DOA and polarization estimation via signal reconstruction with linear polarization-sensitive arrays

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Abstract This paper addresses the problem of direction-of-arrival (DOA) and polarization estimation with polarization sensitive arrays (PSA), which has been a hot topic in the area of array signal processing during the past two or three decades. The sparse Bayesian learning (SBL) technique is introduced to exploit the sparsity of the incident signals in space to solve this problem and a new method is proposed by reconstructing the signals from the array outputs first and then exploiting the reconstructed signals to realize parameter estimation. Only 1-D searching and numerical calculations are contained in the proposed method, which makes the proposed method computationally much efficient. Based on a linear array consisting of identically structured sensors, the proposed method can be used with slight modifications in PSA with different polarization structures. It also performs well in the presence of coherent signals or signals with different degrees of polarization. Simulation results are given to demonstrate the parameter estimation precision of the proposed method.

1. Introduction

The polarization sensitive arrays (PSA) are able to collect the signal energy in different polarization directions and they have been widely used to improve the performance of direction-of-arrival (DOA) estimation (see Refs.1–6 and the references therein). Many PSA with different polarization structures are now available in practical systems, such as the co-centered crossed-dipole pair (CCD) array, the co-centered orthogonal loop and dipole (COLD) array and the electromagnetic vector array (EVA) of super-resolution compact array radiolocation technology (SuperCART). However, the research on DOA estimation with PSA has lagged behind as many shortcomings still remain in the existing methods.

When 2-D DOA estimation and joint polarization estimation are required, the existing methods generally turn to the multi-dimensional searching techniques as the parameters can hardly be separated in the objective functions.7–9 Such a computationally prohibitive searching procedure greatly blocks the theoretical study and practical application of those
methods. Therefore, some of the research focuses on the simpler 1-D estimation problems.\textsuperscript{6,11} In addition, as more than one parameter is usually concerned for each signal with the PSA, a possibly confusion-inducing procedure of variable-pairing is required for successful parameter estimation of multiple signals.\textsuperscript{7,10} Spatial and polarization smoothing techniques have also been combined with the ordinary subspace-based DOA estimation methods, so as to separate coherent signals.\textsuperscript{11,12} However, these techniques can be realized only with particularly designed PSA, thus they have been blocked from widespread applications. Similar constraints lie in the method proposed for DOA estimation when completely and incompletely polarized signals coexist,\textsuperscript{13} as it can be used only on PSA with particular triangular geometries.

The sparsity of the incident signals in space is a comprehensive property in various array applications. Previous research based on the exploitation of such a property in scalar sensor arrays has witnessed significant performance improvements, especially in scenarios with low signal-to-noise ratio (SNR) and much limited snapshots.\textsuperscript{14–19} Among the existing sparsity-based DOA estimation methods, the ones based on the sparse Bayesian learning (SBL) technique\textsuperscript{20,21} exceed their counterparts in DOA estimation precision in adequate scenarios.\textsuperscript{17–19}

In this paper, the SBL technique is introduced to solve the direction and polarization estimation problem with PSA. By exploiting the sparsity of the incident signals, the signal components contained in the differently polarized array measurements are reconstructed first, and then combined with respect to the sources to estimate the parameters of interest. In order for notational facilitation, the proposed method is named reconstruction and combination of polarized signal components and ReCoP for short. It avoids the computationally prohibitive multi-dimensional searching procedure and the confusion-inducing variable-pairing procedure. With very slight modifications, it can be used in various linear PSA consisting of identically structured sensors, such as CCD arrays, COLD arrays and SuperCART, and it is also able to process coherent signals and signals with different degrees of polarization.

The rest of the paper consists of six parts. Section 2 reviews the observation model of the PSA, the SBL technique is introduced in Section 3 to reconstruct the polarized signal components, and those reconstructed signals are combined in Section 4 to estimate the direction and polarization parameters. Based on the differences during method implementation between the proposed method and its counterparts, Section 5 highlights some special properties of the proposed method. Section 6 demonstrates the performance of the proposed method via simulations and Section 7 concludes the whole paper.

2. Model formulation

Suppose that $K$ transverse electromagnetic waves impinge onto an $M$-element PSA, the azimuths of the signals are $\theta_1, \theta_2, \ldots, \theta_K \in [0, \pi]$, which are defined as the projections of the incident signal directions on the $x$-$y$ plane to the $x$ axis, and their elevations, defined as the signal directions to the $z$ axis, are $\varphi_1, \varphi_2, \ldots, \varphi_K \in [0, \pi/2]$. The sketch of a linear COLD array is shown in Fig. 1, where $s(t)$ indicates the incident signal. When only 1-D DOA estimation is concerned, one should set $\varphi_1 = \varphi_2 = \cdots = \varphi_K = \pi/2$.

The output of the PSA at time $t$ is

$$\mathbf{x}(t) = \mathbf{A}\hat{\mathbf{u}}(t) + \mathbf{v}(t)$$

(1)

where $\mathbf{A} = [\mathbf{a}_{1,h}, \mathbf{a}_{1,v} \hat{\mathbf{a}}_{2,h}, \mathbf{a}_{2,v}, \ldots, \hat{\mathbf{a}}_{K,h}, \mathbf{a}_{K,v}], \mathbf{a}_{h,v} = \mathbf{a}_h \otimes \mathbf{a}_v, \hat{\mathbf{a}}_{h,v} \otimes \mathbf{r}_k$, $\mathbf{r}_k$ is the Kronecker product, the subscripts $(\bullet)_h$ and $(\bullet)_v$ are used to indicate the horizontal and vertical components, respectively. $\mathbf{a}_h = [\cos \theta_h \sin \varphi_h, \sin \theta_h \sin \varphi_h, -\cos \theta_h, \sin \theta_h, 0]^T$, $\mathbf{r}_k = \mathbf{v}_k \mathbf{r}_k$. For COLD arrays $\mathbf{u}(t) = [u_1^T(t), u_2^T(t), \ldots, u_K^T(t)]^T$, $\mathbf{v}(t) = \mathbf{v}_k$ $\mathbf{r}_k$ and $\mathbf{s}(t)$ represent the polarized components of the $k$th signal in two orthogonal polarization directions; $\mathbf{v}(t)$ is the additive white Gaussian noise independent of the signals with variance $\sigma^2$. For completely polarized signals, $\mathbf{s}(t)$ and $\mathbf{v}(t)$ are linearly dependent as $\mathbf{u}(t) = [\cos \varphi_h \sin \theta_h, \sin \varphi_h \sin \theta_h, \cdots, \cos \varphi_K \sin \theta_K, \sin \varphi_K \sin \theta_K, 0]^T$, $\mathbf{Z}$ indicates the polarization dimensions of the array selects from the EVA, e.g., $\mathbf{Z} = [1, 0, 0, 0, 0, 0, 0, 0, 1, 0]$. For COLD arrays $\mathbf{A} = [\mathbf{a}_1^T, \mathbf{a}_2^T, \ldots, \mathbf{a}_K^T]^T$.

By separating the outputs of the array elements with identical polarization directions from $\mathbf{x}(t)$, $P$ measurement vectors of $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_P(t) \in \mathbb{C}^{M \times 1}$ can be obtained as follows:

$$\mathbf{x}_p(t) = \sum_{k=1}^{K} a_k (g_k^T, \mathbf{u}_k(t)) + \mathbf{v}_p(t) = \mathbf{A}\mathbf{w}_p(t) + \mathbf{v}_p(t)$$

(2)

where $P$ represents the number of polarization directions of each array sensor, $\mathbf{x}_p(t) = \mathbf{G}_p \mathbf{x}(t)$, $\mathbf{G}_p = \mathbf{I}_M \otimes e_p^T$, $\mathbf{I}_M$ denotes the identity matrix with dimension $M \times M$, $e_p \in \mathbb{C}^{N \times 1}$ stands for a vector with its $p$th element being the only non-zero one of 1, $\mathbf{r}_k = [\hat{\mathbf{r}}_{kh}, \hat{\mathbf{r}}_{kv}] e_p$, $\mathbf{v}_p(t) = g_k^T \mathbf{u}_k(t)$, $\mathbf{w}_p(t) = \mathbf{w}_p \mathbf{w}_p^T$, $\mathbf{v}_p(t) = \mathbf{w}_p \mathbf{w}_p^T$, $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_K]$, $\mathbf{v}_p(t) = \mathbf{G}_p \mathbf{x}_p(t)$. Eq. (2) indicates that the $P$ measurement vectors consist of signals impinging from the same $K$ directions and

Fig. 1 Sketch of a linear COLD array.
independent noise. As $K$ is generally very small, those vectors are much sparse from the perspective of spatial power distribution.

In practical applications, $N$ snapshots are collected by the PSA at time instants of $t = t_1, t_2, \ldots, t_N$, the array output matrices in different polarization directions can thus be denoted by

$$X_p = AW_p + V_p, \quad p = 1, 2, \ldots, P$$

(3)

where $X_p = [x_p(t_1), x_p(t_2), \ldots, x_p(t_N)]$, $W_p = [w_p(t_1), w_p(t_2), \ldots, w_p(t_N)]$, and $V_p = [v_p(t_1), v_p(t_2), \ldots, v_p(t_N)]$. Based on the sparsity of the signal components in the $P$ groups of polarized measurements, the signals will be reconstructed first in Section 3, and then combined for DOA and polarization estimation in Section 4.

3. Reconstruction of polarized signal components

By extending the signal components in Eq. (3) to the potential spatial space of the sources via zero-padding, one can yield the spatially overcomplete formulation of the measurement matrices as follows:

$$X_p = \bar{A}W_p + V_p, \quad p = 1, 2, \ldots, P$$

(4)

where $A = [a(\xi_1), a(\xi_2), \ldots, a(\xi_L)]$, $a(\xi) = \cos \psi \sin \phi$, $a(\xi) = [\rho^{2\theta_0(i)/2}, \rho^{2\theta_0(i)/2}, \ldots, \rho^{2\theta_0(i)/2}]^T$, $\xi_1, \xi_2, \ldots, \xi_L$ are $L$ discrete samples of $\xi$ within $[-1, 1]$ used to indicate the possible incident signals with moderately small quantization errors, which satisfy $\xi_1 < \xi_2 < \ldots < \xi_L$; $W_p = [w_p(t_1), w_p(t_2), \ldots, w_p(t_N)]$, $w_p(t)$ is the zero-padded extension of $w_p(t)$, $W_p$ has nonzero rows only corresponding to the $a(\xi)$’s associated with $\xi = \cos \theta \sin \phi$. As $X_1, X_2, \ldots, X_P$ consist of the signals impinging from the same $K$ directions, $\bar{W}_1, \bar{W}_2, \ldots, \bar{W}_P$ share identical sparsity profiles. The measurement matrices will be decomposed on the discrete grid of $[\xi_1, \xi_2, \ldots, \xi_L]$ in this section via sparse Bayesian reconstruction so as to recover the signal components.

As the signal powers are distributed with unknown ratios in different polarization directions, independent hyperparameters should be introduced during the Bayesian reconstruction process to represent the prior power spectrum of $\bar{W}_1, \bar{W}_2, \ldots, \bar{W}_P$ along their rows, i.e.,

$$\bar{W}_p(t) \sim N(0, \gamma_p) \quad t = t_1, t_2, \ldots, t_N$$

(5)

where $\gamma_p \in \mathbb{R}^{L \times 1}$, with its $l$th element of $\gamma_p$, being inverse-gamma distributed with parameters $(\alpha, \beta_p)$, i.e.,

$$\gamma_p \sim IG(\alpha, \beta_p) \propto (\gamma_p)^{-1}(\alpha + 1) \exp(-\beta_p/\gamma_p)$$

(6)

By combining the above model assumptions with the Gaussian distribution of the additive noise, one can conclude in the probability densities of $X_1, X_2, \ldots, X_P$ with respect to $\gamma_1, \gamma_2, \ldots, \gamma_P, \sigma^2, \alpha$ and $B$ via straightforward mathematical derivations, where $z = [x_1, x_2, \ldots, x_N]^T$, $B = [\beta_1, \beta_2, \ldots, \beta_P]^T$. The $q$th iteration of the SBL technique when used to reconstruct $X_1, X_2, \ldots, X_P$ consists of the following updates, which are derived following the guidelines in,17-21

$$\gamma^{(q+1)}_p = \gamma^{(q)}_p + N$$

(7)

$$\beta^{(q+1)}_p = \frac{\beta^{(q)}_p}{\gamma^{(q)}_p + \sum_n E[N^{(q+1)}(t_n^2)]}$$

(8)

$$\tilde{x}^{(q+1)}_p = \frac{\tilde{x}^{(q)}_p}{|z^{(q+1)}| - 1}$$

(9)

where $w_p(t)$ represents the $l$th element of $w_p(t)$, $w_p^{(q+1)}(t)$ is Gaussian distributed with mean $\mu_p^{(q+1)}(t)$ and variance $\Sigma_p^{(q+1)}$,

$$E[w_p^{(q+1)}(t)^2] = \mu_p^{(q+1)}(t)^2 + (\Sigma_p^{(q+1)})^{+}_{l,l}$$

with $E[\bullet]$ being the expectation operator and $(\bullet)_{l,l}$ indicating the matrix element on the $l$th row and $l$th column, and

$$\mu_p^{(q+1)}(t) = \left((\Sigma_p^{(q)})^{-1} \Sigma_p^{(q+1)} \right)^{-1}$$

(10)

$$\Sigma_p^{(q+1)} = (\Gamma_p^{(q)})^{-1} + \left((\Sigma_p^{(q)})^{-1} \hat{A} \hat{A}^H \right)^{-1}$$

(11)

where $\Gamma_p^{(q)} = \text{diag}(\gamma^{(q)}_p)$ and $\text{diag}(\bullet)$ is the diagonalization operator. The update strategy of $\Sigma_p^{(q)}$ can be obtained similarly as in Ref.19, i.e.,

$$\Sigma^{(q+1)} = \left((\Sigma^{(q)})^{-1} + \frac{1}{MN} \sum_p 2E[||X_p - \bar{A}W_p^{(q+1)}||^2] \right)^{-1}$$

(12)

The updating process approaches a stationary state gradually, and the variable values are exported as their final estimates when a predefined iteration termination condition is achieved. The locations of the significant peaks in $\gamma_p$ indicate the signal directions.

As the spatial grids of $\xi_1, \xi_2, \ldots, \xi_L$ are obtained via discrete sampling, quantization errors will be brought in if the $\xi_l$’s’ corresponding to the locations of the most significant peaks in $\gamma_1, \gamma_2, \ldots, \gamma_p$ are taken as the estimates of $\cos \theta \sin \phi_k$ directly. In order to eliminate such errors, the reconstructed peak clusters around the signal directions generally consisting of several adjacent spectral lines should be replaced with a single line to approximate the signal components more compactly, which results in the following estimator of $\cos \theta \sin \phi_k$:

$$\hat{\gamma}_k = \arg \min \sum_i \left| \ln Q_p + \text{tr}(Q_p^{-1} R_p) \right|, \quad k = 1, 2, \ldots, K$$

(13)

where $R_p = \frac{1}{N} X_p^H X_p$, $\text{tr}(\bullet)$ is the trace operator, $Q_p = J_{p-k} + \tilde{x}^{(p)}(\alpha(\gamma^{(p)}_p))A^H(\gamma^{(p)}_p)$, $A(\alpha)$ is defined similarly as $a(\xi)$ in Eq. (4), $J_{p-k} = \bar{A} \bar{A}^H + \beta^2 I_M$, $\tilde{\gamma}_p = \text{diag}(\gamma^{(p)}_p)$, $\tilde{\gamma}_p$ are the estimates of $\gamma_p$ and $\sigma^2$ when the SBL iteration is terminated, $\bar{A}$ and $\tilde{\gamma}_p$ are equal $\bar{A}$ and $\tilde{\gamma}_p$, respectively, after removing the vectors and elements associated with the $k$th peak cluster (the peak clusters are selected in a similar way as that in Refs.17-19; $\tilde{\gamma}_p(\alpha)$ stands for the coefficient of $a(\alpha)A^H(\gamma^{(p)}_p)$ in $Q_p$ obtained by minimizing the objective function in Eq. (13) for particular $\alpha$, which can be derived following 17-19:

$$\tilde{\gamma}_p(\alpha) = \frac{a^H(\alpha)J_{p-k}^{-1}(R_p - J_{p-k}f_{p-k}^{-1}(\alpha))f_{p-k}^{-1}(\alpha)}{a^H(\alpha)J_{p-k}^{-1}(\alpha)}$$

(14)

By scanning the scope of each peak cluster with required precision, the values of the $\alpha$’s that minimize Eq. (13) for each signal are taken as the estimates of $\{\cos \theta \sin \phi_k\}_{k=1,2,\ldots,K}$, which are denoted by $\hat{\gamma}_1, \hat{\gamma}_2, \ldots, \hat{\gamma}_K$. 
By substituting \( \hat{\tau}_1, \hat{\tau}_2, \ldots, \hat{\tau}_K \) into Eq. (14), one can also obtain the power estimates of the \( K \) signals, denoted by \( \{ \hat{z}_k \}_{k=1}^{K} \), embedded in the \( P \) groups of measurements. The expression of \( \hat{z}_k \) can then be used to represent the \( k \)th signal component in \( Q \), and then substituted into Eq. (10) to recover the corresponding signal waveform within \( X \), which is denoted by \( \hat{s}_k(t) \). 

\[ \text{4. Estimation of direction and polarization parameters} \]

\[ \text{4.1. DOA estimation} \]

The model formulation given in Section 2 indicates that the \( k \)th signal component within \( X \) at time \( t \) is

\[ s_k(t) = c_k^H \psi_{k,b_h, k,r}^a u_k(t) \]

Denoting

\[ z_k(t) = [s_k^H(t), s_{k-1}^H(t), \ldots, s_{k-l}^H(t)]^T \]

and \( Z_k = [z_k(t_1), z_k(t_2), \ldots, z_k(t_N)] \), then

\[ Z_k = [\psi_{k,b_h, k,r}^a]^T \]

As \( z_k(t) = [\cos \phi_k, \sin \phi_k e^{j\phi_0}]^T s_k(t) \) for completely polarized signals, Eq. (17) can be rewritten in a more concrete form in such cases as follows,

\[ Z_k = [\psi_{k,b_h, k,r}^a]^T \cos \phi_k s_k(t) \]

where \( s_k = [s_k(t_1), s_k(t_2), \ldots, s_k(t_N)] \). Eqs. (17) and (18) indicate that whatever the degrees of polarization (with the definition given in Ref. 13), the columns of \( Z_k \) are embedded in the subspace spanned by \( \psi_{k,b_h, k,r}^a \), and the rank of \( Z_k \) is 1 in the case of completely polarized signals. Therefore, by denoting the eigenvector of \( Z_k \) corresponding to the largest eigenvalue by \( c_k \), and \( C_k = [c_k, \psi_{k,b_h, k,r}^a] \), one can easily conclude that \( C_k \) is rank deficient and the smallest eigenvalue of \( C_k^H C_k \) is 0. Such a conclusion holds for completely polarized, incompletely polarized and unpolarized signals, but is usable only in the case of polarization number no smaller than 3. The linear PSA with polarization direction number being 1 or 2 can be used for 1-D DOA estimation only, where the DOA estimates are obtained directly:

\[ \hat{\theta}_k = \cos^{-1}(\hat{\tau}_k) \]

In practical 2-D DOA estimation problems, the only estimates of \( Z_k \) and \( c_k \) obtained from the reconstructed signal components of \( \{ \hat{s}_k^H \}_{k=1}^{K} \), can be obtained from the reconstructed signal components of \( \{ \hat{s}_k^H \}_{k=1}^{K} \). The estimation error within \( \hat{\theta}_k \) deviates \( \hat{\theta}_k = [\hat{\theta}_k, \psi_{k,b_h, k,r}^a] \) from a precise rank-deficient matrix, and the linear dependence among \( \hat{\theta}_k \) and \( \psi_{k,b_h, k,r}^a \) should be tested by checking the distance between the smallest eigenvalue of \( Z_k^H Z_k \) and 0, which leads to the following 2-D DOA estimator when the polarization direction number of the PSA is not smaller than 3,

\[ \hat{\theta}_k = \arg \min_{\theta, \phi} \kappa_{\min}(\hat{C}_k^H \hat{C}_k), \quad \text{subject to } \cos \theta \sin \phi = \hat{\tau}_k \]

where \( \kappa_{\min}(\hat{C}_k^H \hat{C}_k) \) stands for the smallest eigenvalue of \( \hat{C}_k^H \hat{C}_k \). Although both azimuth and elevation parameters are included in the objective function in Eq. (20), they can be estimated via 1-D searching by exploiting the constraint of \( \cos \theta \sin \phi = \hat{\tau}_k \). For example, when scanning \( \theta \) within \([0, \pi] \), the value of \( \phi \) can be determined according to \( \phi = \sin^{-1}(\hat{\tau}_k / \cos \theta) \). The pair of \( \theta \) and \( \phi \) are then used to formulate \( \psi_{k,b_h, k,r}^a \) and \( \psi_{k,b_h, k,r}^a \), and calculate \( \kappa_{\min}(\hat{C}_k^H \hat{C}_k) \), in this way the \( K \) signal directions are obtained from Eq. (20) with \( K \) 1-D searching.

\[ \text{4.2. Polarization estimation} \]

The polarization parameters of completely polarized signals can also be estimated from the reconstructed signal components. In the case of 2-D DOA estimation, one can conclude from Eq. (18) that \( c_k = \Omega^{-1} \times [\psi_{k,b_h, k,r}^a] \cos \phi_k \sin \phi_0 e^{j\phi_0} \), with \( \Omega \) being the normalizing factor, thus

\[ \begin{bmatrix} c_k, \psi_{k,b_h, k,r}^a \end{bmatrix} \begin{bmatrix} 1 \\ -\Omega^{-1} \cos \phi_k \\ -\Omega^{-1} \sin \phi_0 e^{j\phi_0} \end{bmatrix} = C_k r_k = 0 \]

which indicates that \( r_k = [1, -\Omega^{-1} \cos \phi_k, -\Omega^{-1} \sin \phi_0 e^{j\phi_0}] \) is the eigenvector of \( C_k \) associated with the eigenvalue of 0. Such a conclusion can be exploited to estimate the polarization parameters based on the DOA estimates.

Denote the eigenvector of \( C_k^H C_k \) corresponding to the eigenvalue of \( \kappa_{\min}(C_k^H C_k) \) when \( [\theta, \phi] = [\hat{\theta}_k, \hat{\phi}_k] \) by \( \hat{r}_k \), it can be deemed as an estimate of \( r_k \), thus the polarization parameters of the \( k \)th signal can be estimated based on Eq. (21):

\[ \begin{bmatrix} \hat{\theta}_k = \tan^{-1} |\hat{r}_k(3)/\hat{r}_k(2)| \\ \hat{\phi}_k = \text{Arg}(\hat{r}_k(3)/\hat{r}_k(2)) \end{bmatrix} \]

where Arg(\( \cdot \)) is the argument operator.

When PSA with two polarization directions are used for 1-D DOA estimation of completely polarized signals, it also holds that \( c_k = \Omega^{-1} \times [\psi_{k,b_h, k,r}^a] \cos \phi_k \sin \phi_0 e^{j\phi_0} \). By substituting \( \hat{\theta}_k = \cos^{-1}(\hat{\tau}_k) \) into \( \psi_{k,b_h, k,r}^a \), to calculate \( \psi_{k,b_h, k,r}^a \), then

\[ \Omega^{-1} \cos \phi_k \sin \phi_0 e^{j\phi_0} \]

\[ \begin{bmatrix} \hat{\theta}_k = \tan^{-1} |\hat{r}_k(2)/\hat{r}_k(1)| \\ \hat{\phi}_k = \text{Arg}(\hat{r}_k(2)/\hat{r}_k(1)) \end{bmatrix} \]

which finally concludes in the polarization parameter estimates of

\[ \text{5. Further discussions on ReCoP} \]

\[ \text{5.1. Implementation scheme} \]

During the proposition of the new method, many implementation details are buried in the discussions. The implementation scheme is presented in Table 1 to make the method being more comprehensive.
Table 1 Implementation scheme of ReCoP.

<table>
<thead>
<tr>
<th>Implementation scheme of ReCoP</th>
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<tbody>
<tr>
<td><strong>Input:</strong></td>
</tr>
<tr>
<td>( \hat{x}(t_1), \hat{x}(t_2), \ldots, \hat{x}(t_N) ) according to Eqs. (2) and (3)</td>
</tr>
<tr>
<td>(1) Obtain ( \hat{X}_1, \hat{X}_2, \ldots, \hat{X}_K )</td>
</tr>
<tr>
<td>(2) Repeat Eqs. (7)-(12) until convergence</td>
</tr>
<tr>
<td>(3) Estimate ( \hat{t}_1, \hat{t}_2, \ldots, \hat{t}_K ) via Eq. (13) by combining Eq. (14)</td>
</tr>
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</table>

**DOA estimation:**
(4) In 1-D scenarios, estimate source DOA via Eq. (19)
(5) In 2-D scenarios, for \( k = 1, 2, \ldots, K \)
(5-1) Compute \( \hat{s}_p = [\hat{s}_p(t_1), \hat{s}_p(t_2), \ldots, \hat{s}_p(t_N)] \) for \( p = 1, 2, \ldots, P \) in terms of their expectations via Eq. (10)
(5-2) Form \( \mathbf{Z}_k \) according to Eq. (16)
(5-3) Eigen-decompose \( \mathbf{Z}_k \mathbf{Z}_k^H \) to obtain its largest eigenvector \( \mathbf{c}_k \)
(5-4) Form \( \mathbf{\hat{r}}_k \) and \( \mathbf{\hat{r}}_k^* \) for each direction candidate during 1-D directional searching based on the constraint \( \cos \theta \sin \phi = \mathbf{c}_k \) and output \( \mathbf{\hat{r}}_k \) and \( \mathbf{\hat{r}}_k^* \) when the smallest eigenvalue of \( \mathbf{c}_k^H \mathbf{c}_k \) is minimized
(6) In 1-D scenarios, calculate \( \mathbf{\hat{v}}_{i, k} \) and \( \mathbf{\hat{v}}_{i, k}^* \) based on the DOA estimates, combine \( \mathbf{\hat{v}}_{i, k} \) and \( \mathbf{\hat{v}}_{i, k}^* \) with the largest eigenvector of \( \mathbf{Z}_k \mathbf{Z}_k^H \), i.e., \( \mathbf{c}_k \) to compute \( \mathbf{\hat{r}}_k \) according to Eq. (23), finally estimate the polarization parameters via Eq. (24)
(7) In 2-D scenarios, denote the eigenvector corresponding to the minimized smallest eigenvalue of \( \mathbf{c}_k^H \mathbf{c}_k \) by \( \mathbf{r}_k \) (which is obtained in the step of (5-4)), then estimate the polarization parameters via Eq. (22)

5.2. Properties of ReCoP

As the proposed method of ReCoP follows a much different guideline from the previous counterparts, it also has some individual properties that should be highlighted.

(1) Adaptation to different array structures. Given a linear PSA consisting of identically structured sensors, no further constraint is required on the array geometry or polarization for the implementation of ReCoP. The only difference of ReCoP when used in differently structured arrays lies in the variation of \( \mathbf{E} \); it is
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0
\end{bmatrix}
\]
for CCD arrays, and
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix}
\]
for COLD arrays and \( \mathbf{I}_K \) for EMV arrays.
(2) Free of multi-dimensional searching. After reconstructing the polarized signal components from the array outputs, only an 1-D searching procedure is included in ReCoP to realize 2-D DOA estimation based on Eq. (20) and the other procedures are realized via numerical calculations.
(3) Free of variable-pairing. The locations of the spectrum peaks in \( \mathbf{\hat{p}}_1, \mathbf{\hat{p}}_2, \ldots, \mathbf{\hat{p}}_P \) associate the variables of interest with respect to each signal automatically, and the direction and polarization parameters are then estimated by taking the signals into consideration sequentially.
(4) Separation of coherent signals. In ReCoP, the polarized signal components are recovered from the array outputs via sparse reconstruction, which has been shown by previous research to own much enhanced adaptation to inter-signal correlation. Thus ReCoP is expected to perform well in the coherent scenarios.
(5) Adaptation to signals with different degrees of polarization. No assumption on the degree of polarization is introduced for the signals when using ReCoP for DOA estimation; it adapts to completely polarized, incompletely polarized and unpolarized signals inherently without any modifications.

6. Simulation results

Suppose that two equal-power signals impinge onto an 8-sensor linear PSA, and the array geometry is assumed to be uniform and inter-spaced by half-wavelength to facilitate the comparison between different methods. The 1-D DOA and polarization estimation performance of ReCoP is demonstrated first with COLD array by setting the signal parameters as \( \{\vartheta, \phi, \eta\} = \{60^\circ, 30^\circ, 45^\circ\} \). Ten snapshots are collected by the array. During the spatial reconstruction procedure, ReCoP selects the discrete grid as \( \{\vartheta_1, \vartheta_2, \ldots, \vartheta_L\} = \{90^\circ, 89^\circ, \ldots, 90^\circ\} \). The DOA estimation precision is evaluated by the average root-mean-square-error (RMSE) of the two signals, which is defined as
\[
\text{RMSE}_\theta = \sqrt{\frac{1}{K} \sum_{k=1}^{K} \sum_{i=1}^{P} \left( \hat{\vartheta}_k(i) - \vartheta_k \right)^2}
\]
where \( I \) stands for the number of simulations in each particular scenario and \( \hat{\vartheta}_k(i) \) the DOA estimate of the \( k \)th signal in the \( i \)th simulation. The estimation precision of the polarization parameters is evaluated by the RMSE of the angular distance between the location of \( \{\vartheta_k, \phi_k, \eta_k\} \) and its estimate on the sphere. Previous methods of ESPRIT and Quaternion-MUSIC (Q MUSIC) are also implemented for performance comparison.

As the SNR of the two signals vary from 0 dB to 20 dB, the direction and polarization parameter estimation RMSE of ReCoP, ESPRIT, and QMUSIC obtained from 100 simulations are shown in Fig. 2(a) and (b). ReCoP exceeds the other two methods in parameter estimation precision in the given scenarios and the predominance is very significant when the SNR is lower than 10 dB. The average implementation times of the three methods have also been compared during the simulations. The detailed time list has been excluded here due to space limitation, but the statistical results show that ReCoP spends about 10 s to obtain the DOA estimates in each simulation, which is approximately half of the implementation time of QMUSIC, but is much larger than that of ESPRIT (less than 1 s). The QMUSIC method is computationally most expensive because it requires multi-dimensional direction searching, ReCoP and ESPRIT are implemented via 1-D searching, but ReCoP is slower due to the iterative realization of Eqs. (7)-(12).

Then fix the SNR of the two signals at 10 dB and vary their correlation coefficient from 0 to 1, and then the DOA estimation RMSE of the three methods obtained from 100 simulations are shown in Fig. 3. ESPRIT and QMUSIC deteriorate significantly in DOA estimation precision for more heavily correlated signals, and they even fail completely in coherent scenarios. However, ReCoP shows much enhanced adaptation to correlated and coherent signals and it achieves high-precision DOA estimation even when the correlation coefficient approaches 1.
The polarization parameter estimation precisions of the three methods vary in a similar way as the DOA precisions, thus the results are not listed here for conciseness.

In the following group of simulations, the COLD array is replaced by an EMV array, so as to demonstrate the performance of ReCoP in 2-D DOA estimation and separation of coexisted completely and incompletely polarized signals. The signal azimuth and polarization parameters are kept unchanged from the previous simulations, the elevations are set to be 80° and 60°, respectively, and their SNRs are fixed at 10 dB. The degrees of polarization of the two signals are set to be 1 and 0.5, which indicate that they are completely and incompletely polarized signals, respectively. In such a scenario with differently polarized signals, the previous methods are not able to estimate the signal directions with linear PSA. The 2-D DOA estimation results of ReCoP in 30 randomly chosen simulations are shown in Fig. 4, where the red markers indicate the true signal directions. The azimuth estimation errors of ReCoP are all smaller than 1° and the elevation estimation errors are mostly smaller than 3°.

7. Conclusions

(1) The signal components embedded in differently polarized array measurements are reconstructed and combined in this paper, so as to estimate the 1-D and 2-D directions of signals with different degrees of polarization and the polarization parameters of completely polarized signals.

(2) The proposed method is free of multi-dimensional searching and variable-pairing. It adapts to polarization sensitive linear arrays with different polarization directions if only the array sensors have identical structures.

(3) The proposed method also performs well in separating coherent signals and signals with different degrees of polarization.

(4) As is demonstrated by the simulation results, the direction and polarization estimation precisions of the proposed method exceed its counterparts in scenarios with typical settings, especially when the SNR is low.

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