger \in \mathcal{L}(F, E)$ is called the *Moore–Penrose inverse* of T if

$$TT^\dagger T = T, \ T^\dagger TT^\dagger = T^\dagger, \ (TT^\dagger)^* = TT^\dagger \text{ and } (T^\dagger T)^* = T^\dagger T. \tag{2.2}$$

The notation $T^\dagger$ is reserved to denote the Moore–Penrose inverse of T. These properties imply that $T^\dagger$ is unique and $T^\dagger T$ and $TT^\dagger$ are orthogonal projections. Moreover, $Ran(T^\dagger) = Ran(T^\dagger T)$, $Ran(T) = Ran(TT^\dagger)$, $Ker(T) = Ker(T^\dagger T)$ and $Ker(T^\dagger) = Ker(TT^\dagger)$ which lead us to $E = Ker(T^\dagger T) \oplus Ran(T^\dagger T) = Ker(T) \oplus Ran(T^\dagger)$ and $F = Ker(T^\dagger) \oplus Ran(T)$.

Xu and Sheng in [30] have shown that a bounded adjointable operator between two Hilbert C*-modules admits a bounded Moore–Penrose inverse if and only if the operator has closed range. The reader should be aware of the fact that a bounded adjointable operator may admit an unbounded operator as its Moore–Penrose, see [13,28,29] for more detailed information.

Proposition 2.2. *Suppose E, F, G are Hilbert $\mathcal{A}$-modules and $S \in \mathcal{L}(E, F)$ and $T \in \mathcal{L}(F, G)$ are bounded adjointable operators with closed ranges. Then TS has a generalized inverse if and only if $T^\dagger TSS^\dagger$ has. In particular, TS has closed range if and only if $T^\dagger TSS^\dagger$ has.*

Proof. Suppose first that V is a generalized inverse of TS. Then

$$T^\dagger TSS^\dagger (SVT) T^\dagger TSS^\dagger = T^\dagger T\,(SS^\dagger S)\,V\,(TT^\dagger T)\,SS^\dagger = T^\dagger TS\,V\,TSS^\dagger = T^\dagger TSS^\dagger.$$

Similarly, $SVT\,(T^\dagger TSS^\dagger)\,SVT = SVT$ and so SVT is a generalized inverse of $T^\dagger TSS^\dagger$. Conversely, suppose that $U \in \mathcal{L}(F)$ is a generalized inverse of $T^\dagger TSS^\dagger$. Let $P = SS^\dagger$ and $Q = T^\dagger T$ are orthogonal projections onto $Ran(S)$ and $Ker(T)^\perp$, respectively, then $QPUQP = QP$. We set $W = PUQ$, then $PWQ = W$ and $QWP = QP$. The later equality implies that $Q(1 - W)P = 0$, that is, $1 - W$ maps $Ran(P) = Ran(S)$ into $Ker(Q) = Ker(T)$. Consequently, $T(1 - W)S = 0$. Hence,

$$TS\,(S^\dagger WT^\dagger)\,TS = TPWQS = TWS = TS.$$

On the other hand, $S^\dagger WT^\dagger = S^\dagger PUQT^\dagger = S^\dagger SS^\dagger UT^\dagger TT^\dagger = S^\dagger UT^\dagger$ which shows that $(S^\dagger WT^\dagger)\,TS\,(S^\dagger WT^\dagger) = S^\dagger UT^\dagger = S^\dagger WT^\dagger$, i.e. $S^\dagger WT^\dagger$ is a generalized inverse of TS. In particular, TS has closed range if and only if $T^\dagger TSS^\dagger$ has. □

Lemma 2.3. *Let $T \in \mathcal{L}(E, F)$, then T has closed range if and only if $Ker(T)$ is orthogonally complemented in E and T is bounded below on $Ker(T)^\perp$, i.e. $\|Tx\| \geqslant c\|x\|$, for all $x \in Ker(T)^\perp$ for a certain positive constant c.*

The statement directly follows from Proposition 1.3 of [12].

Lemma 2.4. *Let T be a non-zero bounded adjointable operator in $\mathcal{L}(E, F)$, then T has closed range if and only if $Ker(T)$ is orthogonally complemented in E and*

$$\gamma(T) = \inf\{\|Tx\| : x \in Ker(T)^{\perp} \text{ and } \|x\| = 1\} > 0.$$

In this case, $\gamma(T) = \|T^{\dagger}\|^{-1}$ and $\gamma(T) = \gamma(T^*)$.

Proof. The first assertion follows directly from Lemma 2.3. To prove the first equality, suppose T has closed range, $x \in Ker(T)^{\perp} = Ran(T^{\dagger}T)$ and $\|x\| = 1$, then $1 = \|x\| = \|T^{\dagger}Tx\| \leqslant \|T^{\dagger}\|\,\|Tx\|$, consequently, $\|T^{\dagger}\|^{-1} \leqslant \gamma(T)$. Suppose $x \in Ker(T)^{\perp}$ then $\gamma(T)\|x\| \leqslant \|Tx\|$. Suppose $w \in F$ and $x = T^{\dagger}w$ then $x \in Ran(T^{\dagger}) = Ker(T)^{\perp}$, hence,

$$\gamma(T)\|T^{\dagger}w\| \leqslant \|T\,T^{\dagger}w\| \leqslant \|T\,T^{\dagger}\|\,\|w\| \leqslant \|w\|.$$

We therefore have $\gamma(T) \leqslant \|T^{\dagger}\|^{-1}$. To establish the second equality just recall that T has closed range if and only if T^* has. It now follows from the first equality and the fact $\|T^{*\dagger}\| = \|T^{\dagger *}\| = \|T^{\dagger}\|$. □

3. Closeness of the range of the products

Suppose F is a Hilbert $\mathcal{A}$-module and T be a bounded adjointable operator in the unital C*-algebra $\mathcal{L}(F)$, then $\sigma(T)$ and acc $\sigma(T)$ denote the spectrum and the set of all accumulation points of $\sigma(T)$, respectively. According to [17, Theorem 2.4] and [30, Theorem 2.2], a bounded adjointable operator T in $\mathcal{L}(F)$ has closed range if and only if T has a Moore–Penrose inverse, if and only if $0 \notin \text{acc}\,\sigma(T\,T^*)$, if and only if $0 \notin \text{acc}\,\sigma(T^*T)$. In particular, if T is selfadjoint then T has closed range if and only if $0 \notin \text{acc}\,\sigma(T)$. We use these facts in the proof of the following results.

Lemma 3.1. *Suppose F is a Hilbert $\mathcal{A}$-module and P, Q are orthogonal projections in $\mathcal{L}(F)$. Then $P - Q$ has closed range if and only if $P + Q$ has closed range.*

Proof. Following the argument of Koliha and Rakočević [18], for every $\lambda \in \mathbb{C}$ we have

$$(\lambda - 1 + P)(\lambda - (P - Q))(\lambda - 1 + Q) = \lambda(\lambda^2 - 1 + PQ), \tag{3.1}$$

$$(\lambda - 1 + P)(\lambda - (P + Q))(\lambda - 1 + Q) = \lambda((\lambda - 1)^2 - PQ). \tag{3.2}$$

Using the above equations and the facts that $\sigma(P) \subset \{0, 1\}$ and $\sigma(Q) \subset \{0, 1\}$, we obtain that $Ran(P - Q)$ is closed if and only if $0 \notin \text{acc}\,\sigma(P - Q)$, if and only if $1 \notin \text{acc}\,\sigma(PQ)$, if and only if $0 \notin \text{acc}\,\sigma(P + Q)$, if and only if $Ran(P + Q)$ is closed. □

Lemma 3.2. *Suppose F is a Hilbert $\mathcal{A}$-module and P, Q are orthogonal projections in $\mathcal{L}(F)$. Then the following conditions are equivalent:*

(i) *PQ has closed range,*
(ii) *$1 - P - Q$ has closed range,*
(iii) *$1 - P + Q$ has closed range,*
(iv) *$1 - Q + P$ has closed range.*

Proof. Suppose $\lambda \in \mathbb{C} \setminus \{0, 1\}$. In view of the Eq. (3.2), we conclude that $\lambda \in \sigma(P + Q)$ if and only if $(\lambda - 1)^2 \in \sigma(PQ)$.

The above fact together with Remark 1.2.1 of [23] imply that PQ has closed range if and only if $0 \notin \text{acc}\,\sigma(PQP)$, if and only if $0 \notin \text{acc}\,\sigma(P^2Q)$, if and only if $1 \notin \text{acc}\,\sigma(P + Q)$, if and only if $0 \notin \text{acc}\,\sigma(1 - P - Q)$, if and only if $1 - P - Q$ has closed range. This proves the equivalence of (i) and (ii). The statements (ii), (iii) and (iv) are equivalent by Lemma 3.1. □

Remark 3.3. Suppose E, F are two Hilbert $\mathcal{A}$-modules then the set of all ordered pairs of elements $E \oplus F$ from E and F is a Hilbert $\mathcal{A}$-module with respect to the $\mathcal{A}$-valued inner product $\langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle_E + \langle y_1, y_2 \rangle_F$, cf. [26, Example 2.14]. In particular, it can be easily seen that L is a closed submodule of F if and only if $L \oplus \{0\}$ is a closed submodule of $F \oplus F$.

Lemma 3.4. *Suppose P and Q are orthogonal projections on a Hilbert $\mathcal{A}$-module F then the following conditions are equivalent:*

(i) PQ has closed range,
(ii) $Ker(P) + Ran(Q)$ is an orthogonal summand,
(iii) $Ker(Q) + Ran(P)$ is an orthogonal summand.

Proof. Suppose

$$T = \begin{pmatrix} 1 - P & Q \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(F \oplus F).$$

Then $Ran(T) = (Ran(1-P) + Ran(Q)) \oplus \{0\}$ and $Ran(T\,T^*) = Ran(1-P+Q) \oplus \{0\}$. Using Lemmata 2.1, 3.2 and Remark 3.3, we infer that PQ has closed range if and only if $1-P+Q$ has closed range, if and only if $Ran(T\,T^*) = Ran(1-P+Q) \oplus \{0\}$ is closed, if and only if $Ran(T) = (Ran(1-P)+Ran(Q)) \oplus \{0\}$ is closed, if and only if $Ran(1-P)+Ran(Q)$ is closed. In particular, $Ran(1-P+Q) = Ran(1-P)+Ran(Q)$ is an orthogonal summand. This proves that the conditions (i) and (ii) are equivalent. Now, consider the matrix operator

$$\tilde{T} = \begin{pmatrix} 1 - Q & P \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(F \oplus F).$$

A similar argument shows that PQ has closed range if and only if $Ran(1-Q+P) = Ran(1-Q)+Ran(P)$ is closed which shows that conditions (i) and (iii) are equivalent. □

Suppose M and N are closed submodule of a Hilbert $\mathcal{A}$-module E and P_M and P_N are orthogonal projection onto M and N, respectively. Then $P_M P_N = P_M$ if and only if $P_N P_M = P_M$, if and only if $M \subset N$. Beside these, the following statements are equivalent

- P_M and P_N commute, i.e. $P_M P_N = P_N P_M$,
- $P_M P_N = P_{M \cap N}$,
- $P_M P_N$ is an orthogonal projection,
- $P_{M^\perp}$ and P_N commute,
- $P_{N^\perp}$ and P_M commute,
- $P_{M^\perp}$ and $P_{N^\perp}$ commute,
- $M = M \cap N + M \cap N^\perp$.

Proposition 3.5. *Suppose P and Q are orthogonal projections on a Hilbert $\mathcal{A}$-module F and $\overline{Ker(Q) + Ran(P)}$ is an orthogonal summand in F. If R is the orthogonal projection onto the closed submodule $\overline{Ker(Q) + Ran(P)}$ and $PQ \neq 0$ then*

$$\gamma(PQ)^2 + \|(1-P)QR\|^2 \geqslant 1. \tag{3.3}$$

Proof. The inclusion $Ker(Q) \subset \overline{Ker(Q) + Ran(P)}$ implies that the orthogonal projection $1 - Q$ onto $Ker(Q)$ satisfies $(1-Q)R = R(1-Q) = 1 - Q$, consequently, QR is an orthogonal projection and $Ran(QR)$ is orthogonally complemented in F. Since

$$Ran(QP) \subset Ran(QR) \subset \overline{Ran(QP)},$$

we have $\overline{Ran(QP)} = Ran(QR)$ and so $\overline{Ran(QP)}$ is orthogonally complemented. Therefore, $Ker(PQ)^{\perp} = Ran(QR)$. Suppose $x \in Ker(PQ)^{\perp} \subset Ran(Q)$ and $\|x\| = 1$. Then, since $x = QRx = Qx$, we have

$$\begin{aligned}\|PQx\|^2 + \|(1-P)QR\|^2 &\geqslant \|PQx\|^2 + \|(1-P)Qx\|^2 \\ &\geqslant \|\langle PQx, PQx\rangle + \langle (1-P)Qx, (1-P)Qx\rangle\| \\ &= \|\langle Qx, Qx\rangle\| = \|Qx\|^2 = 1.\end{aligned}$$

By definition, the infimum of $\|PQx\|$ is $\gamma(PQ)$. Therefore, $\gamma(PQ)^2 + \|(1-P)QR\|^2 \geqslant 1$. □

Note that as we set $\mathcal{A} = \mathbb{C}$ i.e. if we take F to be a Hilbert space, the inequality (3.3) changes to an equality. In view of this notification, the following problem arises in the framework of Hilbert C*-modules.

Problem 3.6. Suppose P and Q are orthogonal projections on a Hilbert $\mathcal{A}$-module F and $\overline{Ker(Q) + Ran(P)}$ is an orthogonal summand in F. If R is the orthogonal projection onto the closed submodule $\overline{Ker(Q) + Ran(P)}$ and $PQ \neq 0$ then characterize those C*-algebras $\mathcal{A}$ for which the following equality holds:

$$\gamma(PQ)^2 + \|(1-P)QR\|^2 = 1. \tag{3.4}$$

To solve the problem, it might be useful to know that $\gamma(PQ) \leqslant \|PQx\|$ for all $x \in Ker(PQ)^{\perp} \subset Ran(Q)$ of norm $\|x\| = 1$, therefore

$$\gamma(PQ)^2 + \|(1-P)Qx\|^2 \leqslant \|PQx\|^2 + \|(1-P)Qx\|^2 = \|Px\|^2 + \|(1-P)x\|^2.$$

Corollary 3.7. *Suppose P and Q are orthogonal projections on a Hilbert $\mathcal{A}$-module F. If $\delta = \|(1-P)QR\| < 1$ and R is the orthogonal projection onto the orthogonal summand $\overline{Ker(Q) + Ran(P)}$ then PQ has closed range.*

Proof. Suppose $PQ \neq 0$ (in the case $PQ = 0$ the result is clear). According to Proposition 3.5 and its proof, $Ker(PQ)^{\perp} = Ran(QR)$ is orthogonally complimented and $\gamma(PQ)^2 \geqslant 1 - \delta^2 > 0$. Therefore, PQ has closed range by Lemma 2.4. □

Two different concepts of angle between subspaces of a Hilbert space was first introduced by Dixmier and Friedrichs, see [8,14,1] and the excellent survey by Deutsch [7] for more historical notes and information. We generalized Dixmier's definition for the angle between two submodules of a Hilbert C*-module.

Definition 3.8. The Dixmier (or minimal) angle between submodules M and N of a Hilbert C*-module E is the angle $\alpha_0(M, N)$ in $[0, \pi/2]$ whose cosine is defined by

$$c_0(M, N) = \sup\{\|\langle x, y\rangle\| : x \in M,\ \|x\| \leqslant 1\ ,\ y \in N,\ \|y\| \leqslant 1\}.$$

Suppose M and N are submodule of a Hilbert C*-module E, then $(M+N)^{\perp} = M^{\perp} \cap N^{\perp}$. In particular, if $\overline{M + N}$ is orthogonally complemented in E then

$$(M^{\perp} \cap N^{\perp})^{\perp} = (M+N)^{\perp\perp} = \overline{M + N}.$$

Theorem 3.9. *Suppose $S \in \mathcal{L}(E, F)$ and $T \in \mathcal{L}(F, G)$ are bounded adjointable operators with closed range. Then the following three conditions are equivalent:*

(i) TS has closed range,
(ii) $Ker(T) + Ran(S)$ is an orthogonal summand in F,
(iii) $Ker(S^*) + Ran(T^*)$ is an orthogonal summand in F.

Furthermore, if $c_0(Ran(S), Ker(T) \cap [Ker(T) \cap Ran(S)]^{\perp}) < 1$ *and* $\overline{Ker(S^*) + Ran(T^*)}$ *is an orthogonal summand then TS has closed range.*

Proof. Taking $P = T^\dagger T$ and $Q = SS^\dagger$, then

$$Ker(P) = Ker(T)\,,\ Ran(P) = Ran(T^\dagger) = Ran(T^*),$$
$$Ker(Q) = Ker(S^\dagger) = Ker(S^*)\,,\ \text{and}\ Ran(Q) = Ran(S).$$

The equivalence of (i), (ii) and (iii) directly follows from the above equalities and Lemma 3.4. To establish the statement of the second part suppose R is the orthogonal projection onto the orthogonal summand $\overline{Ker(Q) + Ran(P)}$ then $(1-P)R$ is the projection onto

$$\begin{aligned} M = Ker(P) \cap [\,\overline{Ran(P) + Ker(Q)}\,] &= Ker(T) \cap [\,\overline{Ran(T^*) + Ker(S^*)}\,] \\ &= Ker(T) \cap [Ran(T^*)^{\perp} \cap Ker(S^*)^{\perp}\,]^{\perp} \\ &= Ker(T) \cap [Ker(T) \cap Ran(S)]^{\perp}. \end{aligned}$$

If neither M nor $Ran(S)$ is $\{0\}$, by commutativity of R with P and Q, we obtain

$$\begin{aligned} \|(1-P)QR\| &= \|RQ(1-P)\| \\ &= \|Q(1-P)R\| \\ &= \sup\{\|\langle Q(1-P)Rx, y\rangle\| : x, y \in F \text{ and } \|x\| \leqslant 1,\ \|y\| \leqslant 1\} \\ &= \sup\{\|\langle (1-P)Rx, Qy\rangle\| : x, y \in F \text{ and } \|x\| \leqslant 1,\ \|y\| \leqslant 1\} \\ &= \sup\{\|\langle x, y\rangle\| : x \in M,\ y \in Ran(S) \text{ and } \|x\| \leqslant 1,\ \|y\| \leqslant 1\} \\ &= c_0(M, Ran(S)). \end{aligned}$$

The statement is now derived from the above argument and Corollary 3.7. □

Recall that a bounded adjointable operator between Hilbert C*-modules admits a bounded adjointable Moore–Penrose inverse if and only if the operator has closed range. This lead us to the following results.

Corollary 3.10. *Suppose* $S \in \mathcal{L}(E, F)$ *and* $T \in \mathcal{L}(F, G)$ *possess bounded adjointable Moore–Penrose inverses* $S^\dagger$ *and* $T^\dagger$. *Then* $(TS)^\dagger$ *is bounded if and only if* $Ker(T) + Ran(S)$ *is an orthogonal summand, if and only if* $Ker(S^*) + Ran(T^*)$ *is an orthogonal summand. Moreover, if the Dixmier angle between* $Ran(S)$ *and* $Ker(T) \cap [Ker(T) \cap Ran(S)]^{\perp}$ *is positive and* $\overline{Ker(S^*) + Ran(T^*)}$ *is an orthogonal summand then* $(TS)^\dagger$ *is bounded.*

Now, it is natural to ask for the reverse order law, that is, if $S \in \mathcal{L}(E, F)$ and $T \in \mathcal{L}(F, G)$ possess bounded adjointable Moore–Penrose inverses $S^\dagger$ and $T^\dagger$, when does the equation $(TS)^\dagger = S^\dagger T^\dagger$ hold? We will answer this question elsewhere. Note that the above conditions do not ensure the equality.

Recall that a C*-algebra of compact operators is a c_0-direct sum of elementary C*-algebras $\mathcal{K}(H_i)$ of all compact operators acting on Hilbert spaces $H_i,\ i \in I$, i.e. $\mathcal{A} = c_0\text{-}\bigoplus_{i\in I}\mathcal{K}(H_i)$, cf. [2, Theorem 1.4.5]. Suppose $\mathcal{A}$ is an arbitrary C*-algebra of compact operators. Magajna and Schweizer have shown, respectively, that every norm closed (coinciding with its biorthogonal complement, respectively) submodule of every Hilbert $\mathcal{A}$-module is automatically an orthogonal summand, cf. [21,27]. In this situation, every bounded $\mathcal{A}$-linear map $T : E \to F$ is automatically adjointable. Recently further generic properties of the category of Hilbert C*-modules over C*-algebras which characterize precisely the C*-algebras of compact operators have been found in [11–13]. We close the paper with the observation that we can reformulate Theorem 3.9 in terms of bounded $\mathcal{A}$-linear maps on Hilbert C*-modules over C*-algebras of compact operators.

Corollary 3.11. *Suppose $\mathcal{A}$ is an arbitrary C^*-algebra of compact operators, E, F, G are Hilbert $\mathcal{A}$-modules and $S : E \to F$ and $T : F \to G$ are bounded $\mathcal{A}$-linear maps with close range. Then the following conditions are equivalent:*

(i) *TS has closed range,*
(ii) $Ker(T) + Ran(S)$ *is closed,*
(iii) $Ker(S^*) + Ran(T^*)$ *is closed.*

Furthermore, if $c_0(Ran(S), Ker(T) \cap [Ker(T) \cap Ran(S)]^{\perp}) < 1$ *then TS has closed range.*

In view of Corollary 3.11, one may ask about the converse of the last conclusion. To find a solution, one way reader has is to solve Problem 3.6.

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