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Topology of compact space forms from Platonic solids. I.

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ABSTRACT

The problem of classifying, up to isometry, the orientable 3-manifolds that arise by identifying the faces of a Platonic solid was completely solved in a nice paper of Everitt [B. Everitt, 3-manifolds from Platonic solids, Topology Appl. 138 (2004) 253–263]. His work completes the classification begun by Best [L.A. Best, On torsion-free discrete subgroups of $PSL_2(\mathbb{C})$ with compact orbit space, Canad. J. Math. 23 (1971) 451–460], Lorimer [P.J. Lorimer, Four dodecahedral spaces, Pacific J. Math. 156 (2) (1992) 329–335], Prok [I. Prok, Classification of dodecahedral space forms, Beiträge Algebra Geom. 39 (2) (1998) 497–515], and Richardson and Rubinstein [J. Richardson, J.H. Rubinstein, Hyperbolic manifolds from a regular polyhedron, Preprint]. In this paper we investigate the topology of closed orientable 3-manifolds from Platonic solids. Here we completely recognize those manifolds in the spherical and Euclidean cases, and state topological properties for many of them in the hyperbolic case. The proofs of the latter will appear in a forthcoming paper.

1. Introduction and statement of the results

Let $X = \mathbb{S}^3$, $\mathbb{E}^3$ or $\mathbb{H}^3$. A 3-*space form* (or, an X-*manifold*) M is an orbit space X/G, where G is an isometry group acting on X properly discontinuously and without fixed points. This gives a tiling of X. The isometries of G, mapping a distinguished tile onto its neighbours, identify the boundary faces of the tile in pairs. Such isometries generate the fundamental group of M. Of course, M is the quotient space obtained from any distinguished tile via the above pairing of its boundary faces. A *Platonic solid* in X is a polytope P with the combinatorial type of a Platonic solid (convex regular solid), embedded in X, so that all side lengths are equal, as are the interior face angles and dihedral angles. Everitt classified in [5], up to isometry, the orientable 3-space forms that arise from tilings of X by Platonic solids. This completes the work begun by several authors (see [1,9,17,18]). These results, obtained by algebraic and computational methods, follow from the classification of certain subgroups of rank four Coxeter groups. The following theorem summarizes the results of the quoted papers according to Everitt notation (explained after the statement).

Theorem 1.1. *(Everitt [5], Lorimer [9]) The closed orientable spherical* 3*-manifolds arising from Platonic solids as space forms are listed in Table* 1.

The manifold M_1 comes from the tetrahedron with dihedral angle $2\pi/3$, M_2 and M_3 from the cube with angle $2\pi/3$, M_4, M_5 and M_6 from the octahedron with angle $2\pi/3$, and M_7 and M_8 from the dodecahedron with angle $2\pi/3$.

Remark 1. The manifolds M_7 and M_8 were constructed by Lorimer in [9]. The manifold M_3 is the quaternionic space [11, p. 120], and M_6 is the octahedral space [11, p. 117]. The manifolds M_2 and M_5 have the same homology but they

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Table 1

Spherical manifolds	F	E	Homology
M_1	$abab$	$a(--)b(--)aabb$	$\mathbb{Z}_5$
M_2	$ababcc$	$a(++)b(+-)aac(+-)bcd(+-)bcdd$	$\mathbb{Z}_8$
M_3	$abcbca$	$a(++)b(--)c(+-)cd(--)bdabdac$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
M_4	$abcacbdd$	$a(++)b(+-)c(+-)ad(++)cbdacdb$	$\mathbb{Z}_2 \oplus \mathbb{Z}_6$
M_5	$abcacdbd$	$a(++)b(--)c(++)ad(-+)cbcaddb$	$\mathbb{Z}_8$
M_6	$abcdcdab$	$a(++)b(++)c(++)d(++)bcdadabc$	$\mathbb{Z}_3$
M_7	$abcdefefbcda$	$a(-+)b(-+)c(-+)d(-+)e(-+)f(-+)g(-+)h(-+)i(-+)j(-+)idjefagbhcghijfeabcd$	0
M_8	$abcdefbdcfea$	$a(-+)b(-+)c(-+)d(-+)e(-+)f(++)g(++)h(++)i(++)j(++)ajcgbfeidhfhgjieabcd$	$\mathbb{Z}_{15}$

Table 2

Euclidean manifolds	F	E	Homology
M_9	$abacbc$	$a(+++)b(+++)aac(+++)bccbcba$	$\mathbb{Z}_3 \oplus \mathbb{Z}$
M_{10}	$abbcca$	$a(-+-)ab(--+)c(-+-)bacbbacc$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}$
M_{11}	$abccba$	$a(-+-)ab(--+)c(+--)bccbbcaa$	$\mathbb{Z}_4 \oplus \mathbb{Z}_4$
M_{12}	$abcbca$	$a(+++)b(+++)c(+++)bcaaccbba$	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$
M_{13}	$abcbca$	$a(+++)b(+++)c(-+-)cbaacbbca$	$\mathbb{Z}_2 \oplus \mathbb{Z}$
$M_{14}(= M_{10})$	$abcbca$	$a(-+-)b(+--)c(+++)bcaaccbba$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}$

are not homeomorphic (hence non-isometric). This is proved in [5] by algebraic arguments which imply that $\pi_1(M_2) \cong \mathbb{Z}_8$ while $\pi_1(M_5)$ has order 24. We obtain all these facts as particular consequences of our geometric methods. Note that (and this was also pointed out by the referee) the edge identifications given for the manifold M_5 do not quite agree with those for the corresponding one in Table 3 of the paper of Everitt [5] (our paper has $b(--)$ while Everitt's has $b(-+)$). But we have accurately checked the corresponding side pairing of the boundary faces of the octahedron, and can affirm that our sequence is right. This corrects a transcription error in the quoted paper.

Theorem 1.2. *(Everitt [5], Prok [17]) The closed orientable Euclidean* 3*-manifolds arising from Platonic solids as space forms are listed in Table* 2*.*

All these manifolds arise from the familiar cube with dihedral angle $2\pi/4$*.*

Remark 2. The manifold M_{12} is the 3-torus $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. The methods in [5] are not able to distinguish between the manifolds M_{10} and M_{14} (see [5, pp. 260/261]). Prok [17] constructed an affine conjugacy between M_{10} and M_{14}, so they are indeed the same manifold. We prove again this fact in a different way.

Theorem 1.3. *(Richardson and Rubinstein [18]) The closed orientable hyperbolic* 3*-manifolds arising from Platonic solids as space forms are listed in Table* 3*.*

The manifolds $M_{15}, \dots, M_{22}$ *come from the dodecahedron with dihedral angle* $2\pi/5$*, and* $M_{23}, \dots, M_{28}$ *from the icosahedron with angle* $2\pi/3$*.*

Remark 3. For manifolds in Theorem 1.3 with the same first homology, algebraic arguments are provided in [18] to show that they are all distinct. The manifold M_{15} is the Seifert–Weber 3-manifold [21].

Now we recall the Everitt notation for the tables. The columns F and E give the face and edge identifications in the form of an encoded string of letters and $\pm$ signs to be read in conjunction with Fig. 1. The ith and jth faces are paired when the ith and jth positions of the string in column F are occupied by the same letter. Similarly, for the edge identifications, where a string of $\pm$s after a letter indicates whether the corresponding edge is identified with subsequent ones with the orientations matching or reversed. For example, the Seifert–Weber manifold M_{15} arising from the dodecahedron with dihedral angle $2\pi/5$ has face identifications $abcdefefbcda$, where a indicates that faces 1, 12 are identified, b indicates that faces 2, 9 are identified, and so on. It has edge identifications

$$a(-+-+)b(-+-+)c(-+-+)d(-+-+)e(-+-+)$$

$$cdeabf(++++)afbfcfdfecdeabdeabc$$

where a indicates that edges 1, 9, 12, 24 and 28 are identified, and $a(-+-+)$ means edge 1 is identified with edge 9 so that the identifications are reversed, with edge 12 so they match, with edge 24 so they are reversed, and with edge 28 so they match. From the data in these two columns one can reconstruct the side pairing of the boundary faces of the Platonic solid. This completely defines the quotient manifold. From the polyhedral representation, one obtains immediately a finite presentation of the fundamental group and a Heegaard diagram of the quotient manifold.

The purpose of this paper is to investigate the topology of the above manifolds. We recall that a *Seifert manifold* Σ is uniquely characterized by a system of invariants $(\epsilon\ g\ \epsilon' : b\ (\alpha_1, \beta_1)\,(\alpha_2, \beta_2) \cdots (\alpha_r, \beta_r))$, where g is the genus of the base

Table 3

Hyperbolic manifolds	F	E	Homology
M_{15}	$abcdefefbcda$	$a(-+-+)b(-+-+)c(-+-+)d(-+-+)e(-+-+)$ $cdeabf(++++)afbfcfdfecdeabdeabc$	$\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5$
M_{16}	$abcdefdefbca$	$a(++++)b(++++)c(++++)d(++++)e(++++)$ $abcdebf(++++)cfdfefafcdeabbcdea$	$\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5$
M_{17}	$abcdefdefbca$	$a(+-++)b(-+++)c(---+)d(++-+)e(+-++)$ $debaf(+-++)bcfafefcdcfedabeabcd$	$\mathbb{Z}_3 \oplus \mathbb{Z}_3$
M_{18}	$abccadeefbfd$	$a(++--)ab(-+++)ac(-+-+)d(-+++)bab$ $e(+++-)ef(--+-)bfdcaecdfffddcbece$	$\mathbb{Z}_{35}$
M_{19}	$abcdefebfdca$	$a(-+-+)b(-+-+)c(-+-+)d(-+-+)e(-+-+)$ $edacbf(++++)cfefbfafdbdaeceabcd$	$\mathbb{Z}_5 \oplus \mathbb{Z}_{15}$
M_{20}	$abcdeffbdeca$	$a(++++)b(++++)c(++++)d(++++)e(++++)$ $adbcecf(++++)efdfbfafeacdbdeabc$	$\mathbb{Z}_{15} \oplus \mathbb{Z}_{15}$
M_{21}	$abcdebedffca$	$a(+-++)b(+-++)c(---+)d(-+-+)e(-+++)$ $cedaef(--+-)afdfbfcfebdcbacdeab$	$\mathbb{Z}_{48}$
M_{22}	$abbcadefecfd$	$a(+++-)b(++-+)c(--++)ad(-++-)a$ $e(+-+-)dbbeaecf(+--+)acfceffdedbdbfc$	$\mathbb{Z}_{29}$
M_{23}	$abcbdaefghihdefjgcji$	$a(-+)b(+-)c(--)d(--)e(-+)deabf(-+)$ $g(+-)h(-+)i(+-)iaccj(++)jhdebfgfghij$	$\mathbb{Z}_{11} \oplus \mathbb{Z}_{11}$
M_{24}	$abcdebfceghhiijjfgda$	$a(-+)b(-+)c(-+)d(--)e(++)cf(--)ea$ $g(--)ebh(+-)gi(+-)dj(+-)fghhdiifjjabc$	$\mathbb{Z}_9$
M_{25}	$abcdefbdgehiijjhfgca$	$a(++)b(++)c(++)d(++)e(+-)cdf(+-)ad$ $g(+-)bfh(-+)gi(+-)ej(-+)ijgjhehifabc$	$\mathbb{Z}_2 \oplus \mathbb{Z}_{18}$
M_{26}	$abcdaefdgfhihcjjbige$	$a(++)b(+-)bc(+-)d(--)e(+-)baf(--)$ $g(+-)efgh(++)ghci(+-)dj(-+)jjdeiicahf$	$\mathbb{Z}_{35}$
M_{27}	$abcdabefghcijidfjghe$	$a(++)ab(-+)c(++)d(++)e(--)bacf(+-)$ $g(+-)h(+-)ei(-+)j(++)djfidhgihebgjfc$	$\mathbb{Z}_{29}$
M_{28}	$abcdaebdfghicjehjfgi$	$a(++)b(+-)bc(--)d(-+)e(++)bacdef(+-)$ $g(--)h(+-)di(-+)aj(--)ijfehgcighjf$	$\mathbb{Z}_{29}$

orbifold S, $\epsilon = O$ and $\epsilon' = o$ if Σ and S are orientable, respectively, $b = -(e_0 + \sum_{i=1}^{r} \beta_i/\alpha_i) \in \mathbb{Q}$, where e_0 is the rational Euler number of the bundle, and (α_i, β_i) are the Seifert invariants of the ith exceptional fiber. For the theory of Seifert manifolds we refer to the monograph of P. Orlik [12]. Here we state our main results.

Theorem 1.4. *The spherical and Euclidean manifolds obtained from Platonic solids as space forms are homeomorphic to the fibered spaces described in Table* 4.

Theorem 1.5. *For the closed orientable hyperbolic* 3*-manifolds arising from Platonic solids as space forms, the following properties hold*:

(a) *The manifolds M_{15}* (*Seifert–Weber*) *and M_{16} coincide with the manifolds $M_{5,2}$ and $M_{5,1}$, respectively, constructed in* [6]. *They are* 5*-fold strongly cyclic coverings of the* 3*-sphere branched over the Whitehead link. These manifolds have the same homology but they are distinct*;
(b) *The manifold M_{20} is the Lorimer dodecahedral space with homology $\mathbb{Z}_{15} \oplus \mathbb{Z}_{15}$* [9];
(c) *The manifold M_{23} is the Fibonacci manifold M_5* (*of Heegaard genus* 2) *encoded by the standard presentation of the Fibonacci group $F(2,10)$ with generators $x_1, \dots, x_{10}$ and relations $x_i x_{i+1} = x_{i+2}$* (*subscripts* mod 10). *It is the* 5*-fold* (*respectively* 2*-fold*) *cyclic covering of the* 3*-sphere branched over the figure eight knot* (*respectively the knot* 10_{123}) (*see* [7,8]);
(d) *The manifolds M_{24} and M_{25} are* 3*-fold strongly cyclic coverings of the lens space $L(3,1)$ branched over two* (*non-equivalent*) 2*-component links L_{24} and L_{25}, respectively* (*see Fig.* 2);
(e) *The manifolds M_{26}, M_{27}, and M_{28} have Heegaard genus* 2, *and they are* 2*-fold coverings of the* 3*-sphere branched over the π-hyperbolic* 3*-bridge knots K_{26}, K_{27}, and K_{28}, respectively, depicted in Fig.* 3. *The knots K_{26} and K_{28} are chiral and invertible while K_{27} is chiral and non-invertible. The symmetry group of K_{27} and K_{28}* (*respectively K_{26}*) *is $\mathbb{Z}_2$* (*respectively D_2*). *The manifolds M_{27} and M_{28} have the same homology but they are distinct.*

Here we prove Theorem 1.4 while the proof of Theorem 1.5 will be given in the forthcoming paper [4].

2. Spines of closed manifolds

Let M be a closed connected orientable 3-manifold. A *spine* of M is a connected 2-polyhedron $X \subset M$ such that $M \setminus$ (open 3-cell) collapses onto X. It is known that every closed 3-manifold M has a spine with just one vertex. Such a spine corresponds in a natural way with a finite presentation of $\pi_1(M)$. A lot of results on group presentations corre-

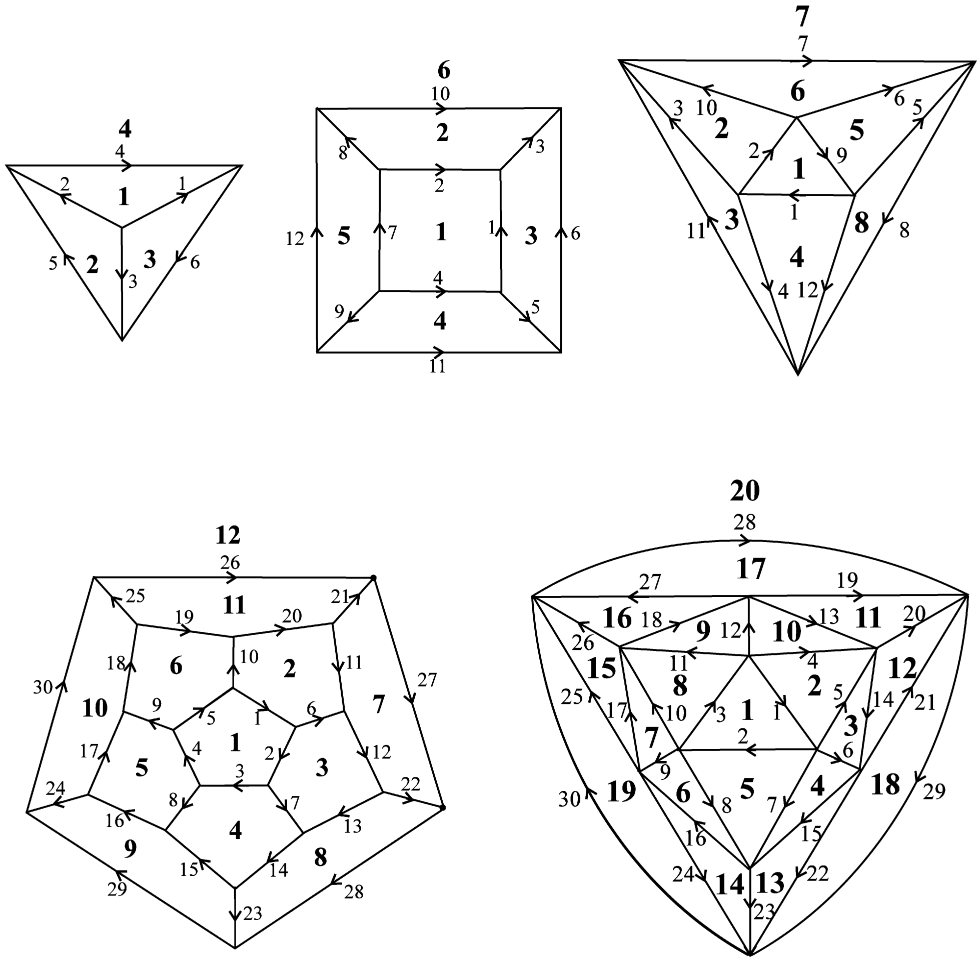

Fig. 1. Platonic solids labelled according to Everitt notation.

Table 4

Spherical manifolds
$M_1 \cong L(5,2)$
$M_2 \cong L(8,3)$
$M_3 \cong \mathbb{S}^3/_{\langle 222\rangle} = (O\ 0\ o\ :-1\ (2,1)(2,1)(2,1))$
$M_4 \cong \mathbb{S}^3/_{Q_8\times\mathbb{Z}_3} = (O\ 0\ o\ :0\ (2,1)(2,1)(2,1))$
$M_5 \cong \mathbb{S}^3/_{D_{24}} = (O\ 0\ o\ :-1\ (2,1)(2,1)(3,2))$
$M_6 \cong \mathbb{S}^3/_{\langle 332\rangle} = (O\ 0\ o\ :-1\ (3,1)(3,1)(2,1))$
$M_7 \cong \mathbb{S}^3/_{P_{120}} = (O\ 0\ o\ :-1\ (2,1)(3,1)(5,1))$
$M_8 \cong \mathbb{S}^3/_{P_{24}\times\mathbb{Z}_5} = (O\ 0\ o\ :-1\ (2,1)(3,2)(3,2))$

Euclidean manifolds
$M_9 \cong T\times I/_{\left(\begin{smallmatrix}0 & 1\\ -1 & -1\end{smallmatrix}\right)} = (O\ 0\ o\ :-1\ (3,1)(3,1)(3,1))$
$M_{10} \cong M_{14} \cong T\times I/_{\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)} = (O\ 0\ o\ :-2\ (2,1)(2,1)(2,1)(2,1))$
$M_{11} \cong (K \underset{\sim}{\times} I)\cup(K \underset{\sim}{\times} I)/_{\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)} = (O\ 1\ n\ :-1\ (2,1)(2,1))$
$M_{12} \cong T\times I/_{\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)} = \mathbb{S}^1\times\mathbb{S}^1\times\mathbb{S}^1$
$M_{13} \cong T\times I/_{\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)} = (O\ 0\ o\ :-1\ (4,1)(4,1)(2,1))$

sponding to spines of compact manifolds can be found in [13–16,20]. We recall two of them which serve in our proof of Theorem 1.4. Let us consider the group presentation for $\pi_1(M)$ given by

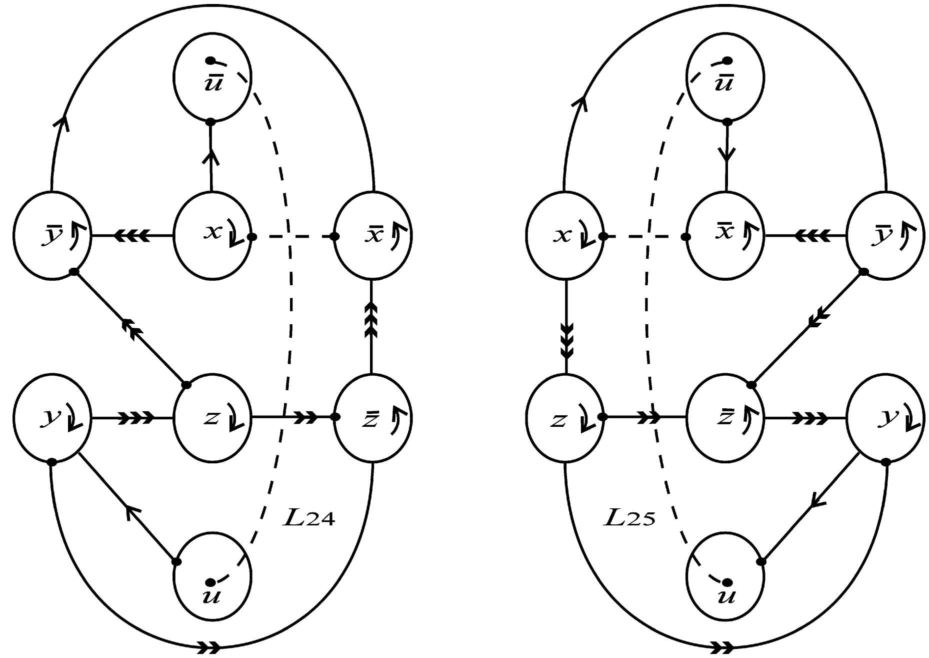

Fig. 2. Heegaard diagrams of the lens space $L(3,1)$; the components of the links L_{24} and L_{25} arise from the dotted edges.

$$\phi = \langle x, y\colon\ x^m y^n = 1,\ x^p y^q = 1\rangle$$

where $m, n, p, q \neq 0$, and $(m, p) = (n, q) = 1$. Following [20, p. 165], we associate the matrix $\left[\begin{smallmatrix} m & n \\ p & q \end{smallmatrix}\right]$ to ϕ. Certain matrix operations yield new matrices corresponding to spines of the same manifold. Among them, there are: (1) interchange of rows or columns, and (2) multiplication of a row or column by -1. These correspond, respectively, to (1) an interchange of relators or generators, and (2) replacement of a relator or generator by its inverse. In addition, we have: (3) subtraction of one row from the other under condition that both entries in at least one column have the same sign.

Theorem 2.1. *(Stevens [20, Theorem 11]) Let $\phi = \langle x, y\colon\ x^m y^n = 1,\ x^p y^q = 1\rangle$, where $m, n, p, q \neq 0$, $(m, p) = (n, q) = 1$, $|m|$ and $|p|$ are not both 1 and $|n|$ and $|q|$ are not both 1. Operate on the matrix $\left[\begin{smallmatrix} m & n \\ p & q \end{smallmatrix}\right]$ using rules (1), (2), and (3) to obtain a matrix of the form $\left[\begin{smallmatrix} 1 & n' \\ 1 & q' \end{smallmatrix}\right]$. Then ϕ corresponds to a spine of a unique closed orientable 3-manifold which is the lens space $L(\lambda, k)$, including $\mathbb{S}^1 \times \mathbb{S}^2$ (respectively $\mathbb{S}^3$) if $\lambda = 0$ (respectively $\lambda = 1$), where $\lambda = |mq - np| = |n' - q'|$, $0 \leqslant k < \lambda$, and $k \equiv n' \pmod{\lambda}$.*

The following result completes the statement of Theorem 3.1 in [16], and the subsequent note (see [16, p. 485]).

Theorem 2.2. *Let φ be the group presentation*

$$\langle x, y\colon\ x^p y^n = 1,\ \left(x^m y^{n+q}\right)^k x^m y^q = 1\rangle$$

where $|p|, |n| > 1$, $k > 0$, and $(p, m) = (n, q) = 1$. Then φ corresponds to a spine of a unique closed connected orientable 3-manifold which is the fiber space defined by the Seifert invariants

$$\left(O\ 0\ o : -1\ (p, m)\,(n, -q)\,(k+1, 1)\right).$$

Proof. We sketch a proof. By Theorem 3.1 of [16] φ corresponds to a spine of a Seifert fiber space M over $\mathbb{S}^2$ with three exceptional fibers. If $\pi_1(M)$ is finite, these fibers are of multiplicity $(p, 2, 2)$, $(3, 3, 2)$, $(4, 3, 2)$, or $(5, 3, 2)$. For these parameters $(p, n, k+1)$, one can draw extended Heegaard diagrams of genus 2 which induce the group presentation φ. All such diagrams are 2-symmetric, that is, they admit an involution with six fixed points. Then by [2,22,23] the represented manifolds are 2-fold branched coverings of the 3-sphere. The singular knots of these branched coverings are equivalent to the Montesinos knots $\mathbf{m}(-1; m/p; -q/n; 1/(k+1))$. So the result follows from [10]. Now we treat the case when M is a large manifold. Let Σ be the Seifert manifold defined by the invariants of the statement. By [12, p. 91] the fundamental group $\pi_1(\Sigma)$ has a finite presentation with generators q_1, q_2, q_3, h and relations $q_1^p h^m = 1$, $q_2^n h^{-q} = 1$, $q_3^{k+1} h = 1$, $[q_i, h] = 1$, $i = 1, 2, 3$, and $q_1 q_2 q_3 = h^{-1}$. Since $(p, m) = (n, q) = 1$, there exist integers a, b, c, d such that $ap + bm = 1$ and $cn + dq = 1$. Define $x := h^a q_1^{-b}$ and $y := h^{-c} q_2^{-d}$. We show that the generators q_i and h can be expressed in terms of x and y. In fact, we have:

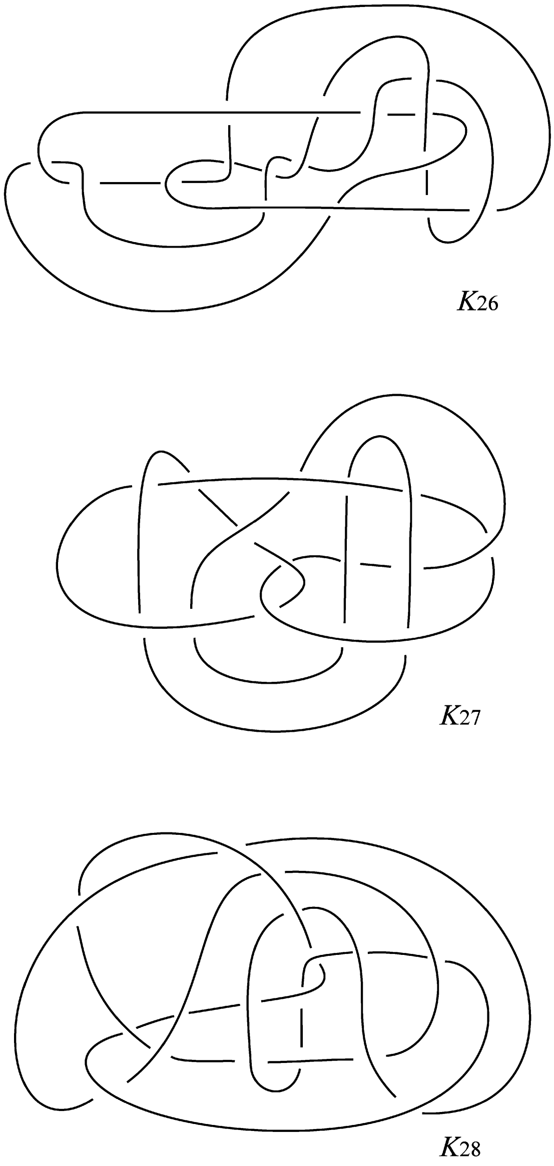

Fig. 3. The π-hyperbolic knots K_{26}, K_{27} and K_{28}.

$$x^p = \left(h^a q_1^{-b}\right)^p = h^{ap} q_1^{-bp} = h h^{-bm} q_1^{-bp} = h\left(q_1^p h^m\right)^{-b} = h,$$
$$x^m = \left(h^a q_1^{-b}\right)^m = h^{am} q_1^{-bm} = h^{am} q_1^{ap} q_1^{-1} = \left(q_1^p h^m\right)^a q_1^{-1} = q_1^{-1},$$
$$y^n = \left(h^{-c} q_2^{-d}\right)^n = h^{-cn} q_2^{-dn} = h^{-1} h^{dq} q_2^{-dn} = h^{-1}\left(q_2^n h^{-q}\right)^{-d} = h^{-1}$$

and

$$y^q = \left(h^{-c} q_2^{-d}\right)^q = h^{-cq} q_2^{-dq} = h^{-cq} q_2^{cn} q_2^{-1} = \left(q_2^n h^{-q}\right)^c q_2^{-1} = q_2^{-1}$$

hence

$$h = x^p = y^{-n}, \qquad q_1 = x^{-m}, \quad \text{and} \quad q_2 = y^{-q}.$$

In particular, we have obtained the relation $x^p y^n = 1$ of φ. From relations $q_3^{k+1} h = 1$ and $q_1 q_2 q_3 = h^{-1}$ of $\pi_1(\Sigma)$, we get

$$q_3^{k+1} h = \left(q_2^{-1} q_1^{-1} h^{-1}\right)^{k+1} h = \left(q_2^{-1} h^{-1} q_1^{-1}\right)^{k+1} h = \left(y^q y^n x^m\right)^{k+1} y^{-n} = y^{n+q}\left(x^m y^{n+q}\right)^k x^m y^{-n} = 1$$

which is the second relation of φ. Since $\pi_1(M) \cong \pi_1(\Sigma)$ and M, Σ are large Seifert manifolds (i.e., $k > 0$ and $1/p + 1/n + 1/(k+1) \leqslant 1$), the result follows from Theorem 4 [12, p. 134]. □

Let now M be a closed orientable 3-manifold obtained by a single polyhedral 3-ball B whose finitely many boundary faces are glued together in pairs. The interior of B becomes in M an open 3-cell whose boundary meets itself in M along an embedded spine of M. If such a spine has a unique vertex, then it is the canonical 2-complex corresponding to a geometric presentation (i.e., induced by a Heegaard diagram) of $\pi_1(M)$. On the other hand, the affine transformations, which identify the faces of B, generate $\pi_1(M)$. Relations between them arise from the cycles of equivalent edges in the above pairing. This group presentation is also geometric, and corresponds to a spine of M, which is dual to the previous one.

3. Spherical manifolds

(M_1) The manifold M_1 comes from the tetrahedron with dihedral angle $2\pi/3$. The face identifications imply that $(1,3)$ and $(2,4)$ are pairs of equivalent faces on the boundary of the tetrahedron. The edge identifications give two classes of equivalent edges, denoted by a ($1\equiv\bar{3}$ and $1\equiv\bar{4}$) and b ($2\equiv\bar{5}$ and $2\equiv\bar{6}$). Here $1\equiv\bar{3}$ means edge 1 is identified with edge 3 so that the identification is reversed. The quotient space has exactly one vertex, so $\pi_1(M_1)$ has a presentation with generators a and b and relations $ab^2=1$ and $a^2b^{-1}=1$. These relations can be read by walking around the oriented boundaries of the faces. The above presentation corresponds to a spine of M_1, and we can associate to it the matrix $\left[\begin{smallmatrix} m & n \\ p & q\end{smallmatrix}\right]=\left[\begin{smallmatrix} 1 & 2 \\ 2 & -1\end{smallmatrix}\right]$. By using rule (3) we get the matrix $\left[\begin{smallmatrix} 1 & n' \\ 1 & q'\end{smallmatrix}\right]=\left[\begin{smallmatrix} 1 & 2 \\ 1 & -3\end{smallmatrix}\right]$. Now Theorem 2.1 implies that M_1 is the lens space $L(\lambda,k)$, where $\lambda=|mq-np|=|n'-q'|=5$, and $0\leqslant k<5$, $k\equiv 2 \pmod 5$, i.e., $M_1\cong L(5,2)$.

(M_2) The manifold M_2 comes from the cube with dihedral angle $2\pi/3$. The face identifications imply that $(1,3)$, $(2,4)$ and $(5,6)$ are pairs of equivalent faces on the boundary of the cube. The edge identifications give four classes of equivalent edges, denoted by a ($1\equiv 3$ and $1\equiv 4$), b ($2\equiv 6$ and $2\equiv\bar{9}$), c ($5\equiv 7$ and $5\equiv\overline{10}$) and d ($8\equiv 11$ and $8\equiv\overline{12}$). Here $1\equiv 3$ (respectively $2\equiv\bar{9}$) means that edge 1 (respectively 2) is identified with edge 3 (respectively 9) so that the identification matches (respectively is reversed). The quotient space has exactly two vertices. Let x_1, x_2, and x_3 denote the isometries which identify the faces in pairs, i.e., $x_1: 1\to 3$, $x_2: 2\to 4$, and $x_3: 5\to 6$. The classes a, b, c, and d of equivalent edges give the relations $x_1^2x_2=1$, $x_1x_3^{-1}x_2^{-1}=1$, $x_1x_2^{-1}x_3^{-1}=1$, and $x_2x_3^{-2}=1$, respectively. Then $\pi_1(M_2)$ has the presentation

$$\left\langle x_1,x_2,x_3\colon\ x_1^2x_2=1,\ x_1x_3^{-1}x_2^{-1}=1,\ x_2x_3^{-2}=1\right\rangle,$$

which corresponds to a spine of M_2, or, equivalently, it is induced by a genus 3 Heegaard diagram of M_2. We see that the curve represented by the relator $x_1^2x_2$ has exactly one point in common with the curve represented by the generator x_2. Then the pair of such curves determines a reducible handle in the diagram, briefly called $(x_1^2x_2,x_2)$-*handle*. Cancelling it yields the presentation $\pi_1(M_2)\cong\langle x_1,x_3\colon\ x_1^3x_3^{-1}=1,\ x_1^2x_3^2=1\rangle$, which arises from a genus 2 Heegaard diagram of M_2. Operating on the associate matrix by rule (3) we get

$$\begin{bmatrix} m & n \\ p & q\end{bmatrix}=\begin{bmatrix} 3 & -1 \\ 2 & 2\end{bmatrix}\to\begin{bmatrix} 1 & -3 \\ 2 & 2\end{bmatrix}\to\begin{bmatrix} 1 & -3 \\ 1 & 5\end{bmatrix}=\begin{bmatrix} 1 & n' \\ 1 & q'\end{bmatrix}.$$

By Theorem 2.1 we see that M_2 is the lens space $L(\lambda,k)$, where $\lambda=|mq-np|=|n'-q'|=8$, and $0\leqslant k<8$, $k\equiv -3 \pmod 8$, i.e., $M_2\cong L(8,5)\cong L(8,3)$.

(M_3) The manifold M_3 comes from the cube with dihedral angle $2\pi/3$. The face identifications imply that $(1,6)$, $(2,4)$, and $(3,5)$ are pairs of equivalent faces on the boundary of the cube, and let x_1, x_2, and x_3 denote the isometries which identify them in that order. The edge identifications give four classes of equivalent edges, denoted by a ($1\equiv 8$ and $1\equiv 11$), b ($2\equiv\bar{6}$ and $2\equiv\bar{9}$), c ($3\equiv 4$ and $3\equiv\overline{12}$), and d ($5\equiv\bar{7}$ and $5\equiv\overline{10}$). The quotient space has exactly two vertices. The classes a, b, c, and d give the relations $x_1x_2^{-1}x_3^{-1}=1$, $x_2x_3^{-1}x_1^{-1}=1$, $x_3x_1^{-1}x_2^{-1}=1$, and $x_1x_2x_3=1$, respectively. Then $\pi_1(M_3)$ has the presentation

$$\langle x_1,x_2,x_3\colon\ x_1x_3=x_2,\ x_2x_1=x_3,\ x_1x_2x_3=1\rangle,$$

which arises from a genus 3 Heegaard diagram of M_3. Cancelling the reducible $(x_1x_2x_3,x_3)$-handle yields the presentation $\pi_1(M_3)\cong\langle x_1,x_2\colon\ x_2^2x_1^2=1,\ (x_2x_1)x_2x_1^{-1}=1\rangle$, which is induced by a genus 2 Heegaard diagram of M_3, and hence it corresponds to a spine of M_3. Now we apply Theorem 2.2 with $p=n=2$, $m=1$, $k=1$, and $q=-1$. Then M_3 is the Seifert manifold defined by the invariants $(O\ 0\ o\ :\ -1\ (2,1)\,(2,1)\,(2,1))$, that is, the well-known quaternionic space $\mathbb{S}^3/_{\langle 222\rangle}$ described in [11, p. 117].

(M_4) The manifold M_4 comes from the octahedron with angle $2\pi/3$. The face identifications imply that $(1,4)$, $(2,6)$, $(3,5)$ and $(7,8)$ are pairs of equivalent faces on the boundary of the octahedron. The edge identifications give four classes of equivalent edges, denoted by a ($1\equiv 4$ and $1\equiv 9$), b ($2\equiv 7$ and $2\equiv\overline{12}$), c ($3\equiv 6$ and $3\equiv\overline{10}$) and d ($5\equiv 8$ and $5\equiv 11$). The quotient space has exactly one vertex. Then $\pi_1(M_4)$ has the presentation

$$\left\langle a,b,c,d\colon\ ba^2=1,\ c^2b^{-1}=1,\ adc^{-1}=1,\ bd^2=1\right\rangle,$$

where the relations are read by walking around the oriented boundaries of the faces. This presentation corresponds to a spine of M_4, and arises from a genus 4 Heegaard diagram of M_4. Cancelling the reducible (adc^{-1},c)-handle we get the presentation $\pi_1(M_4)\cong\langle a,b,d\colon\ ba^2=1,\ (ad)^2b^{-1}=1,\ bd^2=1\rangle$, which comes from a genus 3 Heegaard diagram of M_4. Cancelling the reducible (bd^2,b)-handle yields the presentation $\pi_1(M_4)\cong\langle a,d\colon\ a^2d^{-2}=1,\ (ad)ad^3=1\rangle$, which is induced by a genus 2 Heegaard diagram of M_4, and hence it corresponds to a spine of M_4. Now we apply Theorem 2.2 with $p = n = 2$, $m = k = 1$, and $q = -3$. Then M_4 is the Seifert manifold $(O\ 0\ o\ :\ -1\ (2,1)\,(2,3)\,(2,1))$, which is homeomorphic to $(O\ 0\ o\ :\ 0\ (2,1)\,(2,1)\,(2,1))$ by the observation of [11, p. 146].

(M_5) The manifold M_5 comes from the octahedron with angle $2\pi/3$. The face identifications imply that $(1,4)$, $(2,7)$, $(3,5)$ and $(6,8)$ are pairs of equivalent faces on the boundary of the octahedron. The edge identifications give four classes of equivalent edges, denoted by a ($1 \equiv 4$ and $1 \equiv 9$), b ($2 \equiv \overline{7}$ and $2 \equiv \overline{12}$), c ($3 \equiv 6$ and $3 \equiv 8$) and d ($5 \equiv \overline{10}$ and $5 \equiv 11$). The quotient space has exactly one vertex. Then $\pi_1(M_5)$ has the presentation

$$\langle a, b, c, d\colon\ a^2b^{-1} = 1,\ cdb = 1,\ adc^{-1} = 1,\ dcb^{-1} = 1\rangle,$$

which arises from a genus 4 Heegaard diagram of M_5. Cancelling successively the reducible handles given by the relator-generator pairs (a^2b^{-1}, b) and (adc^{-1}, c), we get the presentation $\pi_1(M_5) \cong \langle a, d\colon\ d^2a^3 = 1,\ (da)da^{-2} = 1\rangle$. This presentation arises from a genus 2 Heegaard diagram of M_5, hence it corresponds to a spine of M_5. Now we apply Theorem 2.2 with $p = 2$, $n = 3$, $k = m = 1$, and $q = -2$. Then M_5 is the Seifert manifold $(O\ 0\ o\ :\ -1\ (2,1)\,(3,2)\,(2,1))$.

(M_6) It is well known that M_6 is the octahedral space $\mathbb{S}^3/_{\langle 332\rangle}$ described in [11, p. 117]. However, we re-prove this fact by using our methods. The face identifications imply that $(1,7)$, $(2,8)$, $(3,5)$ and $(4,6)$ are pairs of equivalent faces on the boundary of the octahedron. The edge identifications give four classes of equivalent edges, denoted by a ($1 \equiv 8$ and $1 \equiv 10$), b ($2 \equiv 5$ and $2 \equiv 11$), c ($3 \equiv 6$ and $3 \equiv 12$), and d ($4 \equiv 7$ and $4 \equiv 9$). The quotient space has exactly one vertex. Then $\pi_1(M_6)$ has the presentation

$$\langle a, b, c, d\colon\ abd = 1,\ bac^{-1} = 1,\ dbc^{-1} = 1,\ adc^{-1} = 1\rangle,$$

which arises from a genus 4 Heegaard diagram of M_6. Cancelling successively the reducible handles given by the relator-generator pairs (abd, d) and (bac^{-1}, c), we get the presentation $\pi_1(M_6) \cong \langle a, b\colon\ b^3a^3 = 1,\ (ba^2)ba^{-1} = 1\rangle$. This presentation comes from a genus 2 Heegaard diagram of M_6, hence it corresponds to a spine of M_6. Now we apply Theorem 2.2 with $p = n = 3$, $k = m = 1$, and $q = -1$. Then M_6 is the Seifert manifold $(O\ 0\ o\ :\ -1\ (3,1)\,(3,1)\,(2,1))$.

(M_7) It is well known that M_7 is the Poincaré homology 3-sphere, and the polyhedral representation as dodecahedral space form is precisely the classical one described in [21] (see also [19, p. 223]). But we re-prove this fact by using our methods. The face identifications imply that $(1,12)$, $(2,9)$, $(3,10)$, $(4,11)$, $(5,7)$ and $(6,8)$ are pairs of equivalent faces on the boundary of the dodecahedron. Let $b, a_1, \dots, a_5$ denote the isometries which identify the faces in pairs according to the order above. The edge identifications give ten classes of equivalent edges, denoted by a ($1 \equiv \overline{16}$ and $1 \equiv 27$), b ($2 \equiv \overline{18}$ and $2 \equiv 28$), c ($3 \equiv \overline{20}$ and $3 \equiv 29$), d ($4 \equiv \overline{12}$ and $4 \equiv 30$), e ($5 \equiv \overline{14}$ and $5 \equiv 26$), f ($6 \equiv \overline{15}$ and $6 \equiv 25$), g ($7 \equiv \overline{17}$ and $7 \equiv 21$), h ($8 \equiv \overline{19}$ and $8 \equiv 22$), i ($9 \equiv \overline{11}$ and $9 \equiv 23$), and j ($10 \equiv \overline{13}$ and $10 \equiv 24$). The quotient space has exactly five vertices. Then $\pi_1(M_7)$ has the presentation

$$\langle b, a_i\ (i = 1, \dots, 5)\colon\ a_ia_{i+3} = b,\ a_ia_{i+2} = a_{i+1}\ (i = 1, \dots, 5)\rangle,$$

where the subscripts are taken mod 5. This presentation corresponds to a spine of M_7. Since $b = a_2a_1a_2^{-1}$, $a_3 = a_1^{-1}a_2$, $a_4 = a_2^{-1}a_1^{-1}a_2$ and $a_5 = a_1a_2^{-1}$, we get the presentation

$$\pi_1(M_7) \cong \langle a_1, a_2\colon\ a_1a_2^2a_1a_2^{-3} = 1,\ a_2a_1a_2a_1^{-1}a_2^{-1}a_1^{-1} = 1\rangle$$

which is induced by the genus 2 Heegaard diagram of M_7 depicted in Fig. 4(a). Setting $u = a_1a_2^2$ and $v = a_2^{-1}$, with inverse relation $a_1 = uv^2$, we get the presentation

$$\pi_1(M_7) \cong \langle u, v\colon\ v^5u^2 = 1,\ (vu)^2vu^{-1} = 1\rangle,$$

which arises from the genus 2 Heegaard diagram of M_7 drawn in Fig. 4(b). It is an exercise to verify that Heegaard diagrams in Fig. 4 are equivalent, i.e., one can be transformed into the other by Singer moves. Now we apply Theorem 2.2 with $p = 5$, $k = n = 2$, $m = 1$, and $q = -1$. Then M_7 is the Seifert manifold $(O\ 0\ o\ :\ -1\ (5,1)\,(2,1)\,(3,1))$.

(M_8) The manifold M_8 is the Lorimer (spherical) dodecahedral space (see [9]). The face identifications imply that $(1,12)$, $(2,7)$, $(3,9)$, $(4,8)$, $(5,11)$ and $(6,10)$ are pairs of equivalent faces on the boundary of the dodecahedron. Let $b, a_1, \dots, a_5$ denote the isometries which identify the faces in pairs according to the above order. The edge identifications give ten classes of equivalent edges, denoted by a ($1 \equiv \overline{11}$ and $1 \equiv 27$), b ($2 \equiv \overline{15}$ and $2 \equiv 28$), c ($3 \equiv \overline{13}$ and $3 \equiv 29$), d ($4 \equiv \overline{19}$ and $4 \equiv 30$), e ($5 \equiv \overline{17}$ and $5 \equiv 26$), f ($6 \equiv 16$ and $6 \equiv 21$), g ($7 \equiv 14$ and $7 \equiv 23$), h ($8 \equiv 20$ and $8 \equiv 22$), i ($9 \equiv 18$ and $9 \equiv 25$), and j ($10 \equiv 12$ and $10 \equiv 24$). The quotient space has exactly five vertices. Then $\pi_1(M_8)$ has a presentation with generators b, $a_1, \dots, a_5$ and relations $a_1^2 = b$, $a_2a_3 = b$, $a_3a_2 = b$, $a_4a_5 = b$, $a_5a_4 = b$, $a_1 = a_2a_4$, $a_2 = a_3^2$, $a_3 = a_4a_1$, $a_4 = a_5^2$ and $a_5 = a_1a_2$. Eliminating the generators $a_1 (= a_3^2a_5^2)$, $a_2 (= a_3^2)$, $a_4 (= a_5^2)$ and $b (= a_1^2 = (a_3^2a_5^2)^2)$ we get the presentation

$$\pi_1(M_8) \cong \langle a_3, a_5\colon\ a_5^3a_3^3 = 1,\ (a_5^2a_3)a_5^2a_3^{-2} = 1\rangle,$$

which arises from a genus 2 Heegaard diagram of M_8, and hence corresponds to a spine of M_8. Now we apply Theorem 2.2 with $p = n = 3$, $m = 2$, $k = 1$, and $q = -2$. Then M_8 is the Seifert manifold $(O\ 0\ o\ :\ -1\ (3,2)\,(3,2)\,(2,1))$.

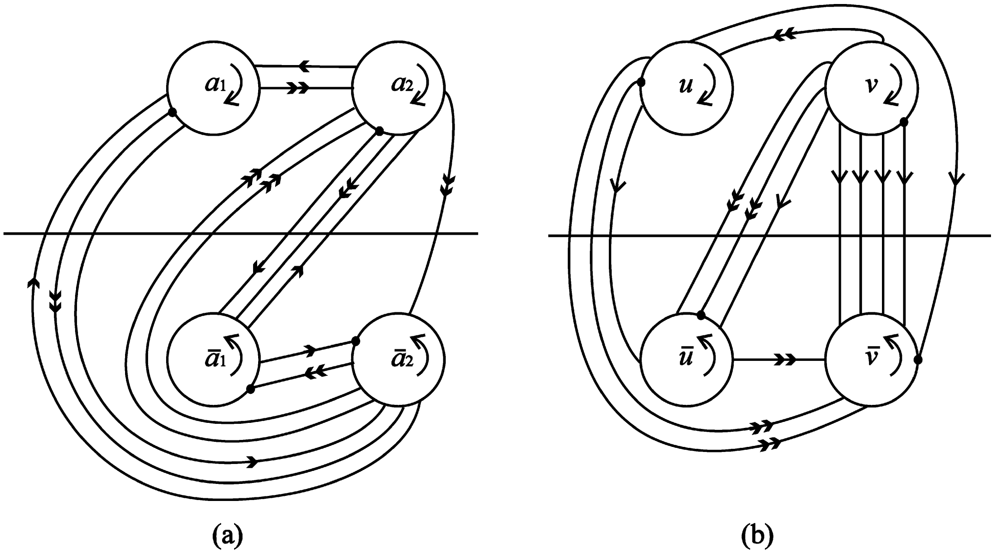

Fig. 4. Heegaard diagrams of the Poincaré homology sphere.

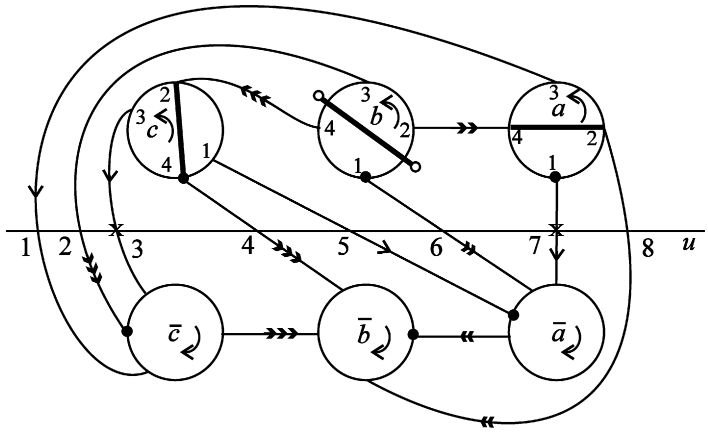

Fig. 5. An extended Heegaard diagram of genus 3 for the manifold M_{10}.

4. Euclidean manifolds

(M_9) The face identifications imply that $(1,3)$, $(2,5)$ and $(4,6)$ are pairs of equivalent faces on the boundary of the cube. The edge identifications give three classes of equivalent edges, denoted by a ($1 \equiv 3$, $1 \equiv 4$ and $1 \equiv 12$), b ($2 \equiv 6$, $2 \equiv 9$ and $2 \equiv 11$) and c ($5 \equiv 7$, $5 \equiv 8$ and $5 \equiv 10$). The quotient space has exactly one vertex, so $\pi_1(M_9)$ has a presentation with generators a, b and c and relations $a^2b^{-1}c^{-1} = 1$, $c^2a^{-1}b^{-1} = 1$ and $b^2c^{-1}a^{-1} = 1$. This presentation corresponds to a spine of M_9, and arises from a genus 3 Heegaard diagram of M_9. Cancelling the reducible handle given by the relator-generator pair $(c^2a^{-1}b^{-1}, b)$, we get the presentation $\pi_1(M_9) \cong \langle a, c\colon\ a^3c^{-3} = 1,\ (ac^{-2})^2ac = 1\rangle$, which is induced by a genus 2 Heegaard diagram of M_9. Now we apply Theorem 2.2 with $p = n = 3$, $m = 1$, $k = 2$, and $q = -1$. Then M_9 is the Seifert manifold $(O\ 0\ o\ :\ -1\ (3,1)\,(3,1)\,(3,1))$.

(M_{10}) The face identifications imply that $(1,6)$, $(2,3)$ and $(4,5)$ are pairs of equivalent faces on the boundary of the cube. The edge identifications give three classes of equivalent edges, denoted by a ($1 \equiv \bar{2}$, $1 \equiv 6$ and $1 \equiv \overline{10}$), b ($3 \equiv \bar{5}$, $3 \equiv \bar{8}$ and $3 \equiv 9$) and c ($4 \equiv \bar{7}$, $4 \equiv 11$ and $4 \equiv \overline{12}$). The quotient space has exactly one vertex, so $\pi_1(M_{10})$ has a presentation with generators a, b and c and relations $a^2c^2 = 1$, $baba^{-1} = 1$ and $bcbc^{-1} = 1$. This presentation comes from the genus 3 Heegaard diagram of M_{10} shown in Fig. 5. Since $H_1(M_{10}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}$, the Heegaard genus of M_{10} is 3. The diagram admits an orientation preserving involution given by the symmetries with respect to the marked axes in the circles a, b, c, and by the symmetry with respect to the axis of ends 3 and 7 in the circle u (see Fig. 5). By a construction described in [2,22,23] the manifold M_{10} is the 2-fold covering of the 3-sphere branched over the 4-component link depicted in Fig. 6(a). By Reidemeister moves we show that such a link is equivalent to the Montesinos link $\mathbf{m}(1/2, 1/2, -1/2, -1/2)$.

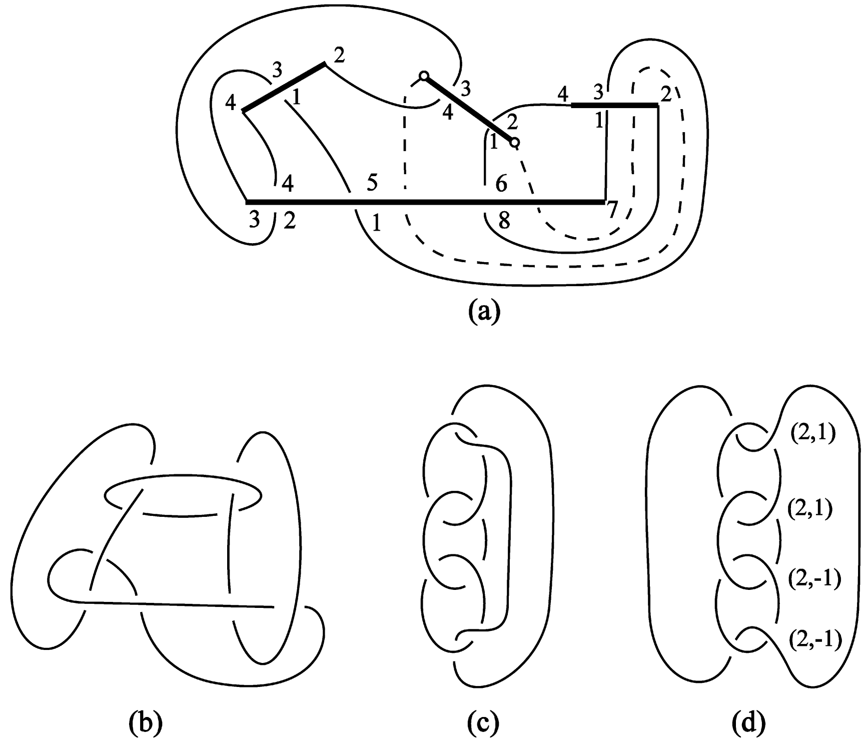

Fig. 6. The Montesinos link $\mathbf{m}(1/2, 1/2, -1/2, -1/2)$.

By Theorem 3.7 of [10] the manifold M_{10} is homeomorphic to the Seifert space $(\mathbb{S}^2\ :\ (2,1)\,(2,1)\,(2,-1)\,(2,-1))$, which is also defined by the Seifert invariants $(O\ 0\ o\ :\ -2\ (2,1)\,(2,1)\,(2,1)\,(2,1))$.

(M_{11}) The face identifications imply that $(1,6)$, $(2,5)$ and $(3,4)$ are pairs of equivalent faces on the boundary of the cube. The edge identifications give three classes of equivalent edges, denoted by a ($1 \equiv \bar{2}$, $1 \equiv 11$ and $1 \equiv \overline{12}$), b ($3 \equiv \bar{5}$, $3 \equiv \bar{8}$ and $3 \equiv 9$) and c ($4 \equiv 6$, $4 \equiv \bar{7}$ and $4 \equiv \overline{10}$). The quotient space has exactly one vertex, so $\pi_1(M_{11})$ has a presentation with generators a, b and c and relations $a^2c^2 = 1$, $bcba^{-1} = 1$ and $babc^{-1} = 1$. This presentation is induced by a genus 3 Heegaard diagram of M_{11}. Cancelling the reducible handle given by the relator-generator pair $(babc^{-1}, c)$, we get the presentation $\pi_1(M_{11}) \cong \langle a, b\colon\ b^2ab^2 = a,\ a^2(bab)^2 = 1\rangle$, which arises from a genus 2 Heegaard diagram of M_{11}. Substituting $b^2a = ab^{-2}$ in the second relation, we obtain

$$a^2(bab)^2 = a^2bab^2ab = a^2ba^2b^{-1} = 1,$$

hence $a^2ba^2 = b$. Thus $\pi_1(M_{11}) \cong \langle a, b\colon\ a^2 = (ab^2)^2,\ a^2 = ba^{-2}b^{-1}\rangle$. Since $H_1(M_{11}) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_4$, the Heegaard genus of M_{11} is 2. Let Σ be the Euclidean Seifert manifold defined by the invariants $(O\ 1\ n\ :\ -1\ (2,1)\,(2,1))$. We recall by [12, p. 91] that $\pi_1(\Sigma) \cong \langle q_0, q_1, q_2, A, h\colon\ AhA^{-1} = h^{-1},\ q_0q_1q_2 = A^2,\ [q_j, h] = 1,\ j = 0,1,2,\ q_0 = h,\ q_1^2h = 1,\ q_2^2h = 1\rangle$. Eliminating the generators $h = q_0\,(= q_1^{-2})$ and $q_2\,(= q_1A^2)$ we get the presentation $\pi_1(\Sigma) \cong \langle q_1, A\colon\ q_1^2 = (q_1A^2)^2,\ Aq_1^{-2}A^{-1} = q_1^2\rangle$. Setting $a := q_1$ and $b := A$ we see that the group $\pi_1(M_{11})$ and $\pi_1(\Sigma)$ are isomorphic. In particular, the element $a^{-2}\,(= q_1^{-2} = h)$ generates an infinite cyclic group $\langle a^2\rangle$, which is the nontrivial center of $\pi_1(M_{11})$. Suppose M_{11} is not prime. Since its genus is 2, M_{11} can be decomposed in a connected sum $M_{11} = L_1 \# L_2$, where L_i is a nontrivial lens space, and $\pi_1(M_{11}) \cong \pi_1(L_1) * \pi_1(L_2)$ with $\pi_1(L_i) \neq 1$. But a free product of nontrivial groups admits only a trivial center and here $\pi(M_{11})$ has a nontrivial one (generated by a^2). This implies that M_{11} is prime, and hence it is irreducible since $M_{11} \ncong \mathbb{S}^1 \times \mathbb{S}^2$ (in fact, $\pi_1(M_{11}) \ncong \mathbb{Z}$ as $H_1(M_{11}) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_4$). By [3] M_{11} is a Seifert fibered manifold. Since M_{11} and Σ are Euclidean manifolds (hence $\pi(M_{11}) \cong \pi_1(\Sigma)$ is infinite), M_{11} and Σ are large Seifert manifolds. By [12, p. 134], M_{11} is homeomorphic to Σ.

(M_{12}) The manifold M_{12} is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ since the polyhedral representation is precisely the standard one of the 3-torus.

(M_{13}) The face identifications imply that $(1,6)$, $(2,4)$ and $(3,5)$ are pairs of equivalent faces on the boundary of the cube. The edge identifications give three classes of equivalent edges, denoted by a ($1 \equiv 6$, $1 \equiv 7$ and $1 \equiv 12$), b ($2 \equiv 5$, $2 \equiv 9$ and $2 \equiv 10$) and c ($3 \equiv \bar{4}$, $3 \equiv 8$ and $3 \equiv \overline{11}$). The quotient space has exactly one vertex, so $\pi_1(M_{13})$ has a presentation with generators a, b and c and relations $c^{-1}ab^{-1}a^{-1} = 1$, $bcb^{-1}c^{-1} = 1$ and $bac^{-1}a^{-1} = 1$. This presentation comes from a genus 3 Heegaard diagram of M_{13}. Cancelling the reducible handle given by the relator-generator pair $(c^{-1}ab^{-1}a^{-1}, c)$, we get the presentation $\pi_1(M_{13}) \cong \langle a, b\colon\ a^2 = ba^2b,\ bab^{-1}a^{-1}b^{-1}aba^{-1} = 1\rangle$, which is induced by a genus 2 Heegaard diagram of M_{13}. Setting $u = a$ and $v = b^{-1}a^{-2}$, with inverse relation $b = u^{-2}v^{-1}$, the first relation becomes $u^4v^2 = 1$, and the second relation becomes $(uv)^3uv^{-1} = 1$. Hence we have $\pi_1(M_{13}) \cong \langle u, v\colon\ u^4v^2 = 1,\ (uv)^3uv^{-1} = 1\rangle$, which arises from a genus 2 Heegaard diagram of M_{13} (one can directly verify that this diagram is equivalent to the previous one via Singer moves). Now we apply Theorem 2.2 with $p = 4$, $n = 2$, $m = 1$, $k = 3$, and $q = -1$. Then M_{13} is the Seifert manifold $(O\ 0\ o\ :\ -1\ (4,1)\,(2,1)\,(4,1))$.

(M_{14}) The manifold M_{14} in Table 2 coincides with M_{10}, as proved in [17]. We re-obtain this result by using our methods. The face identifications imply that $(1,6)$, $(2,4)$ and $(3,5)$ are pairs of equivalent faces on the boundary of the cube. The edge identifications give three classes of equivalent edges, denoted by a $(1 \equiv \bar{6},\ 1 \equiv 7$ and $1 \equiv \overline{12})$, b $(2 \equiv 4,\ 2 \equiv \overline{10}$ and $2 \equiv \overline{11})$ and c $(3 \equiv 5,\ 3 \equiv 8$ and $3 \equiv 9)$. The quotient space has exactly one vertex, so $\pi_1(M_{14})$ has a presentation with generators a, b and c and relations $aba^{-1}b^{-1} = 1$, $bcbc^{-1} = 1$ and $acac^{-1} = 1$. Setting $u = ac$, $v = c^{-1}$ and $w = b$, with inverse relation $a = uv$, the third relation becomes $u^2v^2 = 1$, the second relation becomes $wvwv^{-1} = 1$, and the first relation becomes $uvwv^{-1}u^{-1}w^{-1}$, or, equivalently, $wuwu^{-1} = 1$ as $vw = w^{-1}v$. Then we have $\pi_1(M_{14}) \cong \langle u, v, w\colon\ u^2v^2 = 1,\ wuwu^{-1} = 1,\ wvwv^{-1} = 1\rangle$, hence $\pi_1(M_{14})$ is isomorphic to $\pi_1(M_{10})$. Since $H_1(M_{14}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}$, the Heegaard genus of M_{14} is 3. Reasoning as in the proof of M_{11} we see that M_{14} is irreducible. Since $\pi_1(M_{14}) \cong \pi_1(M_{10})$ has nontrivial center, M_{14} is a Seifert fibered manifold. Since M_{10} and M_{14} are Euclidean fibered manifolds with infinite fundamental groups, they are large Seifert manifolds. By [12, Theorem 8, p. 134], M_{14} is homeomorphic to M_{10}. By [12, Theorem 7, p. 133], there is also a Seifert bundle isomorphism between such manifolds. Finally, they are isometric.

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