Suppose $X$ is a Banach space. We introduce the concept of quasi-distribution semigroups on $X$, as a generalization of the concept of distribution semigroups introduced by Lions in [19]. In our approach, the generator $A$ of a quasi-distribution semigroup may not be densely defined. Also introduced is a functional calculus for $A$ in terms of the Fourier transform. For fixed $r > 0$, $k \in \mathbb{N} \cup \{0\}$, define order $(r, k)$ for a quasi-distribution semigroup and define an $F_{r,k}$ functional calculus. We prove that $A$ generates a $(k+1)$-times integrated semigroup of exponential type $r$ with local Lipschitz continuity if and only if $A$ generates a quasi-distribution semigroup of order $(r, k)$ if and only if $A$ has an $F_{r,k}$ functional calculus. When $A$ is densely defined, a quasi-distribution semigroup reduces to a distribution semigroup in the sense of Lions.

1. INTRODUCTION

In [19], Lions introduced the class of densely defined operators on Banach spaces that generate distribution semigroups. (Regularized) distribution semigroups and, as a special case, (regularized) distribution groups were discussed in [2–4, 6, 7, 9–13, 15]. It is worthwhile to emphasize that the theory of (regularized) distribution semigroups and groups is closely related to the theory of (regularized) functional calculi, which has been extensively developed by deLaubenfels et al. (See, for instance, [7, 9–13, 15]). On the other hand, it was observed by Arendt and deLaubenfels et al. that an interesting feature of the theory of (regularized) distribution semigroups and groups is its link with the theory of integrated (regularized) semigroups and groups (see, for instance, [2–4, 7, 9–13, 15]). Unfortunately, in almost all the literature cited here (except for [9, 10, 15]), etc.), a essential assumption is the density of the domain of the generating operator $A$. But there has been a great deal of interest recently in operators that may not be densely defined. For example, in many population models being constructed, the operator under consideration is not densely defined. In this paper we prove that $A$ generates a $(k+1)$-times integrated semigroup of exponential type $r$ with local Lipschitz continuity if and only if $A$ generates a quasi-distribution semigroup of order $(r, k)$ if and only if $A$ has an $F_{r,k}$ functional calculus. When $A$ is densely defined, a quasi-distribution semigroup reduces to a distribution semigroup in the sense of Lions.

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shall construct a kind of distribution semigroup, which will be called a quasi-distribution semigroup, and functional calculus without assuming the generating operator $A$ to be densely defined. In order to remove the density assumption it is necessary to follow another path.

The paper is organized as follows. In Section 2, equivalent conditions for $C_{k+1}(\tau)$ to be well-posed are studied; the definition of $C_{k+1}(\tau)$ can be found in [2]. The importance of these equivalent conditions will be shown in the next two sections. In Section 3, we define the concept of a quasi-distribution semigroup and generalize [2, Theorem 7.2] to a more general case. Section 4 is devoted to the construction of a special case of a quasi-distribution semigroup, which is called of order $(r,k)$ and a functional calculus, which is called the $F_{r,k}$ functional calculus, to precisely characterize a given $(k+1)$-times integrated semigroup of exponential type $r$ with local Lipschitz continuity. Some equivalent conditions are obtained.

2. BASIC DEFINITIONS AND PROPERTIES

Throughout this paper $A$ denotes a closed operator on a Banach space $X$, $B(X)$ the algebra of all bounded linear operators on $X$, and $D(A)$ the domain of $A$. $D(A)$, equipped with the graph norm $\|x\|_A = \|x\| + \|Ax\|$, will be denoted by $[D(A)]$. If $Y$ is another Banach space, we use $B(X,Y)$ to denote the set of all bounded linear operators from $X$ into $Y$.

Let $0 < \tau \leq \infty$. We consider the following Cauchy problem [2],

$$C_0(\tau) \begin{cases} u \in C([0, \tau]; [D(A)]) \cap C^1([0, \tau); X), \\ u'(t) = Au(t), & t \in [0, \tau), \\ u(0) = x, \end{cases}$$

and the following $(k+1)$-times integrated Cauchy problem [2],

$$C_{k+1}(\tau) \begin{cases} v \in C([0, \tau]; [D(A)]) \cap C^1([0, \tau); X), \\ v'(t) = Av(t) + \frac{t^k}{k!} x, & t \in [0, \tau), \\ v(0) = 0. \end{cases}$$

For $k \in \mathbb{N} \cup \{0\}$, $0 < \tau \leq \infty$, the Cauchy problem $C_{k+1}(\tau)$ is well posed if for all $x \in X$ there exists a unique solution of $C_{k+1}(\tau)$.

According to [2], for $\alpha > 0$, $\beta > 0$, define the exponential region $E(\alpha, \beta)$ by

$$E(\alpha, \beta) = \{ \lambda \in C : \text{Re}(\lambda) \geq \beta, |\text{Im}(\lambda)| \leq e^{\alpha \text{Re}(\lambda)} \}. $$
**Proposition 2.1** [2, Theorem 2.1]. Let $k \in \mathbb{N}, 0 < \tau < \infty$. Assume that $C_{k + 1}(\tau)$ is well posed. Then for all $0 < \infty < \tau$ there exist $\beta > 0$, $M > 0$ such that

$$E(\in, \beta) \subset \rho(A) \quad \text{and} \quad \|R(\lambda; A)\| \leq M \|\lambda\|^k, \quad \lambda \in E(\in, \beta).$$

**Definition 2.2** [1, Definition 3.2]. Let $k \in \mathbb{N}, 0 < \tau < \infty$. Suppose $\{W(t)\}_{t \in [0, \tau]} \in \mathcal{B}(X)$ is a strongly continuous family of operators. $\{W(t)\}_{t \in [0, \tau]}$ is called a (local) $k$-times integrated semigroup if

(i) $W(0) = 0$;

(ii) $W(t) W(s) x = (1/(k - 1)!) \left[ (s + t - r)^{k-1} W(r) x dr \right]$

$\forall s, t \in [0, \tau]$.

A (local) strongly continuous semigroup defined on $[0, \tau]$ is called a (local) $0$-times integrated semigroup.

For $k \in \mathbb{N} \cup \{0\}$, the (local) $k$-times integrated semigroup $\{W(t)\}_{t \in [0, \tau]}$ is called nondegenerate if $W(t) x = 0$, for all $0 \leq t \leq \tau$, implies that $x = 0$. For a nondegenerate (local) $k$-times integrated semigroup we may define its generator in the following way. Let $D(A)$ be the set of all $x \in X$ such that there exists $y \in X$ satisfying

$$W(t) x - \frac{t^k}{k!} x = \int_0^t W(s) y ds \quad \forall t \in [0, \tau];$$

then $Ax = y$. For simplicity, we shall remove the term “local” for the case $0 < \tau < \infty$ throughout.

There is another definition for $A$ generating an integrated semigroup in [2, Proposition 2.3]. But it is somewhat ambiguous, unless one verifies that such an $A$ is unique. This is not indicated until [2, Proposition 3.1(d)] is proven.

More generally, for a given injective operator $C \in \mathcal{B}(X)$ [7, 8] (see also [23]) introduced the concept of a mild integrated $C$-existence family for $A$. By definition, a mild integrated $I$-existence family (which we will call a mild integrated existence family) for $A$ is in fact the integrated semigroup generated by $A$ defined in [2, Proposition 2.3]. In order to keep more consistent with the terminology in the literature on integrated semigroups, we prefer the former to the latter and introduce the following definition, in which we also remove the term “local” for the case $0 < \tau < \infty$.

**Definition 2.3** [7, 8, 23]. Suppose $A$ is a closed operator and $k \in \mathbb{N} \cup \{0\}, 0 < \tau < \infty$. A strongly continuous family $\{W(t)\}_{t \in [0, \tau]} \subset \mathcal{B}(X)$
is a mild $k$-times integrated existence family for $A$ if for any $x \in X$, $0 \leq t < \tau$, 
\[ \int_0^t W(s) x \, ds \in D(A) \] and 
\[ A^k \int_0^t W(s) x \, ds = W(t) x - \frac{t^k}{k!} x. \] 
(2.1)

**Theorem 2.4** [23, Theorem 2.4; 2, Proposition 2.3]. Let $k \in \mathbb{N} \cup \{0\}$, $0 < \tau \leq \infty$. The following are equivalent.

(i) $C_{k+1}(\tau)$ is well posed.

(ii) All solutions of $C_{k+1}(\tau)$ are unique and there exists a mild $k$-times integrated existence family for $A$.

(iii) $A$ generates a nondegenerate $k$-times integrated semigroup.

**Proof.** Most of (i) $\Rightarrow$ (ii) can be found in the proof of [2, Proposition 2.3] except for the definition of the seminorms $P_n$ on the space $C([0, \tau], [D(A)])$, where
\[ P_n(v) = \sup_{0 \leq \tau \leq \tau - 1/n} \| v(t) \|_A. \] 
(2.2)

Clearly, $\tau$ needs to be finite in (2.2). Instead, we define
\[ P_n(v) = \sup_{\tau \in [0, \tau_n]} \| v(t) \|_A \quad \forall v \in C([0, \tau), [D(A)]), \] 
(2.3)
where $\tau > 0$ and $\tau_n \uparrow \tau$. In (2.3), $\tau$ may be finite or not.

(ii) $\Rightarrow$ (i) is clear. If suffices to set $v(t) = \int_0^t W(s) x \, ds \forall x \in X$.

(ii) $\Rightarrow$ (iii). We first prove that (see [14, Theorem 3.3])
\[ W(t) A \subseteq A W(t) \quad \forall t \in [0, \tau). \] 
(2.4)

Fix $x \in D(A)$. We will show that
\[ W(t) x = \int_0^t W(s) A x \, ds + \frac{t^k}{k!} x. \] 
(2.5)
To show (2.5), define
\[ \bar{W}(t) x = \int_0^t W(s) A x \, ds + \frac{t^k}{k!} x. \]
Since $A$ is closed and $\{W(t)\}_{t \in [0, \tau)}$ is a mild $k$-times integrated existence family for $A$, it follows that $\int_0^t W(s) x \, ds$ and $\int_0^t W(t) A x \, dt$ are in the domain of $A$, with
\[ A \int_{t_0}^{t} \tilde{W}(s) x ds = \int_{t_0}^{t} A \left( \int_{t_0}^{r} W(r) A x dr \right) ds + \frac{t^{k+1}}{(k+1)!} A x \]
\[ = \int_{t_0}^{t} W(s) A x ds \]
\[ = \tilde{W}(t) x - \frac{t^k}{k!} x. \]

By the uniqueness of the solutions of \( C_{k+1}(\tau) \), one has \( \tilde{W}(t) x = W(t) x \), proving (2.5).

Assertion (2.5) and the fact that \( \{ W(t) \}_{t \in [0, \tau]} \) is a mild \( k \)-times integrated existence family for \( A \) now imply that
\[ A \int_{t_0}^{t} W(s) x ds = \int_{t_0}^{t} W(s) A x ds. \] (2.6)

Since \( A \) is closed, differentiate both sides of (2.6) to conclude that \( W(t) x \in D(A) \) and \( AW(t) x = W(t) Ax \), proving (2.4).

We now prove that (iii). We confine the proof to the case \( k \in \mathbb{N} \). That of \( k = 0 \) is similar. From (2.5),
\[ \frac{d}{dt} W(t) x = \frac{t^{k-1}}{(k-1)!} x + W(t) A x \quad \forall x \in D(A), \quad t \in [0, \tau). \] (2.7)

Now assume \( x \in X \), \( 0 \leq r \leq t \). (2.7) implies that
\[ \frac{d}{dr} \left[ W(t-r) \int_{t_0}^{r} W(u) x du \right] = \frac{(t-r)^{k-1}}{(k-1)!} \int_{t_0}^{r} W(u) x du - W(t-r) A \int_{t_0}^{r} W(u) x du + W(t-r) W(r) x \]
\[ = -\frac{(t-r)^{k-1}}{(k-1)!} \int_{t_0}^{r} W(u) x du + \frac{r^k}{k!} W(t-r) x. \]

Integrating in \( r \) from 0 to \( s \) for \( s \leq t \) yields
\[ W(t-s) \int_{0}^{s} W(u) x du \]
\[ = \int_{0}^{s} \left( \frac{(t-r)^{k-1}}{(k-1)!} \left( \int_{t_0}^{r} W(u) x du \right) dr + \int_{r-s}^{r} \frac{(t-r)^k}{k!} W(r) x dr. \]
Integrate the last integral by parts and apply \( A \) to both sides of the resulting equality to find

\[
W(t-s) \ W(s) \ x = \frac{s^k}{k!} \ W(t-s) \ x + W(t-s) \ A \left[ \int_0^t W(s) \ x \ ds \right]
\]

\[
= \frac{s^k}{k!} \ W(t-s) \ x - \frac{s^k}{k!} \left[ \int_0^t \frac{(t-r)^k}{(k-1)!} \ W(r) \ x - \frac{r^k}{k!} x \ dr \right]
\]

\[
+ \frac{s^k}{k!} \left[ \int_0^t (t-r)^k \ x \ dr \right]
\]

\[
= \frac{s^k}{k!} \left[ \int_0^t (t-r)^k \ W(r) \ x \ dr \right]
\]

\[
- \int_0^t (t-r)^k \ W(r) \ x \ dr.
\]

(2.8)

Here we make use of the fact that the coefficient of \( x \) equals

\[
\frac{1}{k!} \left[ \frac{(t-r)^k}{(k-1)!} \ x \right.\left. \frac{r^k}{k!} \right] \ dr + \frac{s^k}{k!} \ W(t-s) \ x
\]

\[
= - \frac{1}{k!} \frac{d}{dr} \left[ \frac{(t-r)^k}{k!} \ x \right.\left. \frac{r^k}{k!} \right] \ dr + \frac{s^k}{k!} \ x = 0.
\]

Replacing \( t \) by \( t+s \) in (2.8) implies that \( \{ W(t) \}_{t \in [0, \tau]} \) is a \( k \)-times integrated semigroup. \( \{ W(t) \}_{t \in [0, \tau]} \) is clearly nondegenerate by (2.1).

That \( A \) is the generator of \( \{ W(t) \}_{t \in [0, \tau]} \) follows from \([2, \text{Proposition 3.1 (d)}]\).

(iii) \( \Rightarrow \) (ii). We also confine the proof to the case \( k \in \mathbb{N} \). For \( 0 \leq s \leq t \leq \tau, \ x \in X \), Definition 2.2(ii) implies that

\[
W(t) \left[ \int_0^t W(r) \ x \ dr \right]
\]

\[
= \frac{1}{(k-1)!} \int_0^t \left[ \left( \int_0^{t-r} - \int_0^r \right) (t+r-u)^{k-1} W(u) \ x \ du \right] \ dr
\]

\[
= \frac{1}{k!} \left[ \left( \int_0^{t-s} - \int_0^r \right) (t+s-u)^k W(u) \ x \ du \right] + \frac{r^k}{k!} \int_0^t W(u) \ x \ du.
\]
Hence

\[
W(t) \int_0^t W(u) x \, du - \frac{t^k}{k!} \int_0^t W(u) x \, du
\]

\[
= \frac{1}{k!} \left[ \left( \int_t^s + \int_s^0 \right) (t + s - u)^k W(u) \, du \right].
\]

Interchange the role of \( s \) and \( t \) to find

\[
W(s) \int_0^s W(u) x \, du - \frac{s^k}{k!} \int_0^s W(u) x \, du
\]

\[
= \frac{1}{k!} \left[ \left( \int_s^t + \int_t^s \right) (t + s - u)^k W(u) \, du \right].
\]

Since \( W(t), W(s) \) commute by Definition 2.2(ii), we have

\[
W(t) \int_0^t W(u) x \, du - \frac{t^k}{k!} \int_0^t W(u) x \, du
\]

\[
= W(s) \int_0^s W(u) x \, du - \frac{s^k}{k!} \int_0^s W(u) x \, du
\]

\[
= \int_0^s W(u) \left[ W(s) - \frac{s^k}{k!} x \right] \, du,
\]

hence

\[
\int_0^t W(u) x \, du \in D(A) \quad \text{and} \quad A \int_0^s W(u) x \, du = W(s) - \frac{s^k}{k!} x
\]

by the definition of \( A \). Equation (2.1) is thus proved.

All that remains to show is the uniqueness of the solutions of \( C_{k+1}(\tau) \).

Clearly, it suffices to show that \( v(t) = 0 \) is the only solution of the homogeneous equation

\[
v'(t) = Av(t), \quad v(0) = 0 \quad \forall 0 \leq t < \tau \quad (2.9)
\]

in \( C^1([0, \tau]; X) \cap C([0, \tau]; \left[ D(A) \right]) \). Let \( v \in C^1([0, \tau]; X) \cap C([0, \tau]; \left[ D(A) \right]) \) be a solution of (2.9). Then for \( 0 \leq s \leq t < \tau, \)

\[
\frac{d}{ds} W(t-s) v(s) = W(t-s) v'(s) - W(t-s) Av(s) - \frac{(t-s)^{k-1}}{(k-1)!} v(s)
\]

\[
= - \frac{(t-s)^{k-1}}{(k-1)!} v(s). \quad (2.10)
\]
Integrating (2.10) in $s$ from 0 to $t$ yields

$$
\int_0^t \frac{(t-s)^{k-1}}{(k-1)!\,v(s)} \, ds = 0.
$$

Differentiable in $t$ $k$ times to obtain $v(t) = 0$. (ii) is proved.

Remark 2.5. We shall use the equivalence of (i) and (iii) to claim the “multiplication” property for quasi-distribution semigroups instead of using $D_\varphi = \bigcap \{ D(A^m) : m = 1, 2, 3, \ldots \}$ and the semigroup $T(t)$ on $D_\varphi$ as shown in [2, p. 23]. The latter relies strongly on the assumption that the generating operator $A$ is densely defined. We will see that our approach is efficient for both cases of $A$, densely defined or not.

3. QUASI-DISTRIBUTION SEMIGROUPS

In this section we will remove several conditions for a distribution semigroup defined in [19] and introduce the concept of a quasi-distribution semigroup. From Theorem 3.8 and its corollaries, we will see that among the conditions in [19, Definition 1.1] being removed in this paper, density of the domain of the generating operator is the most important, and plays an essential role in [19].

Let $D$ be the space of all infinitely differentiable complex-valued functions on $\mathbb{R}$ with compact support and let $D_0$ be the subspace of those $\varphi$’s with $\text{supp} \varphi \subset [0, \infty)$. We designate as

$$(\varphi * \psi)(t) = \int_0^t \varphi(t-s) \psi(s) \, ds \quad \forall \varphi, \psi \in D_0, \quad t \in \mathbb{R} \quad (3.1)$$

the convolution product on $D_0$.

There are two ways to extend the above convolution product to $D$. One is defined by

$$(\varphi * \psi)(t) = \int_{-\infty}^\infty \varphi(t-s) \psi(s) \, ds \quad \forall \varphi, \psi \in D, \quad t \in \mathbb{R} \quad (3.2)$$

and another is

$$(\varphi * \psi)(t) = \int_0^t \varphi(t-s) \psi(s) \, ds \quad \forall \varphi, \psi \in D, \quad t \in \mathbb{R}. \quad (3.3)$$

Throughout we shall use (3.3) but not (3.2) as the convolution product on $D$ for our purpose.
For \( \varphi \in \mathcal{D} \), let
\[
\varphi + (t) = \varphi(t) \ Y(t),
\] (3.4)
where \( Y(\cdot) \) is the Heaviside function
\[
Y(t) = \begin{cases} 
1, & \text{if } t \geq 0; \\
0, & \text{if } t < 0.
\end{cases}
\]
Let \( \mathcal{D}_+ \) be the set of all \( \varphi_+ \)'s defined in (3.4). Sometimes, we shall also use \( \varphi, \psi \), etc. to denote the elements in \( \mathcal{D}_+ \) for simplicity. For \( \varphi \in \mathcal{D}_+ \), \((d/dt)^k \varphi(t)|_{t=0} \) means the \( k \) th right derivative. The convolution product of \( \varphi, \psi \) in \( \mathcal{D}_+ \) is the same as in (3.1), (3.3); that is,
\[
(\varphi * \psi)(t) = \int_0^t \varphi(t-s) \psi(s) \, ds \quad \forall \varphi, \psi \in \mathcal{D}_+, \ t \in \mathbb{R}. \tag{3.5}
\]

**Proposition 3.1.** For \( \varphi, \psi \in \mathcal{D} \), the following hold.

(i) \( \varphi * \psi \in \mathcal{D} \) and \( \text{supp}(\varphi * \psi) \subseteq \text{supp} \varphi + \text{supp} \psi \).

(ii) \( (\varphi * \psi)_+ = \varphi * \psi_+ = \varphi_+ * \psi = \varphi * \psi_+ \).

(iii) \( (\varphi' * \psi)(t) = (\varphi * \psi')(t) + \varphi(t) \psi(0) - \varphi(0) \psi(t), \ t \in \mathbb{R} \).

**Proof:** (i) is an immediate consequence of (3.3).

(ii) Since \( Y(t) = Y(t-s) \ Y(s) = Y(t-s) - Y(s) \) for all \( t \in \mathbb{R} \setminus \{0\} \) with \( s \in (0, t) \) when \( t > 0 \) and \( s \in (t, 0) \) when \( t < 0 \), we have
\[
\int_0^t Y(t) \varphi(t-s) \psi(s) \, ds = \int_0^t Y(t-s) \varphi(t-s) Y(s) \psi(s) \, ds \\
= \int_0^t Y(t-s) \varphi(t-s) \psi(s) \, ds \\
= \int_0^t Y(s) \varphi(t-s) \psi(s) \, ds.
\]

(iii) Integration by parts yields
\[
(\varphi' * \psi)(t) = \int_0^t \varphi'(s) \psi(t-s) \, ds \\
= \varphi(0) \varphi(t) - \varphi(0) \psi(t) + \int_0^t \varphi(s) \psi'(t-s) \, ds \\
= (\varphi * \psi')(t) + \varphi(0) \varphi(t) - \varphi(0) \psi(t).
The following definition of distribution semigroups was given in [19] (see also [2, Definition 7.1]).

**Definition 3.2.** A distribution semigroup on $X$ is a continuous linear map $G : \mathcal{D} \mapsto \mathcal{B}(X)$ satisfying the following properties.

(i) $\text{supp } G \subseteq [0, \infty)$.
(ii) $G(\varphi \ast \psi) = G(\varphi) G(\psi)$ \(\forall \varphi, \psi \in \mathcal{D}_0\).
(iii) $\bigcup \{ \text{Im}(G(\varphi)) \mid \varphi \in \mathcal{D}_0\}$ is dense in $X$.
(iv) $\bigcap \{ \ker(G(\theta)) \mid \theta \in \mathcal{D}_0\}$ is trivial.
(v) If $x = G(\varphi) y$ for some $\varphi \in \mathcal{D}_0$ and $y \in X$, then there exists $u \in C([0, \infty), X)$ such that $u(0) = x$, and

$$G(\varphi) x = \int_0^\infty \psi(t) u(t) \, dt \quad \forall \psi \in \mathcal{D}_0.$$  

From [19, pp. 143–145 and p. 145 (2.7)], the generator $A$ of the distribution semigroup $G$ is the closure of the following operator:

$$A_0 G(\varphi) x = -G(\varphi') x \quad \forall \varphi \in \mathcal{D}_0, \ x \in X. \quad (3.6)$$

We now introduce the following.

**Definition 3.3.** A quasi-distribution semigroup on $X$ is a continuous linear map $G : \mathcal{D} \mapsto \mathcal{B}(X)$ satisfying the following properties.

(i) $G(\varphi \ast \psi) = G(\varphi) G(\psi)$ \(\forall \varphi, \psi \in \mathcal{D}\).
(ii) $\bigcap \{ \ker(G(\theta)) \mid \theta \in \mathcal{D}_0\}$ is trivial.

**Remark 3.4.** (i) A quasi-distribution semigroup will be written as QDSG.

(ii) Definition 3.3(ii) implies that $\text{supp } G \subset [0, \infty)$.

(iii) From [22, Theorem XXX III], if $\varphi, \psi \in \mathcal{D}$ satisfy $\varphi(t) = \psi(t)$ for $t \geq 0$, then $G(\varphi) = G(\psi)$. Hence $G$ can be regarded as a continuous linear map from $\mathcal{D}_+$ into $\mathcal{B}(X)$. We will take this into consideration throughout.

For a given QDSG $G$, define the operator $A_1$ by

$$D(A_1) = \bigcup \{ \text{Im}(G(\varphi)) \mid \varphi \in \mathcal{D}\};$$

$$A_1 G(\varphi) x = -G(\varphi') x - \varphi(0) x \quad \forall x \in X, \ \varphi \in \mathcal{D}.$$  

**Proposition 3.5.** $A_1$ is well defined and closable.
Proof. Assume $G(\varphi) x = G(\psi) y$ for some $\varphi, \psi \in \mathcal{D}$, $x, y \in X$. Let $\theta \in \mathcal{D}_0$.

From Proposition 3.1(iii),

$$G(\theta) G(\varphi') x = G(\theta + \varphi') x = -\varphi(0) G(\theta) x + G(\theta') G(\varphi) x,$$

hence

$$G(\theta)[ -G(\varphi') x - \varphi(0) x] = -G(\theta') G(\varphi) x.$$

Similarly,

$$G(\theta)[ -G(\psi') y - \psi(0) y] = -G(\theta') G(\psi) y.$$  

Definition 3.3(ii) implies that

$$-G(\varphi') x - \varphi(0) x = -G(\psi') y - \psi(0) y.$$

$A_1$ is thus well defined.

To prove that $A_1$ is closable, let $x_n \in D(A_1)$ be such that $x_n \to 0$, $A_1 x_n \to y$. Write $x_n = G(\varphi_n) z_n$ for some $\varphi_n \in \mathcal{D}$, $z_n \in X$. Then for every $\theta \in \mathcal{D}_0$,

$$G(\theta) y = \lim_{n \to \infty} G(\theta) A_1 x_n = \lim_{x \to \infty} G(\theta) A_1 G(\varphi_n) z_n$$

$$= -\lim_{n \to \infty} G(\theta)[ G(\varphi_n') z_n + \varphi_n(0) z_n]$$

$$= -\lim_{n \to \infty} G(\theta') G(\varphi_n) z_n = -\lim_{n \to \infty} G(\theta') x_n = 0.$$

This implies that $y = 0$ by Definition 3.3(ii).

**Definition 3.6.** The closure of $A_1$, denoted by $A$, is called the generator of the QDSG $G$.

**Proposition 3.7.** For every $\varphi \in \mathcal{D}$, $G(\varphi) A \subseteq AS(\varphi)$.

**Proof.** For $\varphi, \psi \in \mathcal{D}$, $x \in X$,

$$A G(\varphi) G(\psi) x = -G(\varphi') G(\psi) x - \varphi(0) G(\psi) x$$

$$= -G(\varphi) G(\psi') x - \psi(0) G(\psi) x$$

$$+ \varphi(0) G(\psi) x - \varphi(0) G(\psi) x$$

$$= G(\varphi)[ -G(\psi') x - \psi(0) x]$$

$$= G(\varphi) A G(\psi) x. \quad (3.7)$$
Since $D(A)$ is a core for $A$, (3.7) implies that $A\mathcal{G}(\phi)x = \mathcal{G}(\phi)Ax$ for all $x \in D(A)$.

The following theorem, without assuming $A$ to be densely defined, is a generalization of [2, Theorem 7.2].

**Theorem 3.8.** Let $A$ be a closed operator. Then the following are equivalent.

(i) There exist $k_0 \in \mathbb{N}$ and $\tau_0 > 0$ such that $C_{k_0+1}(\tau_0)$ is well defined.

(ii) $A$ generates a QDSG.

**Proof.** (ii) $\Rightarrow$ (i). This part is almost the same as “(ii) $\Rightarrow$ (i)” of [2, Theorem 7.2]. We give it here for completeness. Since $\mathcal{G}$ is the QDSG generated by $A$, for every $\phi \in \mathcal{D}$, $x \in X$, $A\mathcal{G}(\phi)x = -\mathcal{G}(\phi')x - \phi(0)x$.

Hence $\mathcal{G}$ is a continuous linear map from $\mathcal{D}$ into the space $B(X, [D(A)])$. Let $\tau > 0$. It follows from the regularity theorem for distributions (see [16, Theorem 8.1.5]) that there exists a continuous function $W : [-\tau, \tau] \rightarrow B(X, [D(A)])$ and $k \in \mathbb{N}$ such that

$$\mathcal{G}(\phi)x = (-1)^k \int_{-\tau}^{\tau} \phi^{(k)}(t) W(t)x \, dt \quad (3.8)$$

for $x \in X, \phi \in \mathcal{D}$ with supp $\phi \subset (-\tau, \tau)$. Since supp $\mathcal{G} \subset [0, \infty)$, one has $W(t) = 0$ for $t \leq 0$ by [16, Remark 8.1.6]. From Definition 3.6, (3.8) implies

$$(-1)^k \int_{0}^{\tau} \phi^{(k)}(t) AW(t)x \, dt = A\mathcal{G}(\phi)x$$

$$= (-1)^{k+1} \int_{0}^{\tau} \phi^{(k+1)}(t) W(t)x \, dt - \phi(0)x \quad (3.9)$$

for $x \in X, \phi \in \mathcal{D}$ with supp $\phi \subset (-\tau, \tau)$. Integration by parts yields

$$\int_{0}^{\tau} \phi^{(k+1)}(t) \left[ \int_{0}^{t} AW(s)x \, ds - W(t)x \right] dt = 0$$

for $x \in X, \phi \in \mathcal{D}_0$ with supp $\phi \subset [0, \tau)$. From [16, Lemma 8.1.1], there exist $B_j \in B(X)$ $(j = 0, 1, \ldots, k)$ such that

$$\int_{0}^{\tau} AW(s)x \, ds - W(t)x = \sum_{j=0}^{k} t^j B_j x \quad \forall t \in [0, \tau). \quad (3.10)$$
From (3.9),
\[
(-1)^{k} \int_{0}^{T} \varphi^{(k+1)}(t) \left[ \int_{0}^{t} A W(x) \, ds - W(t) \, x \right] \, dt = \varphi(0) \, x
\]  
(3.11)
for \( x \in X \), \( \varphi \in D \) with \( \text{supp} \varphi \subset (-\infty, \tau) \). Introducing (3.10) into (3.11) gives us
\[
(-1)^{k} \int_{0}^{T} \varphi^{(k+1)}(t) \sum_{j=0}^{k} t^{j} B_{j} \, dt = \varphi(0) \, I.
\]  
(3.12)
Since
\[
\int_{0}^{T} t^{j} \varphi^{(k+1)}(t) \, dt = (-1)^{j+1} j! \varphi^{(j)}(0)
\]
and \( \varphi^{(j)}(0) \) may be arbitrary for \( j = 0, 1, \ldots, k \), (3.12) implies \( B_{j} = 0 \) for \( j = 0, \ldots, k-1 \) and \( B_{k} = -(1/k!) \, I \). Consequently,
\[
\int_{0}^{T} A W(x) \, ds = W(t) \, x - \frac{t^{k}}{k!} x, \quad t \in [0, \tau).
\]
Equation (3.8) and Proposition 3.7 show that \( W(t) \, A \subseteq AW(t) \) for \( t \in [0, \tau) \). Then the argument for the uniqueness given in (iii) \( \Rightarrow \) (ii) of Theorem 2.4 works for the present case. Theorem 2.4 now implies that \( C_{k+1}(\tau) \) is well posed.

(i) \( \Rightarrow \) (ii). From [2, Theorem 4.1] (in that theorem, the authors do not assume that \( A \) is densely defined), for every \( k \in \mathbb{N} \), \( k \geq k_{0} \) there exist \( \tau_{k} \) such that \( \tau_{k+1} \geq \tau_{k} \), \( \lim_{-\infty} \tau_{k} = \infty \) and \( C_{k+1}^{+}(\tau_{k}) \) is well posed. Denote by \( \{ W_{\tau}(t) \}_{\tau \in [0, \tau_{k}]} \) the \( k \)-times integrated semigroup generated by \( A \). Here we make use of Theorem 2.4. Then \( W_{\tau}(t) = W_{\tau}^{(m)}(t) \) for \( t \in [0, \tau_{k}) \) (see [2, (4.1)]), \( m \in \mathbb{N} \) and \( k \geq k_{0} \). Thus the following definition is independent of \( k \in \mathbb{N} \). For \( \varphi \in D \), choose \( k \geq k_{0} \) such that \( \text{supp} \varphi \subset (-\infty, \tau_{k}] \). Define
\[
\mathcal{G}(\varphi) \, x = (-1)^{k} \int_{0}^{T} \varphi^{(k)}(t) \, W_{\tau}(t) \, x \, dt \quad \forall x \in X.
\]  
(3.13)
Then \( \mathcal{G} \) is a continuous linear map from \( D(X) \) into \( B(X) \).

To prove Definition 3.3(i), let \( \varphi, \psi \in D \) and assume that \( (\text{supp} \varphi + \text{supp} \psi) \cup \text{supp} \varphi \cup \text{supp} \psi \subseteq (-\infty, \tau_{k}) \) for some \( k \geq k_{0} \). Then for \( x \in D(A^{k}) \),
\[
\mathcal{G}(\varphi) \, \mathcal{G}(\psi) \, x = (-1)^{k} \int_{0}^{T} \varphi^{(k)}(t) \, W_{\tau}(t) \, \mathcal{G}(\psi) \, x \, dt.
\]
The calculations
\[
W_{\lambda}(t) \mathcal{B}(\psi) x = (-1)^{k} \int_{0}^{\infty} \psi(s) W_{\lambda}(s) x ds
\]

and
\[
W_{\lambda}(t) (W_{\lambda}(s) x)^{(k)} = W_{\lambda}(t+s) x - \sum_{j=0}^{k-1} \frac{t^{j}}{j!} (W_{\lambda}(s) x)^{(j)}
\]

show that
\[
\mathcal{B}(\varphi) \mathcal{B}(\psi) x = (-1)^{k} \int_{0}^{\infty} \varphi^{(k)}(t)
\]

\[
\times \int_{0}^{\infty} \psi(s) \left[ W_{\lambda}(t+s) x - \sum_{j=0}^{k-1} \frac{t^{j}}{j!} (W_{\lambda}(s) x)^{(j)} \right] ds dt
\]

\[
= \int_{0}^{\infty} \varphi(t) \int_{0}^{\infty} \psi(s-t) (W_{\lambda}(s) x)^{(k)} ds dt
\]

\[
= \int_{0}^{\infty} (\varphi * \psi)(s) (W_{\lambda}(s) x)^{(k)} ds
\]

\[
= \mathcal{B}(\varphi * \psi) x.
\]

From Proposition 2.1, $\rho(A)$ is nonempty, let $\lambda \in \rho(A)$. Equation (3.14) shows that
\[
\mathcal{B}(\varphi) \mathcal{B}(\psi)(\lambda - A)^{-k} x = \mathcal{B}(\varphi * \psi)(\lambda - A)^{-k} x \quad \forall x \in X.
\]

From Proposition 3.7, $\mathcal{B}(\theta)$ commutes with $(\lambda - A)^{-k}$ for every $\theta \in \mathcal{D}$, hence $\mathcal{B}$ satisfies Definition 3.3(i).

We now prove Definition 3.3(ii). Let $x \in X$ be such that $\mathcal{B}(\theta) x = 0$ for all $\theta \in \mathcal{D}_{0}$. Let $k \geq k_{0}$ be fixed. Then
\[
\mathcal{B}(\theta) x = (-1)^{k} \int_{0}^{\infty} \theta^{k}(t) W_{\lambda}(t) x dt = 0
\]
for all $\theta \in \mathcal{S}_0$ with $\text{supp} \theta \subset [0, \tau_k)$. From [16, Lemma 8.1.1],

$$W_{\theta}(t) x = \sum_{j=0}^{k-1} t^j z_j, \quad t \in [0, \tau_k), \ z_j \in X \ (0 \leq j \leq k-1). \quad (3.15)$$

Hence $z_0 = W_{\theta}(0) x = 0$. Differentiating both sides of (3.15) up to $(k-1)$ times and using the equality

$$W_{\theta}(t) x = \mathcal{A} \int_0^t W_{\theta}(s) x \, ds + \frac{t^k}{k!} x, \quad (3.16)$$

we find successively

$$z_1 = [W_{\theta}(t) x]_{t=0} = \mathcal{A} W_{\theta}(0) x = A z_0 = 0;$$

$$z_2 = \frac{1}{2!} [W_{\theta}(t) x]_{t=0} = \frac{1}{2!} A \mathcal{A} [W_{\theta}(t) x]_{t=0} = 0 = \frac{1}{2!} A z_1 = 0;$$

$$\vdots$$

$$z_{k-1} = \frac{1}{(k-1)!} [W_{\theta}(t) x]_{t=0}^{(k-1)} = \frac{1}{(k-1)!} A [W_{\theta}(t) x]_{t=0}^{(k-2)} = \frac{1}{(k-1)!} A z_{k-2} = 0.$$  

Therefore $W_{\theta}(t) x = 0$ for $t \in [0, \tau_k)$. \{ $W_{\theta}(t)$ \}_{t \in [0, \tau_k]} being nondegenerate, one obtains $x = 0$.

It remains to prove that $A$ is the generator of $\mathcal{G}$. Let $\varphi \in \mathcal{D}$ be such that $\text{supp} \varphi \subset (-\infty, \tau_k]$. Then for every $x \in X$,

$$\mathcal{G}(\varphi^0) x = (-1)^k \int_0^{\infty} \varphi^{(k+1)}(t) W_{\theta}(t) x \, dt$$

$$= (-1)^k \int_0^{\infty} \varphi^{(k+1)}(t) \left[ \mathcal{A} \int_0^t W_{\theta}(s) x \, ds + \frac{t^k}{k!} x \right] \, dt$$

$$= (-1)^k \mathcal{A} \int_0^{\infty} \varphi^{(k+1)}(t) \int_0^t W_{\theta}(s) x \, ds \, dt$$

$$+ (-1)^k \int_0^{\infty} \varphi^{(k+1)}(t) \frac{t^k}{k!} x \, dt$$

$$= -\mathcal{A} \left\{ (-1)^k \int_0^{\infty} \varphi^{(k)}(t) W_{\theta}(t) x \, dt \right\} - \varphi(0) x.$$
Consequently,

\[ A\mathcal{G}(\varphi) x = -\mathcal{G}(\varphi^\prime) x - \varphi(0) x \]

and hence \( A \) is an extension of the generator of \( \mathcal{G} \). Let \( B \) be the generator of \( \mathcal{G} \). From the proof of "(ii) \( \Rightarrow \) (i)" and Proposition 2.1, there exists \( \alpha > 0, \beta > 0 \) such that \( E(\alpha, \beta) \cap \rho(B) \). Since \( C_k(\tau_k) \) is well posed for \( A \), Proposition 2.1 also implies that \( E(\alpha, \beta) \cap \rho(A) \). Here the values of \( \alpha, \beta > 0 \) may have to be changed when necessary. We now have \( \rho(A) \cap \rho(B) \neq \phi \). This implies \( A = B \). The theorem follows.

**Corollary 3.9.** Let \( D_0 \) be as in Remark 2.5, \( \mathcal{G} \) a QDSG. Then for every \( x \in D_0 \) there exists \( u \in C([0, \infty), X) \) such that

\[
\begin{align*}
\mathcal{G}(\varphi) x &= \left\{ \int_0^\infty \varphi(t) u(t) \, dt \quad \forall \varphi \in \mathcal{D}_0, \right. \\
\end{align*}
\]

**Proof.** From the foregoing we have

\[
[ W_k(t) x ]^{(k)} = [ W_{k+m}(t) x ]^{(k+m)} \quad \forall m \in \mathbb{N}, \ t \in [0, \tau_k), \ k \geq k_0.
\]

Define \( u \in C([0, \infty), X) \) by

\[
u(t) = [ W_k(t) x ]^{(k)} \quad \forall t \in [0, \tau_k), \ k \geq k_0.
\]

Then \( u \) is independent of \( k \).

Let \( \varphi \in \mathcal{D}_0 \). There exists \( k \geq k_0 \) such that \( \text{supp} \varphi \subset [0, \tau_k) \). From (3.13),

\[
\begin{align*}
\mathcal{G}(\varphi) x &= (-1)^k \int_0^\infty \varphi^{(k)}(t) W_k(t) x \, dt \\
&= \int_0^\infty \varphi(t)[ W_k(t) x ]^{(k)} dt \\
&= \int_0^\infty \varphi(t) u(t) dt.
\end{align*}
\]

\( u(0) = x \) is an immediate consequence of (3.16) and (3.18).

**Corollary 3.10.** Suppose \( \mathcal{G} \) is a QDSG and \( x = \mathcal{G}(\psi) y \) for some \( \psi \in \mathcal{D}_0, \ y \in X \). Then there exists \( u \in C([0, \infty), X) \) such that (3.17) holds.
Proof. Since $\psi^{(j)}(0) = 0$ for $j \in N \cup \{0\}$, we have by induction $x \in D(A')$ and $A'x = (-1)^j \varphi(\psi^{(j)}) y \forall j \in N$. Thus $x \in D_\omega$ and the conclusion follows from Corollary 3.9.

**Corollary 3.11.** Let $A$ be closed. Then $A$ generates a distribution semigroup in the sense of Lions if and only if $A$ generates a QDSG with $A$ densely defined.

**Proof. Necessity.** It suffices to prove Definition 3.3(i). In fact, from [2, Theorem 7.2], (i) of Theorem 3.8 is true, then Theorem 3.8 (ii) is also. This implies Definition 3.3(i).

**Sufficiency.** Since $\rho(A) \not= \emptyset$ and $D(A)$ is dense in $X$, $D_\omega$ is also by [2, Proposition 6.2]. Let $x \in D_\omega$. Then (3.17) holds for some $u \in C([0, \infty), X)$ and any $\varphi \in D_0$. Suppose $\rho_n \in D_0$ satisfies $\int_0^\infty \varphi(t) dt = 1$ and $\text{supp} \rho_n \subset (0, 1/n)$. Then (3.17) implies that

$$G(\rho_n) x \rightarrow u(0) = x, \quad as \quad n \rightarrow \infty.$$  

Therefore $\bigcup \{ \text{Im}(G(\varphi)) | \varphi \in D_0 \}$ is dense in $D_\omega$, hence in $X$. This, together with Corollary 3.10, shows that Definition 3.2(iii), (v) are true, so that $G$ is a distribution semigroup in the sense of Lions.

From Definition 3.6,

$$A G(\varphi) x = -G(\varphi^\prime) x \quad \forall \varphi \in D_0.$$  

$A$ is thus an extension of the generator $B$ of $G$ as a distribution semigroup in the sense of Lions. A similar argument as in the last part of the proof of Theorem 3.8 shows that $\rho(A) \cap \rho(B) \not= \emptyset$, hence $A = B$.

### 4. A GENERALIZATION OF STONE'S THEOREM

In this section we shall give some more precise characterization for (global) exponentially bounded $k$-times integrated semigroups over $[0, \infty)$. To this end, we introduce the following

**Definition 4.1.** For given $r > 0$, $k \in N \cup \{0\}$, let $p_{r,k}$ be the norm

$$p_{r,k}(\varphi) = \sum_{j=0}^k c^j \left( \frac{d^j}{dt} \right)^k \varphi(t) \left. \right|_{L^1(0, \infty)}$$  

for $\varphi \in D_\omega$. Let $D_{r,k}$ be the completion of $D_\omega$ for $p_{r,k}$. For every $\varphi \in D_{r,k}$, we define $\varphi(t) = 0 \forall t < 0$. 


We recall that for $\varphi \in \mathcal{D}_+$, $(d/dt)^j \varphi(t)|_{t=0}$ means the right derivative at $t = 0$ for $j = 1, 2, 3, \ldots$. Moreover, from Definition 4.1, for every $\varphi \in \mathcal{D}_{r,k}$, $\varphi, \varphi', \ldots, \varphi^{(k-1)}$ are continuous on $[0, \infty)$.

**Lemma 4.2.** Given $r > 0$. Then for every $\varphi \in \mathcal{D}_+$ and every integer $j \geq 0$,

$$
\int_0^\infty e^{rt} |\varphi^{(j)}(t)| \, dt \leq \frac{1}{r} \int_0^\infty e^{rt} |\varphi^{(j+1)}(t)| \, dt. \tag{4.2}
$$

**Proof.** From

$$
\int_0^\infty e^{rt} |\varphi^{(j)}(t)| \, dt = \int_0^\infty e^{rt} \left| \int_0^\infty \varphi^{(j+1)}(s) \, ds \right| \, dt
$$

$$
\leq \int_0^\infty e^{rt} \int_0^\infty |\varphi^{(j+1)}(s)| \, ds \, dt
$$

$$
= \int_0^\infty |\varphi^{(j+1)}(t)| \int_0^t e^{rt} \, dt \, ds
$$

$$
\leq \frac{1}{r} \int_0^\infty e^{rt} |\varphi^{(j+1)}(t)| \, dt,
$$

(4.2) follows.

**Corollary 4.3.** For $\varphi \in \mathcal{D}_+$, let

$$
q_{r,k}(\varphi) = \|e^{rt} \varphi^{(k)}(t)\|_{L^1(0, \infty)}.
$$

Then $q_{r,k}$ can be extended to $\mathcal{D}_{r,k}$ and the extension is equivalent to $p_{r,k}$ on $\mathcal{D}_{r,k}$.

**Proof.** From Lemma 4.2 and Definition 4.1, there exists $M > 0$ such that

$$
p_{r,k}(\varphi) \leq M q_{r,k}(\varphi)
$$

for every $\varphi \in \mathcal{D}_+$. This, together with the inequality $q_{r,k}(\varphi) \leq p_{r,k}(\varphi)$, shows that $q_{r,k}$ and $p_{r,k}$ are equivalent on $\mathcal{D}_+$, hence on $\mathcal{D}_{r,k}$.

**Corollary 4.4.** $\mathcal{D}_+ \hookrightarrow \mathcal{D}_{r,k} \hookrightarrow L^1(0, \infty)$ for every integer $k \geq 0$. Here $\hookrightarrow$ means the continuous embedding.

Throughout the rest of this section, instead of $p_{r,k}$ we shall use $q_{r,k}$.  

Lemma 4.5. For given $r > 0$, $k \in \mathbb{N}$, there exists $M > 0$ such that
\[
\sum_{j=0}^{k-1} |\varphi^{(j)}(0)| \leq M q_{r,k}(\varphi)
\]
for every $\varphi \in \mathcal{D}_{r,k}$.

Proof. From
\[
\varphi(0) = \varphi(t) - \int_0^t \varphi'(s) \, ds,
\]
one has
\[
|\varphi(0)| \leq \left| \int_0^1 |\varphi(0)| \, dt \right| \leq \left| \int_0^1 |\varphi(t)| \, dt \right| + \left| \int_0^1 |\varphi'(s)| \, ds \, dt \right| \leq K q_{r,1}(\varphi)
\]
for some $K > 0$ by Lemma 4.2. The lemma follows by induction.

Denote $h_\lambda(t) = e^{-it\lambda}Y(t)$. Then $h_\lambda \in \mathcal{D}_{r,k}$ for $\lambda \in \mathbb{C}$ with $\text{Re} \lambda > r$.

Proposition 4.6. $\mathcal{D}_{r,k}$ has the following properties.

(i) $\mathcal{D}_{r,k}$ is an algebra for the convolution product (3.5), in which $\varphi, \psi \in \mathcal{D}_{r,k}$.

(ii) The span of the set $\{h_\lambda | \lambda \in \mathbb{C}, \text{Re} \lambda > r \}$ is dense in $\mathcal{D}_{r,k}$.

Proof. (i) We confine the proof to the case $k \in \mathbb{N}$. That of $k = 0$ is immediate. Since, for $\varphi, \psi \in \mathcal{D}_{r,k}$, 
\[
\left(\frac{d}{dt}\right)^k (\varphi * \psi)(t) = \left(\frac{d}{dt}\right)^k \int_0^t \varphi(t-s) \psi(s) \, ds
\]
and
\[
\int_0^{\infty} e^{rt} \left| \int_0^t |\varphi^{(k)}(t-s) \psi(s)| \, ds \right| \, dt
\]
\[
= \int_0^{\infty} e^{rs} |\psi(s)| \left( \int_s^{\infty} e^{r(t-s)} |\varphi^{(k)}(t-s)| \, dt \right) \, ds,
\]
\[
= q_{r,k}(\varphi) q_{r,1}(\psi),
\]
there exists $M > 0$ such that
\[ q_{r,k}(\varphi * \psi) \leq M q_{r,k}(\varphi) q_{r,k}(\psi) \]
by Lemmas 4.2 and 4.5. Clearly, the above inequality remains true for
$\varphi, \psi \in \mathcal{D}_{r,k}$.

(ii) We also confine the proof to the case $k \in \mathbb{N}$. Define the map
\[ \Phi: \mathcal{D}_{r,k} \mapsto L^1(0, \infty) \]
by
\[ [\Phi(\varphi)](t) = e^{\lambda t} \varphi^{(k)}(t) \quad \forall \varphi \in \mathcal{D}_{r,k}. \]
We show that $\Phi$ is surjective. Let $\psi \in L^1(0, \infty)$. Define
\[ \varphi(t) = -\frac{1}{(k-1)!} \int_0^t (t-s)^{k-1} e^{-\lambda s} \psi(s) \, ds \quad \forall t \geq 0. \]
From $\varphi^{(k)}(t) = e^{-\lambda t} \varphi(t)$, we have
\[ \int_0^\infty e^\lambda |\varphi^{(k)}(t)| \, dt = \int_0^\infty |\psi(t)| \, dt < \infty. \]
Hence $\varphi \in \mathcal{D}_{r,k}$ and $\Phi(\varphi) = \psi$, so that $\Phi$ is surjective. Since $\|\psi\|_{L^1(0, \infty)} = \|\Phi(\varphi)\|_{L^1(0, \infty)} = q_{r,k}(\varphi)$, $\mathcal{D}_{r,k}$ is isometrically isomorphic to $L^1(0, \infty)$. By
the uniqueness of the Laplace transform, if $u \in L^\infty(0, \infty)(=(L^1(0, \infty))^*)$ satisfies
\[ \int_0^\infty u(t) e^{-\lambda t} \, dt = 0 \quad \forall \lambda \in \mathbb{C} \text{ with } \text{Re} \lambda > 0, \]
then $u(t) = 0$ a.e., (ii) follows.

**Definition 4.7.** A QDSG $\mathcal{G}$ is said to be of order $(r, k)$, if $\mathcal{G}$, regarded as defined on $\mathcal{D}_+$, can be extended to a continuous linear map from $\mathcal{D}_{r,k}$ into $B(L^1)$.

Clearly, Definition 4.7 is equivalent to the facts that $\mathcal{G}|_{\mathcal{D}_+}$ is continuous in the topology induced by $q_{r,k}$ on $\mathcal{D}_+$.

**Proposition 4.8.** Suppose $\mathcal{G}$ is a QDSG of order $(r, k)$ generated by $A$. Then for any $\varphi \in \mathcal{D}_{r,k}$,
\[ \mathcal{G}(\varphi) A \subseteq A \mathcal{G}(\varphi). \]

**Proof.** An immediate consequence of Proposition 3.7.
Definition 4.9. Suppose \( A \) is a closed operator and \( \mathcal{A} \) is a Banach algebra of complex-valued functions, on the real line, that contains \( g_*(s) = (\lambda - s)^{-1} \), for some complex \( \lambda \). Then an \( \mathcal{A} \) functional calculus for \( A \) is a linear continuous map \( \mathcal{E} \), from \( \mathcal{A} \) into \( B(X) \), such that

(i) \( \mathcal{E}(f) \mathcal{E}(g) = \mathcal{E}(fg) \) for all \( f, g \in \mathcal{A} \), and

(ii) whenever \( g_* \in \mathcal{A} \), for some complex \( \lambda \), then \( \lambda \in \rho(A) \), and

\( \mathcal{E}(g_*) = (\lambda - A)^{-1} \).

DeLaubenfels [9, 10] introduced a kind of functional calculus by means of the Laplace transform (see [9, Definition 2.1; 10, Definition 2.1] for details). We now introduce an analogous functional calculus in terms of the Fourier transform.

Definition 4.10. We write \( \mathcal{F}_{r,k} \) for the set of all Fourier transforms of \( D_{r,k} \) functions; that is, \( f \in \mathcal{F}_{r,k} \) if and only if

\[
\mathcal{F}(s) = \int_{-\infty}^{\infty} e^{-i\omega t} \varphi(t) \, dt
\] (4.4)

for some \( \varphi \) in \( D_{r,k} \). \( \varphi \) is referred to as the Fourier determining function for \( f \).

In \( \mathcal{F}_{r,k} \), we define the norm

\[
\| f \|_{r,k} = q_{r,k}(\varphi)
\]

where \( f, \varphi \) are as in (4.4) and \( q_{r,k}(\varphi) \) is defined in Corollary 4.3.

From Proposition 4.6, we have

Proposition 4.11. \( \mathcal{F}_{r,k} \) has the following properties.

(i) \( \mathcal{F}_{r,k} \) is an algebra for the pointwise product of functions and is isometrically isomorphic to \( D_{r,k} \).

(ii) The span of the set \( \{ g_* \mid \lambda \in \mathbb{C}, \text{Im}(\lambda) > r \} \) is dense in \( \mathcal{F}_{r,k} \).

To prove our main Theorem 4.13, we need to make use of [1, Corollary 1.2] which was not proved. We shall offer a proof for our purpose. We first state the following.

Lemma 4.12. Let \( f \) be a complex-valued function on \([a, b]\) satisfying

\[
\limsup_{h \to 0} \frac{1}{h} |f(t + h) - f(t)| \leq M
\] (4.5)

for some \( M > 0 \) and all \( t \in [a, b] \). Then \( f \) is Lipschitz continuous on \([a, b]\). In (4.5) we assume \( f(t) = f(b) \) for \( t > b \).
Proof. Let $t_1, t_2 \in [a, b]$ with $t_1 < t_2$. From (4.5), there exist $t' \in (t_1, t_2]$ such that
\[
|f(t') - f(t_1)| \leq (M + 1)(t' - t_1). \tag{4.6}
\]
Let $t_0 = \max\{t' \in (t_1, t_2) \mid t' \text{ satisfies (4.6)}\}$. Then $t_0 = t_2$. Suppose the contrary, then there exists $t'_0 \in (t_0, t_2]$ such that
\[
|f(t'_0) - f(t_0)| \leq (M + 1)(t'_0 - t_0).
\]
This, along with (4.6), gives us
\[
|f(t'_0) - f(t_1)| \leq (M + 1)(t'_0 - t_1).
\]
Thus (4.6) holds for $t' = t'_0$. This contradicts the definition of $t_0$, since $t'_0 > t_0$. Hence $t_0 = t_2$ and
\[
|f(t_2) - f(t_1)| \leq (M + 1)(t_2 - t_1).
\]
$t_1, t_2 \in [a, b]$ with $t_1 < t_2$ being arbitrary, the lemma follows.

**Theorem 4.13.** Let $A$ be a closed operator and let $k \in \mathbb{N} \cup \{0\}$, $r > 0$, $M > 0$. Then the following are equivalent.

(i) $(r, \infty) \subset \rho(A)$ and
\[
\left| \left( \frac{d^n}{d\lambda^n} \right) [R(\lambda; A)/\lambda^k] \right| \leq \frac{Mn!}{(\lambda - r)^{n+1}} \tag{4.7}
\]
for $\lambda > r$ and $n = 0, 1, 2, ...$

(ii) $A$ generates a nondegenerate $(k+1)$-times integrated semigroup $\{W(t)\}_{t \geq 0}$ satisfying
\[
\lim \sup_{h \downarrow 0} \frac{1}{h} \|W(t + h) - W(t)\| \leq Me^r \quad \forall t \geq 0. \tag{4.8}
\]

(iii) There exists a $\mathcal{F}_{r,k}$ functional calculus $\mathcal{E}$ for $iA$.

(iv) $A$ generates a QDSG $\mathcal{G}$ of order $(r, k)$.

Proof. The equivalence of (i) and (ii) was stated in [1] with a proof for (i) $\Rightarrow$ (ii). We now give a proof for (ii) $\Rightarrow$ (i) for later use. Let $x \in X$, $x^* \in X^*$. From Lemma 4.12, $\langle W(t)x, x^* \rangle$ is Lipschitz continuous on every bounded interval $[a, b] \subset [0, \infty)$, hence $\langle W(t)x, x^* \rangle$ is differentiable almost everywhere. (4.8) implies that $|\langle d/dt \langle W(t)x, x^* \rangle \rangle| \leq Me^r \|x\| \|x^*\|$. 


From the nondegeneration of \( \{ W(t) \}_{t \geq 0} \) and [1, Theorem 3.1], \((r, \infty) \subseteq \rho(A)\) and for \( \lambda > r \),
\[
\frac{\langle R(\lambda; A) x, x^* \rangle}{\lambda^2} = \int_0^\infty e^{-\lambda t} \langle W(t) x, x^* \rangle \, dt
\]
\[
= \int_0^\infty e^{-\lambda t} \frac{d}{dt} \langle W(t) x, x^* \rangle \, dt.
\]
Hence
\[
\left| \left( \frac{d}{dt} \right)^n \left[ \frac{\langle R(\lambda; A) x, x^* \rangle}{\lambda^2} \right] \right| \leq M \int_0^\infty e^{-(\lambda - r) t} \, dt \| x \| \| x^* \|
\]
\[
= \frac{M n!}{(\lambda - r)^{n+1}} \| x \| \| x^* \|. 
\]
Thus (i) holds.

(ii) \( \Rightarrow \) (iv). Let \( \varphi \in \mathcal{D}, x \in X \). Define
\[
\mathcal{G}(\varphi) x = (-1)^k \int_0^\infty \varphi^{(k)}(t) \, dW(t) x.
\] (4.9)
The definition of the integral in (4.9) can be found in [15]. Let \( x^* \in X^* \). (4.8) implies that
\[
|\langle \mathcal{G}(\varphi) x, x^* \rangle| \leq \int_0^\infty |\varphi^{(k)}(t)| \, |d\langle W(t) x, x^* \rangle|
\]
\[
\leq M \int_0^\infty e^r |\varphi^{(k)}(t)| \, dt \| x \| \| x^* \|
\]
\[
= Mq_{r,k}(\varphi) \| x \| \| x^* \|.
\]
Here we note that \( q_{r,k}(\varphi) \) also makes sense for \( \varphi \in \mathcal{D} \) by the definition of \( q_{r,k} \). Thus \( \mathcal{G}(\varphi) \) is a bounded operator on \( X \) and
\[
\|\mathcal{G}(\varphi)\| \leq Mq_{r,k}(\varphi) \quad \forall \varphi \in \mathcal{D}.
\] (4.10)
From (4.9), \( \mathcal{G} \) can be regarded as defined on \( \mathcal{D}_+ \); from (4.10), \( \mathcal{G} \) can be extended from \( \mathcal{D}_{r,k} \) into \( B(X) \), and (4.9), (4.10) remain true for \( \varphi \in \mathcal{D}_{r,k} \).
We now prove that
\[
\mathcal{G}(\varphi) \mathcal{G}(\psi) = \mathcal{G}(\varphi * \psi) \quad \forall \varphi, \psi \in \mathcal{D}_{r,k}.
\] (4.11)
For $x \in X$, $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) > r$, (4.9) and [1, Theorem 3.1] imply that

$$\mathcal{G}(h_{\lambda}) x = \lambda^{k+1} \int_{0}^{\infty} e^{-\lambda t} W(t) x \, dt = (\lambda - A)^{-1} x. \tag{4.12}$$

Let $\mu \in \mathbb{C}$ with $\text{Re}(\mu) > r$. The equality $(\mu - \lambda)(\lambda - A)^{-1} (\mu - A)^{-1} = (\lambda - A)^{-1} - (\mu - A)^{-1}$ gives us

$$(\mu - \lambda) \mathcal{G}(h_{\lambda}) \mathcal{G}(h_{\mu}) x = \left[ \mathcal{G}(h_{\lambda}) - \mathcal{G}(h_{\mu}) \right] x.$$ 

This, combined with

$$(\mu - \lambda)(h_{\lambda} * h_{\mu})(t) = h_{\lambda}(t) - h_{\mu}(t), \tag{4.13}$$

shows that

$$\mathcal{G}(h_{\lambda}) \mathcal{G}(h_{\mu}) x = \mathcal{G}(h_{\lambda} * h_{\mu}) x.$$ 

The application of Proposition 4.6 (ii) gives (4.11). This implies Definition 3.3(i).

It is interesting to note that a similar argument as in Theorem 3.8 can be used to prove Definition 3.3(i) for $\mathcal{G}$. But we prefer the above easy argument.

To prove Definition 3.3(ii), let $x \in X$ be such that $\mathcal{G}(\theta) = 0 \quad \forall \theta \in \mathcal{D}_{0}$. Then (4.9) implies that

$$\int_{0}^{\infty} \theta^{k+1}(t) W(t) x \, dt = 0.$$ 

As with the proof of (i) $\Rightarrow$ (ii) of Theorem 3.8 (see (3.15), (3.16)), we have $W(t) x = 0$ for $t \in [0, \infty)$. Hence $x = 0$ by the nondegeneration of $\{ W(t) \}_{t > 0}$.

The fact that $A$ is the generator of $\mathcal{G}$ can be shown as in Theorem 3.8.

(iv) $\Rightarrow$ (iii). Let $\mathcal{G}$ be a QDSG of order $(r, k)$ generated by $A$. From Proposition 2.1 and Theorem 3.8, $E(x, \beta) \subseteq p(A)$. For $f, \varphi$ as in (4.4), define

$$E(f) = \mathcal{G}(\varphi). \tag{4.14}$$
Definition 3.3(i) implies that 

$$\delta(f)\delta(g) = \delta(fg) \quad \forall f, g \in \mathcal{F}_{r,k}.$$ 

Hence Definition 4.9 (i) follows.

Let $\varphi \in D_0$. A direct computation shows that 

$$(\varphi' \ast h_\lambda)(t) = \varphi(t) - \lambda(\varphi \ast h_\lambda)(t).$$ 

Hence for $x \in X$, $\lambda \in E(\alpha, \beta)$ with $\text{Re}(\lambda) > r$, 

$$A \mathcal{G}(\varphi) \mathcal{G}(h_\lambda) x = -\mathcal{G}(\varphi') \mathcal{G}(h_\lambda) x$$ 

$$= -\mathcal{G}(\varphi' \ast h_\lambda) x$$ 

$$= \lambda \mathcal{G}(\varphi) \mathcal{G}(h_\lambda) x - \mathcal{G}(\varphi) x,$$

or equivalently,

$$(\lambda - A) \mathcal{G}(\varphi) \mathcal{G}(h_\lambda) x = \mathcal{G}(\varphi) x.$$ 

This gives $\mathcal{G}(\varphi) \mathcal{G}(h_\lambda) x = \mathcal{G}(\varphi)(\lambda - A)^{-1} x$ by Proposition 4.8. Consequently, 

$$\mathcal{G}(h_\lambda) x = (\lambda - A)^{-1} x \quad \forall x \in X \quad (4.15)$$

by Definition 3.3(ii). (4.15) implies that $\mathcal{G}(h_\lambda)$ is injective for $\lambda \in E(\alpha, \beta)$ with $\text{Re}\lambda > r$. This and (4.13) imply that $\mathcal{G}(h_\lambda)$ is a resolvent for $\lambda \in \mathbb{C}$ with $\text{Re}\lambda > r$. Hence $\lambda \in \rho(A)$ and (4.15) remains true for all $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) > r$.

Let $\mu \in \mathbb{C}$ with $\text{Im}\mu > r$. Write $\mu = i\omega$, then $\text{Re}\lambda > r$. From 

$$g_\mu(s) = g_{\omega}(s) = (-i) \int_0^\infty e^{-i\omega}h_\lambda(t)\,dt$$

and (4.14), (4.15), we have $g_\mu \in \mathcal{F}_{r,k}$ and 

$$\delta(g_\mu) = \delta(g_{\omega}) = (-i)(\lambda - A)^{-1} = (i\omega - iA)^{-1} = (\mu - iA)^{-1}.$$ 

Definition 4.9 (ii) follows and $iA$ has a $\mathcal{F}_{r,k}$ functional calculus.

(iii) $\Rightarrow$ (i). Assume $\lambda > r$. Let $\delta$ be a $\mathcal{F}_{r,k}$ functional calculus for $iA$. Since $(\lambda - A)^{-1} = i\delta(g_{\omega})$, the calculations...
\[ \left( \frac{d}{dt} \right)^n \left[ (\lambda - A)^{-1}/\lambda^k \right] = \left( \frac{d}{dt} \right)^n \left[ g_{\nu_0}/\lambda^k \right] \]
\[ \leq M \pi_{r, k} \left( \frac{d}{dt} \right)^n \left[ g_{\nu_0}/\lambda^k \right] \]
\[ = M \left( \frac{d}{dt} \right)^n \left[ h_\nu/\lambda^k \right] \]
\[ = M \left\| e^{it} \left( \frac{d}{dt} \right)^n (e^{-it}/\lambda^k) \right\|_{L^1(0, \infty)} \]
\[ = M \left\| e^{it} \left( \frac{d}{dt} \right)^n (e^{-it}/\lambda^k) \right\|_{L^1(0, \infty)} \]
\[ = M \int_0^\infty t^e e^{-(-\nu) t} \, dt = \frac{M n!}{(\lambda - r)^{n+1}} \]

show that (i) holds, where \( M > 0 \) is a constant. Thus the theorem is proved.

From Proposition 4.6(ii) and (4.12), if one of the equivalent conditions (i)-(iv) in Theorem 4.13 holds, then both \( G \) and \( \epsilon \) are uniquely determined by \( A \).

The following consequences of Theorem 4.13 are analogues of Corollaries 3.9 and 3.10.

**Corollary 4.14.** If one of the equivalent conditions (i)-(iv) in Theorem 4.13 holds, then for every \( x \in D(A^{k+1}) \) there exists \( u \in C([0, \infty), X) \) such that

\[
\begin{aligned}
    u(0) &= x \\
    \mathcal{G}(\varphi) \, x &= \int_0^\infty \varphi(t) \, u(t) \, dt \quad \varphi \in \mathcal{D}_0,
\end{aligned}
\]

(4.16)

**Proof.** Let \( x \in D(A^{k+1}) \). (4.9) implies that

\[
\mathcal{G}(\varphi) \, x = \int_0^\infty \varphi(t) \, [W(t) \, x]^{(k+1)} \, dt.
\]

Set \( u(t) = [W(t) \, x]^{(k+1)} \) \((t \geq 0)\) to complete the proof.

**Corollary 4.15.** With the same hypothesis as in Corollary 4.14, if \( x = \mathcal{G}(\varphi) \, y \) for some \( \varphi \in \mathcal{D}_0 \), \( y \in X \), then there exists \( u \in C([0, \infty), X) \) such that (4.16) holds.

**Proof.** It suffices to note that \( x \in D(A^{k+1}) \).
We close this section with an observation for the special case of $A$ being closed and densely defined.

**Theorem 4.16.** Let $A$ be a closed operator and $k \in \mathbb{N} \cup \{0\}$, $r > 0$, $M > 0$. Then the following are equivalent.

(i) $A$ is densely defined, $(r, \infty) \subset \rho(A)$, and (4.7) holds for $\lambda > r$, $n = 0, 1, 2, ...$.

(ii) $A$ is densely defined and generates a nondegenerate $k$-times integrated semigroup $\{W(t)\}_{t \geq 0}$ satisfying

\[ \|W(t)\| \leq Me^r, \quad \text{for } t \geq 0. \]

(iii) There exists a $\mathcal{F}_{r,k}$ functional calculus $\mathcal{E}$ for $iA$ and

\[ \mathcal{R} = \bigcup \{ \text{Im } \mathcal{E}(f) \mid f \text{ is defined in (4.4) with } \varphi \in \mathcal{D}_0 \} \]

is dense in $X$.

(iv) $A$ generates a QDSG $\mathcal{G}$ of order $(r,k)$ and

\[ \mathcal{R} = \bigcup \{ \text{Im } \mathcal{G}(\varphi) \mid \varphi \in \mathcal{D}_0 \} \]

is dense in $X$.

(v) $A$ generates a distribution semigroup $\mathcal{G}$ in the sense of Lions such that $\mathcal{G} | \mathcal{D}_0$ is continuous in the topology induced by the norm $q_{r,k}$ on $\mathcal{D}_0$.

**Proof.** (i) $\Rightarrow$ (ii). From Theorem 4.13, $A$ generates a QDSG $\mathcal{G}$ of order $(r,k)$. Since $A$ is densely defined, the proof of Corollary 3.11 implies that $\mathcal{R}$ is dense in $X$.

(iv) $\Rightarrow$ (i). Since $\mathcal{R} (\subseteq D(A))$ is dense in $X$, $D(A)$ is also.

(iv) $\Rightarrow$ (v). From the hypotheses, there exists $M_1 > 0$ such that

\[ \|\mathcal{G}(\varphi)\| \leq M_1 q_{r,k}(\varphi) \quad \forall \varphi \in \mathcal{D}_0. \]

\[ (4.17) \]

Let $\theta_j(j = 0, 1, ..., k - 1)$ be fixed functions in $\mathcal{D}$ enjoying the following properties.

(a) $\text{Supp } \theta_j \subset [-1, 1]$ for $j = 0, 1, ..., k - 1$.

(b) $\theta_j(t) = t^j/j!$ for $t$ in a fixed neighborhood of 0 and $j = 0, 1, ..., k - 1$. 

Given $\varphi \in \mathcal{D}$. Let
\begin{equation}
\psi(t) = \varphi(t) - \sum_{j=0}^{k-1} \varphi^{(j)}(0) \theta_j(t).
\end{equation}
(4.18)

Then
\begin{equation}
\psi^{(j)}(0) = 0 \quad \forall j = 0, 1, ..., k - 1.
\end{equation}
(4.19)

Assume that $\text{supp } \psi \subset (-a, a)$ for some $a > 0$.

We now prove that there exists $\psi_n \in \mathcal{D}_0$ so that
\begin{equation}
q_{r,t}(\psi_n - \psi) \to 0 \quad \text{as } n \to \infty.
\end{equation}
(4.20)

To do this, let $\rho_n \in \mathcal{D}_0$ satisfy
\[ \rho_n(t) \geq 0; \quad \int_0^\infty \rho_n(t) \, dt = 1; \quad \text{supp } \rho_n \subset \left(0, \frac{1}{n}\right). \]

Define
\begin{equation}
\psi_n(t) = \int_0^t \rho_n(s) \psi(t-s) \, ds \quad t \in \mathbb{R}.
\end{equation}

Then $\psi_n \in \mathcal{D}_0$, $\text{supp } \psi_n \subset [0, a]$ for $n$ sufficiently large. From (4.19),
\begin{equation}
\psi_n^{(k)}(t) = \int_0^t \rho_n(s) \psi^{(k)}(t-s) \, ds.
\end{equation}

Let $t \in (0, a]$. Then
\begin{align*}
|\psi_n^{(k)}(t) - \psi^{(k)}(t)| &
\leq \int_0^t \rho_n(s) |\psi^{(k)}(t-s) - \psi^{(k)}(t)| \, ds \\
&
\leq \int_0^t \rho_n(s) |\psi^{(k)}(t-s) - \psi^{(k)}(t)| \, ds \to 0 \quad \text{as } n \to \infty.
\end{align*}

Since $\{\psi_n^{(k)}(t)\}$ is uniformly bounded, (4.20) follows by dominated convergence theorem. (4.17) and (4.20) imply that $\|\psi(\psi)\| \leq M_1 q_{r,t}(\psi)$. From (4.18) and Lemma 4.5, there exists $M_2 > 0$, that is independent of $\psi$, so that
\begin{equation}
q_{r,t}(\psi) \leq M_2 q_{r,t}(\varphi).
\end{equation}
This implies that
\[ \| \Theta(\varphi) \| \leq M_{q_r}(\varphi) \]
by Lemma 4.5 again.
Definition 3.3(i) can be shown as with the necessity of Corollary 3.11.

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REFERENCES


