Tangential Differential Equations for Dynamical Thin/Shallow Shells

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We present the mathematical construction of a dynamical second order operational differential equation for thin/shallow shells from elasticity by using the tangential differential calculus and the oriented boundary distance function. This model extends to thin/shallow shells the “natural theory” and the theory of Love-Kirchhoff of plates. We specify the appropriate function spaces, and give existence and uniqueness theorems.

1. Introduction

The main objective of this paper is to present equations for dynamical $(N-1)$-dimensional thin/shallow shells in $\mathbb{R}^N$ using the tangential differential calculus, the oriented boundary (resp. algebraic or signed) distance function, and linear elasticity. Under a simple rheological law it takes the form of a second order operational differential equation involving a positive $V$-$H$ coercive elliptic operator for an appropriate choice of Hilbert spaces $V$ and $H$ as in Lions [1, Chap. 4]. However the constructions are quite general and more complex rheologies can be considered. The model is an extension to thin/shallow shells of the “natural theory” and the

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Love–Kirchhoff theory of plates (cf. for instance Germain [1] or Zole [1]).

The main advantage of this new model is that it is mathematically more tractable than currently available models which use local coordinates systems and Christoffel symbols. Our approach was first presented for a static model in previous papers (cf. Delfour and Zolesio [4, 5]). In order to illustrate the method and to simplify the computations we made two simplifying assumptions: the Lamé coefficient $\lambda$ was zero in the rheological law (first paper), and in both papers the strain tensor was approximated by an affine expression in the thickness variable $z$, $|z| < h$ for a shell of (possibly variable) thickness $2h > 0$.

In this paper we present a more complete and refined version of the static model. It includes the Lamé coefficient $\lambda$ and uses an approximation of the strain tensor by a quadratic expression in the thickness variable. This slight increase in complexity is motivated by the fact that in so doing we first obtain a Korn's inequality for shells. Secondly the kernel of the "approximate" strain tensor coincides with the analogue of the "rigid displacements" as in 3-D elasticity. Our previous model only gave an approximation of this mechanical property. The two models coincide when they are specialized to plates and the difference is only apparent for shells. Finally the Love–Kirchhoff theory of shells comes out of the analysis as a special case of the natural theory by looking at the same variational equation over some closed linear subspace of the Hilbert space $\mathcal{V}$ associated with the natural theory.

The dynamical model is obtained by subtracting the kinetic energy from the sum of the strain energy and the work of the body forces. The resulting second order operational differential equation involves the positive $V$-$H$ coercive operator of the static model. Hence standard results can be used to obtain the existence, uniqueness and smoothness of solutions. The Love–Kirchhoff theory is obtained by considering the same equation on some appropriate closed linear subspaces of the Hilbert spaces $\mathcal{V}$ and $H$ of the natural theory. Our results specialize to plates by using the fact that second and higher order derivatives of the oriented boundary distance function are zero. We now have a new tool to study control and optimal design problems for thin/shallow shells along the lines of the work of Lagnese and Lions [1] and others.

2. Notation, Definitions and Mathematical Description of the Shell

In order to make the paper as self contained as possible, we first recall and expand definitions, notation, and key technical results from Delfour and Zolesio [5]. Let $\mathbb{R}^N$ be the $N$-dimensional Euclidean space for some integer
$N > 1$ (in practice $N = 3$). Let $\Omega$ be a subset of $\mathbb{R}^N$ with a boundary $\partial \Omega$ which is a $C^2$ $(N - 1)$-dimensional submanifold of $\mathbb{R}^N$. In general the mean surface of the shell $\Gamma$ will be modelled as an open bounded Lipschitzian domain in the submanifold $\partial \Omega$. Roughly speaking the shell is an open domain around $\Gamma$ of (possibly variable) thickness $2h$.

2.1. Oriented Boundary Distance Function, Curvatures, and Projection

Associate with $\Omega$ the oriented boundary distance function

$$b_{\partial}(x) = d_{\partial}(x) - d_{\partial}(x)$$

where $\partial \Omega = \{ x \in \mathbb{R}^N : x \notin \Omega \}$ and $d_{\partial}$ is the usual distance function to a subset $A$ of $\mathbb{R}^N$. This function captures all the geometrical properties of the boundary $\partial \Omega$. For $k \geq 2$ a domain $\Omega$ has a $C^k$ boundary $\partial \Omega$ if and only if in each point $X \in \partial \Omega$ there exists a bounded open neighbourhood $N(X)$ of $X$ such that $b_{\partial} \in C^k(N(X))$.

At each point $X$ of $\partial \Omega$, its gradient $\nabla b_{\partial}(X)$ coincides with the unitary exterior normal field $n$ to $\partial \Omega$ and the eigenvalues of the symmetrical matrix of second order partial derivatives $D^2 b_{\partial}$ are $0$ and the principal curvatures, $k_i$, $1 \leq i \leq N - 1$, of the surface $\partial \Omega$. The trace of $D^2 b_{\partial}(X)$ is the mean curvature

$$H(X) \overset{\text{def}}{=} \text{tr}(D^2 b_{\partial}(X)) = \kappa b_{\partial}(X),$$

up to a multiplying factor which is used as a normalization factor to make the mean curvature of the unit sphere equal to one in all dimensions. The trace of the matrix of cofactors $M(D^2 b_{\partial})$ is the total or Gaussian curvature

$$K(X) \overset{\text{def}}{=} \text{tr}(M(D^2 b_{\partial}(X))).$$

The reader is referred to Delfour and Zolézio [1] for more details on the properties of the function $b_{\partial}$ and to Gilbarg and Trudinger [1] for the study of curvature via distance functions.

Since the domain $\Omega$ is fixed throughout this paper, from now on the function $b_{\partial}$ will be denoted by $b$. For each $X \in \partial \Omega$, the projection mapping $p : N(X) \to \partial \Omega$ is obtained directly from the oriented distance function $b$ as

$$p(x) = x - b(x) \nabla b(x) = x - \frac{1}{2} \nabla b^2(x).$$

This definition is independent of the choice of $N(X)$ and $X$. It only uses the fact that $\nabla b(x)$ exists. Its Jacobian matrix is given by

$$Dp(x) = I - b(x) \, D^2 b(x) - \nabla b(x) * \nabla b(x),$$
where $\nabla b(x)$ is the transposed of the vector $\nabla b(x)$ and $I$ is the identity matrix. For $x \in N(X)$, the linear projector onto the tangential plane $T_{p(x)} \partial \Omega$ at the point $p(x)$ of $\partial \Omega$ is given by
\[
P(x) = I - \nabla b(x) \nabla b(x),
\]
and we have the identities
\[
Dp(x) = P(x) - b(x) D^2 b(x) \quad \text{and} \quad Dp(x) \nabla b(x) = 0.
\]

2.2. Definition of the Shell

A shell is characterized by its mean surface $\bar{I}$ and its thickness (function) $h$. The mean surface $\bar{I}$ of the shell is a bounded open domain in the $(N-1)$-submanifold $\partial \Omega$ of $\mathbb{R}^N$. When $\bar{I} = \partial \Omega$ (hence $\partial \Omega$ is compact), the shell has no boundary. When $\bar{I} \subset \partial \Omega$, the (relative) boundary $\partial_{\Omega} \bar{I}$ is assumed to be uniformly Lipschitzian in $\partial \Omega$.

Since $\bar{I}$ is bounded and $\partial \Omega$ is $C^2$, there exists a bounded neighbourhood $N(\bar{I})$ of $\bar{I}$ such that $b \in C(\bar{N}(\bar{I}))$. For each $X \in \Gamma$, we can introduce the quantities
\[
0 < h^-(X) \overset{\text{def}}{=} \sup \{ -b(x) : x \in N(\Gamma), p(x) = X \} < \infty,
\]
\[
0 < h^+(X) \overset{\text{def}}{=} \sup \{ b(x) : x \in N(\Gamma), p(x) = X \} < \infty.
\]
The thickness of the shell is a Lipschitz continuous function $\tilde{h} : \Gamma \rightarrow \mathbb{R}^+$ such that
\[
\forall X \in \Gamma, \quad 0 < \tilde{h}(X) \leq \min \{ h^-(X), h^+(X) \}.
\]

Given $\bar{I}$ and $\tilde{h}$ the shell is the set
\[
S_{\tilde{h}} = \{ x \in \mathbb{R}^N : p(x) \in \bar{I}, |b(x)| < \tilde{h}(p(x)) \}.
\]
In view of the assumptions on $\bar{I}$ and $\tilde{h}$, the set $S_{\tilde{h}}$ is a bounded open domain in $\mathbb{R}^N$ with a Lipschitzian boundary.

When $\bar{I} \subset \partial \Omega$, $S_{\tilde{h}}$ has a lateral boundary
\[
\Sigma_{\tilde{h}} = \{ x \in \mathbb{R}^N : p(x) \in \partial_{\Omega} \bar{I}, |b(x)| < \tilde{h}(p(x)) \}
\]
which is an $(N-1)$-dimensional surface normal to the mean surface $\bar{I}$.

In practice the mean surface $\bar{I}$ will be given first and the underlying assumption will be the existence of an appropriate domain $\Omega$ with the above properties. It is important to keep in mind that we use the distance function $b = b_{\Omega}$ and not the distance function to $\bar{I}$. 


2.3. Flow of the Gradient of b and Local Coordinates

Let $h > 0$ be a constant such that

$$S_h \equiv \{ x \in \mathbb{R}^N : p(x) \in \Gamma, |b(x)| < h \} \subset N(\Gamma). \tag{2.14}$$

Since $\nabla b \in C^1(S_h)$, consider the flow mapping $T_z = T_z(\nabla b)$, defined by

$$T_z(X) = x(z), \begin{cases} \frac{dx}{dz}(z) = \nabla b(x(z)), & |z| < h, \\ x(0) = X. \end{cases} \tag{2.15}$$

It is a homeomorphism from $\Gamma$ onto $\Gamma_z = \{ x \in \mathbb{R}^N : b(x) = z, p(x) \in \Gamma \}$. In particular

$$T_z(X) = X + z \nabla b(X) \tag{2.16}$$

for $|z| < h$. This induces a “curvilinear coordinate system” $(X, z) \in \Gamma \times ]-h, h[$ in $S_h$. The points on the level set $\Gamma_z$ are given by $\{ X + z \nabla b(X) : X \in \Gamma \}$ and for each $(X, z) \in \Gamma \times ]-h, h[$

$$\nabla b(T_z(X)) = \nabla b(X + z \nabla b(X)) = \nabla b(X).$$

We have the following identities and properties on $\Gamma$:

$$p \cdot T_z = p \quad \text{and} \quad b \cdot T_z = z \tag{2.17}$$

$$DT_z = I + z D^2 b \tag{2.18}$$

$$\frac{\partial}{\partial z} DT_z = D^3 b \cdot T_z DT_z = D^3 b \tag{2.19}$$

$$\frac{\partial}{\partial z} \det DT_z = \det DT_z \cdot \det DT_z \tag{2.20}$$

$$D^2 b = D^2 b \cdot T_z [I + z D^2 b] \quad D^3 b \cdot T_z = [I - z D^2 b \cdot T_z] D^3 b \tag{2.21}$$

$$[I + z D^2 b]^{-1} = I - z D^2 b \cdot T_z. \tag{2.22}$$

In particular $\det DT_z(X)$ is a polynomial of degree at most $N - 1$ and

$$\det DT_z(X) = \begin{cases} 1 + z \det b(X), & \text{for } N = 2 \\ 1 + z \det b(X) + z^2 \det M(D^2 b(X)), & \text{for } N = 3 \end{cases} \tag{2.23}$$

where $M(D^2 b(X))$ is the cofactor matrix of the matrix $D^2 b(X)$. Moreover

$$^{\ast} DT_z^{-1}(X) m(X) = m(X) \Rightarrow ^{\ast} DT_z^{-1}(X) m(X) = 1. \tag{2.24}$$
It will be useful to introduce the notation
\[
\det DT_z(X) = \sum_{i=0}^{N-1} K_i z^i
\]  
(2.25)

where the \( K_i \)'s are functions of \( X \) on \( T \), \( K_0 = 1 \), \( K_1 = H \) for \( N \geq 2 \), and \( K_{N-1} = K \) for \( N \geq 3 \).

From (2.21) if \( \kappa_1, \kappa_2, ..., \kappa_{N-1}, 0 \), are the eigenvalues of \( D^2 b(X) \) at \( X \in \partial \Omega \), then
\[
\kappa_i(z) = \frac{K_i}{1 + zK_i}, \quad 1 \leq i \leq N - 1,
\]  
(2.26)

and 0 are the eigenvalues of \( D^2 b(T_z(X)) \). It is also instructive to make the connection between the second fundamental form associated with the submanifold \( \partial \Omega \) of \( \mathbb{R}^N \) and the matrix \( D^2 b \) (cf. M. Bernadou [1] for definitions and notation). For \( N = 3 \) associate with a bounded open subset \( A \) of \( \mathbb{R}^2 \) and a \( C^2 \)-mapping the mean surface \( \Gamma \)
\[
(\xi_1, \xi_2) \mapsto \Phi(\xi_1, \xi_2) : A \subset \mathbb{R}^2 \to \Gamma = \Phi(A) \subset \mathbb{R}^3.
\]
Define the tangent vectors \( \langle \tilde{a}_1, \tilde{a}_2 \rangle \) and the unit normal vector \( \tilde{a}_3 \)
\[
\tilde{a}_s = \frac{\partial \Phi}{\partial \xi_s}, \quad s = 1, 2, \quad \tilde{a}_3 = \frac{\tilde{a}_2 \times \tilde{a}_1}{|\tilde{a}_1 \times \tilde{a}_2|}.
\]
The normal vector coincides with our normal to the boundary and since the normal to \( \partial \Omega \) can be defined either by \( \Omega \) or its complement, we choose \( \Omega \) such that

\[
\tilde{a}_3 = -\nabla b.
\]

Then the elements of the second fundamental form are defined as
\[
b_{s\beta} \overset{\text{def}}{=} -\tilde{a}_s \cdot \frac{\partial}{\partial x} \tilde{a}_\beta = \tilde{a}_s \cdot \frac{\partial}{\partial x} \nabla b = D^2 b \tilde{a}_s \cdot \tilde{a}_\beta = b_{s\beta}.
\]

In view of the fact that \( D^2 b \nabla b = 0 \), for all \( (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \)
\[
D^2 b \tilde{c}^\alpha_\gamma = \xi_1 D^2 b \tilde{a}_1 + \xi_2 D^2 b \tilde{a}_2 + \xi_3 D^2 b \tilde{a}_3 = \xi_1 D^2 b \tilde{a}_1 + \xi_2 D^2 b \tilde{a}_2,
\]
where \( \tilde{c} = \xi_1 \tilde{a}_1 + \xi_2 \tilde{a}_2 + \xi_3 \tilde{a}_3 \) is an arbitrary vector in \( \mathbb{R}^3 \). In particular
\[
\forall \tilde{c}, \tilde{c} \in \mathbb{R}^3 \quad b_{s\beta} \xi^\gamma \xi^\beta = D^2 b \tilde{c} \cdot \tilde{c}.
\]
and the second fundamental form coincides with the bilinear form generated by $D^2b$. Of course this extends to $\mathbb{R}^N$. Similarly the third fundamental form

$$c_{\alpha\beta} = b^*_x b_{x\beta}$$

coincides with the bilinear form generated by $(D^2b)^2$:

$$(D^2b)^2 \xi \cdot \zeta = D^2b^*_x \cdot D^2b^*_\zeta = D^2b^*_x \cdot \left( D^2b^*_\zeta \cdot a^x \cdot \left( \left( D^2b^*_x \cdot a^x \right) a^\zeta \cdot \left( \left( D^2b^*_\zeta \cdot a^\zeta \right) a^x \right) \right) \right)$$

$$= D^2b^*_x \cdot a^x \cdot D^2b^*_\zeta \cdot a^\zeta \cdot \left( \left( D^2b^*_x \cdot a^x \right) a^\zeta \cdot \left( \left( D^2b^*_\zeta \cdot a^\zeta \right) a^x \right) \right)$$

where $\{a^i\}$ is the contravariant basis. The matrices $I$, $D^2b$, and $(D^2b)^2$ will play a fundamental role in the paper.

2.4. **Tangential Differential Operators**

For any scalar function $w: \Gamma \to \mathbb{R}$, denote by $\nabla_T w$ the **tangential gradient**

$$\nabla_T w = \nabla W \mid_T \frac{\partial W}{\partial \nu}$$

(2.27)

defined in terms of an extension $W$ of $w$ to $S_b$. It can be shown that this definition is independent of the choice of the extension $W$ and that $\nabla_T w(X)$ is the projection of $W$ onto the tangent plane $T_x \Gamma$ to $\Gamma$ in $X$. It is easy to check that

$$\nabla (w \cdot p) = Dp(\nabla_T w) \cdot p = \left[ I - b \cdot D^2b \right] \nabla_T w \cdot p$$

(2.28)

and that $\nabla (w \cdot p) = \nabla_T w$ on $\Gamma$. The **tangential Jacobian matrix** of a vector $v: \Gamma \to \mathbb{R}^N$ is defined through its transposed

$$*D_T v = (\nabla_T v_1, \ldots, \nabla_T v_N)$$

(2.29)

to the column tangential gradients. In particular

$$*D(v \cdot p) = Dp(*D_T v) \cdot p$$

$$= Dp(\left[ I - b \cdot D^2b \right] *D_T v) \cdot p$$

(2.30)

$$D(v \cdot p) = (D_T v) \cdot p \cdot Dp = (D_T v) \cdot p \left[ I - b \cdot D^2b \right]$$

(2.31)

and on $\Gamma$, $D(v \cdot p) = D_T v$. If, in addition, $v$ is a tangential vector field, that is, the inner product of $v(X)$ and $\nabla b(X)$ is equal to zero,
\[ v(X) \cdot \nabla b(X) = 0, \quad \forall X \in \Gamma, \quad (2.32) \]
\[ \Rightarrow v \cdot p \cdot \nabla b \cdot p = 0, \quad \forall x \in \Omega_b, \quad (2.33) \]
\[ \Rightarrow *D(v \cdot p) \nabla b + Dp[D^2 b \cdot p] v \cdot p = 0. \quad (2.34) \]

But the term \([D^2 b \cdot p] v \cdot p\) is necessarily tangential and
\[ Dp[D^2 b \cdot p] v \cdot p = [I - b D^2 b][D^2 b \cdot p] v \cdot p \]
and using identity (2.30) for the first term in (2.34) we get from (2.34)
\[ [I - b D^2 b] [*D_F(v \cdot p) \nabla b + [D^2 b \cdot p] v \cdot p] = 0 \]
\[ *D_F v \nabla b + D^2 b v = 0 \quad \text{on } \Gamma. \quad (2.35) \]

When \(v\) is not tangential this last identity becomes
\[ *D_F v \nabla b + D^2 b v = 0 \quad \text{on } \Gamma. \quad (2.36) \]

To see this apply identity (2.35) to the tangential component of \(v\)
\[ *D_F (v - v \cdot \nabla b) = -D^2 b (v - v \cdot \nabla b) = -D^2 b v \]
\[ *D_F v \nabla b = *D_F (v \cdot \nabla b) \nabla b - D^2 b v \]
\[ = *[\nabla b *\nabla_F (v \cdot \nabla b) + v \cdot \nabla b D^2 b] \nabla b - D^2 b v \]
\[ = \nabla_F (v \cdot \nabla b) - D^2 b v \]

since \(\nabla b = \nabla b \cdot p\) on \(\Gamma\),
\[ D^2 b = D(\nabla b) = D(\nabla b \cdot p) = D_F (\nabla b) \cdot p Dp = D_F (\nabla b) \cdot p[I - b D^2 b], \]
and \(D_F(\nabla b) = D^2 b\) on \(\Gamma\).

In the same way define the **tangential divergence** as
\[ \text{div}_T v \overset{\text{def}}{=} \text{tr} D_F v \quad (2.37) \]
or equivalently in term of an extension \(V\) of \(v\) to a neighbourhood of \(\Gamma\)
\[ \text{div}_T v \overset{\text{def}}{=} \text{div} V|_\Gamma - D V n \cdot n. \quad (2.38) \]

It is easy to verify that
\[ \text{div}(v \cdot p) = \text{div}_T v \cdot p - b \text{tr}[D_F v \cdot p D^2 b] \quad (2.39) \]
and \( \text{div}(v \cdot p)_r = \text{div}_r v \). Similarly the tangential strain tensor is defined as

\[
\varepsilon_r(v) \triangleq \frac{1}{2} (D_r v + *D_r v)
\]

(2.40)

\[
\varepsilon(v \cdot p) = \frac{1}{2} (D(v \cdot p) + *D(v \cdot p))
\]

\[
= \varepsilon_r(v) \cdot p - \frac{b}{2} [D_r v \cdot p D^2 b + D^2 b \cdot *D_r v \cdot p]
\]

(2.41)

\[
\varepsilon_r(v) = \varepsilon(v \cdot p)_r.
\]

(2.42)

In view of identities (2.28), (2.31) and (2.39) the composition of \( \text{div}_r \) and \( \nabla_r \) yields the Laplace–Beltrami operator

\[
A_r w \triangleq \text{div}_r(\nabla_r w) \quad (= A(w \cdot p)_r).
\]

(2.43)

Similarly the matrix of tangential second order derivatives is defined as

\[
D^2_r w \triangleq D_r(\nabla_r w) \quad (= D^2(w \cdot p)_r = D^2_r w - D^2 b \nabla_r w \cdot *\nabla_r)
\]

(2.44)

and it is readily seen that the second order tangential derivatives are not symmetrical and that \( D^2_r w \) and its transpose differ by first order terms \( D^2_r w - D^2 b \nabla_r w \cdot *\nabla_r \) \( = D^2_r w - \nabla b(\nabla_r w) \). Moreover

\[
\varepsilon(\nabla(w \cdot p))_r = \varepsilon_r(\nabla_r(w)) - \frac{1}{2} [D^2 b \nabla_r w \cdot *\nabla_r + \nabla b \cdot *\nabla_r w D^2 b].
\]

(2.45)

The tangential operators are directly related to the classical covariant derivatives for tangent vector fields \( v: \Gamma \to \mathbb{R}^3 \), \( v \cdot \nabla b = 0 \) on \( \Gamma \). They naturally extend to vector fields \( v: \Gamma \to \mathbb{R}^N \) with \( a_N = -\nabla b \). By definition of the partial derivatives

\[
v_{\gamma} = \tilde{a}_\gamma \cdot D_r v_{\gamma} + v \cdot D_r \tilde{a}_\gamma, \quad \text{and} \quad v_{\gamma}^{\alpha} = \tilde{a}^\alpha \cdot D_r v_{\gamma} + v \cdot D_r \tilde{a}^\alpha,
\]

where the indices can take the value \( N \). The definitions for \( N \)-dimensional vector fields become

\[
v_{\gamma} \mid = v_{\gamma} - \sum_{\lambda=1}^{N-1} \Gamma_{\nu \gamma}^{\lambda} e_\lambda \quad \text{and} \quad v^\alpha \mid = v^\alpha + \sum_{\lambda=1}^{N-1} \Gamma^\alpha_{\nu \gamma} e^\lambda
\]

where

\[
\Gamma^\alpha_{\beta \gamma} \triangleq \tilde{a}^\alpha \cdot D_r \tilde{a}_\beta \tilde{a}_\gamma = -D_r \tilde{a}^\alpha \tilde{a}_\gamma \tilde{a}_\beta \quad 1 \leq \alpha, \beta, \gamma \leq N
\]
are the Christoffel symbols. Then it can be verified that
\[ v^i|_{\Gamma} = \tilde{a}_x \cdot (D_x v + v_N D^2 b) \tilde{a}_x \quad v^i|_{\Gamma} = \tilde{a}_x \cdot (D_x v + v_N D^2 b) \tilde{a}_x \]
\[ = v^i - b_{xy} v^y = \tilde{a}_x \cdot D_x v \tilde{a}_x \quad \text{and} \quad v^i|_{\Gamma} - b_{xy} v^y = \tilde{a}_x \cdot D_x v \tilde{a}_x. \]

For more details the reader is referred to Delfour and Zolésio [6].

2.5. Decomposition of the integration over \( S_\Gamma \) along the level sets of \( b \)

The next ingredient is the use of Federer's decomposition of the measure on \( S_\Gamma \) along the level curves of \( b \): for any sufficiently smooth function \( f \), say in \( H^1(\tilde{S}_\Gamma) \),
\[ \int_{\tilde{S}_\Gamma} f(x) \, dx = \int_{-\infty}^{\infty} \int_{f^{-1}(x)} \chi_{\tilde{S}_\Gamma} d\Gamma_z \, dz, \quad (2.46) \]
where \( d\Gamma_z \) is the surface measure of \( \Gamma_z = \{ x \in \mathbb{R}^N : b(x) = z \} \) and \( \chi_{\tilde{S}_\Gamma} \) is the characteristic function of \( \tilde{S}_\Gamma \) in (2.12). The decomposition of a measure along the level sets has been used by Temam [1] and Zolésio [2] in the context of Plasma Physics problems. Since \( |\{ b \}| = 1 \), after a change of variable using the transformation \( T_z \) we get
\[ \int_{-\infty}^{\infty} \int_{f^{-1}(x)} d\Gamma_z f \chi_{\tilde{S}_\Gamma} \, dz = \int_{-\infty}^{\infty} \int_{T_z f^{-1}(x)} d\Gamma_{T_z} f \chi_{T_z \tilde{S}_\Gamma} \, |j(z)| \]
\[ = \int_{T_z f^{-1}(x)} d\Gamma_{T_z} \int_{-\infty}^{\infty} dz \, |j(z)| f \chi_{T_z \tilde{S}_\Gamma} \, T_z \]
\[ = \int_{T_z f^{-1}(x)} d\Gamma_{T_z} \int_{-\infty}^{\infty} dz \, f \chi_{T_z \tilde{S}_\Gamma} \, |j(z)|, \quad (2.47) \]
where in view of (2.24) and (2.25)
\[ j(z) = \det(DT_z) \, |*DT_z^{-1}n| = \det(DT_z) = \sum_{j=0}^{N-1} K_z z^j \quad \text{on} \, \Gamma. \quad (2.48) \]
Thus the function \( j(z) \) is a polynomial of degree at most \( N - 1 \). The area density on the boundary \( \partial \Omega_z \) of \( \Omega_z = \{ x \in \mathbb{R}^N : b(x) = z \} \) is given by
\[ d\Gamma_z = |j(z)| \, d\Gamma. \quad (2.49) \]
In particular from (2.23)
\[ j(z) = \begin{cases} 
1 + zH, & \text{for} \quad N = 2 \\
1 + zH + z^2 K, & \text{for} \quad N = 3.
\end{cases} \quad (2.50) \]
Note that \( j(0) = 1 \) and that for \( z \) sufficiently small \( |j(z)| = j(z) \). Since \( b \) is of class \( C^2 \) and \( \Gamma \) is bounded, this is always true for \( h \) sufficiently small.

3. Displacement Field and Strain Tensor

For simplicity we shall work with a shell \( S_h \) of constant thickness, but almost everything we shall say applies to shells \( S_h \) with variable thickness under appropriate assumptions on the function \( h \). As in our previous paper we first make the following assumption on the displacement vector as in the “natural theory” of plates (cf. for instance Germain [1]).

Assumption 1. At each point \( x \) of the shell the displacement vector \( V(x) \) is of the form

\[
V(x) = e \cdot p(x) + b(x) \cdot p(x), \quad x \in S_h, \quad (3.1)
\]

for vector-valued mappings \( e \) and \( l \) in \( H^1(\Gamma) \).

Assumption \( \Gamma' \). In addition to Assumption 1, assume that \( l(X) \) is a tangential vector, that is \( l(X) \) belongs to the tangent space \( T_X \Gamma \) at \( X \) for each \( X \) on \( \Gamma \) or equivalently

\[
l(X) \cdot \nabla b(X) = 0, \quad \forall X \in \Gamma, \quad (3.2)
\]

where \( \cdot \) denotes the inner product in \( \mathbb{R}^N \).

For \( X \) on \( \Gamma \), \( z = b(X) = 0 \) and \( V = e \) along \( \Gamma \). For \( x \in \Gamma^*\), \( z = b(x) \neq 0 \) and there is an additional tangential displacement \( zl(p(x)) \) along the level curve \( \Gamma^* \) which is proportional to \( z = b(x) \).

3.1. The Strain Tensor

From (3.1), a direct computation yields the following expressions in \( S_h \)

\[
DV = [D_f e \cdot p + b D_f l \cdot p][I - b D^2 b] + l \cdot p * \nabla b, \quad (3.3)
\]

\[
DV = [D_f e \cdot p + b D_f l \cdot p + l \cdot p * \nabla b][I - b D^2 b] \quad (3.4)
\]

\[
DV^* b = l \cdot p, \quad (3.5)
\]

where \( D_f e \) and \( D_f l \) are the tangential Jacobian matrices as defined in (2.29).

Consider the strain tensor \( e(V) \) over \( S_h \) associated with the displacement field \( V(x) \). From (3.4)

\[
2e(V) = D(V) + * D(V)
\]

\[
= [D_f e \cdot p + b D_f l \cdot p + l \cdot p * \nabla b][I - b D^2 b] + [I - b D^2 b] *[D_f e \cdot p + b D_f l \cdot p + l \cdot p * \nabla b] \quad (3.6)
\]
in $S_h$. In particular

$$2\varepsilon(V) \nabla b = (I - b D^2 b) \ast [D_r e \cdot p + b D_r l \cdot p + l \cdot p \ast \nabla b] \nabla b + l \cdot p,$$

and since $l$ is tangential from (2.35)

$$2\varepsilon(V) \nabla b = (I - b D^2 b) \ast [2\varepsilon_r(e) \cdot p \nabla b - b D^2 b \cdot p l \cdot p] + l \cdot p,$$  (3.7)

where $\varepsilon_r(e)$ is the *tangential strain tensor* defined in (2.40). But $\nabla b = \nabla b \cdot p$ on $\Gamma$ and

$$D^2 b = D(\nabla b) = D(\nabla b \cdot p) = D_r(\nabla b) \cdot p \, D p$$

$$= D_r(\nabla b) \cdot p[I - b D^2 b] = D^2 b \cdot p[I - b D^2 b],$$  (3.8)

since $D_r(\nabla b) = D^2 b$ on $\Gamma$. So by symmetry $[I - b D^2 b] D^2 b \cdot p = D^2 b$ and finally

$$2\varepsilon(V) \nabla b = (I - b D^2 b) \ast [2\varepsilon_r(e) \cdot p \nabla b + l \cdot p]$$

(3.9)

$$\varepsilon(V) \nabla b \cdot \nabla b = l \cdot p \cdot V b \quad (= 0 \text{ for Assumption 1'}).$$  (3.10)

Note that in terms of its curvilinear coordinates $(X, z)$, the tensor $\varepsilon(V) \cdot T_z$ is almost affine in $z$, that is the sum of a tensor in $X$ and $z$ times another tensor in $X$

$$D V \cdot T_z = [D_r e + z D_r l + l \ast \nabla b][I + z D^2 b]^{-1}$$  (3.11)

$$2\varepsilon(V) \cdot T_z = [D_r e + z D_r l + l \ast \nabla b][I + z D^2 b]^{-1}$$

$$+ [I + z D^2 b]^{-1} \ast [D_r e + z D_r l + l \ast \nabla b].$$  (3.12)

$$2\varepsilon(V) \cdot T_z \nabla b = [I + z D^2 b]^{-1} \ast [2\varepsilon_r(e) \nabla b + l],$$  (3.13)

where we have used identity (2.22) in (3.9). The nonlinear part is contained in the matrix $[I + z D^2 b]^{-1}$. So for $\|z D^2 b\|$ small

$$[I + z D^2 b]^{-1} = \sum_{i=0}^{\infty} (-D^2 b)^i z^i,$$  (3.14)

and we get

$$D V \cdot T_z = D_r e + l \ast \nabla b + [D_r l - D_r e \, D^2 b] z$$

$$+ \sum_{i=2}^{\infty} [D_r l - D_r e \, D^2 b] (-D^2 b)^{i-1} z^i.$$  (3.15)
With this we can also write the strain tensor as an infinite sum
\[
2\varepsilon(V) \cdot T_z = 2\varepsilon_r(e) + l \ast \nabla b + \nabla \ast l \\
+ \{2\varepsilon_r(l) - [D_r e \ D^2 b + D^2 b \ast D_r e]\} z \\
+ \sum_{i=2}^{\infty} \{[D_r l - D_r e \ D^2 b](-D^2 b)^{i-1} \\
+ (-D^2 b)^{i-1} \ast [D_r l - D^2 b \ast D_r e]\} z^i.
\]
(3.16)

At this juncture it is convenient to introduce the notation
\[
2\varepsilon^0 = D_r e + l \ast \nabla b + \nabla \ast l \\
2\varepsilon^1 = D_r l - D_r e \ D^2 b + \ast D_r l - D^2 b \ast D_r e \\
2\varepsilon^2 = [D_r l - D_r e \ D^2 b](-D^2 b) \\
+ (-D^2 b) \ast [D_r l - D^2 b \ast D_r e] \\
2\varepsilon^n = [D_r l - D_r e \ D^2 b](-D^2 b)^{n-1} \\
+ (-D^2 b)^{n-1} \ast [D_r l - D^2 b \ast D_r e]
\]
(3.17)

for \(n \geq 3\). The following result is central to the subsequent development of the model.

**Theorem 3.1.** Let \(e\) and \(l\) be in \(H^1(\Gamma)^N\). (i)
\[
e(V) \cdot T_z = 0, \quad \forall |z| < h, \quad \forall X \in \Gamma
\]
(3.21)

if and only if
\[
\varepsilon^0 = \varepsilon^1 = \varepsilon^2 = 0 \quad \text{on } \Gamma,
\]
(3.22)

if and only if there exists a vector \(a \in \mathbb{R}^N\) and an \(N \times N\) matrix \(B\) such that
\[
e(X) = a + BX, \quad l(X) = B \nabla b(X), \quad \forall X \in \Gamma,
\]
(3.23)

where \(B\) is antisymmetric
\[
B + \ast B = 0.
\]
(3.24)

In particular (3.23) and (3.24) imply that \(l\) is tangential. (ii) For all \(z, |z| < h, \ X \in \Gamma,\)
\[
e(V) \cdot T_z = e^0 + [I + z \ D^2 b]^{-1} \{z[\varepsilon^1 + \varepsilon^1 \ D^2 b + z \ D^2 \varepsilon^1] \\
+ z^2 \varepsilon^2] [I + z \ D^2 b]^{-1}.
\]
Remark 3.1. (i) From Lemma 3.1 below condition (3.22) is equivalent to
\[ e^0 = e^1 = \frac{1}{2} [ *D_r l D^2 b + D^2 b D_r l ] = 0 \text{ on } \Gamma. \]  
(3.25)

(ii) Conditions (3.23) and (3.24) correspond to the rigid displacements as in 3-D elasticity. Our previous model only yielded an approximation of this mechanical property (cf. Delfour ad Zolésio [5, Remark 5.1]).

Proof. (i) Define the matrices \( A = D_r l - D_r e D^2 b \) and \( B = - D^2 b \). The equivalence of (3.21) and (3.22) is a direct consequence of the following lemma.

**Lemma 3.1.** Let \( e \) and \( l \) be vector functions in \( H^1(\Gamma)^N \).

\[ e^2 = e^1 B + Be^1 - Be^0 B + \frac{1}{2} [ *D_r l D^2 b + D^2 b D_r l ] \]  
(3.26)

and for \( n \geq 2 \)

\[ e^{n+1} = e^n B + Be^n - Be^{n-1} B. \]  
(3.27)

Proof. (i) We first prove (3.26)

\[
\]

But

\[
-(*AB + BA) = [ *D_r l - D^2 b *D_r e ] D^2 b + D^2 b [ D_r l - D_r e D^2 b ]
\]

\[
= *D_r l D^2 b + D^2 b D_r l - D^2 b [ D_r e + *D_r e ] D^2 b
\]

\[
= *D_r l D^2 b + D^2 b D_r l + D^2 b [ l * \nabla b + \nabla b * l ] D^2 b - B2e^0 B.
\]

However

\[
D^2 b [ l * \nabla b + \nabla b * l ] D^2 b = 0
\]

\[
\Rightarrow 2e^2 = 2e^1 B + B2e^1 + [ *D_r l D^2 b + D^2 b D_r l ] - B2e^0 B.
\]

(ii) By definition, for \( n \geq 1 \),

\[
2e^n = AB^{n-1} + B^{n-1} * A.
\]

First we prove (3.27) for \( n = 2 \):

\[
\]

\[
\]
For $n > 2$

$$2e^{n+1} = AB^n + B^n A = [2e^n - B^{n-1} A] B + B[2e^n - AB^{n-1}]$$

$$= 2e^n B + B2e^n - B[2e^{n-2} A + AB^{n-2}] B$$

$$= 2e^n B + B2e^n - B2e^{n-1} B.$$

Assume that (3.21) is verified. Then from (3.1) and Temam [2, Lemma 1.1, p. 18], there exist a vector $a \in \mathbb{R}^N$ and an antisymmetrical $N \times N$ matrix $B (B + * B = 0)$ such that for all $x \in S_h$

$$[e + p + bl \cdot p](x) = V(x) = a + Bx. \quad (3.28)$$

For $X \in \Gamma$, $|z| < h$, and $x = T_r(X)$

$$e(X) + zl(X) = a + BT_r(X) = a + B(X + zVb(X))$$

$$\Rightarrow e(X) = a + BX \quad \text{and} \quad l(X) = B\nabla b(X).$$

To complete the characterization $a$ and $B$ must be chosen in such a way that the three identities (3.22) are verified. By direct computation

$$D_r e = B - B \nabla b \ast \nabla b \quad \text{and} \quad D_r l = BD^2 b$$

$$\epsilon^0 (e, l) = 0, \quad D_r l - D_r e \ D^2 b = B \ D^2 b - (B - B \nabla b \ast \nabla b) \ D^2 b = 0$$

$$\epsilon^i (e, l) = \epsilon^i (e, l) = 0.$$

Also $2l \cdot \nabla b = 2B \ \nabla b \cdot \nabla b = (B + * B) \ \nabla b \cdot \nabla b = 0$ on $\Gamma$. This yields (3.23) and (3.24). Conversely it is readily seen that conditions (3.22) will be verified under conditions (3.23) and (3.24).

(ii) Denote $e(V) \cdot T_\Gamma$ by $e$. By definition

$$e = \epsilon^0 + ze^1 + z^2 e^2 + \sum_{m=3}^\infty z^m e^m.$$

From (3.27)

$$z^m e^m = z^{m-1} e^{m-1} Z B + z B z^{m-1} e^{m-1} - z B z^{m-2} e^{m-2} Z B$$

and

$$\sum_{m=3}^\infty z^m e^m = \left( \sum_{m=2}^\infty z^m e^m \right) Z B + z B \left( \sum_{m=2}^\infty z^m e^m \right) - z B \left( \sum_{m=1}^\infty z^m e^m \right) Z B$$

$$= (\epsilon - \epsilon^0 - ze^1) Z B + z B (\epsilon - \epsilon^0 - ze^1) - z B (\epsilon - \epsilon^0) Z B.$$
Hence

\[
\varepsilon - \varepsilon B - z B \varepsilon + z B \varepsilon z B = \varepsilon^0 - \varepsilon^0 B - z B \varepsilon^0 + z B \varepsilon^0 B + z \varepsilon^1 - z \varepsilon^1 B - z B \varepsilon^1 + z^2 \varepsilon^2
\]

or by rewriting

\[
(I - z B) \varepsilon (I - z B) = (I - z B) \varepsilon^0 (I - z B) + z \varepsilon^1 - z \varepsilon^1 B - z B \varepsilon^1 + z^2 \varepsilon^2.
\]

3.2. Approximate Strain Tensor and Korn’s Inequality

Assumption 2. The equivalence of the conditions \( \varepsilon (V) \cdot T_z = 0 \) and \( \varepsilon^0 = \varepsilon^1 = 0 \) suggests to use an approximation of the strain tensor which is quadratic in \( z \):

\[
\tilde{\varepsilon} (V) \cdot T_z = \varepsilon^0 + \varepsilon^1 z + \varepsilon^2 z^2.
\] (3.29)

It corresponds to the assumption that the dimensionless quantity \( ||z D^3 b(X)|| \) is small or equivalently that the shell is either thin (\( h \) small) or shallow (\( ||D^2 b(X)|| \) small) or both.

Remark 3.2. In view of Remark 3.1(i), we could also choose

\[
\tilde{\varepsilon} (V) \cdot T_z = \varepsilon^0 + \varepsilon^1 z + \frac{1}{2} [ * D_f I D^2 b + D^2 D_f I ] z^2
\] (3.29a)

and everything we shall say will remain true when replacing the term \( \varepsilon^2 \) by

\[
\frac{1}{2} [ * D_f I D^2 b + D^2 D_f I ].
\]

It is natural to associate with the above approximation the Hilbert spaces

\[
\mathcal{H} = \{ (e, l) \in L^2(\Gamma)^N \times L^2(\Gamma)^N \}
\]

(3.30)

\[
\mathcal{V} = \{ (e, l) \in \mathcal{H} : e \in L^2(\Gamma)^N, 0 \leq i \leq 2 \}
\]

(3.31)

\[
\mathcal{N} = \{ (e, l) \in \mathcal{V} : e = 0 \text{ on } \Gamma, 0 \leq i \leq 2 \}
\]

(3.32)

with norms

\[
| (e, l) |_{\mathcal{H}}^2 = |e|_{L^2(\Gamma)^N}^2 + |l|_{L^2(\Gamma)^N}^2
\]

(3.33)

\[
| (e, l) |_{\mathcal{V}}^2 = |(e, l)|_{\mathcal{H}}^2 + \sum_{i=0}^{2} \| e^i (e, l) \|_{L^2(\Gamma)^N}^2.
\]

(3.34)

In view of Theorem 3.1 (i)

\[
\mathcal{N} = \{ (e, l) \in \mathcal{V} : l(X) = B \nabla b(X), e(X) = a + B X, \forall a \in \mathbb{R}^N, \forall B \text{ an } N \times N \text{ matrix such that } B + * B = 0 \}.
\]

(3.35)
When working with Assumption 1 we use the following closed subspaces
\[ \mathcal{H}^t = \{ (e, l) \in \mathcal{H} : l \cdot n = 0 \text{ on } \Gamma \}, \quad \mathcal{V}^t = \{ (e, l) \in \mathcal{V} : l \cdot n = 0 \text{ on } \Gamma \} \]
and \[ \mathcal{N}^t = \{ h(e, l) \in \mathcal{N} : l \cdot n = 0 \text{ on } \Gamma \} \] coincides with \( \mathcal{N} \). The following considerations will equally apply to \( \mathcal{H}, \mathcal{V}, \mathcal{N} \) and \( \mathcal{H}^t, \mathcal{V}^t, \mathcal{N}^t \). This is equivalent to the definition (3.32). The subspace \( \mathcal{N} \) characterizes the rigid displacements as in the case of 3-D elasticity. So by accepting a slightly more complex model, we recover a fundamental mechanical property. Note that the seminorm on the space \( \mathcal{V} \)
\[ \| (e, l) \| = \left( \sum_{i=0}^{2} \| e_i (e, l) \|_{2, \Gamma}^2 \right)^{1/2} \] (3.36)
becomes a norm on the quotient space \( \mathcal{H}/\mathcal{N} \).

To obtain a better characterization of the spaces \( \mathcal{H} \) and \( \mathcal{V} \), we need an extension to shells of the Korn's inequality for plates. But \( S_h \) is an open bounded Lipschitzian domain in \( \mathbb{R}^N \) since \( \partial \Omega \) is of class \( C^2 \) and \( \Gamma \) is a bounded open Lipschitzian domain in the \((N-1)\)-submanifold \( \partial \Omega \). Therefore Korn's inequality can be applied to the displacement vector
\[ V = e \circ p + h l \circ p. \] (3.37)
Specifically there exists a constant \( c = c(S_h) > 0 \) which only depends on \( S_h \) such that for all \( V \in L^2(S_h)^N \) such that \( e_h(V) \in L^2(S_h) \)
\[ \sum_{i,j=1}^{N} \int_{S_h} [\partial_l V_j]^2 \, dx \leq c^2 \int_{S_h} |V|^2 + \sum_{i,j=1}^{N} |e_{ij}(V)|^2 \, dx \] (3.38)
(cf. for instance Temam [2, Prop. 1.1, p. 16]). By using the decomposition of the integration over \( S_h \) along the level sets of the function \( h \), we obtain a natural extension of this inequality for thin/shallow shells.

**Theorem 3.2.** Assume that \( \Gamma \) is a bounded open domain in the \( C^2 \) \((N-1)\)-dimensional submanifold \( \partial \Omega \) of \( \mathbb{R}^N \) with a Lipschitzian boundary \( \partial \Omega \) in \( \partial \Omega \). As \( h \) goes to zero, there exists a constant \( c = c(S_h, D^2 h) > 0 \) such that for all \( (e, l) \in \mathcal{V} \) (resp. \( \mathcal{V}^t \))
\[ \int_{\Gamma} 2h \left[ |l|^2 + \| D_l e_l \|^2 \right] + \frac{2h^3}{3} \| D_l l \|^2 \, d\Gamma \leq c^2 \int_{\Gamma} 2h |e|^2 + \frac{2h^3}{3} |l|^2 \]
\[ + 2h \| e_0 (e, l) \|^2 + \frac{2h^3}{3} \| e_0 (e, l) \|^2 + \frac{2h^5}{5} \| e_0 (e, l) \|^2 \, d\Gamma, \] (3.39)
where
\[ |A|^2 = A \cdot A = \sum_{i,j=1}^{N} A_i A_j, \quad |a|^2 = \sum_{i=1}^{N} a_i^2. \]

In particular
\[ \mathcal{V} = H^1(\Gamma)^N \times H^1(\Gamma)^N \]  
(3.40)

Proof. We first provide estimates for each term in (3.38). From (3.37)
\[ \int_{S_h} |V(x)|^2 \, dx = \int_{R} d\Gamma \int_{-h}^{h} dz \, |V \cdot T_z|^2 j(z) = \int_{R} d\Gamma \int_{-h}^{h} dz \, |e + zl|^2 j(z). \]

Recall that
\[ j(z)(X) = \det[I + zD^2b(X)] \]  
(3.41)

and, since \( b \in C^2(S_h) \) and \(|z| < h\), there exist \( \tilde{h} > 0 \) and \( c > 0 \) such that
\[ |j(z)(X)| < c, \quad \forall z, \ |z| < \tilde{h}, \ \forall X \in \Gamma. \]

Therefore
\[ \int_{S_h} |V(x)|^2 \, dx \leq \int_{R} d\Gamma \int_{-h}^{h} dz \left[ |e|^2 + z^2 |l|^2 + 2ze \cdot l \right] j(z) \leq c \int_{R} d\Gamma \left[ 2h |e|^2 + 2h^3/3 |l|^2 \right]. \]

Similarly
\[ \int_{S_h} |e(V)|^2 \, dx \leq \int_{R} d\Gamma \int_{-h}^{h} dz \, ||e(V) \cdot T_z||^2 j(z) \leq c^2 \int_{R} d\Gamma \int_{-h}^{h} dz \, ||e(V) \cdot T_z||^2. \]

From Theorem 3.1 (ii) with the notation \( e = e(V) \cdot T_z \)
\[ e = e^0 + [I + zD^2b]^{-1} \left\{ z(e^1 + e^1 zD^2b + zD^2b^1) + z^2e^2 [I + zD^2b]^{-1} \right\}. \]
and there exist $\hat{h} > 0$ and $c > 0$ such that for all $h$, $0 < h \leq \hat{h}$

$$
\int_{\Gamma} d\Gamma \int_{-h}^{h} dz \|z\|^2 \leq c^2 \int_{\Gamma} d\Gamma \int_{-h}^{h} dz \sum_{m=0}^{2} \|z^m e^m\|^2 d\Gamma

\leq 3c^2 \int_{\Gamma} 2h \|z^0\|^2 + 2 \frac{h^3}{3} \|e^1\|^2 + 2 \frac{h^5}{5} \|e^2\|^2 d\Gamma
$$

which holds for all $(e, l)$. Recall from (3.11) that

$$DV: T_r = [D_r e + l^* \nabla b + z D_r l][I + z D^2 b]^{-1}.$$ 

But

$$\|D_r e + l^* \nabla b + z D_r l\|^2 = \|l\|^2 + \|D_r e + z D_r l\|^2 + 2l \cdot [D_r e + z D_r l] \nabla b$$

$$= \|l\|^2 + \|D_r e + z D_r l\|^2$$

$$= \|l\|^2 + \|D_r e\|^2 + z^2 \|D_r l\|^2 + 2z D_r e \cdot D_r l,$$

where

$$D_r e \cdot D_r l = \sum_{i,j=1}^{N} (D_r e)_i (D_r l)_j.$$ 

For $\hat{h}$ sufficiently small there exists constants $\beta_0 > 0$ and $\beta_1 > 0$ such that for all $h$, $0 < h \leq \hat{h}$,

$$0 < \beta_0 \leq 1 - \hat{h} \|D^2 b\| \leq \|I + z D^2 b\| \leq 1 + \hat{h} \|D^2 b\| \leq \beta_1$$

$$0 < \beta_0 < \beta(\hat{h})$$

and for some constant $\alpha > 0$

$$\alpha \int_{\Gamma} d\Gamma \int_{-h}^{h} dz \left[ \|l\|^2 + \|D_r e\|^2 + z^2 \|D_r l\|^2 + 2z D_r e \cdot D_r l \right]$$

$$\leq \int_{S_h} \|DV\|^2 dx.$$ 

Finally

$$\alpha \int_{\Gamma} 2h \left[ \|l\|^2 + \|D_r e\|^2 \right] + 2 \frac{h^3}{3} \|D_r l\|^2 d\Gamma \leq \int_{S_h} \|DV\|^2 dx.$$ 


Now (3.40) is a direct consequence of (3.39). From Lemma 3.1 with $l = 0$, we get

\[ e^2 = -e^1 D^2 b - D^3 b e^1 - D^3 b e^0 D^2 b \]
\[ 2e^1 = -D e D^2 b - D^2 b \star D e \text{ and } e^0 = e_r(e). \]

### 3.3. Love–Kirchhoff Theory

We conclude Section 3 by introducing the underlying mathematical assumption in the Love–Kirchhoff theory. From (3.29), (3.17) to (3.19) we have

\[ 2\epsilon(V) \cdot T_z \nabla b = [I - z D^2 b + z^2 (D^2 b)^2] [2x_r(e) \nabla b + l]. \tag{3.42} \]

When

\[ l = -2e_r(e) \nabla b \tag{3.43} \]

identities (3.42) and (3.13) yield

\[ \epsilon(V) \cdot T_z \nabla b = 0 \text{ in } S_h \tag{3.44} \]
\[ e(V) \cdot T_z \nabla b = 0 \text{ in } S_h. \tag{3.45} \]

Conditions (3.43) and (3.44) (resp. (3.45)) are in fact equivalent. Identity (3.44) (resp. (3.45)) means that if $n_z$ is the unit exterior normal to the level curve $\Gamma_z$

\[ \epsilon(V) \cdot T_z n_z = 0 \text{ (resp. } e(V) \cdot T_z n_z = 0) \text{ on } \Gamma_z. \tag{3.46} \]

Assumption (3.43) is the natural extension to thin/shallow shells of the underlying assumption in the Love–Kirchhoff theory of plates

\[ l = -\nabla_r(e \cdot n) \tag{3.47} \]

(cf. Germain [1] and Delfour and Zolésio [5, Remark 4.1]). This case will be studied in more detail in Section 5.

### 4. Computation of the Energies and the Work

As in Lagnese and Lions [1] first compute the total energy made up of the strain energy plus the work of the external forces minus the kinetic
energy. External forces mean forces and torques applied to the shell. In this section they are distributed within $S_h$, but more general loadings will be considered in Section 7. The dynamical shell equation corresponds to a stationary point of this total energy functional.

4.1. Strain Energy and Work of the External Forces

Define the strain energy $\mathcal{P}$ and the work of the external forces $\mathcal{W}$ as follows:

$$
\mathcal{P} = \frac{1}{2} \int_{S_h} \sigma : (V) \, dx \quad \mathcal{W} = \int_{S_h} \left[ F : V + M : (l \cdot p) \right] \, dx,
$$

(4.1)

where $\sigma$ is the stress tensor, $F$ the loading and $M$ the moment applied to the shell, and

$$
A \cdot B = \sum_{i,j=1}^N A_{ij} B_{ij}
$$

denotes the double contraction of the two $N \times N$ tensors $A$ and $B$. Assume that $F$ and $M$ belong to $H^1(S_h)^N$ so that the trace is well-defined on the surface $\Gamma$. Furthermore assume that $S_h$ is made up of an homogeneous isotropic elastic material which obeys the following rheological law

$$
\sigma = \lambda \text{tr} (V) I + 2\mu (V), \quad \lambda \geq 0 \text{ and } \mu > 0
$$

(4.2)

where $\lambda \geq 0$ and $\mu > 0$ are the Lamé coefficients. Hence

$$
\sigma \cdot T_z \cdot (V) \cdot T_z = \left[ \lambda \text{tr} (\tilde{V}) \cdot T_z \cdot I + 2\mu (V) \cdot T_z \right] \cdot (V) \cdot T_z = \\
= \lambda \left( \text{tr} (\tilde{V}) \cdot T_z \right)^2 + 2\mu \| (V) \cdot T_z \|^2.
$$

(4.3)

Recall expressions (3.1) and (3.29):

$$
V \cdot T_z = e + zl
$$

(4.4)

$$
\tilde{V} \cdot T_z = e^\theta + ze^1 + z^2 e^2
$$

(4.5)

Recall also the measure decomposition (2.46)–(2.47). From (4.1) and (4.2)

$$
\mathcal{P} = \frac{1}{2} \int_{-h}^h d\sigma \int_{\Gamma} d\Gamma \left[ \lambda \| (V) \cdot T_z \|^2 + 2\mu \| (V) \cdot T_z \|^2 \right] j(z),
$$

(4.6)

$$
\mathcal{W} = \int_{-h}^h d\sigma \int_{\Gamma} d\Gamma \left[ F \cdot V \cdot T_z + M \cdot T_z \cdot (l \cdot p \cdot T_z) \right] j(z).
$$

(4.7)
Using identities (4.3) to (4.5) and (2.48) for the polynomial $j(z)$

$$
\sigma(V) + T_z \cdot \hat{\sigma}(V) \cdot T_z j(z)
= \sum_{n=0}^{4} z^n A_n \sum_{i=0}^{N-1} K_i z^i = \sum_{n=0}^{4} \left( \sum_{i=0}^{N-1} K_i z^{i+n} \right) A_n,
$$

and

$$
A_0 = 2\mu \, |e_0|^2 + \hat{\lambda} \, |tr \, e_0|^2
$$
$$
A_1 = 2\mu 2e_0 \cdot e_1 + i2 \, tr \, e_0 \, tr \, e^1
$$
$$
A_2 = 2\mu [ \, |e_1|^2 + 2e_0 \cdot e_2 \, ] + \hat{\lambda} \, [ \, |tr \, e_1|^2 + 2 \, tr \, e_0 \, tr \, e^1 \, ]
$$
$$
(4.8)
$$
$$
A_3 = 2\mu 2e_1 \cdot e_2 + i2 \, tr \, e_1 \, tr \, e^2
$$
$$
A_4 = 2\mu \, |e_2|^2 + \hat{\lambda} \, |tr \, e^2|^2,
$$

where

$$
tr \hat{\sigma}(V) T_z = tr \, e^0 + z \, tr \, e^1 + z^2 \, tr \, e^2
$$
(4.9)
$$
tr \, e^0 = tr \, e_p(e),
$$
(4.10)
$$
tr \, e^1 = tr \, e_p(l) - tr(D_e D^2 b)
$$
(4.11)
$$
tr \, e^2 = - tr \, \{ [ D_e l - D_e D^2 b ] D^2 b \}
$$
$$
tr \, e^3 = - tr \, \{ [ D_e l D^2 b ] + tr \, [ D_e (D^2 b)^2 ] \}.
$$
(4.12)

Similarly assuming that the force $F$ and the moment $M$ are of the form

$$
F = f \cdot p \quad M = m \cdot p \quad on \quad S_h, \quad f, m \in L^2(\Gamma)^N, \quad m \cdot \nabla b = 0
$$
(4.13)

the integrand of (4.7) is equal to

$$
= \left[ f \cdot (e + zl) + m \cdot l \right] j(z) = \left[ f \cdot (e + zl) + m \cdot l \right] \sum_{i=0}^{N-1} K_i z^i
$$
$$
= \left[ f \cdot e + m \cdot l \right] \sum_{i=0}^{N-1} K_i z^i + f \cdot l \sum_{i=0}^{N-1} K_i z^{i+1},
$$
(4.14)
The final expressions for the strain energy $\mathcal{P}$ and the work $\mathcal{W}$ of the external forces and torques become

$$\mathcal{P} = \frac{1}{2} \int_{-h}^{h} \sum_{n=0}^{N-1} A_n z^n \sum_{i=0}^{n} K_i z^i \, dz \, d\Gamma = \frac{1}{2} \sum_{n=0}^{N-1} \mathfrak{x}_n(h) A_n \, d\Gamma$$

$$\mathcal{W} = \int_{-h}^{h} dz \left[ \int_{-h}^{h} d\Gamma \left[ (F \cdot T_z) - (V \cdot T_z) + (M \cdot T_z) \cdot ((l \cdot p \cdot T_z) \right] j(z) \right]$$

$$= \int_{-h}^{h} \mathfrak{x}_n(h) (f \cdot e + m \cdot l) + \mathfrak{x}_n(h) f \cdot l \, d\Gamma,$$

where the $\mathfrak{x}_n$'s result from the integration with respect to $z$ from $-h$ to $h$

$$\mathfrak{x}_n(h) = \int_{-h}^{h} z^n \sum_{i=0}^{N-1} K_i z^i \, dz$$

$$= h^{n+1} \sum_{i=0}^{N-1} [1 - (-1)^{n+i+1}] \frac{h^i}{n+i+1} K_i, \quad 0 \leq n \leq 4. \quad (4.15)$$

They are polynomials in odd powers of $h$ or more explicitly

$$\mathfrak{x}_0(h) = 2h \left[ 1 + \frac{h^2}{2} \sum_{i=2}^{N-1} [1 - (-1)^{i+1}] \frac{h^{i-2}}{i+1} K_i \right]$$

$$= 2h(1 + h^2 c_0(h))$$

$$\mathfrak{x}_1(h) = 2 \frac{h^3}{3} \left[ H + 3 \frac{h^2}{2} \sum_{i=3}^{N-1} [1 - (-1)^{i+2}] \frac{h^{i-3}}{i+2} K_i \right]$$

$$= 2 \frac{h^3}{3} (H + h^2 c_1(h))$$

$$\mathfrak{x}_2(h) = 2 \frac{h^3}{3} \left[ 1 + 3 \frac{h^2}{2} \sum_{i=2}^{N-1} [1 - (-1)^{i+3}] \frac{h^{i-2}}{i+3} K_i \right]$$

$$= 2 \frac{h^3}{3} (1 + h^2 c_2(h))$$

$$\mathfrak{x}_3(h) = 2 \frac{h^3}{5} \left[ H + 5 \frac{h^2}{2} \sum_{i=3}^{N-1} [1 - (-1)^{i+4}] \frac{h^{i-3}}{i+4} K_i \right]$$

$$= 2 \frac{h^3}{5} (H + h^2 c_3(h))$$

$$\mathfrak{x}_4(h) = 2 \frac{h^3}{5} \left[ 1 + 5 \frac{h^2}{2} \sum_{i=2}^{N-1} [1 - (-1)^{i+5}] \frac{h^{i-2}}{i+5} K_i \right]$$

$$= 2 \frac{h^3}{5} (1 + h^2 c_4(h)).$$
The spaces $H$, $V$, and $N$, and their associated norms and seminorm have been defined in (3.30) to (3.34). The following also applies to the subspaces $H$ and $V$: for all $(e, I)$ and $(\tilde{e}, \tilde{I})$ in $H$ and $e' = e'(e, I)$, $\tilde{e}' = \tilde{e}'(\tilde{e}, \tilde{I})$

$$\langle A(e, l), (\tilde{e}, \tilde{I}) \rangle_H = \sum_{n=0}^{4} \int_{I} a_{n}(e, l, (\tilde{e}, \tilde{I})) \, d\Gamma,$$  (4.17)

where

$$\begin{align*}
a_{0}(e, l, (\tilde{e}, \tilde{I})) &= 2\mu \delta \cdot \delta \bar{\delta} + \lambda \bar{\delta} \delta \delta \\
a_{1}(e, l, (\tilde{e}, \tilde{I})) &= 2\mu [\delta \delta \bar{\delta} + \bar{\delta} \delta \delta \delta] + \lambda [\bar{\delta} \delta \delta \bar{\delta} + \delta \bar{\delta} \delta \delta] \\
a_{2}(e, l, (\tilde{e}, \tilde{I})) &= 2\mu [\delta \delta \delta \delta + \delta \delta \delta \delta + \delta \delta \delta \delta + \delta \delta \delta \delta] \\
a_{3}(e, l, (\tilde{e}, \tilde{I})) &= 2\mu [\delta \delta \delta + \delta \delta \delta + \delta \delta \delta + \delta \delta \delta] + \lambda \bar{\delta} \delta \delta \delta + \delta \bar{\delta} \delta \delta \\
a_{4}(e, l, (\tilde{e}, \tilde{I})) &= 2\mu \delta \delta \delta + \delta \delta \delta + \lambda \delta \delta \delta.
\end{align*}$$  (4.18)

Note that $A_{n} = a_{n}(e, l, (e, l))$, $0 \leq n \leq 4$. Similarly introduce the continuous linear operator

$$\mathcal{B}: U \overset{\text{def}}{=} L^{2}(\Gamma)^{N} \times L^{2}(\Gamma)^{N} \rightarrow H'$$

$$\langle \mathcal{B}(f, m), (e, l) \rangle_H = \int_{I} \left[ \xi_{0}(f + \bar{\delta}e + \delta \bar{\delta}l) + \xi_{1}(f + \delta \bar{\delta}l) \right] \, d\Gamma.$$  (4.19)

By construction $A$ is symmetrical and positive

$$\langle A(e, l), (\tilde{e}, \tilde{I}) \rangle_H = \langle A(\tilde{e}, \tilde{I}), (e, l) \rangle_H \quad \langle A(e, l), (e, l) \rangle_H \geq 0.$$  (4.20)

**Lemma 4.1.** There exists $\hat{h} > 0$ and $\alpha > 0$ such that for all $0 < h < \hat{h}$ and all $(e, l) \in H$ (resp. $V'$)

$$\langle A(e, l), (\tilde{e}, \tilde{I}) \rangle_{H} \geq 2\mu \alpha \sum_{n=0}^{2} \hat{h}^{2n} \| e(e, l) \|^{2}.$$  (4.20)

**Proof.** From (4.6)

$$\langle A(e, l), (e, l) \rangle_{H} \geq \int_{I_{h}} dz \int_{I} d\Gamma 2\mu \| \bar{e}(V) \cdot \bar{T}_{z} \|^{2} j(z) \, d\Gamma \geq \sum_{n=0}^{4} \int_{I} \xi_{n} A_{n} \, d\Gamma,$$
where the $A_n^*$'s, $0 \leq n \leq 4$, are defined by (4.8) with $\lambda = 0$. Hence

$$
\frac{1}{2\mu} \left\langle A(e, l), (e, l) \right\rangle_f \\
\geq \alpha_d(h) \|e^0\|^2 + \alpha_1(h) 2e^0 \cdot e^1 + \alpha_2(h) 2x_2(h) \|x_2(h)\|^2 + 2e^0 \cdot e^2 + \alpha_3(h) \|e^2\|^2$

From (4.16), note that there exist $\hat{h} > 0$ sufficiently small and $c > 0$ such that

$$
\forall n, \forall h, 0 < h < \hat{h}, \quad |c_n(h)| < c, \quad 1 - ch^2 > 0.
$$

Therefore

$$
\frac{1}{2\mu} \left\langle A(e, l), (e, l) \right\rangle_f \geq 2h(1 - ch^2)\|e^0\|^2 - \frac{h^3}{3} \left( |H| + ch^2 \right) 2\|e^0\|\|e^1\| \\
+ 2\frac{h^3}{3} (1 - ch^2)\|e^1\|^2 - \frac{h^3}{2} (1 - ch^2) 2\|e^0\|\|e^2\| \\
- \frac{h^5}{2} \left( |H| + ch^2 \right) 2\|e^1\|\|e^2\| + 2\frac{h^5}{5} (1 - ch^2)\|e^2\|^2.
$$

But

$$
- \frac{h^3}{3} 2\|e^0\|\|e^2\| = -2 \left( \frac{2h^3}{3} \frac{3}{h^2} \right) \|e^0\| \frac{h^3}{3} \|e^2\| \\
\geq -\frac{4}{3} h \|e^0\|^2 - \frac{h^5}{3} \|e^2\|^2
$$

$$
- \frac{h^3}{3} 2\|e^0\|\|e^1\| = -2 h \|e^0\| \frac{2h^2}{3} \|e^1\| \\
\geq -h^2 \|e^0\|^2 - \frac{4h^4}{9} \|e^1\|^2
$$

$$
- \frac{h^5}{2} 2\|e^1\|\|e^2\| = -2h^2 \|e^1\| \frac{2h^3}{5} \|e^2\| \\
\geq -h^4 \|e^1\|^2 - \frac{4h^6}{25} \|e^2\|^2.
$$
Finally
\[
\frac{1}{2\mu} \langle A(e, l), (e, l) \rangle_v \geq \left[ \frac{2}{3} h(1 - ch^2) - h^2 [ \frac{2}{9} h^3 [ |H| + ch^2 ] \right] \|e^0\|^2 \]
\[+ \left[ \frac{2}{3} h(1 - ch^2) - \frac{13}{9} h^4 [ |H| + ch^2 ] \right] \|e^1\|^2 \]
\[+ \left[ \frac{1}{15} h^3 (1 - ch^2) - \frac{4}{25} h^5 [ |H| + ch^2 ] \right] \|e^2\|^2.
\]

The leading term in each square bracket is a low order term in \(h\). Thus as \(h\) goes to 0, it absorbs the higher order terms and the conclusion of the theorem follows.

If the elements of the dual \(\mathcal{H}'\) of \(\mathcal{H}\) are identified with those of \(\mathcal{H}\), then from the above Lemma \(A\) is a \(\mathcal{V} - \mathcal{H}\) coercive operator.

**Theorem 4.1.** Given \(\tilde{h} > 0\) as specified in Lemma 4.1, \(0 < h \leq \tilde{h}\), and assuming that the following condition is verified for all \((e, l) \in \mathcal{V}\)
\[
\int_{\Gamma} \bar{\sigma}_d f \cdot e + m \cdot l \right] + \bar{\tau}_1 f \cdot l \, d\Gamma = 0,
\]
then for all \(0 < h \leq \tilde{h}\), there exists a unique solution \((\bar{e}, \bar{l}) \in \mathcal{V}')\) to the variational equation: for all \((e, l) \in \mathcal{V}\)
\[
\langle A(\bar{e}, \bar{l}), (e, l) \rangle_v + \langle \mathcal{M}(f, m), (e, l) \rangle_{\mathcal{H}} = 0.
\]
The same conclusions hold with \(\mathcal{H}'\) and \(\mathcal{V}'\) when the elements of the dual \((\mathcal{H}')\)' of \(\mathcal{H}'\) are now identified with those of \(\mathcal{H}'\).

**4.2. Kinetic Energy**

To complete the model it remains to compute the kinetic energy. Assume that \(V(x, t), e(x, t), \) and \(l(x, t)\) depend on the time \(t\) and denote by \(V', e', \) and \(l'\) the derivatives with respect to the time \(t\):
\[
\mathcal{K} = \int_{\mathcal{S}_s} \frac{1}{2} \rho |V'|^2 \, dx = \int_{\mathcal{S}_s} \frac{1}{2} \rho |e' - p + b\ell' - p'|^2 \, dx
\]
\[= \frac{1}{2} \rho \int_{\Gamma} \bar{\sigma}_d (h) |e'|^2 + \bar{\tau}_1 (h) 2e' \cdot l + \bar{\tau}_2 (h) |l'|^2 \, d\Gamma,
\]
where \(\rho\) is the volume density and \(e'\) and \(l'\) are the time derivatives of \(e\) and \(l\). Define the operator \(\bar{\mathcal{M}}: \mathcal{H} \rightarrow \mathcal{H}'\) as follows:
\[ \langle \mathcal{A}(e, l), (\tilde{e}, \tilde{l}) \rangle_\mathcal{X} = p \int_I \left( \alpha_0(h) e \cdot \tilde{e} + \alpha_3(h) [e \cdot \tilde{l} + \tilde{e} \cdot l] \right) + \alpha_3(h) \tilde{l} \cdot d\Gamma. \]  

(4.24)

For \( h \) sufficiently small, \( \mathcal{A} \) is symmetrical, positive and invertible.

### 4.3. Dynamical Shell Equation

Identify the elements of the dual \( \mathcal{H}' \) of \( \mathcal{H} \) with those of \( \mathcal{H} \) and denote by \( A : \mathcal{H}' \to \mathcal{H} \) the corresponding canonical isomorphism. Then the linear operators

\[ M = A A', \quad B = A B' \]  

are bounded from \( \mathcal{H} \) to \( \mathcal{H} \) and from \( U \) to \( \mathcal{H} \). For \( h \) sufficiently small, \( M \) is a positive symmetrical continuous and invertible linear operator. A stationary point \((e, l)\) of the total energy \( \mathcal{P} + \mathcal{W} - \mathcal{K} \) verifies the following second order dynamical equation

\[ \frac{d^2}{dt^2} \left( M \begin{bmatrix} e(t) \\ l(t) \end{bmatrix}, \begin{bmatrix} \phi \\ \psi \end{bmatrix} \right)_\mathcal{X} + \left( A \begin{bmatrix} e(t) \\ l(t) \end{bmatrix}, \begin{bmatrix} \phi \\ \psi \end{bmatrix} \right)_\mathcal{X} + \left( B \begin{bmatrix} f(t) \\ m(t) \end{bmatrix}, \begin{bmatrix} \phi \\ \psi \end{bmatrix} \right)_\mathcal{X} = 0 \]  

(4.26)

for all \((\phi, \psi) \in \mathcal{X}'\). Standard existence and uniqueness theorems (cf. for instance Lions [1, Thm 1.1 and Rem. 1.3, p. 294]) can now be used.

**Theorem 4.2.** Given \( \tilde{h} > 0 \) as specified in Lemma 4.1, \( 0 < h \leq \tilde{h} \), there exists a unique solution

\[ (e, l) \in C^1([0, T]; \mathcal{X}) \cap C([0, T]; \mathcal{X}') \]  

(4.27)

to equation (4.26) verifying the initial conditions

\[ (e(0), l(0)) = (e_0, l_0) \quad \text{in} \quad \mathcal{X}, \]  

(4.28)

\[ (e'(0), l'(0)) = (e_1, l_1) \quad \text{in} \quad \mathcal{H}. \]  

(4.29)

The same constructions and conclusions hold for \( \mathcal{H}' \) and \( \mathcal{X}' \) when the elements of the dual \((\mathcal{H}')'\) are now identified with those of \( \mathcal{H}' \).

#### 4.4. Homogeneous Dirichlet Boundary Conditions

For a shell with boundary and homogeneous Dirichlet boundary conditions the results are analogous to the ones of Theorem 4.1.

**Theorem 4.3.** Given \( \tilde{h} > 0 \) as specified in Lemma 4.1 and \( h, 0 < h \leq \tilde{h} \), there exists a unique solution \((\tilde{e}, \tilde{l}) \in \mathcal{X}_0 \) to the variational equation: for all \((e, l) \in \mathcal{X}_0 \)

\[ \langle A(\tilde{e}, \tilde{l}), (e, l) \rangle_\mathcal{X} + \langle B(f, m), (e, l) \rangle_\mathcal{X} = 0, \]  

(4.30)
where

\[ \mathcal{V}_0 = H^1_0(\Gamma)^N \times H^1(\Gamma)^N \]  

(4.31)

(resp. \( \mathcal{V}'_0 = \{(e, l) \in H^1_0(\Gamma)^N \times H^1(\Gamma)^N : l \cdot n = 0 \} \)).

This follows by the same arguments as in the proof of Theorem 4.1 and the following Poincaré inequality for the shell.

**Lemma 4.2.** Assume that \( \Gamma \) is a bounded open domain in the \( C^2 \) \((N-1)\)-dimensional submanifold \( \partial \Omega \) of \( \mathbb{R}^N \) with a Lipschitzian boundary \( \partial \Gamma = \partial \Omega \Gamma \) in \( \partial \Omega \). For \( h \) sufficiently small, there exists a constant \( c = c(S, D^3b) > 0 \) such that for all \((e, l) \in \mathcal{V}_0 \) (resp. \( \mathcal{V}'_0 \))

\[
\int_{\Gamma} 2h|e|^2 + 2\frac{h^3}{3}|l|^2 + 2h[|l|^2 + \|D\varepsilon(e)\|^2] + 2\frac{h^3}{3}\|D_l l\|^2 d\Gamma \\
\leq c^2 \left[ \int_{\Gamma} 2h\|\varepsilon(e, l)\|^2 + 2\frac{h^3}{3}\|\varepsilon(e, l)\|^2 + 2\frac{h^3}{3}\|\varepsilon(e, l)\|^2 d\Gamma \right] (4.32)
\]

and \( \|\varepsilon_l(e)\| + \|\varepsilon'(e, 0)\| \) is an equivalent norm on \( H^1_0(\Gamma)^N \).

**Proof.** Same techniques as in Theorem 3.2 using Poincaré inequality on \( S_k \) for the displacement vector \( V \) defined in (3.1) with \( V = 0 \) on the lateral boundary \( \Sigma_k \)

\[
\sum_{i=1}^{N} \int_{S_k} |V_i|^2 d\sigma \leq \sum_{i,j=1}^{N} \int_{S_k} |\partial_i V_j|^2 d\sigma \\
\leq c^2 \left[ \sum_{i,j=1}^{N} \int_{S_k} |\varepsilon_{ij}(V)|^2 d\sigma \right] (4.33)
\]

From (4.32) the sum of the norms of the \( \varepsilon'(e, l)'s \) is an equivalent norm on \( \mathcal{V}_0 \) in Theorem 4.30. Hence from (3.26) with \( l = 0 \), \( \|\varepsilon_l(e)\| + \|\varepsilon'(e, 0)\| \) is also equivalent.

**Remark 4.1.** Of course we also have the analogue of Theorem 4.2 for homogeneous Dirichlet boundary conditions.

5. The Love–Kirchhoff Theory

We have seen in Section 3.3 that the Love–Kirchhoff theory is characterized by identity (3.43)

\[ l = -2\varepsilon_l(e) \nabla b. \]  

(5.1)

As a result

\[ l = -[D_l e + \ast D_l e] \nabla b. \]
But
\[ D_f e \nabla b = 0 \Rightarrow l = -^* D_f e \nabla b \Rightarrow l \cdot \nabla b = - D_f e \nabla b = 0. \quad (5.2) \]

In other words identity (5.1) already contains the fact that \( l \) is a tangent vector and we can repeat the constructions of § 4 with the following subspaces of \( \mathcal{H} \) and \( \mathcal{V}^1 \):

\[ \mathcal{H}_1 = \{(e, l) \in L^2(\Gamma)^N \times L^2(\Gamma)^N : l + 2 \varepsilon_f(e) \nabla b = 0 \text{ on } \Gamma \} \quad (5.3) \]
\[ \mathcal{V}^1_1 = \{(e, l) \in \mathcal{H} : \varepsilon'(e, l) \in L^2(\Gamma)^N \times N, 0 \leq i \leq 2 \}, \quad (5.4) \]
\[ \mathcal{N}_1 = \{(e, l) \in \mathcal{V} : \varepsilon'(e, l) = 0 \text{ on } \Gamma, 0 \leq i \leq 2 \}. \quad (5.5) \]

Endowed with the norms (3.33) and (3.34), they are also Hilbert spaces. The subspace \( \mathcal{N}_1 \) coincides with \( \mathcal{N} \) as defined in (3.32) and corresponds to the rigid displacements (3.35). From the Korn's inequality (3.39) in Theorem 3.2,

\[ \mathcal{V}^1_1 = \{(e, l) \in H^1(\Gamma)^N \times H^1(\Gamma)^N : l + 2 \varepsilon_f(e) \nabla b = 0 \}. \]

The seminorm on the space \( \mathcal{V}^1 \)

\[ ||(e, l)|| = \left\{ \sum_{j=0}^{2} \int_{\Gamma} \varepsilon'(e, l)^2 d\Gamma \right\}^{1/2} \quad (5.6) \]

again becomes a norm on the quotient space \( \mathcal{H}_1 / \mathcal{N} \). Lemma 4.1 remains true on \( \mathcal{V}^1 \) and \( A \) is \( \mathcal{V}^1 \)-coercive.

We now have the equivalent of Theorems 4.1 and 4.2.

**Theorem 5.1.** Given \( h > 0 \) as specified in Lemma 4.1, \( 0 < h \leq \tilde{h} \), and assuming that the following condition is verified for all \( (e, l) \in \mathcal{N} \):

\[ \int_{\Gamma} \alpha_0 (\tilde{f} \cdot e + m \cdot l) + \alpha_1 f \cdot l d\Gamma = 0, \quad (5.7) \]

then for all \( 0 < h \leq \tilde{h} \) there exists a unique solution \((\hat{e}, \hat{l}) \in \mathcal{V}^1_1 / \mathcal{N} \) to the variational equation: for all \( (e, l) \in \mathcal{V}^1_1 \)

\[ \langle A(\hat{e}, \hat{l}), (e, l) \rangle_{\mathcal{H}_1} + \langle B (f, m), (e, l) \rangle_{\mathcal{H}} = 0. \quad (5.8) \]

As in Section 4, identify the elements of the topological dual \( \mathcal{H}^*_1 \) of \( \mathcal{H}_1 \) with those of \( \mathcal{H}^*_1 \) and denote by \( A_1 : \mathcal{H}^*_1 \to \mathcal{H}_1 \) the corresponding canonical isomorphism. Let

\[ M_1 = A_1 \mathcal{H}_1, \quad B_1 = A_1 \mathcal{B}. \quad (5.9) \]
Theorem 5.2. Given \( \hat{h} > 0 \) as specified in Lemma 4.1, \( 0 < h \leq \hat{h} \), there exists a unique solution \( (e, l) \in C^1([0, T]; \mathcal{X}_i) \cap C([0, T]; \mathcal{Y}_i) \). (5.10)

to the equation

\[
\begin{align*}
\frac{d^2}{dt^2} \left( \begin{array}{c}
M_1 \left[ \begin{array}{c}
\varphi \\
l(t)
\end{array} \right] \\
A \left[ \begin{array}{c}
\varphi \\
l(t)
\end{array} \right]
\end{array} \right) + \left( \begin{array}{c}
B_1 \left[ \begin{array}{c}
f(t) \\
m(t)
\end{array} \right] \\
0
\end{array} \right) = 0
\end{align*}
\]

(5.11)

for all \( (\varphi, \psi) \in \mathcal{Y}_i \), verifying the initial conditions

\[
\begin{align*}
(e(0), l(0)) &= (e_0, l_0) \quad \text{in } \mathcal{Y}_i, \quad (l_0 = 2e_0(e_0) n) \\
(e'(0), l'(0)) &= (e_1, l_1) \quad \text{in } \mathcal{X}_i, \quad (l_1 = 2e_1(e_1) n).
\end{align*}
\]

Remark 5.1. The Love-Kirchhoff identity (5.1) can be written in the form

\[
l = D^2be - \nabla_f (e \cdot n)
\]

(5.14)

(cf. Delfour and Zolésio [5, Remark 5.1]). To see this we use identity (2.36)

\[
l = -2e_f(e) \nabla b = -D_f e \nabla b = D^2b e - \nabla_f (e \cdot n).
\]

This expression only uses first order derivatives of \( e \cdot n \) in expression for \( l \) instead of first order derivatives of \( e \). The price to pay for this new expression of \( l \) is some additional smoothness on the function \( b \) and hence on the smoothness of the boundary \( \Gamma \) since

\[
l = D^2be - \nabla_f (e \cdot n) \in H^1(\Gamma).
\]

If we now assume that \( \partial \Omega \) is of class \( C^3 \), then the spaces \( \mathcal{X}_i \) and \( \mathcal{Y}_i \) can be characterized as follows:

\[
\mathcal{X}_i = \{ (e, D^2be - \nabla_f (e \cdot n)) \in L^2(\Gamma)^N \times L^2(\Gamma)^N; \nabla_f (e \cdot n) \in L^2(\Gamma)^N \}
\]

(5.16)

\[
\mathcal{Y}_i = \{ (e, D^2be - \nabla_f (e \cdot n)) \in H^1(\Gamma)^N \times H^1(\Gamma)^N; D^2_f(e \cdot n) \in L^2(\Gamma)^N \}
\]

(5.17)

where \( D^2_f(e \cdot n) \) is the matrix of tangential second order derivatives as defined in (2.44) and \( \mathcal{Y}_i = \mathcal{X}_i \) in (3.35). Such spaces where the normal component of \( e \) is one degree smoother than \( e \) itself are typical in the Theory.
of Plates where $\partial \Omega$ is of class $C^\infty$ (cf. §6). It must be emphasized that the assumption that $\partial \Omega$ be of class $C^3$ is not a necessary mathematical assumption of the Love–Kirchhoff theory resulting from identity (5.1). It only arises from a rearrangement of the variables in the form (5.14). This suggests to introduce the spaces

\begin{align}
H_1 &= \{ e \in L^2(\Gamma)^N; \nabla_\Gamma (e \cdot n) \in L^2(\Gamma)^N \} \\
V_1 &= \{ e \in H^1(\Gamma)^N; D^2_\Gamma (e \cdot n) \in L^2(\Gamma)^{N \times N} \} \\
N &= \{ e; e(X) = a + BX, \ \forall a \in \mathbb{R}^N, \end{align}

\begin{equation}
\forall B \text{ an } N \times N \text{ matrix such that } B + B^* = 0 \}. \tag{5.20}
\end{equation}

and use identity (5.14) to substitute for $l$ in the previous expressions.

Remark 5.2. Theorems 5.1 and 5.2 correspond to homogeneous Neumann boundary conditions when the shell has a boundary. They remain true with obvious changes for homogeneous Dirichlet boundary conditions without condition (5.7) (cf. Theorem 4.3).

6. Specialization to Plates

In the case of plates, most of the previous expressions simplify. Here we assume that $\Gamma$ is a bounded open subset of a linear $(N - 1)$-dimensional submanifold $\partial \Omega$ in $\mathbb{R}^N$ with a Lipschitzian boundary $\partial \Omega$. The distance function is chosen as $b = b_\Omega$, where $\Omega$ is one of the half spaces determined by $\partial \Omega$. The shell is defined as

\begin{equation}
S_b = \{ x \in \mathbb{R}^N; |b_\Omega(x)| < h \text{ and } p_\Omega(x) \in \Gamma \} \tag{6.1}
\end{equation}

as in (2.25). With this definition

\begin{align}
V &= e \cdot p + bl \cdot p, \quad V \cdot T_z = e + zl \\
DV &= D e \cdot p + b D l \cdot p + l \cdot p \ast \nabla b, \quad DV \nabla b = l \cdot p \\
\varepsilon(V) &= \varepsilon_\Gamma (e) \cdot p + b \varepsilon_\Gamma (l) \cdot p + \frac{1}{2} [ l \cdot p \ast \nabla b + l \cdot p \ast \nabla b ] \\
\varepsilon(V) \cdot T_z &= \varepsilon^0 + z e^1, \quad \varepsilon(V) \nabla b = \varepsilon_\Gamma (e) \cdot p \nabla b + \frac{1}{2} l \cdot p \\
\varepsilon^0 &= \varepsilon_\Gamma (e) + \frac{1}{2} [ l \ast \nabla b + \nabla b \ast l ], \quad e^1 = \varepsilon_\Gamma (l) \\
e'' &= 0, \quad n \geq 2, \quad \text{tr} \ v^0 = \text{tr} \ v^0_{\varepsilon_\Gamma (e)}.
\end{align}

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Moreover

\[ \langle A(e, l), (\hat{e}, \hat{l}) \rangle_x = \int_{\Gamma} 2\beta \left[ \nabla e \cdot \nabla \hat{e} + \lambda \nabla \cdot \nabla \hat{e} \right] + \frac{2}{3} \lambda \nabla \cdot \nabla \hat{e} \dI \]

\[ \langle B(f, m), (e, l) \rangle_x = \int_{\Gamma} 2h \left[ f \cdot e + m \cdot l \right] \dI \]

\[ \langle \mathcal{M}(e, l), (\bar{e}, \bar{l}) \rangle_x = \rho \int_{\Gamma} 2he \cdot \bar{e} + \frac{2}{3} \lambda \bar{e} \cdot \bar{l} \dI. \]

From the extended Korn's inequality

\[ \mathcal{K} = L^2(\Gamma)^N \times L^2(\Gamma)^N, \]

\[ \mathcal{K}' = \{(e, l) \in L^2(\Gamma)^N \times L^2(\Gamma)^N : l \cdot n = 0 \text{ on } \Gamma \}, \]

\[ \mathcal{V} = H^1(\Gamma)^N \times H^1(\Gamma)^N, \]

\[ \mathcal{V}' = \{(e, l) \in H^1(\Gamma)^N \times H^1(\Gamma)^N : l \cdot n = 0 \text{ on } \Gamma \}, \]

\[ \mathcal{N} = \{(e, l) : e(X) = a + BX, \ l(X) = B \nabla b(X), \ \forall a \in \mathbb{R}^N, \] \[ \forall B \text{ an } N \times N \text{ matrix such that } B + \ast B = 0 \}. \]

We can now explicit the variational equation (4.22) by using the following identities

\[ \varepsilon^0(\hat{e}, \hat{l}) \cdot \varepsilon^0(e, l) = \varepsilon_{\nabla}(\hat{e}) \cdot \varepsilon_{\nabla}(e) + \varepsilon_{\nabla}(\hat{e}) \nabla b \cdot l + \hat{l} \cdot \varepsilon_{\nabla}(e) \nabla b + \frac{1}{2} \hat{l} \cdot l \]

\[ \nabla \cdot \varepsilon^0(e, l) = \nabla \cdot \varepsilon_{\nabla}(e) = \text{div}_{\nabla} e. \]

Moreover by introducing the normal component \( w = e \cdot \nabla b \) and the tangential component \( u = e - w \nabla b \) of \( e \) and using (2.36)

\[ \varepsilon_{\nabla}(e) = \varepsilon_{\nabla}(u) + \frac{1}{2} \left[ \nabla b \ast \nabla_{\nabla} w + \nabla r w \ast \nabla b \right] \]

\[ \text{div}_{\nabla}(e) = \text{div}_{\nabla}(u) \quad \text{and} \quad \varepsilon_{\nabla}(u) \nabla b = 0 \]

\[ \varepsilon_{\nabla}(e) \nabla b = \varepsilon_{\nabla}(u) \nabla b + \frac{1}{2} \nabla r w = \frac{1}{2} \nabla r w \]

and

\[ \varepsilon^0(\hat{e}, \hat{l}) \cdot \varepsilon^0(e, l) = \varepsilon_{\nabla}(\hat{u}) \cdot \varepsilon_{\nabla}(u) + \frac{1}{2} \left( \hat{l} + \nabla_{\nabla} \hat{w} \right) \cdot \left( l + \nabla r w \right). \]
Hence (4.22) reduces to

\[
    \int_I 2h\left(2\mu[\varepsilon_F(\hat{\mu}) \cdot \varepsilon_F(u) + \frac{1}{2}(\hat{l} + \nabla \hat{\nu} \cdot (l + \nabla w)] + \lambda \text{div}_F \hat{\mu} \text{div}_F u \right) \\
    + \frac{h^3}{2} \left\{2\mu \varepsilon_F(l) \cdot \varepsilon_F(l) + \lambda \text{div}_F \hat{l} \text{div}_F l \right\} \\
    + 2h[f \cdot u + f \cdot \nabla w + m \cdot l] \, d\Gamma = 0.
\]

This yields three variational equations

\[
    \int_I 2h\left(2\mu \varepsilon_F(\hat{\mu}) \cdot \varepsilon_F(u) + \lambda \text{div}_F \hat{\mu} \text{div}_F u + f \cdot u \right) \, d\Gamma = 0 \quad (6.3)
\]

\[
    \int_I 2\mu \left[ 2 \frac{h^3}{3} \varepsilon_F(l) \cdot \varepsilon_F(l) + 2h \frac{1}{2} (\hat{l} + \nabla \hat{\nu} \cdot l) \right] \\
    + \lambda 2 \frac{h^3}{3} \text{div}_F \hat{l} \text{div}_F l + 2hm \cdot l \, d\Gamma = 0 \quad (6.4)
\]

\[
    \int_I 2h \left[ 2\mu \frac{1}{2} (\nabla \hat{\nu} + \hat{l}) \cdot \nabla w + f \cdot \nabla w \right] \, d\Gamma = 0. \quad (6.5)
\]

All this is under the following condition on the functions \((f, m)\) appearing in (6.3)–(6.5)

\[
    \forall (e, l) \in \mathcal{N}, \quad \int_I f \cdot e + m \cdot l \, d\Gamma = 0.
\]

This condition can be made more explicit by using (2.28) and (2.41) with \(D^2b = 0\) and by observing that

\[
    0 = l = -\nabla \hat{w} \Rightarrow \nabla (w \cdot p) = 0 \text{ in } S_a \Rightarrow w = \text{constant on } \Gamma
\]

\[
    0 = \hat{e}(e, 0) = \varepsilon_F(e) = \varepsilon_F(u + w \nabla b) = \varepsilon_F(u) + \frac{1}{2} [\nabla w \cdot \nabla b + \nabla b \cdot \nabla w]
\]

\[
    \Rightarrow \varepsilon_F(u) = 0 \Rightarrow u = u_0 + U_1 \cdot \hat{x}, \quad U_1 + *U_1 = 0, \quad U_1 \nabla b = 0, \quad U_1 \cdot \nabla b = 0.
\]

Finally the condition becomes

\[
    \int_I f \cdot \nabla b \, d\Gamma = 0
\]

\[
    \int_I f \cdot u \, d\Gamma = 0, \quad \forall u \in H^1_0(\Gamma)^N, \quad \varepsilon_F(u) = 0 \text{ and } u \cdot \nabla b = 0
\]
From Section 5 the Love–Kirchhoff theory is characterized by the relation (5.14) with \( D^2 b = 0 \) and from identities (2.44) and (2.45)

\[
I = -\nabla_F (e \cdot n) \quad \text{and} \quad \varepsilon_F (l) = -\varepsilon_F (\nabla_F (e \cdot n)) = -D^2_F (e \cdot n) \tag{6.6}
\]

with the associated spaces \( \mathscr{H}_1, \mathscr{V}_1 \) and \( \mathscr{V}_1 \) defined by (5.16), (5.17) and (3.35). We can also express everything in terms of \( e \) and use the spaces \( H_1, V_1 \) and \( N_1 \) defined in (5.18) to (5.20). Using expression (6.3) \( I \) can be eliminated and the \( \lambda \)'s can be written in term of \( e \)

\[
e^0 = \varepsilon_F (e) - \frac{1}{2} [\nabla_F (e \cdot n) \ast \nabla b + \nabla b \ast \nabla_F (e \cdot n)] ,
\]

\[
e^1 = -D^2_F (e \cdot n) . \tag{6.7}
\]

By using the normal and tangential components of \( e \) the variational equation (5.8) becomes

\[
\int_F 2 h [2 \varepsilon_F (\hat{u}) \cdot \varepsilon_F (u) + \lambda \text{div}_F \hat{u} \text{div}_F u + f \cdot u ] \, d\Gamma = 0 \tag{6.8}
\]

\[
\int_F \frac{2 h^3}{3} [2 \varepsilon_F (l) \cdot \varepsilon_F (l) + 2 h \frac{1}{2} [\hat{I} + \nabla_F \hat{w}] \cdot (l + \nabla_F w)]
+ \lambda 2 h^3 \text{div}_F \hat{I} \text{div}_F l + 2 h (m \cdot f + \nabla b w) \, d\Gamma = 0 . \tag{6.9}
\]

Therefore for the Love–Kirchhoff theory the equation for \( u \) is the same as the one for the natural theory. As for the two equations for \( l \) and \( w \), they combine into a fourth order equation for \( w \) and the Love–Kirchhoff condition on \( l \):

\[
\int_F 2 h [2 \varepsilon_F (\hat{u}) \cdot \varepsilon_F (u) + \lambda \text{div}_F \hat{u} \text{div}_F u + f \cdot u ] \, d\Gamma = 0 \tag{6.10}
\]

\[
\int_F \frac{h^3}{3} [2 \varepsilon_F (\hat{w}) \cdot \varepsilon_F (w) + \lambda \text{A}_F \hat{w} \cdot \text{A}_F w] + 2 h (f \cdot \nabla b w - m \cdot \nabla_F w) \, d\Gamma = 0 \tag{6.11}
\]

\[
\hat{I} + \nabla_F \hat{w} = 0. \tag{6.12}
\]
Remark 6.1. For plates the present model coincides with the one used in Delfour and Zolésio [5, Sect. 4, Eq. 4.24, Sect. 5, Eq. 5.4, Remark 5.1]. The kernel of the strain tensor are the same in both cases.

Remark 6.2 (Asymptotic behaviour). When \( h \) is constant, the term \( 2h \) can be dropped in (6.10) and its solution is independent of \( h \). In particular \( \hat{u}(h) = \hat{u}(0) \). If the pair \( (f, m) \) is of the form

\[
f \cdot \nabla b = \frac{h^2}{3} f_n \quad \text{and} \quad \text{div}_\Gamma m = \frac{h^2}{3} m_n, \quad m_n \in H^2_0(\Gamma),
\]

then the solution \( \hat{\nu}(h) \) of (6.11) is also independent of \( h \) and \( \hat{\nu}(h) = \hat{\nu}(0) \) and \( \hat{\nu}(h) \) is equal to the solution \( \hat{\nu}_0 \) of the following fourth order variational equation

\[
\int_\Gamma 2\mu D^2_\Gamma (\hat{\nu}_0) : \cdot D^2_\Gamma (w) + \lambda \mu_\Gamma \hat{\nu}_0 w \cdot w + f_n w d\Gamma + \langle m_n, w \rangle_{H^2_\Gamma(\Gamma)} = 0.
\]

(6.14)

7. Boundary Conditions

The specification of the boundary conditions for a shell with boundary is not necessarily difficult but the final expressions are more complex. Dirichlet conditions are usually incorporated in the definition of the spaces and do not require extra work. To find the exact form of Neumann boundary conditions requires “integration by parts” formulae. The analysis of this section will be performed with generic terms to illustrate the type of mathematical results one can expect. A more detailed analysis would be required to introduce mechanically meaningful terms on the right-hand side of the equations.

7.1. Integration by Parts Formulae

It will be convenient to use the notation \( \partial \Gamma \) instead of \( \partial_{\partial \Omega} \Gamma \). Assume that \( \partial \Omega \) is a \( C^4 \) \((N-1)\)-submanifold in \( \mathbb{R}^N \). For simplicity we further assume that \( N = 3 \) and that \( \partial \Gamma \) is the finite union of \( C^3 \) closed curves in \( \mathbb{R}^3 \). Two formulæ will be used. First for a scalar function \( f: \Gamma \to \mathbb{R} \) and a vector function \( g: \Gamma \to \mathbb{R}^3 \) of appropriate smoothness

\[
\int_\Gamma \nabla f \cdot g + f \text{div}_\Gamma g \ d\Gamma = \int_\Gamma \mu_\Gamma g \cdot n \ d\Gamma + \int_{\Gamma \setminus \Gamma} fg \cdot v \ ds
\]

(7.1)

where \( v \) is the exterior unit normal to \( \partial \Gamma \) tangent to \( \partial \Omega \) and orthogonal to \( n = \nabla b, \ H = \Delta b, \) and the tangential divergence \( \text{div}_\Gamma v \) of a vector function
v: \mathcal{I} \rightarrow \mathbb{R}^N is defined in (2.37). Secondly for e: \mathcal{I} \rightarrow \mathbb{R}^N and G: \mathcal{I} \rightarrow \mathbb{R}^{N \times N} of appropriate smoothness

\[ \int_{\mathcal{I}} D\varepsilon(e) \cdot G + e \cdot \operatorname{div} G \, d\mathcal{I} = \int_{\mathcal{I}} H \varepsilon \cdot \varepsilon \mathrm{d}\mathcal{I} + \int_{\partial \mathcal{I}} e \cdot G \varepsilon \, ds, \]

(7.3)

where for two \(N \times N\) matrices or tensors \(A\) and \(B\)

\[ A \cdot \cdot B = \sum_{i,j=1}^{N} A_{ij} B_{ij} \]

(7.4)

and the tangential vectorial divergence of a matrix or tensor \(A\) is defined as

\[ (\operatorname{div}_T A)_i = \operatorname{div}_T A_{ii}. \]

(7.5)

7.2. The Natural Theory

In this section we only give the strong form of the equations under Assumption 1, that is for the space \(Y^t\) and \(H^t\) when \(l\) is tangential. This means that we use \(l - l \cdot \nabla b \nabla b\) as a test function in the variational equation. For the spaces \(Y^t\) and \(H^t\) associated with Assumption 1 the same computations can be repeated and the strong equations will be the same up to a few terms of the form \(F \cdot \nabla b \nabla b\) which will disappear in (7.11) and (7.12).

Recall definition (4.17) of the operator \(A\) and note that

\[ \operatorname{tr} \varepsilon' = I \cdot \varepsilon' \]

(I, the identity matrix in \(\mathbb{R}^N\)). By rearranging the terms

\[ \langle A(e, l), (\varepsilon, \tilde{l}) \rangle_{\mathcal{I}} = \sum_{i=0}^{2} \int_{\mathcal{I}} A'_{i} \cdot \varepsilon' \, d\mathcal{I}, \]

where for \(0 \leq i \leq 2\)

\[ A'_i = 2\mu \varepsilon' + \lambda \operatorname{tr} \varepsilon' I \quad \text{and} \quad E' = \sigma e^0 + \sigma_{i+1} e^i + \sigma_{i+2} e^{i+2} \]

\[ A'_i = \sum_{j=0}^{2} \sigma_{i+j}[2\mu \varepsilon^j + \lambda \operatorname{tr} \varepsilon^j I] \]

(7.6)

\[ = \sum_{j=0}^{2} \sigma_{i+j} \sigma^j \quad \text{and} \quad \sigma^j = 2\mu \varepsilon^j + \lambda \operatorname{tr} \varepsilon^j I. \]
Then

\[ \langle A(e, l), (\tilde{e}, \tilde{l}) \rangle \rangle = \sum_{i,j=0}^{2} \int_{\Gamma} x_{i,j} \sigma' \cdot e' d\Gamma \]

\[ = \sum_{j=0}^{2} \int_{\Gamma} \sigma' \cdot \varepsilon_{r}(\tilde{e}) + \sigma' \cdot \nabla b \cdot \tilde{l} \]

\[ + x_{j+1} \left[ \sigma' \cdot \varepsilon_{r}(\tilde{l}) - \sigma' \cdot D_{r}(\tilde{e}) D^{2}b \right] \]

\[ + x_{j+2} \left[ - \sigma' \cdot D_{r}(\tilde{l}) D^{2}b + \sigma' \cdot D_{r}(\tilde{e})(D^{2}b)^{2} \right] d\Gamma \]

\[ = \sum_{i=0}^{2} \int_{\Gamma} \sigma' [x_{i+1} I - x_{i+2} D^{2}b] \cdot D_{r}(\tilde{l}) + \sigma' \nabla b \cdot \tilde{l} d\Gamma. \]

We get

\[ \langle A(e, l), (\tilde{e}, \tilde{l}) \rangle \rangle = \sum_{i=0}^{2} \int_{\Gamma} \{ H \sigma' [x_{i+1} I - x_{i+2} D^{2}b] \} \nabla b \]

\[ - \frac{\partial}{\partial x_{r}} \left[ \sigma' (x_{i+1} I - x_{i+2} D^{2}b) \right] \cdot \tilde{e} \]

\[ + \left[ \sigma' (x_{i+1} I - x_{i+2} D^{2}b) \right] \nabla b \]

\[ - \frac{\partial}{\partial x_{r}} \left[ \sigma' (x_{i+1} I - x_{i+2} D^{2}b) \right] \cdot \tilde{d}\Gamma \]

\[ + \int_{\Sigma_{x}} \sigma' [x_{i+1} I - x_{i+2} D^{2}b] v \cdot \tilde{d}\Gamma \]

\[ + \sigma' [x_{i+1} I - x_{i+2} D^{2}b] v \cdot \tilde{l} ds. \quad (7.7) \]

Assume that the variational equation is of the form

\[ \langle A(e, l), (\tilde{e}, \tilde{l}) \rangle \rangle + \langle M(f, m), (\tilde{e}, \tilde{l}) \rangle = \int_{\Sigma_{x}} 2bg \cdot \tilde{e} + 2 \frac{h^{3}}{3} q \cdot \tilde{l} ds \quad (7.8) \]

for some appropriate vector functions \( g: \partial \Gamma \rightarrow \mathbb{R}^{n} \) and \( q: \partial \Gamma \rightarrow \mathbb{R}^{n} \) such that \( q \cdot n = 0 \) on \( \Gamma \). The coefficients in \( h \) have been added to indicate the order of the terms which would result from the total work along the lateral boundary \( \Sigma_{x} \). Recall (4.19)

\[ \langle M(f, m), (\tilde{e}, \tilde{l}) \rangle \rangle = \int_{\Gamma} x_{0} (f \cdot \tilde{e} + m \cdot \tilde{l}) + x_{1} f \cdot \tilde{l} d\Gamma. \]
\[
\sum_{i=0}^{2} \left\{ H\sigma'(x_i I - x_{i+1} D^2 b + x_{i+2} (D^2 b)^2) \right\} \nabla b
\]

\[
- \nabla \cdot \left[ \sigma'(x_i I - x_{i+1} D^2 b + x_{i+2} (D^2 b)^2) \right] + \pi_0 f = 0 \quad \text{on } \Gamma \quad (7.9)
\]

\[
\sum_{i=0}^{2} \left\{ \sigma'(x_i I - x_{i+1} D^2 b + x_{i+2} (D^2 b)^2) \right\} v = 2 h g \quad \text{on } \partial \Gamma \quad (7.10)
\]

\[
\sum_{i=0}^{2} \left\{ \sigma'(x_i I + H(x_{i+1} I - x_{i+2} D^2 b)) \right\} \nabla b
\]

\[
- \sigma'(x_i I + H(x_{i+1} I - x_{i+2} D^2 b)) \nabla b \cdot \nabla b
\]

\[
- \left\{ \sigma'(x_{i+1} I - x_{i+2} D^2 b) \right\} + \nabla \cdot \left( \sigma'(x_{i+1} I - x_{i+2} D^2 b) \right) \nabla b \nabla b
\]

\[
+ \pi_0 m + \pi_1 \left[ f - f \cdot \nabla b \nabla b \right] = 0 \quad \text{on } \Gamma \quad (7.11)
\]

\[
\sum_{i=0}^{2} \left\{ \sigma'(x_{i+1} I - x_{i+2} D^2 b) \right\} v - \sigma'(x_{i+1} I - x_{i+2} D^2 b) v \cdot \nabla b \nabla b
\]

\[
= 2 \frac{h^3}{3} q \quad \text{on } \partial \Gamma \quad (7.12)
\]

There are obviously a number of additional simplifications: \( D^2 b \nabla b = 0 \) and

\[
\epsilon^i \nabla b \cdot \nabla b = 0 \quad (\text{tr } \epsilon^i) I \nabla b \cdot \nabla b = \text{tr } \epsilon^i \quad \sigma^i \nabla b \cdot \nabla b = \text{tr } \epsilon^i
\]

where

\[
\text{tr } \epsilon^0 = \text{tr } \epsilon_r(e) = \nabla \cdot \epsilon_r(e)
\]

\[
\text{tr } \epsilon^i = \text{tr } \left[ \epsilon_r(l) - \frac{1}{2} (D_r e \cdot D^2 b) * D_r e \right]
\]

\[
= \nabla \cdot \epsilon_r(l) - D^2 b \cdot \epsilon_r(e)
\]

\[
\text{tr } \epsilon^2 = - \frac{1}{2} \text{tr } \left[ (D_r I - D_r e \cdot D^2 b) D^2 b + D^2 b (D_r I - D^2 b) * D_r e \right]
\]

\[
= - D^2 b \cdot \epsilon_r(l) + (D^2 b)^2 \cdot \epsilon_r(e).
\]

Also since \( \Gamma \) is of class \( C^3 \) for all \( v: \Gamma \to \mathbb{R}^N \)

\[
2 \pi_r(v) \nabla b = \nabla_r(v \cdot \nabla b) - D^2 b \cdot v
\]
and
\[ \varepsilon^0 \nabla b = \frac{1}{2} [ I - D^2 b e + \nabla e \cdot (e \cdot \nabla b) ] \]
\[ \varepsilon^1 \nabla b = -\frac{1}{2} D^2 b [ I - D^2 b e + \nabla e \cdot (e \cdot \nabla b) ] \]
\[ \varepsilon^2 \nabla b = \frac{1}{2} (D^2 b)^2 [ I - D^2 b e + \nabla e \cdot (e \cdot \nabla b) ] . \]

Similarly terms of the form \( \nabla b \cdot v \) in (7.12) are zero. All this can be used in specific cases.

7.3. The Love–Kirchhoff Theory

We have seen that for the natural theory the variational equation (7.8) is equivalent to

\[ \int_{\Gamma} V_1 \cdot \bar{e} + V_2 \cdot \bar{l} d\Gamma + \int_{\partial\Gamma} v_1 \cdot \bar{e} + v_2 \cdot \bar{l} ds = 0, \quad (7.13) \]

where from (7.7)

\[ V_1 = \sum_{i=0}^{2} \{ H \sigma (\alpha_i I - \alpha_{i+1} D^2 b + \alpha_{i+2} (D^2 b)^2) \} \nabla b \]
\[ - \text{div} \int_{\Gamma} [ \sigma (\alpha_i I - \alpha_{i+1} D^2 b + \alpha_{i+2} (D^2 b)^2) ] + x_0 f \] \quad (7.14)

\[ V_2 = \sum_{i=0}^{2} \{ \sigma [ \alpha_i I + H (\alpha_{i+1} I - \alpha_{i+2} D^2 b) ] \} \nabla b \]
\[ - \text{div} \int_{\Gamma} [ \sigma (\alpha_{i+1} I - \alpha_{i+2} D^2 b) ] + x_0 m + x_1 f \] \quad (7.15)

\[ v_1 = \sum_{i=0}^{2} \{ \sigma [ \alpha_i I - \alpha_{i+1} D^2 b + \alpha_{i+2} (D^2 b)^2 ] v \} - 2 h g \] \quad (7.16)

\[ v_2 = \sum_{i=0}^{2} \{ \sigma [ \alpha_{i+1} I - \alpha_{i+2} D^2 b ] v \} - \frac{2 h^3}{3} q. \] \quad (7.17)

The system of equations (7.9) to (7.12) is equivalent to

\[ \begin{cases} V_1 = 0 \quad \text{and} \quad V_2 - V_2 \cdot \nabla b \nabla b = 0 \quad \text{on} \quad \Gamma \\ v_1 = 0 \quad \text{and} \quad v_2 - v_2 \cdot \nabla b \nabla b = 0 \quad \text{on} \quad \partial \Gamma. \end{cases} \] \quad (7.18)
For the Love–Kirchhoff theory the solution and the test functions both verify the identity
\[ l + 2\varepsilon_r(e) \nabla b = 0. \]

When \( \partial \Omega \) is \( C^3 \) this can be rewritten
\[ l = D^2 b \ e - \nabla r(e \cdot \nabla b). \]

It is therefore convenient to introduce the normal and tangential components of \( e \)
\[ w = e \cdot \nabla b \in H^2(\Gamma) \]
\[ u = e - w \nabla b \in H^1(\Gamma), \quad (u \cdot \nabla b = 0). \]

Note that
\[ l = D^2 bu - \nabla r w. \]

Identity (7.13) becomes
\[
\int_{\Gamma} V_1 \cdot (\bar{u} + \bar{w} \nabla b) + V_2 \cdot (D^2 b \bar{u} - \nabla r \bar{w}) \, d\Gamma \\
+ \int_{\partial \Gamma} V_1 \cdot (\bar{u} \nabla b) + V_2 \cdot (D^2 b \bar{u} - \nabla r \bar{w}) \, ds \\
= \int_{\Gamma} \left[ V_1 + D^2 b V_2 \right] \cdot \bar{u} + V_1 \cdot \nabla b \bar{w} - V_2 \cdot \nabla r \bar{w} \, d\Gamma \\
+ \int_{\partial \Gamma} \left[ v_1 + D^2 b v_2 \right] \cdot \bar{u} + v_1 \cdot \nabla b \bar{w} - v_2 \cdot \nabla r \bar{w} \, ds.
\]

We use integration by parts for the term \( V_2 \cdot \nabla r \bar{w} \)
\[
\int_{\Gamma} \left[ V_1 + D^2 b V_2 \right] \cdot \bar{u} + \left[ V_1 \cdot \nabla b + \text{div} \, V_2 - H V_2 \cdot \nabla b \right] \bar{w} \, d\Gamma \\
+ \int_{\partial \Gamma} \left[ v_1 + D^2 b v_2 \right] \cdot \bar{u} + \left[ v_1 \cdot \nabla b - V_2 \cdot \nabla b \right] \bar{w} - v_2 \cdot \nabla r \bar{w} \, ds = 0 \quad (7.19)
\]
for all \( \bar{u} \in H^1(\Gamma) \), all \( \bar{w} \in H^2(\Gamma) \) such that \( \bar{u} \cdot \nabla b = 0. \)
This yields

\[ V_1 + D^2bV_2 - [V_1 + D^2bV_2] \cdot \nabla b \quad \text{on } \Gamma \quad (7.20) \]
\[ V_1 \cdot \nabla b + \text{div}_r V_2 - HV_2 \cdot \nabla b = 0 \quad \text{on } \Gamma \quad (7.21) \]
\[ v_1 + D^2b v_2 - [v_1 + D^2b v_2] \cdot \nabla b = 0 \quad \text{on } \partial \Gamma \quad (7.22) \]
\[ v_1 \cdot \nabla b - V_2 \cdot v = 0 \quad \text{on } \partial \Gamma \quad (7.23) \]
\[ v_2 = 0 \quad \text{on } \partial \Gamma \quad (7.24) \]

and after rearranging and using the fact that \( D^2b \nabla b = 0 \) equations (7.24) and (7.22) yield

\[ v_1 - v_1 \cdot \nabla b \nabla b = 0, \]

we get

\[ V_1 + D^2bV_2 - V_1 \cdot \nabla b = 0 \quad \text{on } \Gamma \quad (7.25) \]
\[ V_1 \cdot \nabla b + \text{div}_r V_2 - HV_2 \cdot \nabla b = 0 \quad \text{on } \Gamma \quad (7.26) \]
\[ v_1 - v_1 \cdot \nabla b \nabla b = 0 \quad \text{on } \partial \Gamma \quad (7.27) \]
\[ v_1 \cdot \nabla b - V_2 \cdot v = 0 \quad \text{on } \partial \Gamma \quad (7.28) \]
\[ v_2 = 0 \quad \text{on } \partial \Gamma \quad (7.29) \]

or simply

\[ V_1 + D^2bV_2 + (\text{div}_r V_2 - HV_2 \cdot \nabla b) \nabla b = 0 \quad \text{on } \Gamma \quad (7.30) \]
\[ v_1 - V_2 \cdot v \nabla b = 0 \quad \text{on } \partial \Gamma \quad (7.31) \]
\[ v_2 = 0 \quad \text{on } \partial \Gamma \quad (7.32) \]
\[ l = D^2be - \nabla f(e \cdot \nabla b). \quad (7.33) \]

We don’t write the details. We have the choice of using the variable \((e, l)\) with (7.30)-(7.33) or to make the change of variable to \((u, w)\), incorporate into (7.33), and solve (7.30)-(7.32). Finally note the presence of the operator

\[ \text{div}_r(\text{div}_r G) \]

for a symmetric \(N \times N\) tensor \(G\). As for other tangential operators

\[ \text{div}_r(\text{div}_r G) = \text{div}(\text{div}_r(G \cdot p))|_{r^*}, \quad \text{div}(\text{div}_r G) = \sum_{i,j=1}^{N} \partial_{ij}^2 G_{ij}. \]
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Note added in proof. It is important to observe that all the constructions and results in this paper also hold in closed subspaces of the pair $\mathcal{E}', \mathcal{E}$. For instance recent discussions with M. Bernadou indicate that a model of the Naghdi (resp. Koiter) type can be obtained by using instead of the tangential condition $\mathcal{I}_n \cdot n = 0$ on $\mathcal{I}$ the condition $\mathcal{R}_n \cdot n = 0$ (resp. $\mathcal{R}_n = 0$) on $\mathcal{I}$, in the second case we recover the Love-Kirchhoff assumption for the tangential part of $\mathcal{I}$ while its normal component is equal to a constant. So there is a wide spectrum of models with different mechanical properties, but a single underlying mathematical model.

References


