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Slowly Varying Functions and Asymptotic Relations

R. BOJANIC*

Department of Mathematics, Ohio State University, Columbus, Ohio 43210

AND

E. SENETA

Australian National University

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1. Introduction and Results

We shall say that L is a slowly varying (SV) function if L is a real-valued, positive, and measurable function on $[A, \infty)$, $A > 0$, and if

$$\lim_{x \to \infty} \frac{L(\lambda x)}{L(x)} = 1 \tag{1.1}$$

for every $\lambda > 0$. The most important properties of SV functions may be stated as follows:

UNIFORM CONVERGENCE THEOREM. *If L is a* SV *function, then for every $[a, b]$, $0 < a < b < \infty$, the relation* (1.1) *holds uniformly with respect to $\lambda \in [a, b]$.*

REPRESENTATION THEOREM. *If L is a* SV *function, then there exists a positive number $B \geqslant A$ such that for all $x \geqslant B$ we have*

$$L(x) = \exp\left(\eta(x) + \int_B^\infty \frac{\epsilon(t)}{t}\, dt\right),$$

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where η and ϵ are bounded measurable functions on $[B, \infty)$ such that

$$\eta(x) \to c \ (|c| < \infty) \qquad \textit{and} \qquad \epsilon(x) \to 0 \ (x \to \infty).$$

The functions η and ϵ in the Representation Theorem are clearly not uniquely determined. We shall need here the Representation Theorem with ϵ defined by

$$\epsilon(x) = \frac{1}{\log \lambda_0} \log \left(\frac{L(\lambda_0 x)}{L(x)}\right), \qquad x \geqslant B, \tag{1.2}$$

where λ_0 is an arbitrary fixed number >1.

Both of these two theorems were first obtained by J. Karamata [1, 2] for continuous SV functions. The Uniform Convergence Theorem for measurable SV functions was proved by T. van Aardenne-Ehrenfest, N. G. de Bruijn, and J. Korevaar [3], H. Delange [4], and W. Matuszewska [5, 6]. In addition to this, a proof of the Uniform Convergence Theorem, due to A. S. Besicovitch, was given in Ref. [7]. A close examination of proofs in Refs. [5] and [7] shows that they are valid for continuous, but not necessarily for measurable, SV functions (see Refs. [6], [8], and [9]). The Representation Theorem for SV functions L such that $\log L$ is integrable on every compact subinterval of (A, ∞) was proved in Ref. [3]. Finally, the Representation Theorem in the present form, for arbitrary measurable SV functions, was established by N. G. de Bruijn [10].

In this paper we shall give a simple and reasonably general condition for a SV function L in order that the asymptotic relation

$$\lim_{x \to \infty} \frac{L(xL^\alpha(x))}{L(x)} = 1 \tag{1.3}$$

hold for every real number α. This relation does not hold for every SV function L as the following example shows:

For $x \geqslant 1$ let $L(x) = \exp((\log x)^\beta)$, $0 < \beta < 1$. Then for any real number $\alpha \neq 0$ we have

$$\lim_{x \to \infty} \frac{L(xL^\alpha(x))}{L(x)} = \begin{cases} 1 & \text{if} \quad 0 < \beta < \frac{1}{2}, \\ \exp(\alpha\beta) & \text{if} \quad \beta = \frac{1}{2}, \\ 0 & \text{if} \quad \alpha < 0 \ \text{ and } \ \frac{1}{2} < \beta < 1, \\ \infty & \text{if} \quad \alpha > 0 \ \text{ and } \ \frac{1}{2} < \beta < 1. \end{cases}$$

The asymptotic relation (1.3) appears in a paper of A. Békéssy [11] in connection with the inversion of asymptotic relations involving SV functions. It is contained implicitly also in a paper of J. Karamata [12].

Our first result in this direction can be stated as follows:

THEOREM 1. *Let L be a* SV *function.*[1] *If*

$$\left(\frac{L(\lambda_0 x)}{L(x)} - 1\right) \log L(x) \to 0 \qquad (x \to \infty) \tag{1.4}$$

for a fixed $\lambda_0 > 1$, *then* (1.3) *holds for every real number* α.

It is not difficult to find other conditions more special than (1.4) for the validity of (1.3). For instance, (1.3) will hold for a SV function L if there exists $\lambda_0 > 1$ such that

$$\frac{L(\lambda_0 x)}{L(x)} = 1 + O\left(\frac{1}{\log x}\right) \qquad (x \to \infty).$$

In this case, the condition (1.4) is satisfied since for any SV function L we have $\log L(x)/\log x \to 0$ $(x \to \infty)$. If L is a positive and continuously differentiable function such that

$$\left(\frac{xL'(x)}{L(x)}\right) \log L(x) \to 0 \qquad (x \to \infty)$$

the asymptotic relation (1.3) holds again. In this case, the condition (1.4) is satisfied for *every* $\lambda_0 > 1$. Another even more restrictive condition for the validity of (1.3),

$$\frac{xL'(x)}{L(x)} = O\left(\frac{1}{\log x}\right) \qquad (x \to \infty),$$

is mentioned in Ref. [10].

In addition to Theorem 1 we shall prove here a slightly more general result.

THEOREM 2. *Let L be a* SV *function such that for a fixed* $\lambda_0 > 1$

$$\left(\frac{L(\lambda_0 x)}{L(x)} - 1\right) \log T(x) \to 0 \qquad (x \to \infty), \tag{1.5}$$

where T is a positive function.

If $x^\gamma T(x)$ *is an eventually increasing function for some* $\gamma \in (0, 1)$, *we have for any* $0 < \delta < 1/\gamma$

$$\lim_{x \to \infty} \frac{L(xT^\delta(x))}{L(x)} = 1. \tag{1.6}$$

[1] In the first version of this paper it was assumed also that $L(x) \to \infty$ $(x \to \infty)$. We are obliged to the Referee for the remark that this condition is unnecessary.

If, on the other hand, $x^{-\gamma}T(x)$ is an eventually decreasing function for some $\gamma \in (0, 1)$ we have for any $0 < \delta < 1/\gamma$

$$\lim_{x\to\infty} \frac{L(xT^{-\delta}(x))}{L(x)} = 1. \tag{1.7}$$

In Section 2 we shall give proofs of Theorems 1 and 2. The remaining Sections 3 and 4 are of complementary nature. In Section 3 we shall consider the problem of inversion of asymptotic relations involving SV functions and its applications in the theory of branching stochastic processes. In Section 4, by a small modification of the arguments used in Ref. [3], we shall give short proofs of the Uniform Convergence and Representation Theorems with ϵ as in (1.2).

2. Proofs of Theorems 1 and 2

We shall show first that Theorem 1 is a corollary to Theorem 2. By the Representation Theorem for $x \geqslant B$ we have $L(x) = c(x)\,\ell(x)$, where

$$c(x) = \exp(\eta(x)) \qquad \text{and} \qquad \ell(x) = \exp\left(\int_B^x \frac{\epsilon(t)}{t}\,dt\right).$$

Since $c(x) \to e^c (x \to \infty)$, $|c| < \infty$, from (1.4) it follows that

$$\left(\frac{L(\lambda_0 x)}{L(x)} - 1\right) \log \ell(x) \to 0 \qquad (x \to \infty).$$

Moreover, since $\epsilon(x) \to 0$ $(x \to \infty)$ it follows that $x^{\gamma}\ell(x)$ is eventually increasing and $x^{-\gamma}\ell(x)$ is eventually decreasing for every fixed $\gamma \in (0, 1)$. Hence, by Theorem 2, with $T = \ell$, we find that for every real number α we have

$$\lim_{x\to\infty} \frac{L(x\ell^{\alpha}(x))}{L(x)} = 1.$$

On the other hand, since $L(x)/\ell(x) \to e^c (x \to \infty)$ with $|c| < \infty$, we have, by the Uniform Convergence Theorem (since $x\ell^{\alpha}(x) \to \infty$)

$$\lim_{x\to\infty} \frac{L(xL^{\alpha}(x))}{L(x\ell^{\alpha}(x))} = 1.$$

Finally,

$$\frac{L(xL^{\alpha}(x))}{L(x)} = \frac{L(xL^{\alpha}(x))}{L(x\ell^{\alpha}(x))} \cdot \frac{L(x\ell^{\alpha}(x))}{L(x)}$$

and (1.3) follows.

Thus, it remains only to prove Theorem 2. From the Representation Theorem it follows that for $x \geqslant B$ we have the inequality

$$\left| \log \left(\frac{L(xT^{\delta}(x))}{L(x)} \right) \right| \leqslant | \eta(xT^{\delta}(x)) - c | + | \eta(x) - c | + \left| \int_{x}^{xT^{\delta}(x)} \frac{\epsilon(t)}{t} \, dt \right|.$$

Since $x^{\gamma} T(x)$ is increasing for $x \geqslant X_1 \geqslant B$ and $0 < \delta\gamma < 1$, we have

$$xT^{\delta}(x) \geqslant (x^{\gamma} T(x))^{\delta} x^{1-\delta\gamma} \geqslant (X_1{}^{\gamma} T(X_1))^{\delta} x^{1-\delta\gamma} \to \infty \qquad (x \to \infty).$$

Hence

$$\limsup_{x \to \infty} \left| \log \left(\frac{L(xT^{\delta}(x))}{L(x)} \right) \right| \leqslant \limsup_{x \to \infty} \left| \int_{x}^{xT^{\delta}(x)} \frac{\epsilon(t)}{t} \, dt \right|. \tag{2.1}$$

We shall show next that

$$I(x) = \int_{x}^{xT^{\delta}(x)} \frac{\epsilon(t)}{t} \, dt \to 0 \qquad (x \to \infty). \tag{2.2}$$

We note first that the hypothesis (1.5) and the Representation Theorem with ϵ as in (1.2) imply that

$$\epsilon(x) = \frac{1}{\log \lambda_0} \log \left(\frac{L(\lambda_0 x)}{L(x)} \right) = o \left(\frac{1}{| \log T(x) |} \right) \qquad (x \to \infty). \tag{2.3}$$

Hence, given $\epsilon > 0$, we can find $X_{\epsilon} \geqslant X_1$ such that

$$| \epsilon(u) | \leqslant \frac{\epsilon}{| \log T(u) |} \qquad \text{for all } u \geqslant X_{\epsilon}. \tag{2.4}$$

We shall show next that for all $x \geqslant X_{\epsilon}$ and all u between x and $xT^{\delta}(x)$ we have the inequality

$$| \epsilon(u) | \leqslant \frac{1}{1 - \gamma\delta} \frac{1}{\log T(x)}. \tag{2.5}$$

Suppose first that $T(x) \geqslant 1$ for an $x \geqslant X_{\epsilon}$. Since $0 < \delta\gamma < 1$ and $x^{\gamma} T(x)$ is increasing, we have for any $u \in [x, xT^{\delta}(x)]$

$$(xT^{\delta}(x))^{\gamma} T(u) \geqslant u^{\gamma} T(u) \geqslant x^{\gamma} T(x),$$

i.e.,

$$T(u) \geqslant T^{1-\gamma\delta}(x) \geqslant 1.$$

Hence

$$| \log T(u) | = \log T(u) \geqslant (1 - \gamma\delta) \log T(x) = (1 - \gamma\delta) \, | \log T(x) |$$

and (2.5) follows from (2.4).

If $0 < T(x) < 1$ for this $x \geqslant X_\epsilon$ we have, likewise, for any $u \in [xT^\delta(x), x]$

$$(xT^\delta(x))^\gamma\, T(u) \leqslant u^\gamma T(u) \leqslant x^\gamma T(x),$$

i.e.,

$$T(u) \leqslant T^{1-\gamma\delta}(x) < 1.$$

Hence

$$|\log T(u)| = \log\left(\frac{1}{T(u)}\right) \geqslant (1-\gamma\delta)\log\left(\frac{1}{T(x)}\right) = (1-\gamma\delta)\,|\log T(x)|$$

and (2.5) follows again from (2.4).

Thus, (2.5) holds for each $x \geqslant X_\epsilon$ and all u between x and $xT^\delta(x)$.

Next, using the inequality (2.5) we find that for $x \geqslant X_\epsilon$ we have

$$\begin{aligned} |I(x)| &\leqslant \left| \int_x^{xT^\delta(x)} \frac{|\epsilon(u)|}{u}\, du \right| \\ &\leqslant \frac{\epsilon}{(1-\gamma\delta)\,|\log T(x)|} \left| \int_x^{xT^\delta(x)} \frac{du}{u} \right| \\ &\leqslant \frac{\delta\epsilon}{1-\gamma\delta} \end{aligned}$$

and (2.2) follows. Finally, (1.6) follows in view of (2.1) and (2.2).

If $x^{-\gamma}T(x)$ is eventually decreasing, then $x^\gamma(1/T(x))$ is eventually increasing and (1.7) follows from (1.6).

3. Inversion of Asymptotic Relations and Applications

The problem of inversion of asymptotic relations involving SV functions in its simplest form can be stated as follows:

Suppose that L is a SV function and that there exists a positive function f such that $f(t) \to \infty$ $(t \to \infty)$ and

$$f(t)\,L^\alpha(f(t)) \simeq Ct^\beta \qquad (t \to \infty)\,^2 \tag{3.1}$$

where $\beta > 0$, $C > 0$. It is required to find conditions under which (3.1) may be inverted to give a simple asymptotic expression for f. This problem can be solved by means of the following theorem of N. G. de Bruijn [10]:

THEOREM. *If L is a* SV *function, then there exists a* SV *function L^* such that*

$$L^*(xL(x))\,L(x) \to 1 \qquad (x \to \infty), \tag{3.2}$$

$$L(xL^*(x))\,L^*(x) \to 1 \qquad (x \to \infty). \tag{3.3}$$

Moreover, L^* is asymptotically uniquely determined.

[2] $f(x) \simeq g(x)(x \to \infty)$ means that $\lim_{x\to\infty} f(x)/g(x) = 1$.

Since $f(t) \to \infty$ $(t \to \infty)$, using (3.1), (3.2) (with L replaced by L^α) and the Uniform Convergence Theorem we find that

$$\frac{1}{L^\alpha(f(t))} \simeq L^*(f(t) L^\alpha(f(t))) \simeq L^*(t^\beta) \qquad (t \to \infty)$$

and it follows that

$$f(t) \simeq Ct^\beta L^*(t^\beta) \qquad (t \to \infty). \tag{3.4}$$

De Bruijn's theorem is an existence result and if we want to obtain more information about L^* we have to restrict appropriately the class of SV functions. If, for instance, the function L has the property that

$$\frac{L(xL^{-\alpha}(x))}{L(x)} \to 1 \qquad (x \to \infty), \tag{3.5}$$

then $L^\alpha(xL^{-\alpha}(x)) L^{-\alpha}(x) \to 1$ $(x \to \infty)$ and (3.3) (with L replaced by L^α) implies that $L^*(x) \simeq L^{-\alpha}(x)$ $(x \to \infty)$. Hence the relation (3.4) can be replaced by

$$f(t) \simeq Ct^\beta L^{-\alpha}(t^\beta) \qquad (t \to \infty). \tag{3.6}$$

The asymptotic relation (3.1) can be inverted also by means of the following argument of A. Békéssy [11] and J. Karamata [12]. If the SV function L satisfies the condition

$$\frac{L(xL^\alpha(x))}{L(x)} \to 1 \qquad (x \to \infty), \tag{3.7}$$

then (3.1), (3.7) and the Uniform Convergence Theorem imply that

$$1 \simeq \frac{L(f(t) L^\alpha(f(t)))}{L(f(t))} \simeq \frac{L(t^\beta)}{L(f(t))} \qquad (t \to \infty)$$

and (3.6) follows from (3.1).

The results of J. Karamata [2] and A. Békéssy [11] are of considerable relevance in the theory of branching stochastic processes. Part of Karamata's theorem has been recently rediscovered in this context by R. S. Slack (see Ref. [13, Lemma 2]). We shall give here a brief description of the part of Slack's paper which is related to the problem of inversion of asymptotic relations.

Let (Z_n) be the simple Galton–Watson process initiated by a single ancestor with offspring distribution generated by

$$F(s) = E[s^{Z_1}] = \sum_{j=0}^{\infty} p_j s^j, \qquad s \in [0, 1].$$

Here $p_j \geqslant 0$ and $\sum_{j=0}^{\infty} p_j = 1$ (see T. E. Harris, Ref. [14, Chap. 1]). Then the probability distribution of the number of individuals in the n-th generation is given by

$$F_n(s) = E[s^{Z_n}], \qquad s \in [0, 1],$$

where $F_n(s)$ is the n-th functional iterate of $F(s)$. We are concerned here with the critical process when $F'(1-) = 1$, $F(0) > 0$; hence $F_n(0) < 1$ and $F_n(0) \nearrow 1$ $(n \to \infty)$ so that the distribution of the extinction time N is given by

$$P[N > n] = 1 - F_n(0), \qquad n = 0, 1, 2,... \quad (F_0(0) = 0).$$

The properties of the process depend heavily on the behavior of the generating function $F(s)$ as $s \to 1-$; in the cited paper, R. S. Slack has studied the particular form

$$F(s) = s + (1-s)^{1+\theta} L\left(\frac{1}{1-s}\right)$$

where $0 < \theta \leqslant 1$ and L is a SV function. It follows that

$$1 - F_{n+1}(0) = 1 - F_n(0) - (1 - F_n(0))^{1+\theta} L\left(\frac{1}{1-F_n(0)}\right),$$
$$n = 0, 1, 2,... \qquad (F_0(0) = 0)$$

and consequently,

$$(1 - F_n(0))^{\theta} L\left(\frac{1}{1-F_n(0)}\right) \simeq \frac{1}{\theta n} \qquad (n \to \infty),$$

i.e.,

$$\frac{1}{1-F_n(0)} L^{-1/\theta}\left(\frac{1}{1-F_n(0)}\right) \simeq \theta^{1/\theta} n^{1/\theta} \qquad (n \to \infty), \tag{3.8}$$

(see Ref. [12, p. 52], or Ref. [13, p. 141]). This relation, however, does not in general enable us to obtain a simple explicit relation for

$$P[N > n] = 1 - F_n(0) \qquad \text{as} \qquad n \to \infty,$$

although such an inversion is clearly desirable.

If we assume that L satisfies the condition (3.7) with $\alpha = -1/\theta$ then the asymptotic relation (3.8) in view of (3.1) and (3.6) implies immediately that

$$\frac{1}{1-F_n(0)} \simeq \theta^{1/\theta} n^{1/\theta} L^{1/\theta}(n^{1/\theta}) \qquad (n \to \infty).$$

Hence

$$1 - F_n(0) \simeq \theta^{-1/\theta} n^{-1/\theta} L^{-1/\theta}(n^{1/\theta}) \qquad (n \to \infty). \tag{3.9}$$

In the example given by Slack,

$$F(s) = s + (1 - s)^2 \left(\frac{1}{2} + \frac{1}{4} \log \frac{1}{1 - s}\right),$$

we have $\theta = 1, L(x) = \frac{1}{2} + \frac{1}{4} \log x \simeq \frac{1}{4} \log x$ $(x \to \infty)$ and the condition (3.7) is clearly satisfied with $\alpha = -1/\theta = -1$. Hence

$$1 - F_n(0) \simeq \frac{4}{n \log n} \qquad (n \to \infty).$$

If we assume that SV function L satisfies the condition

$$\frac{L(\lambda_0 x)}{L(x)} = 1 + o\left(\frac{1}{\log x}\right) \qquad (x \to \infty)$$

for some $\lambda_0 > 1$, then (3.7) will hold by Theorem 1 for any real number α and (3.9) will follow again. However, in this case, by Theorem 2 with $T(x) = x$, we find that for $0 < \theta \leqslant 1$

$$\frac{L(x^{1/\theta})}{L(x)} \to 1 \qquad (x \to \infty).$$

Consequently, the formula (3.9) can be replaced by

$$1 - F_n(0) \simeq \theta^{-1/\theta} n^{-1/\theta} L^{-1/\theta}(n) \qquad (n \to \infty).$$

Finally, it is worth remarking that precisely analogous nonexplicitness problems arose in the parallel treatment of the continuous parameter homogeneous Markov branching processes by V. M. Zolotarev [15] and may be treated in exactly the same way.

4. Proofs of the Uniform Convergence and Representation Theorems

The Uniform Convergence Theorem is equivalent to

Lemma 1. *If L is a* SV *function, then for every $\lambda > 1$ we have*

$$\sup_{1 \leqslant t \leqslant \lambda} \left| \log \left(\frac{L(tx)}{L(x)}\right) \right| \to 0 \qquad (x \to \infty).$$

In order to show this, observe first that for any $0 < a < b < \infty$ we have, by Lemma 1,

$$\sup_{a \leqslant t \leqslant b} \left| \log \left(\frac{L(tx)}{L(x)} \right) \right| = \sup_{1 \leqslant t \leqslant b/a} \left| \log \left(\frac{L(tax)}{L(x)} \right) \right|$$

$$\leqslant \sup_{1 \leqslant t \leqslant b/a} \left| \log \left(\frac{L(tax)}{L(ax)} \right) \right| + \left| \log \left(\frac{L(ax)}{L(x)} \right) \right| \to 0 \qquad (x \to \infty).$$

This and the inequality $| y - 1 | \leqslant | \log y | \exp(| \log y |)$ shows that

$$\sup_{a \leqslant t \leqslant b} \left| \frac{L(tx)}{L(x)} - 1 \right| \to 0 \qquad (x \to \infty)$$

and the Uniform Convergence Theorem follows.

For the proof of the Representation Theorem with ϵ defined by (1.2) in addition to Lemma 1, we need the following results:

LEMMA 2. *If L is a positive and measurable function on $[A, \infty)$, $A > 0$, and if*

$$\sup_{1 \leqslant t \leqslant 2} \left| \log \left(\frac{L(tx)}{L(x)} \right) \right| = O(1) \qquad (x \to \infty)$$

then there exists $B \geqslant A$ such that $\log L$ is bounded and consequently integrable on every finite interval $[a, b]$ if $a \geqslant B$.

LEMMA 3. *If L is a positive and measurable function on $[A, \infty)$, $A > 0$, and if $\log L$ is integrable on every finite subinterval of $[B, \infty)$, $B \geqslant A$, we have for $x \geqslant B$*

$$L(x) = \exp \left(\eta(x) + \int_B^x \frac{\epsilon(t)}{t} \, dt \right) \tag{4.1}$$

where, for arbitrary fixed $\lambda_0 > 1$, the functions η and ϵ are defined on $[B, \infty)$ by

$$\eta(x) = \frac{1}{\log \lambda_0} \int_B^{\lambda_0 B} \log L(t) \frac{dt}{t} + \frac{1}{\log \lambda_0} \int_1^{\lambda_0} \log \left(\frac{L(tx)}{L(x)} \right) \frac{dt}{t}, \tag{4.2}$$

$$\epsilon(x) = \frac{1}{\log \lambda_0} \log \left(\frac{L(\lambda_0 x)}{L(x)} \right). \tag{4.3}$$

In order to prove the Representation Theorem, suppose that L is a SV function and $\lambda_0 > 1$. Then by Lemmas 1 and 2 there exists $B \geqslant A$ such that $\log L$ is integrable on every compact interval $[a, b]$ with $a \geqslant B$, and

$$\int_1^{\lambda_0} \log \left(\frac{L(tx)}{L(x)}\right) \frac{dt}{t} \to 0 \qquad (x \to \infty).$$

By Lemma 3 the relation (4.1) holds for $x \geqslant B$ with the functions η and ϵ defined by (4.2) and (4.3) respectively. The functions η and ϵ are clearly bounded measurable functions on $[B, \infty)$ with $\eta(x) \to c$ $(x \to \infty)$, $|c| < \infty$, $\epsilon(x) \to 0$ $(x \to \infty)$.

The proof of Lemma 1, due to P. O. Frederickson [16], is based essentially on the same ideas as the proof of Theorem 3 in Ref. [3]. The Lemma 2, which shows that $\log L$ is integrable on every compact subinterval of $[B, \infty)$, makes it possible to prove the Representation Theorem by methods used in Ref. [3] for SV functions L such that $\log L$ is integrable on every compact subinterval of $[A, \infty)$. Finally, Lemma 3 and its proof, except for notation, are the same as Theorem 6 and its proof in Ref. [3].

Proof of Lemma 1. In order to prove Lemma 1, let L be a SV function, $\ell = \log \lambda$, $\lambda > 1$, and let f be defined by

$$f(x) = \begin{cases} \log L(e^x) & \text{if} \quad x \geqslant \log A, \\ 0 & \text{if} \quad x < \log A. \end{cases}$$

Then, as is easy to see, f is a measurable function on R and

$$f(x + \mu) - f(x) \to 0 \qquad (x \to \infty) \tag{4.4}$$

for every $\mu \in R$. If we show that

$$\sup_{0 \leqslant \mu \leqslant \ell} |f(x + \mu) - f(x)| \to 0 \qquad (x \to \infty) \tag{4.5}$$

the Lemma 1 will clearly be proved.

Suppose that (4.4) holds and that (4.5) is not true. Then we can find $\delta > 0$ and sequences (μ_n) and (x_n) such that

$$\mu_n \in [0, \ell], \qquad x_n \geqslant n, \qquad |f(x_n + \mu_n) - f(x_n)| \geqslant \delta, \qquad n = 1, 2,... \tag{4.6}$$

Let $0 < \epsilon < \delta/4$ and

$$M_n = \{t : \sup_{x \geqslant n} |f(x + t) - f(x)| \leqslant \epsilon\}.$$

Let m^* be the outer measure of subsets of R. Since $(M_n \cap [0, 3\ell])$ is an increasing sequence of subsets of R converging to $[0, 3\ell]$, we have

$$\lim_{n\to\infty} m^*(M_n \cap [0, 3\ell]) = 3\ell$$

(see Ref. [17, p. 20]). Hence we can find N such that $m^*(M_n \cap [0, 3\ell]) \geqslant \frac{5}{2}\ell$. Let

$$S = \{t : |f(t) - f(x_N)| \leqslant \epsilon\} \cap [x_n, x_N + 4\ell],$$
$$T = \{t : |f(t) - f(x_N + \mu_N)| \leqslant \epsilon\} \cap [x_N, x_N + 4\ell].$$

Clearly, S and T are disjoint measurable subsets of $[x_N, x_N + 4\ell]$; if they had a point in common we would immediately obtain a contradiction to (4.6). Hence

$$m(S) + m(T) \leqslant 4\ell. \tag{4.7}$$

On the other hand, if we denote by X and Y the set $M_N \cap [0, 3\ell]$ translated by x_N and $x_N + \mu_N$, respectively, i.e., if

$$X = M_N \cap [0, 3\ell] \oplus \{x_N\},$$
$$Y = M_N \cap [0, 3\ell] \oplus \{x_N + \mu_N\},$$

then it is easy to see that $X \subseteq S$ and $Y \subseteq T$. Consequently,

$$\tfrac{5}{2}\ell \leqslant m^*(M_N \cap [0, 3\ell]) = m^*(X) \leqslant m(S),$$
$$\tfrac{5}{2}\ell \leqslant m^*(M_N \cap [0, 3\ell]) = m^*(Y) \leqslant m(T),$$

and so $m(S) + m(T) \geqslant 5\ell$, which is impossible in view of (4.7).

Proof of Lemma 2. Choose $B \geqslant A$ so that

$$x \geqslant B \Rightarrow \sup_{1\leqslant t\leqslant 2} \left| \log\left(\frac{L(tx)}{L(x)}\right) \right| \leqslant C. \tag{4.8}$$

Take any $[a, b]$ with $a \geqslant B$ and choose n such that $1 \leqslant b/a \leqslant 2^n$. We have then

$$\begin{aligned}
\sup_{a\leqslant t\leqslant b} |\log L(t)| &= \sup_{1\leqslant t\leqslant b/a} |\log L(at)| \\
&\leqslant \sup_{1\leqslant t\leqslant 2^n} |\log L(at)| \\
&\leqslant \sum_{k=0}^{n-1} \sup_{1\leqslant t\leqslant 2} |\log L(2^k at)| \\
&\leqslant \sum_{k=0}^{n-1} \sup_{1\leqslant t\leqslant 2} \left| \log\left(\frac{L(2^k at)}{L(2^k a)}\right) \right| + \sum_{k=0}^{n-1} |\log L(2^k a)|.
\end{aligned}$$

But if $a \geqslant B$, then $2^k a \geqslant B$, $k = 0, 1, 2, \ldots$, and so, by (4.8),

$$\sup_{a \leqslant t \leqslant b} |\log L(t)| \leqslant nC + \sum_{k=0}^{n-1} |\log L(2^k a)| < \infty.$$

Proof of Lemma 3. Let $\lambda_0 > 1$, $x \geqslant B$, and

$$\delta(x) = \frac{1}{\log \lambda_0} \int_1^{\lambda_0} \log\left(\frac{L(tx)}{L(x)}\right) \frac{dt}{t}.$$

We then have

$$\log L(x) = \delta(x) + \frac{1}{\log \lambda_0} \int_1^{\lambda_0} \log L(tx) \frac{dt}{t}.$$

Since

$$\begin{aligned} \int_1^{\lambda_0} \log L(tx) \frac{dt}{t} &= \int_x^{\lambda_0 x} \log L(t) \frac{dt}{t} \\ &= \int_B^{\lambda_0 x} \log L(t) \frac{dt}{t} - \int_B^x \log L(t) \frac{dt}{t} \\ &= \int_B^{\lambda_0 B} \log L(t) \frac{dt}{t} + \int_{\lambda_0 B}^{\lambda_0 x} \log L(t) \frac{dt}{t} - \int_B^x \log L(t) \frac{dt}{t} \\ &= \int_B^{\lambda_0 B} \log L(t) \frac{dt}{t} + \int_B^x \log\left(\frac{L(\lambda_0 t)}{L(t)}\right) \frac{dt}{t} \end{aligned}$$

it follows that

$$\log L(x) = \frac{1}{\log \lambda_0} \int_B^{\lambda_0 B} \log L(t) \frac{dt}{t} + \delta(x) + \frac{1}{\log \lambda_0} \int_B^x \log\left(\frac{L(\lambda_0 t)}{L(t)}\right) \frac{dt}{t}$$

and the lemma is proved.

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