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On the Maintenance of Oscillations of nth Order Equations under the Effect of a Small Forcing Term

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INTRODUCTION

One of the major, and generally unstudied, problems in the theory of oscillation of nonlinear equations, is the problem of maintaining oscillations under the effect of a forcing term. Namely, and more generally, we study here the oscillation problem of even order equations of the form

$$x^{(n)} + P(t, x, x',..., x^{(n-1)}) = Q(t, x, x',..., x^{(n-1)}), \qquad n \geqslant 2, \qquad (*)$$

where it is not assumed that $x_1[P(t, x_1 ,..., x_n) - Q(t, x_1 ,..., x_n)] \geqslant 0$ for every $x_1 \neq 0$. Problems of this type have already been studied for second-order equations by several authors. For example, Bhatia [1], Kiguradge [8], the author [4, 5], Legatos and the author [9], and Bobisud [2] have studied the oscillation of solutions of equations

$$x'' + P(t) G(x) = 0, \qquad (E_1)$$

where $xG(x) \geqslant 0$ for $x \neq 0$, and $P(t)$ is not necessarily positive for all large t. Moreover, Bobisud [3] has given sufficient conditions for the oscillation of all solutions of

$$x'' + a(t, x, x')x' + f(t, x, x') = 0 \qquad (E_2)$$

with a small damping $a(t, x, x')$, while the author has given some results in [7] for even order equations of the type $(*)$. It should be mentioned, however, that only the results of the author in [7] contain as a special case the equation

$$x^{(n)} + P(t, x, x',..., x^{(n-1)}) = Q(t) \qquad (**)$$

under conditions different or more restrictive than the ones considered in this paper.

Let us first show that the oscillation of all solutions of the equation

$$x^{(n)} + P(t)\,G(x) = 0 \tag{$**$}$$

is not generally maintained if one considers the "forced" equation, by adding the term $Q(t)$, which is small in the sense that $\int^{+\infty} t^{n-1}\,|Q(t)|\,dt < +\infty$. In fact, consider the equation

$$x^{(4)} + (\sin t + 2)\,t^{-4}x^3 = [(\sin t + 2)t^{-5}]^{(4)} + (\sin t + 2)^4 t^{-19}. \tag{1}$$

Here we have $P(t) > 0$ and $\int^{+\infty} t^3 P(t)\,dt = +\infty$, which (according to Theorem 2 of [6]) shows the oscillation of all solutions of

$$x^{(4)} + (\sin t + 2)\,t^{-4}x^3 = 0, \tag{2}$$

but $x(t) = (\sin t + 2)t^{-5}$ is a solution of (1) which is nonoscillatory, and not even monotonic for all large t.

The above example is instructive in that it suggests that we must impose more on the function $Q(t)$ in ($**$), in order to ensure oscillation of all solutions. This is done here for equations of the form ($**$) with $x_1 P(t, x_1, \ldots, x_n) > 0$ for $x_1 \neq 0$. Moreover, we study the effect of conditions of the form $\int^{+\infty} t^m P(t, u, u', \ldots, u^{(n-1)})\,dt = +\infty$ (m integer, $0 \leqslant m \leqslant n-2$) on the bounded solutions (if any) of ($*$). Our results are, in some cases, extensions of or related to several results in the references.

In what follows, all functions considered are continuous on their domains (although this assumption can be dropped in some cases), and the functions P, Q in ($*$) will be assumed smooth enough to ensure the extendability of solutions on rays of the form $[T_x, +\infty)$ ($T_x \geqslant t_0$ (t_0 fixed) and depending on the particular solution $x(t)$). Let $\mathscr{F}$ be the family of all such extended solutions. A solution $x \in \mathscr{F}$ is said to be oscillatory if it has an unbounded set of zeros on $[T_x, +\infty)$, and it is said to be bounded if $|x(t)| \leqslant K$ for every $t \in [T_x, +\infty)$, where K is a positive constant. All theorems are given for $n =$ even. The case $n =$ odd is covered by the remarks at the end of the paper.

1. Theorem 1. *For the equation* ($*$) *assume the following*:

(i) $P : I \times \mathbf{R}^n \to \mathbf{R} = (-\infty, +\infty)$, $I = [t_0, +\infty)$, $t_0 \geqslant 0$;

(ii) $Q : I \times \mathbf{R}^n \to \mathbf{R}$;

(iii) $\int^{+\infty} t^m[P(t, u(t), \ldots, u^{(n-1)}(t)) - Q(t, u(t), \ldots, u^{(n-1)}(t))]\,dt = +\infty\ (-\infty)$ *for every* $u \in C^n[t_0, +\infty)$ *which is bounded between two positive* (*negative*) *constants for all large* t, *and for some integer* m *such that* $0 \leqslant m \leqslant n-2$.

Then, if $x \in \mathscr{F}$ is bounded, it satisfies $\liminf_{t\to+\infty} |x(t)| = 0$.

Proof. Suppose that $x(t)$ is a bounded solution of (*) which does not oscillate. Then it must be of one sign for all large t. Let $0 < L \leqslant |x(t)| \leqslant M$ for all large t (L, M constants). Without any loss of generality, we assume that there exists $t_1 \geqslant T_x$ such that $0 < L \leqslant x(t) \leqslant M$ for every $t \in [t_1, +\infty)$. The case of an eventually negative $x(t)$ can be carried out in a similar way. Now, by differentiation of the function $t^m x^{(n-1)}(t)$, $t \in [t_1, +\infty)$, we obtain

$$[t^m x^{(n-1)}(t)]' = -t^m[P(t, x(t),..., x^{(n-1)}(t) - Q(t, x(t),..., x^{(n-1)}(t))] + mt^{m-1}x^{(n-1)}(t). \tag{3}$$

If $m = 0$, then integration of (3) gives $\lim_{t\to+\infty} x^{(n-1)}(t) = -\infty$, a contradiction to the positivity of $x(t)$. Assume that $m \geqslant 1$; then integration of (3) gives

$$t^m x^{(n-1)}(t) = t_1{}^m x^{(n-1)}(t_1) - \int_{t_1}^t s^m[P - Q]\, ds + m \int_{t_1}^t s^{m-1}x^{(n-1)}(s)\, ds, \tag{4}$$

which in view of (iii) implies

$$\lim_{t\to+\infty} \left[t^m x^{(n-1)}(t) - m \int_{t_1}^t s^{m-1}x^{(n-1)}(s)\, ds\right] = -\infty, \tag{5}$$

or

$$\lim_{t\to+\infty} [tq'(t) - mq(t)] = -\infty, \qquad q(t) \equiv \int_{t_1}^t s^{m-1}x^{(n-1)}(s)\, ds. \tag{6}$$

We show that (6) is impossible. In fact, let us consider the three possible cases:

Case I. $q(t)\, q'(t) > 0$ for all large t.

Case II. $q(t)\, q'(t) < 0$ for all large t.

Case III. $q(t)\, q'(t)$ is oscillatory.

Case I. Assume $q(t) > 0$, $q'(t) > 0$ for $t \in [t_2, +\infty)$, $t_2 \geqslant t_1$. Then (6) implies that

$$\lim_{t\to+\infty} \int_{t_2}^t s^{m-1}x^{(n-1)}(s)\, ds = +\infty,$$

and since $x^{(n-1)}(t) > 0$, we must have $(-1)^k x^{(k)}(t) < 0$ for every $k = 1, 2,..., n-1$ and every $t \in [t_2, +\infty)$. (See, e.g., [6, Theorem 1]). Consequently, from

$$\int_{t_2}^t s^{m-1}x^{(n-1)}(s)\, ds = s^{m-1}x^{(n-2)}(s)]_{t_2}^t - (m-1) \int_{t_2}^t s^{m-2}x^{(n-2)}(s)\, ds \tag{7}$$

we have

$$\lim_{t\to+\infty} \int_{t_2}^{t} s^{m-2}x^{(n-2)}(s)\, ds = -\infty, \tag{8}$$

and, finally,

$$\lim_{t\to+\infty} \int_{t_2}^{t} x^{(n-m)}(s)\, ds = \begin{cases} +\infty & \text{if } m \text{ is odd} \\ -\infty & \text{if } m \text{ is even,} \end{cases} \tag{9}$$

from which we obtain a contradiction to the boundedness of $x(t)$:

$$\lim_{t\to+\infty} [x^{(n-m-1)} - x^{(n-m-1)}(t_2)] = \pm\infty.$$

Now suppose $q(t) < 0$, $q'(t) < 0$ on $[t_2, +\infty)$. Then $tq'(t) \to -\infty$ as $t \to +\infty$ and this yields

$$\lim_{t\to+\infty} \int_{t_2}^{t} s^m x^{(n-1)}(s)\, ds = -\infty \tag{10}$$

and, finally,

$$\int_{t_2}^{+\infty} x^{(n-m-1)}(s)\, ds = \begin{cases} +\infty & \text{if } m \text{ is odd} \\ -\infty & \text{if } m \text{ is even} \end{cases} \tag{11}$$

or

$$\lim_{t\to+\infty} [x^{(n-m-2)}(t) - x^{(n-m-2)}(t_2)] = \pm\infty,$$

a contradiction to the boundedness of $x(t)$.

Case II. Obviously impossible.

Case III. Assume that $q'(t)$ oscillates; then for every sequence $\{t_n\}$ such that $q'(t_n) = 0$ and $\lim_{n\to+\infty} t_n = +\infty$, we must have $\lim_{n\to\infty} q(t_n) = +\infty$, which implies that $\lim_{t\to+\infty} q(t) = +\infty$. Thus, we have

$$\lim_{t\to+\infty} \int_{t_2}^{t} s^{m-1}x^{(n-1)}(s)\, ds$$
$$= \lim_{t\to+\infty} \left[t^{m-1}x^{(n-2)}(t) - (m-1)\int_{t_2}^{t} s^{m-2}x^{(n-2)}(s)\, ds\right] = +\infty. \tag{12}$$

If $x^{(n-2)}(t)$ is eventually positive or negative we obtain a contradiction as in Case I. Assume that $x^{(n-2)}(t)$ is oscillatory; then (12) implies similarly,

$$\int_{t_2}^{t} s^{m-2}x^{(n-2)}(s)\, ds = -\infty \tag{13}$$

and, eventually,

$$\lim_{t\to+\infty}\left[tx^{(n-m)}(t)-(n-m-1)\int_{t_2}^{t}x^{(n-m)}(s)\,ds\right]=\begin{cases}+\infty & \text{if } m \text{ is odd}\\ -\infty & \text{if } m \text{ is even,}\end{cases} \tag{14}$$

provided that none of the derivatives

$$x^{(k)}(t), \qquad k=n-m+1, n-m+2,\dots, n-1$$

is of one sign for all large t, otherwise the contradiction would follow as in Case I. Thus,

$$\lim_{t\to+\infty}[tx^{(n-m)}(t)-(n-m-1)\,x^{(n-m-1)}(t)]=\pm\infty. \tag{15}$$

If $x^{(n-m)}(t)$ is oscillatory then $\lim_{t\to+\infty}x^{(n-m-1)}(t)=\pm\infty$, which contradicts the boundedness of $x(t)$. If on the other hand, $x^{(n-m)}(t)$ is eventually of one sign, then we have

$$\text{either } \lim_{t\to+\infty}\int_{t_3}^{t}sx^{(n-m)}(s)\,ds=\pm\infty, \qquad \text{or} \qquad \lim_{t\to+\infty}x^{(n-m-1)}(t)=\pm\infty,$$

both implying a contradiction to the boundedness of $x(t)$. Thus, our assertion is true.

COROLLARY. *Suppose that* $P\equiv P_0(t)\,G(x,x',\dots,x^{(n-1)})$, *where*

$$x_1G(x_1,\dots,x_n)>0 \qquad \text{for} \quad x_1\neq 0,$$

and G is bounded between two positive or negative constants whenever the first variable is bounded in the same way. Moreover,

$$\int_{t_0}^{+\infty}t^m[\mu P_0^+(t)+P_0^-(t)]\,dt=+\infty \qquad \text{for every } \mu>0,$$

and

$$\int_{t_0}^{+\infty}t^mQ_0(t)\,dt<+\infty,$$

where $P_0^+(t)=\max\{P_0(t),0\}$ *and* $P_0^-(t)=\min\{P_0(t),0\}$, *and* $|Q(t,x_1,\dots,x_n)|\leqslant Q_0(t)$ *for every* $t\in[t_0,+\infty)$. *Then for every bounded solution* $x\in\mathscr{F}$ *we have* $\liminf_{t\to+\infty}|x(t)|=0$.

Proof. In fact, if for two constants K,L we have $0<K<x(t)<L$, then

there exist K_1, L_1 such that $0 < K_1 < G(x(t), x'(t),..., x^{(n-1)}(t)) < L_1$, by use of which we obtain

$$\begin{aligned}\int_{t_0}^{t} s^m[P - Q]\, ds &\geqslant \int_{t_0}^{t} s^m[P_0^+(s)G + P_0^-(s)G]\, ds - \int_{t_0}^{t} s^m Q_0(s)\, ds \\ &\geqslant \int_{t_0}^{t} s^m[K_1 P_0^+(s) + L_1 P_0^-(s)]\, ds - \int_{t_0}^{t} s^m Q_0(s)\, ds \\ &= L_1 \int_{t_0}^{t} s^m[(K_1/L_1)\, P_0^+(s) + P_0^-(s)]\, ds \\ &\quad - \int_{t_0}^{t} s^m Q_0(s)\, ds \to +\infty \qquad \text{as} \quad t \to +\infty. \qquad (16)\end{aligned}$$

An analogous situation appears if $x(t)$ is bounded between two negative constants.

2. This section is devoted to equations of the form $(*)$ with $x_1 P(t, x_1, ..., x_n) > 0$ for every $(t, x_1, ..., x_n) \in I \times \mathbf{R}^n$ with $x_1 \neq 0$, and $Q \equiv Q(t)$. Theorem 2 concerns itself with the bounded solutions of $(*)$, and Theorem 3 ensures the oscillation of all solutions of $(*)$ by further restricting the functions P, Q.

THEOREM 2. *Assume that* $(**)$ *satisfies*

(i) $P : I \to \mathbf{R}$, *and there exist four functions* P_i, G_i, $i = 1, 2$ *such that*

$$P_i : I \to \mathbf{R}_+ \cup \{0\},$$
$$G_1 : \mathbf{R}_+ \times \mathbf{R}^{n-1} \to \mathbf{R}_+,$$
$$G_2 : \mathbf{R}_- \times \mathbf{R}^{n-1} \to \mathbf{R}_- \qquad (\mathbf{R}_- = (-\infty, 0))$$

and, moreover,

$$P(t, x_1, ..., x_n) \begin{cases} \geqslant P_1(t)\, G_1(x_1, ..., x_n) & \text{if} \quad x_1 > 0 \\ \leqslant P_2(t)\, G_2(x_1, ..., x_n) & \text{if} \quad x_1 < 0; \end{cases}$$

(ii) $Q : I \to \mathbf{R}$, *and for some* $R : I \to \mathbf{R}$ *with* $R^{(n)}(t) = Q(t)$, *we have*

$$\lim_{t \to +\infty} R^{(k)}(t) = 0, \qquad k = 0, 1, 2, ..., n - 1;$$

then the condition

$$\int_{t_0}^{+\infty} t^{n-1} P_i(t) = +\infty, \qquad i = 1, 2$$

is sufficient for all bounded $x \in \mathscr{F}$ *to oscillate or satisfy* $\lim_{t \to +\infty} |x(t)| = 0$.

Proof. Suppose that $x(t)$ is a solution of $(**)$ with the property $0 < x(t) < M$ for every $t \in [t_1, +\infty)$, $t_1 \geqslant t_0$. Then the function $u(t) \equiv x(t) - R(t)$, $t \geqslant t_1$ is bounded, $u \in C^n[t_1, +\infty)$, and satisfies

$$u^{(n)} + P(t, u + R(t), u' + R'(t),..., u^{(n-1)} + R^{(n-1)}(t)) = 0. \tag{17}$$

We shall show that (17) cannot have bounded solutions $u(t)$ such that $u(t) + R(t) > 0$, $t \in [t_1, +\infty)$ unless $u(t) < 0$, $t \in [t_1, +\infty)$. In fact, if $u + R > 0$, then from (17) we obtain $u^{(n)}(t) \leqslant 0$, $t \in [t_1, +\infty)$, which implies $(-1)^k u^{(k)}(t) \leqslant 0$ for $k = 1, 2,..., n - 1$ and $u(t) \gtrless 0$ for every $t \geqslant t_1$. Assume now $u(t) > 0$ and $\lim_{t\to+\infty} u(t) = \alpha > 0$. Then, given a positive $\epsilon < G_1(\alpha, 0, 0,..., 0)$, there exists $t_2 \geqslant t_1$ such that

$$\begin{aligned} A = G_1(\alpha, 0,..., 0) - \epsilon < G_1(u(t) + R(t),..., u^{(n-1)}(t) + R^{(n-1)}(t)) \\ < G_1(\alpha, 0,..., 0) + \epsilon \end{aligned} \tag{18}$$

by use of which we get (as in [6, Theorem 1])

$$\begin{aligned} t^{n-1}u^{(n-1)}(t) &= t_2^{n-1}u^{(n-1)}(t_2) \\ &\quad - \int_{t_2}^{t} s^{n-1}P(s, u(s) + R(s),..., u^{(n-1)}(s) + R^{(n-1)}(s))\, ds \\ &\quad + (n-1)\, t^{n-2}u^{(n-1)}(t) \\ &\leqslant t_2^{n-1}u^{(n-1)}(t_2) - A \int_{t_2}^{t} s^{n-1}P_1(s)\, ds + (n-1)\, t^{n-2}u^{(n-1)}(t) \end{aligned} \tag{19}$$

This, by use of (ii), implies

$$\lim_{t\to+\infty} \left[t^{n-1}u^{(n-1)}(t) - \int_{t_2}^{t} s^{n-2}u^{(n-1)}(s)\, ds\right] = +\infty, \tag{20}$$

and the desired contradiction follows as in Theorem 1 in [6]. Thus $u(t) < 0$, i.e., $x(t) < R(t)$ for all large t. A similar proof holds in the case $-M < x(t) < 0$ (M = a positive constant) and our theorem is true.

THEOREM 3. *In Theorem 2 assume instead of* (ii) *that* $\lim_{t\to+\infty} R(t) = 0$, *and that* $G_i = G_i(x)$, *increasing and such that*

$$\int_{\epsilon}^{+\infty} [G_1(s)]^{-1}\, ds < +\infty, \qquad \int_{-\epsilon}^{-\infty} [G_2(s)]^{-1}\, ds < +\infty \qquad \textit{for every } \epsilon > 0.$$

Then every solution of $(**)$ *is oscillatory or such that* $\lim_{t\to+\infty} |x(t)| = 0$.

Proof. If $x(t)$ is a bounded solution of $(**)$, then the conclusion follows from Theorem 2. Suppose that $x(t)$ is positive and unbounded, i.e., $x(t) > 0$, $t \in [t_1, +\infty)$ and $\lim_{t\to+\infty} x(t) = +\infty$; then there exist $t_2 \geqslant t_1$ and $\epsilon > 0$ such that $u(t) > 0$, and $0 < u(t) - \epsilon < u(t) + R(t)$ for every $t \geqslant t_2$. Consequently,

$$\begin{aligned} 0 &= u^{(n)}(t) + P(t, u(t) + R(t),..., u^{(n-1)}(t) + R^{(n-1)}(t)) \\ &\geqslant u^{(n)}(t) + P_1(t)G_1(u(t) + R(t)) \qquad (21) \\ &\geqslant u^{(n)}(t) + P_1(t)G_1(u(t) - \epsilon),\ t \in [t_2, +\infty). \end{aligned}$$

Thus, since $u^{(n)}(t) \leqslant -P_1(t)\, G_1(u(t) - \epsilon) \leqslant 0$, there are two possible cases (cf. Theorem 2 in [6]):

Case I. $(-1)^k u^{(n-k)}(t) \leqslant 0$ for every $k = 1, 2,..., n-1$. Then if $F(t) = t^{n-1}u^{(n-1)}(t)/G(u(t) - \epsilon)$, we obtain

$$\begin{aligned} F(t) \leqslant F(t_1) &- \int_{t_2}^{t} s^{n-1}P_1(s)\, ds + (n-1) \int_{t_2}^{t} s^{n-2}u^{(n-1)}(s)\, ds/G(u(s) - \epsilon) \\ &+ \int_{t_2}^{t} s^{n-1}u^{(n-1)}(s)\, d[1/G(u(s) - \epsilon)], \qquad (22) \end{aligned}$$

where the last integral is considered in the Riemann–Stieltjes sense, since the function $1/G(u(t) - \epsilon)$ is decreasing w.r.t. t. For the same reason, this integral is $\leqslant 0$, and this implies that

$$\lim_{t\to+\infty} \left[F(t) - \int_{t_2}^{t} s^{n-2}u^{(n-1)}(s)\, ds/G(u(s) - \epsilon)\right] = -\infty,$$

and the proof follows as in Theorem 2 in [6], by working with Riemann–Stieltjes integrals as above.

Case II. For some integer $n - 2i$ ($n - 2i \geqslant 2$), we have $u^{(n-2i)}(t) \geqslant 0$, $(-1)^k\, u^{(n-k)}(t) \geqslant 0$ for $k = 1, 2,..., 2i - 1$. This case can be carried out in the same way as in the above-mentioned theorem with attention to the use of Riemann–Stieltjes integrals. Thus, our theorem is true.

COROLLARY. *If, in addition to the assumptions made in Theorem* 2 (*Theorem* 3), $R(t)$ *is oscillatory, then every bounded solution* (*every solution*) *of* $(*)$ *is oscillatory.*

The proof is obvious, since for a positive (negative) solution of $(*)$ we cannot have $x < R$ ($x > R$) for all large t.

Remarks. It is evident that analogous results hold in the case $n =$ odd. In fact, Theorem 1 and its corollary hold as they are stated, while in

Theorems 2 and 3 one should conclude that every solution considered oscillates or tends monotonically to zero. One could also show that the integral condition on the P_i's in Theorem 3 is necessary for the theorem to hold. In fact, it suffices to show that the equation

$$u^{(n)} + P(t, u + R(t), \ldots, u^{(n-1)} + R^{(n-1)}(t)) = 0$$

with P as in Theorem 3 has a solution $u(t)$, such that $\lim_{t \to +\infty} u(t) = k$ $(0 \neq k < +\infty)$, provided that $\int_{t_0}^{+\infty} t^{n-1} P(t)\, dt < +\infty$, and this can be done by use of a functional analytic method of Svec [11] employed there for odd order equations. One could also extend Theorems 2 and 3 to the cases considered by Ryder and Wend in [10].

It remains an open question whether one could state a result like Theorem 1 for all solutions of $(*)$. It can be shown (by working with the function $F(t) = t^{n-1} x^{(n-1)}(t)/G(x(t)))$ that there are no positive (negative) solutions of $(*)$ such that $x'(t) \geqslant 0$, $x^{(n-1)}(t) \geqslant 0$ $(x'(t) \leqslant 0, x^{(n-1)}(t) \leqslant 0)$ for all large t, provided that $P \equiv P_0(t)\, G(x)$ with $xG(x) \neq 0$, increasing, $\int_{\epsilon}^{+\infty} [G(s)]^{-1}\, ds < +\infty$, $\int_{-\epsilon}^{+\infty} [G(s)]^{-1}\, ds < +\infty$, and $\int_{t_0}^{+\infty} t^{n-1}[P(t) - |Q/G(x(t))|]\, dt = +\infty$ for every $x(t) \neq 0$. This also suggests the study of the condition $\int^{+\infty} t^{n-1}[\mu P^{+}(t) + P^{-}(t)]\, dt = +\infty$, which is not included in (iii) of Theorem 1.

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