Toward a Hybrid-Trefftz element with a hole for elasto-plasticity?

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Abstract

The paper deals with the modelling of riveted assemblies for full-scale complete aircraft crashworthiness. Many comparisons between experiments and FE computations of bird impacts onto aluminium riveted panels have shown that macroscopic plastic strains were not sufficiently developed (and localised) in the riveted shell FE in the impact area. Consequently, FE models never succeed in initialising and propagating the rupture in the sheet metal plates and along rivet rows as shown by experiments, without calibrating the input data (especially the damage and failure properties of the riveted shell FE). To model the assembly correctly, it appears necessary to investigate on FE techniques such as Hybrid-Trefftz finite element method (H-T FEM). Indeed, perforated FE plates developed for elastic problems, based on a Hybrid-Trefftz formulation, have been found in the open literature. Our purpose is to find a way to extend this formulation so that the super-element can be used for crashworthiness. To reach this aim, the main features of an elastic Hybrid-Trefftz plate are presented and are then followed by a discussion on the possible extensions. Finally, the interpolation functions of the element are evaluated numerically.

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1. Introduction

Finite element simulations of airframe high velocity impacts hardly succeed in representing the failure of the structure when it occurs in riveted joint areas. Computational and experimental results were recently compared for bird impacts onto aluminium riveted panels [3,9]. The analysis has shown that the macroscopic plastic strains were not sufficiently localised within the shell finite elements (without holes), to which link finite elements are connected, so as to initiate and propagate the failure along rivet lines. The structural embrittlement, caused by holes (that are necessary for the riveting process), is not introduced or taken into account in the shell finite element formulation that is used for structural computations. In order to properly analyse strain localisation, it was necessary to introduce a model that makes it possible to measure geometrical defect effects in a “continuous” medium (structural embrittlement) [1]. This model is built up on a strain field distribution description, and particularly allows highlighting the switching from a homogeneous strain field to a heterogeneous strain field. Structural embrittlement models should enable to identify the

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unrecoverable energy dissipated in plasticity and damage mechanisms (i.e., internal energy from a classical standpoint), incipient failure energy and separation energy [10]. To collect the necessary data to build this model, the strain field has to be described accurately around the geometrical defect (hole), which is the main cause of strain field distribution perturbation. The best way to describe this strain field around the geometrical defect is thus to consider a FE model that features a high strain-accuracy level [11].

2. FE and embrittlement effects involved by geometrical defects

There are various methods to take into account the influence of geometrical defects in a finite element approach. One of them consists in modelling the geometrical defects through a really fine mesh. However, when dealing with full-scale structural computations, it becomes impossible to keep using these fine meshes around geometrical defects mainly due to cost effectiveness. Therefore, there is a rising need for alternative FE methods so as to model geometrical defects for full-scale structural computations. Among these methods, one of the most promising is the Hybrid-Trefftz FE approach that enables to build “super-elements” compatible with standard finite elements. To particularly deal with our riveted joint matter, we will focus on perforated shell elements. The main problem rising from the use of Hybrid-Trefftz elements is that they are basically formulated for elastic problems. So, in the following sub-section, a typical formulation of such elements [14,15,18] will be detailed, to aim at a possible extension to plasticity and viscoplasticity problems.

2.1. Hybrid-Trefftz FE method features

The Hybrid-Trefftz finite element method (H-T FEM) mainly relies on:

- a hybrid variational principle (to enforce inter-element compatibility) [31];
- the use of the Trefftz method [8,27].

The Trefftz method is characterised by:

- the building of shape functions that are a priori solution of the governing equations;
- the reduction of interior domain integrals (of the special element) to boundary integrals.

The main difference between standard FEM and FEM featuring Trefftz method mainly stands in the choice of shape functions. For standard FEM, shape functions do not satisfy continuum equations exactly neither in the inner domain nor on the boundaries. For FEM featuring Trefftz method, shape functions are built so as to satisfy the differential equations throughout the inner domain and only boundary conditions have to be approximated [29,30]. For some specific (among them, singularities) and simple (linear and homogenous, for which suitable shape functions can be identified) problems, it was found that Trefftz methods are undoubtedly more economical than standard FEM [29]. Another interesting comparison concerns convergence. For standard FEM, convergence is guaranteed when the element size decreases, although H-T FEM guarantees convergence if the number of terms of the series is infinite [29].

To the best of the authors’ knowledge, the first Hybrid-Trefftz finite element featuring a defect appeared in 1973 [19]. This element was featuring a crack and was the first one to combine a hybrid formulation with the complex variable method. This element was dedicated to the analysis of elastic stress intensity factors for plane cracks. The following H-T elements can be considered as being of the same family, because their common point is the combination of the hybrid method with the complex variable method. The first element featuring a hole appeared in 1982 [14] and was dedicated to plane elasticity isotropic problems [14,15,18]. A similar finite element was developed for aeronautic laminated structures computations [2]. Other elements for plane elasticity isotropic problems were developed, but without the necessary hybrid formulation feature [22,23]. We can notice that a discussion between several authors arose, mainly focusing on the choice of the variational principle [16,17,24,25]. The aim of this discussion was to highlight the need to use a hybrid functional instead of the total potential energy functional to formulate the special element, so as to ensure computation accuracy (thanks to inter-element compatibility). A piezoelectric plate element featuring a hole was developed to reduce the finite element modelling effort in the analysis of the behaviour of piezoelectric media with
2.2. Special trial functions formulation [15]

In order to build special trial functions satisfying a priori the governing equations, one has to solve the problem of the infinite plate (featuring a circular hole) using the Kolossov–Muskelishvili complex potentials \((\Phi, \Psi)\) [13].

First, let us write the plane linear elasticity problem statement [13]:

\[
\begin{align*}
2\mu(u + iv) &= k\Phi(z) - zd\Phi(z) - \overline{\Psi(z)}, \\
\sigma_{xx} + \iota\sigma_{xy} &= d\Phi(z) + \overline{d\Phi(z)} - zd^2\Phi(z) - d\overline{\Psi(z)}, \\
\sigma_{yy} - \iota\sigma_{xy} &= d\Phi(z) + \overline{d\Phi(z)} + zd^2\Phi(z) + d\overline{\Psi(z)}, \\
k\Phi(z) - zd\overline{\Phi(z)} - \overline{\Psi(z)} &= 2\mu(\bar{u} + \bar{v}) \quad \text{on } \Gamma_u,
\end{align*}
\]

and

\[
\Phi(z) + zd\Phi(z) + \overline{\Psi(z)} = i \int (\overline{\bar{T}_x} + i\overline{\bar{T}_y}) \, dS \quad \text{on } \Gamma_1,
\]

where \(k\) is Muskelishvili’s constant, \(u\) and \(v\) are the real and imaginary parts of the analytical displacement, \(\bar{u}\) and \(\bar{v}\) the analytical problem prescribed displacements, \(\bar{T}_x\) and \(\bar{T}_y\) the analytical problem prescribed tractions, \(\Gamma_u\) and \(\Gamma_1\) the analytical problem boundaries on which displacements/tractions are prescribed, \(d()\) denotes complex differentiation and \(\overline{()}\) denotes complex conjugate.

In order to describe easily the free boundary of the hole, it is convenient to map the circular hole onto a unit circle using the mapping function \(z = f(\zeta) = r_0\zeta\).

Then, Eq. (1) becomes [15]:

\[
\begin{align*}
2\mu(u + iv) &= k\Phi - f \frac{d\Phi}{df} - \overline{\Psi}, \\
\sigma_{xx} + \iota\sigma_{xy} &= \frac{d\Phi}{df} + \frac{d\Phi}{df} - \bar{f} \left( \frac{d^2\Phi}{df^2} - \frac{d\Phi}{df} \frac{d^2\Phi}{df^3} \right) - \frac{d\overline{\Psi}}{df}, \\
\sigma_{yy} - \iota\sigma_{xy} &= \frac{d\Phi}{df} + \frac{d\Phi}{df} + \bar{f} \left( \frac{d^2\Phi}{df^2} - \frac{d\Phi}{df} \frac{d^2\Phi}{df^3} \right) + \frac{d\overline{\Psi}}{df}, \\
k\bar{\Phi} - \bar{f} \frac{d\Phi}{df} - \overline{\Psi} &= 2\mu(\bar{u} + \bar{v}) \quad \text{on } \Gamma_u',
\end{align*}
\]

and

\[
\bar{\Phi} + \bar{f} \frac{d\Phi}{df} + \overline{\Psi} = i \int (\overline{\bar{T}_x} + i\overline{\bar{T}_y}) \, dS \quad \text{on } \Gamma_1',
\]

where \(\Gamma_u'\) and \(\Gamma_1'\) are the mapped boundaries \(\Gamma_u\) and \(\Gamma_1\).

It is obvious that the displacement field \((u, v)\) can be found through the identification of the real and imaginary parts of the first equation in relation (2).

Then, let us assume the Kolossov–Muskelishvili potential \(\Phi\) as a Laurent series \(\Phi(\zeta) = \sum_{j=-N}^{M} a_j \zeta^j\) and determine the second term of relation (3) in the whole domain:

\[
\bar{f} \frac{d\Phi}{df} = \bar{\zeta} \sum_{j=-N}^{M} ja_j \zeta^{j-1}.
\]

Finally, let us determine the third term of (3), the Kolossov–Muskelishvili potential \(\Psi\) featuring the stress-free boundary condition along the hole.
It is possible to describe the boundary of the unit circular hole with the following equation:

\[ \zeta^2 = \zeta^2 - \tau \quad \text{on } |\zeta| = 1. \]  

(5)

Expressing homogenous boundary conditions \((\vec{T}_x = \vec{T}_y = 0)\) along the boundary of the hole (5) in Eq. (3) provides \(\Psi\) potential such that:

\[ \Psi(\zeta) = - \sum_{j=-N}^{M} [\tilde{a}_j \zeta^{-j} + a_j j \zeta^{j-2}]. \]  

(6)

Then, let us sum the obtained expressions in order to determine the first equation of (2).

By using \(\zeta^j = R^j \cos j \theta + i R^j \sin j \theta\) and \(a_j = \alpha_j + i \beta_j\), we obtain

\[ 2\mu(u + iv) = \sum_{j=-N}^{M} \alpha_j [(k R^j + R^{-j}) \cos j \theta - j (R^j - R^{-j}) \cos (j - 2) \theta] \]
\[ + \beta_j [-(k R^j + R^{-j}) \sin j \theta + j (R^j - R^{-j}) \sin (j - 2) \theta] \]
\[ - i (\alpha_j [(k R^j + R^{-j}) \sin j \theta + j (R^j - R^{-j}) \sin (j - 2) \theta] \]
\[ + \beta_j [(k R^j + R^{-j}) \cos j \theta + j (R^j - R^{-j}) \cos (j - 2) \theta]]. \]  

(7)

By identifying the real part and the imaginary part of relation (7), one can easily find the displacement field \((u, v)\), which can be expressed in the following way:

\[ \mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{N} \begin{bmatrix} \alpha_j \\ \beta_j \end{bmatrix} = N_{ij} c_j, \]  

(8)

where \(c\) is the unknown parameter’s vector \((\alpha, \beta)\), \(\mathbf{N}\) is the matrix of special shape functions.

Following the same method, one can compute [15] the stress tensor \(\mathbf{\sigma}\) and deduce the traction vector \(\mathbf{T}\) from the two last equations in relation (2).

\[ \mathbf{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} A_{xx} \\ A_{yy} \\ A_{xy} \end{bmatrix} \begin{bmatrix} \alpha_j \\ \beta_j \end{bmatrix} = L_{ij} c_j, \]  

(9)

\[ \mathbf{T} = \begin{bmatrix} \sigma_{xx} n_x + \tau_{xy} n_y \\ \sigma_{yy} n_y + \tau_{xy} n_x \end{bmatrix} = \begin{bmatrix} T_x \\ T_y \end{bmatrix} \begin{bmatrix} \alpha_j \\ \beta_j \end{bmatrix} = P_{ij} c_j. \]

2.3. Hybrid variational principle [12]

When the special trial functions are defined, a variational principle needs to be chosen (total potential energy, Hellinger–Reissner, Hu–Washizu, ...), in order to build our special finite element. It has been discussed previously that it was better to use a hybrid functional for the special finite element, in order to enforce inter-element compatibility. This paragraph describes the use of the Trefftz method and of the hybrid functional. The hypotheses used in these derivations are hereafter highlighted.

Let us start with the well-known total potential energy functional \(II_{\text{PTE}}\) written in our special element domain [12] by the following relation (with \(\mathbf{b}\) the body forces vector and \(\mathbf{T}\) the prescribed traction vector on the FE boundary \(S_t\)):

\[ II_{\text{PTE}}[u_i] = \frac{1}{2} \int_{V} \sigma_{ij}^u u_i u_j \, dV - \int_{V} b_i u_i \, dV - \int_{S_t} \tilde{T}_i u_i \, dS. \]  

(10)

When using the divergence theorem [6] in (10), we obtain the following expression:

\[ II[u_i] = \frac{1}{2} \int_{S} T_i u_i \, dS - \frac{1}{2} \int_{V} (\sigma_{ij,x} + b_i) u_i \, dV - \frac{1}{2} \int_{V} b_i u_i \, dV - \int_{S_t} \tilde{T}_i u_i \, dS. \]  

(11)
Assuming that $\sigma_{ij,j} + b_i = 0$ (special trial functions based on exact analytical solution property) and $b_i = 0$ (body forces are neglected) in (11) leads to

$$II[u_i] = \frac{1}{2} \int_S T_i u_i \, dS - \int_{S_{\hat{T}}} \tilde{T}_i u_i \, dS.$$  \hspace{1cm} (12)

This expression needs only to be integrated on the boundary of the element, contrary to the total potential energy functional, which needs to be integrated through the whole element domain.

Now, let us make this principle an hybrid one ($\Pi_H$), in order to enforce the special element compatibility with conventional elements. This can be performed through adding a potential to our functional [15]:

$$\Pi_H[u_i] = II[u_i] + \int_{S_u+S_i} (\tilde{u}_i - u_i)T_i \, dS,$$  \hspace{1cm} (13)

where $S_u$ is the prescribed displacement boundary, $S_i$ the inter-element boundary, and $\tilde{u}$ the assumed boundary displacement vector.

The expression becomes

$$\Pi_H[u_i] = \frac{1}{2} \int_S T_i u_i \, dS - \int_{S_{\hat{T}}} \tilde{T}_i u_i \, dS + \int_{S_u+S_i} (\tilde{u}_i - u_i)T_i \, dS.$$  \hspace{1cm} (14)

The link between (14) and other functionals [15] is shown in [12]. In particular, after a few developments, the principle (14) can be modified so that the integrations can be performed on the inter-element boundary of the special element only. The stiffness matrix of the special perforated element is finally obtained when assuming the stationary condition of the above functional [12].

2.4. Discussion

The previous Hybrid-Trefftz finite element formulation [14,15,18] was dedicated to quasi-static elastic plate computations.

The extension from a perforated plate element to a perforated shell element may not cause problem. Indeed, in the thin-plate theory, the equations can be decoupled so that the equations off the membrane type problem and the bending type problem can be dealt with separately [7].

Concerning the extension to elastoplasticity of the Hybrid-Trefftz FE method [20,21,26], it appeared that H-T formulation may only be used for geometrically linear domains [20,26]. However, it has to be demonstrated that reference [21] can only deal with geometrically linear domains. Then, the difficulty (and the objective) of the current research is to extend this elastic perforated plate element formulation to a plastic or viscoplastic shell element formulation.

Through the hypothesis used to formulate this element, the following problems arise:

- there is a need for an analytical plastic or viscoplastic solution of the perforated plate problem formulated the same way as Kolossov–Muskelishvili potentials. The main problem is that this solution does not exist. Only an analytical solution formulated using Kolossov–Muskelishvili potentials for perfectly plastic bodies can be found in [5];
- body forces cannot be neglected in crash computations (i.e., problems where inertia effects have to be considered).

Therefore, one can conclude that for now, the integration of the equations along the element boundary is no longer relevant, and that Hybrid-Trefftz perforated plate elements for (general) plasticity or viscoplasticity cannot be formulated.

A possible solution could be to keep the hybrid part of the formulation (in order to enforce inter-element compatibility), and not to use the Trefftz method. Then the use of the hypotheses “shape functions are a priori solution of the problem” and “body forces are neglected” is no longer needed. New problems may arise then. One will have to make sure that the stiffness matrix is symmetrical (this property is ensured due to the Trefftz method used in Ref. [15]). This property may not be a priori obtained when giving up the Trefftz method, and a nonsymmetric stiffness matrix may cause problem for structural assembly and computations algorithm [29]. One may have to use the elastic analytical shape functions within the element in order to perform plastic computations, as the analytical solution in the plastic
domain does not exist. The point then is to know how suitable this hypothesis is, and if a p-refinement for the special element could help (there could be an accuracy problem). Last, there will be a need to split the element interior domain into sub-domains so as to integrate the equations through the interior domain, at least for the body forces.

3. Numerical evaluation of the special element interpolation functions

The Kolossov–Muskelishvili solution (7)–(9) is compared to a fine mesh FE computation. A square perforated plate, which sides are 10 times the hole radius, is finely meshed with 5000 quadratic elements in the FE code ZéBuLoN (developed by EMP and ONERA). The behaviour of the material is described by Hooke’s law \( E = 74,000 \text{ MPa} \), \( v = 0.3 \). The load applied to the plate is either uniaxial tensile or simple shear.

So as to build up the fields \((u, v)\) and \(\sigma_{ij}\) in the whole domain thanks to Kolossov–Muskelishvili solution, the parameters \(\alpha_j\) and \(\beta_j\) need to be determined. These coefficients can be obtained by providing known displacement (8) or stress (9) data. We are focusing on the case in which \(N = M = 4\) with \(j = 0\) rigid body motion terms neglected, which corresponds to an 8-node perforated element interpolation functions. Therefore, eight \(\alpha_j\) and eight \(\beta_j\) unknown parameters need to be determined (i.e., 16 unknowns). These parameters can be determined from values of \((u, v)\) or \((\sigma_{xx}, \sigma_{yy})\) taken at eight points (i.e., 16 input data). Then the system of equations (8) and (9) are solved to compute \(\alpha_j\), \(\beta_j\) from displacement data and stress data, respectively. The parameters \(\alpha_j\) and \(\beta_j\) will be used to compute the analytical solution within the whole domain.

The input data for the computation of \(\alpha_j\) and \(\beta_j\) can be collected either at the data set 1 (which can be considered as the eight nodes of the perforated FE), or at the data set 2 (Fig. 1). Data set 1 appeared unsuitable for an accurate computation of \(\alpha_j\) and \(\beta_j\), mainly due to the appearance of singular values in Eqs. (8) and (9) for \(\theta = k\pi/2\). Results are thus presented for input data collected at data set 2 to avoid these singular values problems.

When the applied load is uniaxial tensile, the analytical solution obtained with \(\alpha_j\) and \(\beta_j\) defined from stress data fits pretty well with the FE computation (Fig. 2(a)). When \(\alpha_j\) and \(\beta_j\) are defined from displacement data (Fig. 2(b)), the stress distribution is underestimated (however, the same tendency is observed). When the applied load is simple shear, the analytical solution obtained with \(\alpha_j\) and \(\beta_j\) defined from displacement data is underestimated compared to the FE computation. The analytical solution from stress data is really bad in that case. These tendencies have been observed for all stress and displacement fields and both applied loads investigated in this numerical comparison. It can be concluded that the analytical solution shows good agreement when \(\alpha_j\) and \(\beta_j\) are computed from displacement data. Fortunately, this is the way the perforated element is activated, through enforcing displacement continuity with
quadratic neighbouring elements (13). Moreover, the compatibility enforced along the whole special element boundary could help determining accurately $\alpha_j$ and $\beta_j$ using more than eight points only. This last point will be investigated later.

4. Conclusion

The paper deals with the formulation of Hybrid-Trefftz finite elements featuring geometrical defects and especially circular holes. A typical formulation of the literature is detailed, followed by a discussion on a possible extension of this formulation to elasto-viscoplastic computations. The discussion mainly concluded that one should avoid using the Trefftz method so as to extend the element of the literature to crashworthiness. Finally, the analytical solution, which is a cornerstone of the Hybrid-Trefftz element and for future developments (elasto-plasticity), is evaluated numerically for elastic stress fields.

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