Wavelet-like bases for thin-wire integral equations in electromagnetics

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Abstract

In this paper, wavelets are used in solving, by the method of moments, a modified version of the thin-wire electric field integral equation, in frequency domain. The time domain electromagnetic quantities, are obtained by using the inverse discrete fast Fourier transform. The retarded scalar electric and vector magnetic potentials are employed in order to obtain the integral formulation. The discretized model generated by applying the direct method of moments via point-matching procedure, results in a linear system with a dense matrix which have to be solved for each frequency of the Fourier spectrum of the time domain impressed source. Therefore, orthogonal wavelet-like basis transform is used to sparsify the moment matrix. In particular, dyadic and $M$-band wavelet transforms have been adopted, so generating different sparse matrix structures. This leads to an efficient solution in solving the resulting sparse matrix equation. Moreover, a wavelet preconditioner is used to accelerate the convergence rate of the iterative solver employed. These numerical features are used in analyzing the transient behavior of a lightning protection system. In particular, the transient performance of the earth termination system of a lightning protection system or of the earth electrode of an electric power substation, during its operation is focused. The numerical results, obtained by running a complex structure, are discussed and the features of the used method are underlined.

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1. Introduction

Wavelet theory is a relevant continuously emerging area in mathematical research. It has been applied in a wide range of engineering disciplines and it has received considerable attention in computational electromagnetics, particularly in solving integral equations. With reference to this mathematical problem, it is well-known that these equations are obtained by applying boundary conditions to an integral representation of the electromagnetic (EM) field. In solving complex scattering EM problems, the integral equations approach has the advantage of reducing the solution domain to a small finite region containing the boundary conditions implicitly; but, as a drawback, in this formulation the global interactions among sources, object and fields have to be specified. The numerical method generally employed to solve integral equations is the method of moments (MoM) [21]. In its direct formulation, MoM results in the solution of a linear system. So operating, when scattering problems from electrically large complex objects have to be approached, a dense linear systems have to be treated.

In this paper, wavelets are used in solving, by MoM, a modified version of the thin-wire electric field integral equation (EFIE), in frequency domain. The final aim is the analysis of the transient behavior of a lightning protection system (LPS) [3]. Thus, the time domain EM quantities, are obtained by using the inverse discrete fast Fourier transform (DFFT). In particular in this paper, the transient performance of the earth termination system of an LPS during its operation or of the earth electrode of an electric power substation is focused [1,2,6–8]. The retarded scalar and vector potentials are employed in order to obtain the integral formulation. By applying the direct MoM via point-matching procedure a linear system with dense matrix must be solved for each significant frequency of the Fourier spectrum of the time domain EM source.

The size of the dense linear system grows by increasing the partitioning of the discretized model. Therefore, in order to lead with large scale problems, a matrix sparsification is really important. It has been found that the application of wavelet transform can result in a very sparse moment matrix [4,5,9,10,14,15,17,22]. As a result, the computational complexity associated with the solution of electromagnetic integral equations is significantly reduced, by applying an iterative solver; however, no improvements are reached in the convergence rate. Hence, preconditioners involving different wavelet transforms have been generated, by exploiting the sparsity of the modified MoM matrix.

The paper is organized as follows. In Section 2, the electromagnetic problem and the discretized model are introduced. Section 3 briefly reviews selected preliminaries on wavelet transform referred to the sparsification and preconditioning of MoM matrices. In Section 4, the numerical results obtained by running a complex earthing structure, as those employed in a LPS or in electric power substations, are discussed and the features of the used method are underlined.

2. Problem formulation

In this section an essential description of the EM problem formulation is sketched. More complete analytical developments and physical considerations can be found in [3] to which the reader is invited to refer.

For the whole LPS made up of interconnected cylindrical straight conductors the well-known thin-wire hypothesis can be adopted. As a consequence, the current $I_a$ on each wire, can be represented by a filament placed on its axis. In order to obtain an integral equation for the longitudinal currents, the
following statement on the surface $S$ of the generic wire can be used as a boundary condition:

$$\bar{u} \cdot (\vec{E}^{\text{inc}} + \vec{E}^{\text{sc}}) = z_s I_a \text{ on } S,$$

(1)

where $z_s$ is the per-unit length surface impedance of the cylindrical wire, and $\bar{u}$ is the unit-vector tangential to $S$ in the direction of the axis of the wire. The left side of (1) represents the tangential component of the total electric field, parallel to the axis of the wire. $\vec{E}^{\text{inc}}$ is the incident electric field, namely the electric field in the absence of the wires structure. An incident field can be due, for example, to the lightning channel. $\vec{E}^{\text{sc}}$ is the scattered electric field due to the presence of the wires structure. $\vec{E}^{\text{sc}}$ can be expressed by means of the retarded electric scalar and magnetic vector potentials:

$$\vec{E}^{\text{sc}} = -(j \omega \vec{A} + \nabla \Phi).$$

(2)

By introducing the integral expression of $\nabla \Phi$ and $\vec{A}$ as functions of the free electric charges and of the electric conduction currents, respectively, by integrating (1) on the surface of the generic wire, the following modified version of the EFIE is obtained:

$$\int_L E^{\text{inc}}(\vec{r}) \, dl - j \omega \int_L \bar{u} \cdot \left[ \bar{u}' \int_\Omega \mu I_a(l') g(\vec{r}, \vec{r}') \, dl' \right] \, dl$$

$$+ \int_L \frac{\partial}{\partial l} \left[ \frac{1}{j \omega 4 \pi \varepsilon} \int_\Omega \frac{dI_a(l')}{dl'} g(\vec{r}, \vec{r}') \, dl' \right] \, dl = z_s \int_L I_a(\vec{r}) \, dl \text{ on } S,$$

(3)

where $\Omega$ is the domain of integration; $L$ is the length of the wire under consideration; $\mu$ is the magnetic permeability, $\sigma$ is the conductivity, $\varepsilon = \varepsilon_r \varepsilon_0 + \sigma / j \omega$ is the complex permittivity of the medium; $\vec{r}$ and $\vec{r}'$ are the position vectors of the observation point and of the source point, respectively, $g(\vec{r}, \vec{r}')$ is the Green’s function for an unbounded region with $\hat{k} = (-\omega^2 / \mu \varepsilon)^{0.5}$ as the wave number.

Eq. (3) has to be solved numerically, in terms of the longitudinal current distribution $I_a$ along the wires of the structure, by introducing an appropriate discretization based on the direct MoM. Thus, by considering the LPS as a set of straight thin-wire branches with arbitrary oriented directions; by subdividing each branch in an integer number of segments having the same length, generally different from one branch to another but much less than the minimum wavelength related to the maximum work frequency, a discrete formulation of Eq. (3) can be obtained. By introducing triangular basis functions, the unknown longitudinal current along the branches is formally piecewise linear and the unknowns are the set of longitudinal currents at both ends of each segment. In this way a linear system is generated, for each frequency of the Fourier spectrum of the impressed source:

$$Z I = E,$$

(4)

where $Z$ is the full moment $N \times N$ matrix, $I$ is the unknown currents $N$ vector and $E$ the known $N$ vector. By solving (4) the longitudinal currents along the wires of the structure are obtained. Then the electric and magnetic fields can be computed in any interesting point into the soil, on the ground surface and on separated electrodes by using $\vec{A}$ and $\Phi$ retarded potentials. The time profiles of the interesting electromagnetic quantities can be obtained by means of the inverse DFFT algorithm.
3. Numerical approach

As already underlined, System (4) has to be solved for each frequency of the Fourier spectrum of the impressed source. Usually, in the electromagnetic environment under study, the impressed source could be an incident electric field propagating in the air or in the soil, or a current source generator that directly drives the earth structure. The time domain profile of these sources, which simulate a lightning stroke condition, is usually a double exponential waveform. This transient waveform has a Fourier spectrum that contains a lot of frequencies that have to be considered for a satisfactory time domain representation.

The moment matrix $Z$ in (4), is a dense matrix and it is often ill-conditioned. Therefore, for each frequency, the linear system solution could be prohibitively expensive for large $N$, and the result could be unacceptable. On the other hand, iterative methods could be a valid computational way for the solution of linear system (4), if the system is opportently treated. Namely, the matrix structure has to be modified by increasing the zero entries, and preconditioners have to be used in order to improve the convergence rate.

3.1. Wavelet-like bases and MoM matrix

With reference to wavelet transform, this enables the MoM matrix to be modified into a sparse one. In this section a brief overview of discrete wavelet transform of vectors and matrices is reported.

3.1.1. Dyadic wavelet transform

Let $s^0$ be a vector of size $N = 2^n$. A wavelet decomposition [12,13] can be expressed in a matrix notation by means of a $N \times N$ matrix $W_n$

$$W_n = \begin{pmatrix} L_n \\ H_n \end{pmatrix},$$

where $L_n$ and $H_n$ are matrices of size $(N/2) \times N$ called low- and high-pass filters, respectively. The two matrices are expressed in terms of scaling $\{h_k\}_{k=1}^{2p+2}$ and wavelet $\{g_k = (-1)^k h_{2p-k+3}\}_{k=1}^{2p+2}$ coefficients. These coefficients can be defined in such a way that the matrix $W_n$ is orthogonal. Moreover, the wavelets are said to have $p$ vanishing moments if

$$\sum_{k=1}^{2p+1} g_k k^j = 0, \quad j = 0, \ldots, p.$$ 

By applying the matrix $W_n$ to the vector $s^0$, the following decomposition is generated. Namely

$$W_n s^0 = [L_n s^0, H_n s^0]^t = [s^1, d^1]^t.$$

The process can be recursively applied $r$ times giving rise to the vectors $s^j, j = 1, \ldots, r$, i.e.

$$T s^0 = W_{n-r} \cdots W_{n-1} W_n s^0 = [s^r, d^r, \ldots, d^1]^t.$$

The matrices $W_{n-j}$ are matrices of the following form:

$$W_{n-j} = \begin{pmatrix} H_{n-j} \\ G_{n-j} \\ I_{N-N/2j} \end{pmatrix}.$$
where $I_{N-N/2}$ is the identity matrix. The reconstruction of vector $s^0$ can be obtained from the decomposition vectors, as follows:

$$s^0 = T^t [s^r, d^r, \ldots, d^1]^t.$$ 

Wavelet transformation of a matrix $A$, is performed by the product $T AT^t$. The product $TA$ results in the wavelet decomposition of the columns of $A$, and the multiplication for $T^t$ results in the wavelet decomposition of the rows of $TA$.

### 3.1.2. M-band wavelet transform

The $M$-band wavelet transform is a generalization of the commonly used dyadic wavelet, and it is based to the case of the dilation scale factor $M > 2$ [11,16,18]. In this context, a wavelet decomposition can be expressed in a matrix notation by means of a $N \times N$ matrix $W_n$ as follows:

$$W_n = \begin{pmatrix} L_n & H_n^{(1)} & \cdots & H_n^{(M-1)} \\ H_n^{(1)} & H_n^{(2)} & \cdots & H_n^{(M-1)} \\ \vdots & \vdots & \ddots & \vdots \\ H_n^{(M-1)} & H_n^{(M-2)} & \cdots & L_n \end{pmatrix},$$

where $L_n$ and $H_n^{(i)}$, $i = 1, \ldots, M - 1$, are matrices of size $(N/M) \times N$. The matrix $L_n$ is the low-pass filter matrix whilst the $M - 1$ matrices $H_n^{(i)}$ are the high-pass filters matrices. Generally, unlike the 2-band case, there is not a simple relationship between the scaling and wavelet filters sequences. In [16,19,20] the scaling and wavelet filters have been characterized and an explicit formula to evaluate the wavelet matrix $T$ has been given. This kind of construction has been used in this paper, and 4-band with $p = 1$ and 2 vanishing moments have been taken into account.

By considering a wavelet orthogonal matrix $T$ involved in the decomposition–reconstruction wavelet algorithm, the system (4) is modified as follows:

$$\tilde{Z} \tilde{I} = \tilde{E},$$

where

$$\tilde{Z} = T^t Z T, \quad \tilde{I} = T^t I \quad \text{and} \quad \tilde{E} = T^t E.$$ (6)

The matrix $\tilde{Z}$ is still full but many of its elements are negligibly small with respect to the others. In order to determine which small elements can be discarded, an appropriate threshold have to be considered. The elements of $\tilde{Z}$ which are smaller than a fixed threshold are set to zero. The following threshold has been taken into account [22,23]:

$$\tau = \frac{d}{N} \| \tilde{Z} \|_{\infty},$$

where $d$ is an opportune constant value and $N$ is the problem size.

In this way the matrix $\tilde{Z}$ becomes a sparse one. The resulting sparse matrix allows rapid matrix–vector multiplications so improving the efficiency of the solving process. Hence, the solution vector $\tilde{I}$ in the transformed domain is obtained and $I$ is generated in the initial domain, i.e.

$$I = T \tilde{I}.$$ (8)
By using an iterative solver for System (5), the computational complexity at each step is reduced but the convergence rate is the same.

It is to be underlined that, for some typical electromagnetic configurations, the matrix $Z$, such as the matrix $\tilde{Z}$, are ill-conditioned, and $\tilde{Z}$ is near to be singular. When these events occur, $\tilde{Z}$ can be used to generate a preconditioner for $Z$, and the vector solution is directly gained in the initial domain.

### 3.2. Wavelet preconditioners

As it is well-known, the convergence rate of an iterative solver can be improved by involving a preconditioner. To achieve good preconditioning effects, a preconditioner $C$ should be close as possible to the inverse of the system matrix

$$
C \approx Z^{-1}.
$$

The previously described sparse matrix $\tilde{Z}$ can be used to generate an efficient wavelet preconditioner $C$.

When the dyadic transform is used $\tilde{Z}$ is near to be a block-tridiagonal matrix, therefore it can be approximated by a block-diagonal matrix $B$. Hence, the matrix $C$ is obtained by involving only the inverse of the diagonal blocks of $B$. The blocks order depends on the skeleton of the $\tilde{Z}$ matrix.

By using an $M$-band transform with $M > 2$, the matrix has the most significant entries on the diagonal blocks and on the left and upper blocks. Thus, it can be approximated by means of an upper arrow block matrix $B$. In this case the order of the blocks is chosen to be $N/M$ (see Fig. 1).

In order to generate an efficient wavelet preconditioner $C$, a re-ordering of the $B$ blocks has been performed by means of permutation matrices $P_1$ and $P_2$ (see Fig. 2).

In this way the inverse of $\hat{B}$ employs only the inversion of the diagonal blocks as in the dyadic case. In the following, the adopted computational process is summarized:

$$
\hat{B}_{qq}^{(1)} = \hat{B}_{qq} - \hat{B}_{q1} \hat{B}_{1q} (\hat{B}_{11})^{-1},
$$

$$
\hat{B}_{qq}^{(2)} = \hat{B}_{qq}^{(1)} - \hat{B}_{q2} \hat{B}_{2q} (\hat{B}_{22})^{-1},
$$

...
Fig. 2. Re-ordering of the matrix $B$.

\[
\hat{B} = P_1BP_2 = \begin{pmatrix}
\hat{B}_{11} & \hat{B}_{1q} \\
\hat{B}_{2q} & \hat{B}_{22} \\
\hat{B}_{3q} & \hat{B}_{33} \\
\hat{B}_{q1} & \hat{B}_{q2} & \hat{B}_{q3} & \hat{B}_{qq}
\end{pmatrix}
\]

\[
\hat{B}_{qq}^{(q-1)} = \hat{B}_{qq}^{(q-2)} - \hat{B}_{q-1q} \hat{B}_{q-1q}^{-1}.
\]

\[
A = (\hat{B}_{nn}^{(n-1)})^{-1} = (\hat{B}_{nn})^{-1},
\]

\[
(\hat{B}_{ni})^{-1} = -A \hat{B}_{ni} (\hat{B}_{ii})^{-1}, \quad i = 1, \ldots, q - 1, \ j \neq i,
\]

\[
i = 1, \ldots, q - 1 \begin{cases}
(\hat{B}_{ii})^{-1} = -(\hat{B}_{ii})^{-1} (I - \hat{B}_{qi} \hat{B}_{iq}) & j = 1, \ldots, q \ j \neq i,
(\hat{B}_{ij})^{-1} = -\hat{B}_{qj} \hat{B}_{iq} (\hat{B}_{ii})^{-1},
\end{cases}
\]

\[
\tilde{Z}^{-1} \approx B^{-1} = P_2 \hat{B}^{-1} P_1.
\]

At this time the preconditioner $C$ for the matrix $Z$ is constructed by transforming the resulting matrix, back to the initial domain:

\[
C = TB^{-1}T^T.
\]

4. Case study

In this section numerical results concerning a complex earthing system are reported, in order to test the performance of the adopted numerical features. In Fig. 3, a three-dimensional thin-wire earth electrode with a lightning current source directly injected in a corner is shown.

The sparsification of the transformed moment matrix has been carried out by means of the dyadic wavelet transform, with $p = 8$ vanishing moments and the 4-band wavelet transform with $p = 1$ and 2. The threshold has been chosen as in (7); the value of the constant $d$ has been selected between $1/5$ and $1/10$. The calculated unknowns by means of (8), show a root-mean square error of around 2%.

In Fig. 4 the sparsity patterns of the matrix $\tilde{Z}$ for the size $N = 256$ are depicted. In Table 1 the zero percentage entries are reported by varying the problem size.

The conjugate gradient method (CG) is the iterative solver used for Systems (4) and (5). The convergence rate is reported in Table 2.

In Fig. 5, the improvements in solving System (4) with CG and preconditioned conjugate gradient (P-CG) methods, generated by means of the preconditioners previously discussed, are reported.
Fig. 3. Simulation example: a square meshed earth electrode directly fed at a corner.

Fig. 4. The sparsity patterns of the matrix $\tilde{Z}$ for $N = 256$.

Table 1
Zero percentage entries of matrix $\tilde{Z}$ by varying problem size

<table>
<thead>
<tr>
<th>$N$</th>
<th>128</th>
<th>256</th>
<th>512</th>
<th>1024</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M = 2, p = 8$</td>
<td>46</td>
<td>77</td>
<td>82</td>
<td>94</td>
</tr>
<tr>
<td>$M = 4, p = 1$</td>
<td>64</td>
<td>84</td>
<td>88</td>
<td>95</td>
</tr>
<tr>
<td>$M = 4, p = 2$</td>
<td>59</td>
<td>82</td>
<td>88.5</td>
<td>96.6</td>
</tr>
</tbody>
</table>

Table 2
Convergence rate of conjugate gradient method by varying problem size

<table>
<thead>
<tr>
<th>$N$</th>
<th>CG iterations number</th>
</tr>
</thead>
<tbody>
<tr>
<td>128</td>
<td>262</td>
</tr>
<tr>
<td>256</td>
<td>357</td>
</tr>
<tr>
<td>512</td>
<td>693</td>
</tr>
<tr>
<td>1024</td>
<td>1634</td>
</tr>
</tbody>
</table>
The best performance are obtained by means of the diagonal preconditioner generated with the dyadic wavelet. Since the blocks order of 2-band P-CG is doubled with respect to the order of the blocks of 4-band preconditioners, a better preconditioning effectiveness is reached.

By using the same blocks order for the best P-CG, i.e. $M = 2$ and 4 with $p = 2$, the 4-band preconditioner accelerates the convergence of the iterative solver. In Fig. 6 the related performance are shown.

### 5. Conclusion

Wavelet transforms have been applied to the MoM matrix for the solution of electromagnetic thin-wire integral equations in frequency domain. The linear system associated with the discretized model has been handled in order to obtain a sparseness in the system matrix. Therefore, iterative solvers are adequate computational kernels to efficiently compute the unknowns.

Dyadic and 4-band wavelet transforms have been adopted so generating different sparse matrix structures. In order to improve the convergence behavior of the iterative CG solver, preconditioners formulated with the wavelet transform have been proposed. The computations obtained with dyadic and 4-band transforms have shown very interesting results. These results encourage the authors to proceed in a future research by employing $M > 4$. 
References