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On the Andreadakis–Johnson filtration of the automorphism group of a free group

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ABSTRACT

Families of non-trivial cohomology classes are given for the discrete groups belonging to the Johnson filtration of the automorphism group of a free group generated by n letters. The methods are (1) to analyze analogous classes for filtrations of a subgroup of the pure symmetric automorphism group of a free group and (2) to analyze features of these classes which are preserved by the Johnson homomorphism with values in the Lie algebra of derivations of a free Lie algebra. One consequence is that the ranks of the cohomology groups in any fixed degree i for $1 \leqslant i \leqslant n-2$ for the Johnson filtrations of IA_n increase without bound. The actual classes constructed are fragile in the sense that they vanish after passage to successive filtrations; furthermore, these classes are all naturally in the image of the Johnson homomorphism on the level of cohomology. The methods are similar to those occurring within the theory of arrangements.
The authors congratulate Corrado De Concini on the occasion of his 60-th birthday. One of the authors would like to thank Corrado for a wonderful time arising from a discussion about cohomology of groups with friends in the basement of Il Palazzone!

1. Introduction and preliminaries

Let $F_n = F[x_1, \ldots, x_n]$ be the free group on generators $x_1, \ldots, x_n$. There is a surjective natural map $Aut(F_n) \to \mathrm{GL}_n(\mathbb{Z})$ which sends an automorphism to the induced map on $H_1(F_n)$. The kernel IA_n

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consists of exactly those automorphisms which induce the identity on $H_1(F_n)$. This is the $Aut(F_n)$-analogue of the *Torelli group*, the subgroup of the mapping class group of a surface which acts trivially on the first homology group of the surface.

The group IA_n is the first in a series of subgroups belonging to the *Andreadakis–Johnson filtration of* $Aut(F_n)$; that is, the descending series $IA_n = J_n^1 \supset J_n^2 \supset \cdots$, where J_n^s consists of those automorphisms of $Aut(F_n)$ which induce the identity map on the level of F_n/Γ_n^{s+1}; here Γ_n^{s+1} denotes the $(s+1)$-st stage of the descending central series for F_n, as given in more detail below [1,12–14].

The subgroup J_n^s is sometimes called the s-th Andreadakis–Johnson filtration, or s-th Johnson filtration below, which gives a decreasing filtration of IA_n. The main goal in this paper is to establish quantitative information about the cohomology of these groups. One implication of the work here is that the ranks of the cohomology groups in any fixed degree i for $1 \leqslant i \leqslant n-2$ increase without bound by going deeper into the Andreadakis–Johnson filtration.

A brief summary is given next for what was known previously about the (co)homology of the Johnson filtrations.

In 1934, Magnus [17] provided a finite generating set for IA_n (see Section 3 below). The first homology and cohomology groups were partially computed by Andreadakis [1]; completely by Kawazumi in [15]. A number of degree two cohomology classes in the image of cup product were computed by Pettet [21]. However it is still unknown whether or not IA_n is finitely presentable for $n \geqslant 4$. The cohomological dimension of IA_n was recently computed to be $2n-3$ by Bestvina, Bux and Margalit [2], where the maximal non-zero homology group was shown not to be finitely generated; this implies in particular that IA_3 is *not* finitely presentable.

Even less is known about the groups J_n^s for $s \geqslant 2$. By adapting a theorem of McCullough and Miller [19], it follows for $n = 3$ and $s \geqslant 2$ that J_3^s is not finitely generated, but it is unknown whether or not its abelianization is finitely generated, and little is known along these lines for $n \geqslant 4$. Satoh [23,22] has studied some abelian quotients of terms of the Johnson filtration via the Johnson homomorphisms (see Section 2 for definitions).

Before stating precise results of this paper, first record some facts and define some notation. Recall the structure of the free Lie algebra defined over the integers $\mathbb{Z}$ generated by a free abelian group V, denoted here by $L[V]$. The free abelian group V is frequently of finite rank. In the case that the rank is q, the notation V_q is used in place of V below.

The Lie algebra $L[V]$ can be described as the smallest sub-Lie algebra of the tensor algebra $T[V]$ which contains V. Thus $L[V]$ is graded with the s-th graded summand given by

$$L_s[V] = L[V] \cap V^{\otimes s}.$$

One classical result is that $L_s[V]$ is a finitely generated free abelian group with ranks given in [10,24,20], with [25] for the graded case.

It is convenient to use the standard dual of $L_s[V]$ given by

$$L_s[V]^* = Hom\big(L_s[V], \mathbb{Z}\big)$$

together with

$$\mathbb{Z} \oplus L_s[V]^*$$

where $\mathbb{Z}$ is concentrated in degree 0, and the finitely generated free abelian group $L_s[V]^*$ is required to be concentrated in degree 1 in the main result stated next.

The s-th Johnson homomorphism, as described in Section 2 below, has domain J_n^s and takes values in the finitely generated free abelian group $Hom(V_n, L_{s+1}[V_n])$; thus is a direct product of finitely many copies of the integers. The integer cohomology of the discrete group $Hom(V_n, L_{s+1}[V_n])$ is therefore a finitely generated exterior algebra. The purpose of this article is to show that the image of the Johnson homomorphism

$$\tau_s : J_n^s \to Hom\big(V_n, L_{s+1}[V_n]\big)$$

on the level of integer cohomology groups has the direct summand in the image described next.

Theorem 1.1. *If $n \geqslant 3$, and $2 \leqslant q \leqslant n-1$, the integral cohomology ring $H^*(J_n^s)$ contains a direct summand which is additively isomorphic to*

$$\bigotimes_{n-q}\big(\mathbb{Z} \oplus L_s[V_q]^*\big)$$

in the image of

$$\tau_s^* : H^*\big(Hom\big(V_n, L_{s+1}[V_n]\big)\big) \to H^*\big(J_n^s\big).$$

Thus if $1 \leqslant i \leqslant n-2$, then the ranks of $H^i(J_n^s)$ increase without bound as s increases.

Remark 1.2. The proof of Theorem 1.1 has direct implications concerning geometric features of the classifying space of J_n^s. Namely, the suspension of the classifying space BJ_n^s has a wedge summand given by a bouquet of spheres corresponding to the cohomology classes in Theorem 1.1. This feature translates into a statement about the cohomology of BJ_n^s for any cohomology theory.

It follows from the theorem that

$$\bigotimes_{n-q} L_s[V_q]^*$$

is a direct summand of the cohomology group $H^{n-q}(J_n^s)$, a fact recorded next, in order to exhibit the rapid growth of the cohomology as the depth of the Johnson filtration s increases.

Corollary 1.3. *For fixed $n > 2$, and $n-2 \geqslant k \geqslant 1$, the cohomology group $H^k(J_n^s)$ contains*

$$\bigotimes_{k}\big(L_s[V_{n-k}]^*\big)$$

as a direct summand. Thus specializing to $k = n-2$, the cohomology group $H^{n-2}(J_n^s)$ contains $\bigotimes_{n-2}(L_s[V_2]^*)$ as a direct summand.

Observe that the rank of $L_s[V_q]^*$ grows rapidly for $q > 1$; estimates of this growth are given at the end of Section 7 using Witt's formula for the ranks in free Lie algebras. Thus the ranks of $H^i(J_n^s)$ for $1 \leqslant i \leqslant n-2$ grow rapidly for n at least 3.

The classes constructed in Theorem 1.1 are fragile in the following sense, proven in Corollary 7.3 below, and stated as follows.

Corollary 1.4. *If $n \geqslant 3$, and $2 \leqslant q \leqslant n-1$, the subalgebra generated by $\bigotimes_{n-q} L_s[V_q]^*$ is in the kernel of the map*

$$H^*\big(J_n^s\big) \to H^*\big(J_n^{s+1}\big).$$

Note that $Aut(F_n/\Gamma^{s+1})$ acts on the cohomology of J_n^s as this last group is the kernel of the homomorphism $Aut(F_n) \to Aut(F_n/\Gamma^{s+1})$. However, a more precise description of algebra generators for $\bigotimes_{n-q}(\mathbb{Z} \oplus L_s[V_q]^*)$ gives that this module is not closed with respect to this action; the closure of this module with respect to the action is in fact much larger, a feature not addressed here.

The organization of this paper is outlined as follows. Section 2 is a review of the filtrations of $Aut(F_n)$ given by descending central series and the Johnson filtration. Subgroups of $Aut(F_n)$ known as McCool's group are the subject of Section 3, and are used to detect cohomology classes. The descending central series for McCool's group is the subject of Section 4. Section 5 gives natural subgroups of McCool's group which provide a considerable clarification of the work here. The values of the Johnson homomorphism on certain subgroups of McCool's group are recorded in Section 6. Equipped with a theorem of Falk and Randell [7, Theorem 3.1] (appearing here as Theorem 4.1), these structures are used to give a family of cohomology classes for certain subgroups of McCool's group. These classes are in the image of the Johnson homomorphism, seen directly by using subgroups of McCool's group as given in Section 7. That step finishes the proof of the main theorem.

The authors (weakly) conjecture the following:

Conjecture. *If $3 \leqslant n$, $2 \leqslant s$ and $1 \leqslant i \leqslant n-2$, the cohomology group $H^i(J_n^s)$ is not finitely generated.*

2. Central series and the Johnson homomorphisms

Recall that the *descending central series* of a group π is the sequence of subgroups

$$\pi = \Gamma^1(\pi) \supset \Gamma^2(\pi) \supset \cdots \supset \Gamma^s(\pi) \supset \cdots$$

with $\Gamma^s(\pi) = [\pi, \Gamma^{s-1}(\pi)]$. It is natural to study the Lie algebra

$$gr_*^{DCS}(\pi) = \bigoplus_{s \geqslant 1} gr_s^{DCS}(\pi)$$

associated to its descending central series with the graded terms given by

$$gr_s^{DCS}(\pi) = \Gamma^s(\pi)/\Gamma^{s+1}(\pi)$$

with bracket given by

$$[-,-] : gr_s^{DCS}(\pi) \otimes gr_t^{DCS}(\pi) \mapsto gr_{s+t}^{DCS}(\pi)$$

induced by the commutator on the level of π given by

$$[x, y] = x^{-1}y^{-1}xy, \quad x, y \in \pi.$$

In case π is residually nilpotent, the descending central series filtration is convergent. Note that the Lie algebra $gr_*^{DCS}(\pi)$ is also graded, but fails to satisfy the axioms for a graded Lie algebra because of sign conventions; this failure can be remedied by doubling all degrees to obtain a graded Lie algebra.

A classical example is that of the free group F_n first investigated by P. Hall and E. Witt. Recall that $V = V_n$ denotes a free abelian group of rank n, the first homology group of F_n. As in Section 1, $L[V_n]$ denotes the free Lie algebra generated by V_n. Hall [10] and Witt [25] proved that

$$L[V_n] = gr_*^{DCS}(F_n)$$

where V_n is a free abelian group of rank n with a choice of basis given by $\{x_1, \dots, x_n\}$, and each x_i is the image under the projection $F_n \to V_n \simeq H_1(F_n)$ of a basis element for F_n.

The graded derivations of a graded Lie algebra inherit the structure of the graded Lie algebra. In the case of the free Lie algebra $L[V_n]$, the Lie algebra of graded derivations

$$Der\bigl(L[V_n]\bigr)$$

is additively isomorphic to the direct sum

$$\bigoplus_{s\geqslant 1} Hom\bigl(V_n, L_{s+1}[V_n]\bigr).$$

Writing

$$Der_s\bigl(L[V_n]\bigr) = Hom\bigl(V_n, L_{s+1}[V_n]\bigr),$$

the Lie bracket is given by a bilinear pairing

$$Der_s\bigl(L[V_n]\bigr) \otimes Der_t\bigl(L[V_n]\bigr) \to Der_{s+t-1}\bigl(L[V_n]\bigr)$$

as developed by M. Kontsevich [16]; see also [6] and T. Jin [11].

Remark 2.1. A natural variation is the Lie algebra

$$\widehat{Der}\bigl(L[V_n]\bigr) = \bigoplus_{s\geqslant 1} Hom\bigl(V_n, L_s[V_n]\bigr)$$

given by the direct sum

$$Der\bigl(L[V_n]\bigr) \oplus Hom(V_n, V_n).$$

The additional group

$$Hom(V_n, V_n) = End(V_n) = \bigoplus_{n^2} \mathbb{Z}$$

is not used in the computations below.

Turn now to the Johnson filtration $\{J_n^s\}$ of $Aut(F_n)$. Recall that the s-th term J_n^s is the kernel of the "reduction map"

$$Aut(F_n) \to Aut\bigl(F_n/\Gamma^{s+1}\bigr).$$

It is well known that the intersection of all the terms of the Johnson filtration is trivial. Furthermore, the successive quotients $gr_s^J(\mathrm{IA}_n) = J_n^s/J_n^{s+1}$ are torsion-free finitely generated abelian groups. The quotients $gr_s^J(\mathrm{IA}_n)$ give a graded Lie algebra which is free as a $\mathbb{Z}$-module, with bracket inherited from the commutator. The direct sum of these quotients

$$gr_*^J(\mathrm{IA}_n) = \bigoplus_{s\geqslant 1} gr_s^J(\mathrm{IA}_n)$$

admits a natural structure of a Lie algebra, called the *Johnson Lie algebra* of the group IA_n [1,4,8,15].

The first *Johnson homomorphism* τ_1 on IA_n is defined by

$$\tau_1 : \mathrm{IA}_n \to Hom\big(V_n, L_2[V_n]\big)$$

by (with some abuse of notation) $\tau_1(\phi)(w)$ equal to the image of $\phi(\tilde{x})\tilde{x}^{-1}$ in $L_2[V_n]$, where $\phi \in \mathrm{IA}_n$, the element $x \in V_n$, and $\tilde{x}$ is a lift of x to $F_n/\Gamma^3 F_n$. It is straightforward to check that the kernel of τ_1 is J_n^2. Define inductively the s-th Johnson homomorphism on the kernel J_n^s of τ_{s-1} by

$$\tau_s : J_n^s \to Hom\big(V_n, L_{s+1}[V_n]\big)$$

with $\tau_s(\phi)(x) = \phi(\tilde{x})\tilde{x}^{-1}$ for $\phi \in J_n^s$, and $x \in H_1(F_n)$ for any lift $\tilde{x} \in F_n/\Gamma^{s+2}F_n$. As the group J_n^{s+1} is precisely the kernel of

$$\tau_s : J_n^s \to Hom\big(V_n, L_{s+1}[V_n]\big)$$

there is an induced map

$$\tau_s : gr_s^J(\mathrm{IA}_n) = J_n^s / J_n^{s+1} \to Hom\big(V_n, L_{s+1}[V_n]\big).$$

Passing to direct sums, there is an induced map

$$\bigoplus_{s\geqslant 1} \tau_s : \bigoplus_{s\geqslant 1} J_n^s \to \bigoplus_{s\geqslant 1} Hom\big(V_n, L_{s+1}[V_n]\big).$$

With the identification of the Lie algebra of graded derivations $Der(L[V_n])$ with $\bigoplus_{s\geqslant 1} Hom(V_n, L_{s+1}[V_n])$, the induced homomorphism

$$J : gr_*^J(\mathrm{IA}_n) \to Der\big(L[V_n]\big)$$

is a morphism of Lie algebras [1,4,8,15].

Thus, there are two natural structures of Lie algebras for IA_n given by $gr_*^{DCS}(\mathrm{IA})$ and $gr_*^J(\mathrm{IA}_n)$.

3. Properties of McCool's group

Recall Magnus's generating set for IA_n [17], consisting of automorphisms

$$\mathcal{M}_n = \{\alpha_{ij} \mid i \neq j\} \cup \{A_{ijk} \mid i \neq j, k;\ \ j < k\}$$

where

$$\alpha_{ij}(x_r) = \begin{cases} x_r, & r \neq i, \\ x_j x_r x_j^{-1}, & r = i, \end{cases}$$

$$A_{ijk}(x_r) = \begin{cases} x_r, & r \neq i, \\ [x_j, x_k]x_i, & r = i. \end{cases}$$

McCool [18] proved that the subgroup $\mathrm{P}\Sigma_n$ of *pure symmetric automorphisms* (or the *McCool group*) of IA_n, consisting of those automorphisms which map each generator x_i to a conjugate of itself, is generated by the subset of Magnus generators

$$\mathrm{P}\Sigma_n = \langle \alpha_{ij} \mid i \neq j \rangle.$$

McCool provided a finite presentation of $\mathrm{P}\Sigma_n$ in terms of these generators. The group $\mathrm{P}\Sigma_n$ is interesting to topologists as it appears as the mapping class group of the complement of n unlinked circles in $\mathbb{R}^3$, and is thus a generalization of the pure braid group (see [9], for example). The pure braid group is itself realized as a subgroup of $\mathrm{P}\Sigma_n$. McCool proved the following theorem.

Theorem 3.1. *A presentation of* $\mathrm{P}\Sigma_n$ *is given by generators* $\alpha_{k,j}$ *together with the following relations.*

(1) $\alpha_{i,j} \cdot \alpha_{k,j} \cdot \alpha_{i,k} = \alpha_{i,k} \cdot \alpha_{i,j} \cdot \alpha_{k,j}$ *for* i, j, k *distinct.*
(2) $[\alpha_{k,j}, \alpha_{s,t}] = 1$ *if* $\{j,k\} \cap \{s,t\} = \phi$.
(3) $[\alpha_{i,j}, \alpha_{k,j}] = 1$ *for* i, j, k *distinct.*
(4) $[\alpha_{i,j} \cdot \alpha_{k,j}, \alpha_{i,k}] = 1$ *for* i, j, k *distinct* (*redundantly*).

Consider the subgroup $\mathrm{P}\Sigma_n^+$ of $\mathrm{P}\Sigma_n$ generated by the subset

$$\mathrm{P}\Sigma_n^+ = \langle \alpha_{ij} \mid 1 \leqslant j < i \leqslant n \rangle.$$

This group is referred to as the *upper triangular McCool group* in [5]. There exists an exact sequence

$$1 \to F_{n-1} \to \mathrm{P}\Sigma_n^+ \to \mathrm{P}\Sigma_{n-1}^+ \to 1$$

where the induced action of $\mathrm{P}\Sigma_{n-1}^+$ on $H_1(F_{n-1})$ is trivial. This fact is recorded in [5] as Lemma 4.2, and it should be compared with the case of the pure braid group. (See [3, pp. 281–283], or [7] for instance.)

The map $\pi : \mathrm{P}\Sigma_n^+ \to \mathrm{P}\Sigma_{n-1}^+$ in [5] is recalled next for the convenience of the reader, as this map is used below:

$$\pi(\alpha_{k,i}) = \begin{cases} \alpha_{k,i} & \text{if } i < n \text{ and } k < n, \\ 1 & \text{if } i = n \text{ or } k = n. \end{cases}$$

Furthermore, there is a natural cross-section for π,

$$\sigma : \mathrm{P}\Sigma_{n-1}^+ \to \mathrm{P}\Sigma_n^+$$

defined by

$$\sigma(\alpha_{k,i}) = \alpha_{k,i}.$$

Thus the group $\mathrm{P}\Sigma_{n-1}^+$ is regarded as a subgroup of $\mathrm{P}\Sigma_n^+$ below.

4. The descending central series

The purpose of this section is to develop properties of the descending central series for $\mathrm{P}\Sigma_n^+$ by using the following theorem of Falk and Randell [7]:

Theorem 4.1. *(See Falk and Randell [7, Theorem 3.1].) Suppose that $1 \to A \to B \to C \to 1$ is a split exact sequence of groups, and the induced conjugation action of C on $H_1(A)$ is trivial (that is, $[A, C] \subset [A, A]$). Then the sequence of induced maps*

$$1 \to \Gamma^s A \to \Gamma^s B \to \Gamma^s C \to 1$$

is split exact for every s.

Furthermore, there is an induced exact sequence on the level of associated graded modules

$$0 \to gr_s^{DCS}(A) \to gr_s^{DCS}(B) \to gr_s^{DCS}(C) \to 0$$

which is additively split.

To apply Theorem 4.1 to the group $\mathrm{P}\Sigma_n^+$, consider a lemma given by the second conclusion of Theorem 1.2 of [5].

Lemma 4.2. *The sequence*

$$1 \to F_{n-1} \to \mathrm{P}\Sigma_n^+ \to \mathrm{P}\Sigma_{n-1}^+ \to 1$$

induced by the map $x_n \mapsto \mathrm{id}_{F_{n-1}}$ is split exact, where F_{n-1} is the free group on the generators

$$\alpha_{n,1}, \dots, \alpha_{n,n-1}.$$

Furthermore, the action of $\mathrm{P}\Sigma_{n-1}^+$ on $H_1(F_{n-1})$ is trivial.

A consequence of Lemma 4.2 follows.

Lemma 4.3. *The sequence*

$$1 \to \Gamma^s F_{n-1} \to \Gamma^s \mathrm{P}\Sigma_n^+ \to \Gamma^s \mathrm{P}\Sigma_{n-1}^+ \to 1 \tag{1}$$

is split exact.

The next proposition then follows directly.

Proposition 4.4. *If $s \geqslant 2$, then there is an additive isomorphism*

$$\bigoplus_{q=2}^{n} L_s[V_{q-1}] \to \Gamma^s \mathrm{P}\Sigma_n^+ / \Gamma^{s+1} \mathrm{P}\Sigma_n^+.$$

Furthermore, each free Lie algebra $L[V_{q-1}]$ is a sub-Lie algebra of $gr_*(\mathrm{P}\Sigma_n^+)$, and there is an induced isomorphism of abelian groups

$$\bigoplus_{q=2}^{n} L[V_{q-1}] \to gr_*\big(\mathrm{P}\Sigma_n^+\big)$$

(*where this isomorphism does not preserve the structure as Lie algebras*).

If $s=1$, then there is an additive isomorphism

$$\bigoplus_{q=2}^{n} V_{q-1} \to \Gamma^1\,\mathrm{P}\Sigma_n^+/\Gamma^2\,\mathrm{P}\Sigma_n^+ = H_1\big(\mathrm{P}\Sigma_n^+\big).$$

5. On a subgroup

The purpose of this section is to define subgroups of $\mathrm{P}\Sigma_n^+$, denoted $H(n,k)$ and $G(n,k,j)$; these groups will appear in the proof of the main theorem.

Definition 5.1. Fix integers $1 \leqslant j \leqslant k-1 \leqslant n-1$, and define

$$G(n,k,j)$$

as the subgroup of $\mathrm{P}\Sigma_n^+$ generated by the elements

$$\alpha_{k,1}, \alpha_{k,2}, \ldots, \alpha_{k,j}.$$

The group

$$H(n,k)$$

is defined to be the direct product

$$G(n,k,k-1) \times G(n,k+1,k-1) \times \cdots \times G(n,n,k-1).$$

Properties of the groups $G(n,k+r,k-1)$, and $H(n,k)$ are recorded next.

Lemma 5.2. *The groups $G(n,k+r,k-1)$ are free subgroups of the group $\mathrm{P}\Sigma_n^+$ for $2 \leqslant k \leqslant k+r \leqslant n$. Furthermore, if*

(1) $x \in G(n,k+s,k-1)$ *for* $0 \leqslant s \leqslant n-k$, *and*
(2) $y \in G(n,k+r,k-1)$ *for* $0 \leqslant s < r \leqslant n-r$, *then*

$$xy = yx,$$

and there are induced homomorphisms

$$\Theta(n,k) : G(n,k,k-1) \times G(n,k+1,k-1) \times \cdots \times G(n,n,k-1) \to \mathrm{P}\Sigma_n^+$$

such that

$$\Theta(n,k)(\alpha_{t,s}) = \alpha_{t,s} \in G(n,k+r,k-1)$$

for

$$0 \leqslant r \leqslant n-k.$$

Proof. Recall from [5] that there are split epimorphisms

$$\pi : \mathrm{P}\Sigma_n^+ \to \mathrm{P}\Sigma_{n-1}^+$$

with kernel given by a free group on $(n-1)$-letters with basis

$$\alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,n-2}, \alpha_{n,n-1}.$$

Thus $G(n,n,n-1)$ is a free subgroup of $\mathrm{P}\Sigma_n^+$ for all $2 \leqslant n$, and the first assertion that each $G(n,k+r,k-1)$ is free follows.

Next, assume that

(1) $x \in G(n,k+r,k-1)$ for $0 \leqslant r \leqslant n-k$, and
(2) $y \in G(n,k+s,k-1)$ for $0 \leqslant r < s \leqslant n-r$.

It suffices to check the next assertion that $xy = yx$ in case

$$x = \alpha_{k+r,u}, \quad r \leqslant n-k,\ u \leqslant k-1,$$

and

$$y = \alpha_{k+s,t}, \quad r < s \leqslant n-k,\ t \leqslant k-1.$$

Observe that either

(1) $u = t$ so that $[\alpha_{k+r,t}, \alpha_{k+s,t}] = 1$ by Theorem 3.1, or
(2) $u \neq t$ in which case $[\alpha_{k+r,u}, \alpha_{k+s,t}] = 1$ by Theorem 3.1 as $t, u \leqslant k-1$ and so the sets $\{k+r,u\}$ and $\{k+s,t\}$ are disjoint. □

The next lemma follows by inspection of the definitions.

Lemma 5.3. *If $2 \leqslant k \leqslant k+r \leqslant n-1$, then the groups $G(n,k+r,k-1)$ are subgroups of the group $\mathrm{P}\Sigma_{n-1}^+ \subset \mathrm{P}\Sigma_n^+$, and*

$$G(n,k+r,k-1) = G(n-1,k+r,k-1).$$

Next, consider the projection maps

$$p : H(n,k) \to G(n,k,k-1) \times G(n,k+1,k-1) \times \cdots \times G(n,n-1,k-1)$$

which deletes the coordinate in $G(n,n,k-1)$. Observe that $G(n,k+1,k-1) \times \cdots \times G(n,n-1,k-1)$ is equal to $H(n-1,k-1) \subset \mathrm{P}\Sigma_{n-1}^+$. Furthermore, the maps

$$\Theta(n,k) : G(n,k,k-1) \times G(n,k+1,k-1) \times \cdots \times G(n,n,k-1) \to \mathrm{P}\Sigma_n^+$$

are compatible with the projection maps of

$$\pi : \mathrm{P}\Sigma_n^+ \to \mathrm{P}\Sigma_{n-1}^+$$

of [5] in the following sense.

Lemma 5.4. *If $2 \leqslant k < n$, the group $\Gamma^s(H(n,k))$ is isomorphic to the direct product*

$$\Gamma^s\big(G(n,k,k-1)\big) \times \Gamma^s\big(G(n,k+1,k-1)\big) \times \cdots \times \Gamma^s\big(G(n,n,k-1)\big).$$

There is a morphism of group extensions

$$\begin{CD}
G(n,n,k-1) @>{\Theta(n,k)}>> F_{n-1} \\
@V{i}VV @VV{i}V \\
H(n,k) @>{\Theta(n,k)}>> \mathrm{P}\Sigma_n^+ \\
@V{p}VV @VV{\pi}V \\
H(n-1,k) @>{\Theta(n-1,k-1)}>> \mathrm{P}\Sigma_{n-1}^+ .
\end{CD}$$

Furthermore, there are induced maps of the level of the s-th stage of the descending central series

$$\begin{CD}
\Gamma^s(G(n,n,k-1)) @>{\Theta(n,k)}>> \Gamma^s(F_{n-1}) \\
@V{i}VV @VV{i}V \\
\Gamma^s(H(n,k)) @>{\Theta(n,k)}>> \Gamma^s(\mathrm{P}\Sigma_n^+) \\
@V{p}VV @VV{\pi}V \\
\Gamma^s(H(n-1,k)) @>{\Theta(n-1,k-1)}>> \Gamma^s(\mathrm{P}\Sigma_{n-1}^+),
\end{CD}$$

and the vertical columns are group extensions.

Proof. The first assertion concerning the product decomposition of $\Gamma^s(H(n,k))$ follows from the fact that $H(n,k)$ is a product. That the first diagram commutes follows from the definition of the map $p : H(n,k) \to H(n-1,k)$.

The third assertion concerning the group extensions as well as stages of the descending central series follows by naturality for the $H(n,k)$ and by the Falk–Randell theorem, stated here as Lemma 4.3. □

Since $H(n,k)$ is a direct product of $(n-k)$ free groups, each of which have $(k-1)$ generators, the next corollary follows at once.

Corollary 5.5. *If $2 \leqslant k \leqslant n$, the Lie algebra $gr_*^{DCS} H(n,k)$ is isomorphic to the direct sum of Lie algebras*

$$\bigoplus_{k \leqslant m \leqslant n} gr_*^{DCS} G(n,m,k-1) \cong \bigoplus_{n-k} L[V_{k-1}]$$

with generators for the m-th summand represented by

$$\alpha_{m,1}, \alpha_{m,2}, \dots, \alpha_{m,k-1}$$

for all $n \geqslant m \geqslant k$.

Lemma 5.6. *If* $2 \leqslant k \leqslant n$, *the map*

$$\Gamma^s\big(H(n,k)\big) \xrightarrow{\Theta(n,k)} \Gamma^s\big(\mathrm{P}\Sigma_n^+\big)$$

of Lemma 5.4 *induces a monomorphism of Lie algebras*

$$gr_*^{DCS}\big(H(n,k)\big) \xrightarrow{gr_*(\Theta(n,k))} gr_*^{DCS}\big(\mathrm{P}\Sigma_n^+\big)$$

which is a split monomorphism of abelian groups.

Proof. Observe that $gr_s^{DCS}(\mathrm{P}\Sigma_n^+)$ was computed in Proposition 4.4. Corollary 5.5 states that if $2 \leqslant k \leqslant n$, the Lie algebra $gr_*^{DCS}H(n,k)$ is isomorphic to the direct sum of Lie algebras $\bigoplus_{k \leqslant m \leqslant n} gr_*^{DCS}G(n, m, k-1) \cong \bigoplus_{n-k} L[V_{k-1}]$ with generators for the m-th summand represented by

$$\alpha_{m,1}, \alpha_{m,2}, \dots, \alpha_{m,k-1}$$

for all $m \geqslant k$.

Since the map $\Theta(n,k) : H(n,k) \to \mathrm{P}\Sigma_n^+$ restricts to a map

$$\Theta(n,k) : G(n, k+r, k-1) \to \mathrm{P}\Sigma_{k+r}^+$$

for every $0 \leqslant r \leqslant n-k$, it suffices to check that the induced map $G(n,n,k-1) \to F_{n-1}$, where F_{n-1} is the kernel of $\pi : \mathrm{P}\Sigma_n^+ \to \mathrm{P}\Sigma_{n-1}^+$, induces a split monomorphism on the level of Lie algebras.

Note that $G(n,n,k-1)$ is the subgroup of $\mathrm{P}\Sigma_n^+$ generated by the elements

$$\alpha_{n,1}, \alpha_{n,2}, \dots, \alpha_{n,k-1}.$$

Thus the inclusion $G(n,n,k-1) \to F_{n-1}$ is a split monomorphism on the level of free groups, and hence on the level of Lie algebras. Thus, the induced map

$$gr_s^{DCS}H(n,k) \to gr_s^{DCS}\big(\mathrm{P}\Sigma_n^+\big)$$

is a split monomorphism as it is a direct sum of maps which are monomorphisms of Lie algebras each of which is split as abelian groups. □

The statement and proof of the next standard fact are recorded for the convenience of the reader.

Lemma 5.7. *Let* $\Gamma^s(F_q)$ *denote the s-th stage of the descending central series for the free group* F_q. *Then*

$$\mathbb{Z} \oplus L_s[V_q]^*$$

is a direct summand of the cohomology of $\Gamma^s(F_q)$.

Proof. Recall that the Lie algebra attached to the descending central series of F_q is the free Lie algebra $L[V_q]$ with the s-th graded direct summand given by

$$L_s[V_q] = \Gamma^s(F_q)/\Gamma^{s+1}(F_q).$$

Thus there is a group extension

$$1 \to \Gamma^{s+1}(F_q) \to \Gamma^s(F_q) \to L_s[V_q] \to 1.$$

Since $L_s[V_q]$ is a finitely generated free abelian group, it has a basis over the integers. That basis depends on both s and q. Thus let $\mathbb{S}(s,q)$ denote a set which indexes this basis. Fix a choice of basis b_α, $\alpha \in \mathbb{S}(s,q)$, and let F denote a free group with this choice of basis.

There is a choice of lift of b_α to $\gamma_\alpha \in \Gamma^s(F_q)$ for each $\alpha \in \mathbb{S}(s,q)$. Thus there is an induced homomorphism

$$\Theta : F \to \Gamma^s(F_q)$$

with the property that the composite

$$F \xrightarrow{\Theta} \Gamma^s(F_q) \to L_s[V_q]$$

is an epimorphism.

Thus this composite map induces an isomorphism

$$H_1(F) \to H_1\big(L_s[V_q]\big) = L_s[V_q]$$

as well as an isomorphism

$$H^1\big(L_s[V_q]\big) = L_s[V_q]^* \to H^1(F).$$

It follows that $H^1(L_s[V_q]) = L_s[V]^*$ injects in $H^1(\Gamma^s(F_q))$ by inspection. Furthermore, this injection is split by the map induced in cohomology from $\Theta : F \to \Gamma^s(F_q)$. Thus this gives a direct summand. □

Remark 5.8.

(1) The cohomology of $F_q/\Gamma^s(F_q)$ is not yet well understood for large q. One classical result of Hopf's theorem about the second homology group of a discrete group is an isomorphism

$$H_2\big(F_q/\Gamma^s(F_q)\big) \cong L_{s+1}[V_q].$$

(2) One feature concerning Lemma 5.7 is developed next. The group $\Gamma^s(F_q)$ is a free group. However, control of the generators or even the first homology group is tenuous (as seen from the formulae of Witt for the ranks). Indeed, the first homology group is generally much larger than the group $L_s[V]$. However, this last group $L_s[V]$ is a direct summand of $H_1(\Gamma^s(F_q))$, which allows easy manipulation in this context and forces the rapid growth of the cohomology of J_n^s.

The utility of Lemma 5.7 is as follows.

Corollary 5.9. *Assume that $2 \leqslant k \leqslant n$.*

(1) *The integral cohomology algebra of $H(n,k)$ is isomorphic to*

$$\bigotimes_{n-k} \big(\mathbb{Z} \oplus L_1[V_{k-1}]^*\big) = \bigotimes_{n-k} \big(\mathbb{Z} \oplus V_{k-1}^*\big).$$

(2) *The integral cohomology algebra of $\Gamma^s H(n,k)$ contains a subalgebra which is isomorphic to*

$$\bigotimes_{n-k} \big(\mathbb{Z} \oplus L_s[V_{k-1}]^*\big),$$

which is the image of the natural map

$$H^*\big(gr_s^{DCS} H(n,k)\big) \to H^*\big(\Gamma^s H(n,k)\big).$$

6. Values of the Johnson homomorphism on certain subgroups

The purpose of this section is to derive the values of the Johnson homomorphism for certain subgroups of $\mathrm{P}\Sigma_n^+$. Recall from Section 3 that $\mathrm{P}\Sigma_n^+$ is generated by the subset of Magnus generators

$$\{\alpha_{ij} \mid 1 \leqslant j < i \leqslant n\}.$$

For ease of notation, fix q and write

$$w_r = \alpha_{qr}$$

for $q > r$. A fact observed in [5] is that the elements $w_r = \alpha_{qr}$ for $1 \leqslant r \leqslant q-1$ give a basis for a free group in $\mathrm{P}\Sigma_n^+$.

Throughout the remainder of this section, it will be tacitly assumed that

$$1 \leqslant r \leqslant q-1.$$

The formulae are verified as follows. First consider the action of

$$w_r = \alpha_{qr}$$

for $q > r$ on x_t:

$$w_r(x_t) = \begin{cases} x_t & \text{if } t \neq q, \\ x_r x_q x_r^{-1} & \text{if } t = q. \end{cases}$$

Thus for example

$$1 \leqslant r_i \leqslant q-1,$$

$$(w_{r_1} w_{r_2})(x_t) = w_{r_1}\big(w_{r_2}(x_t)\big) = \begin{cases} w_{r_1}(x_t) = x_t & \text{if } t \neq q, \\ w_{r_1}(x_{r_2} x_q x_{r_2}^{-1}) = x_{r_1}(x_{r_2} x_q x_{r_2}^{-1}) x_{r_1}^{-1} & \text{if } t = q. \end{cases}$$

Next, consider a product given by

$$\mathcal{W} = w_{r_1}^{\epsilon_1} \cdot w_{r_2}^{\epsilon_2} \cdots w_{r_m}^{\epsilon_m}$$

for $\epsilon_i = \pm 1$ for $1 \leqslant r_i \leqslant q-1$. We begin to record some formulae essential to our computations in the sequel.

Lemma 6.1. *If*

$$\mathcal{W} = w_{r_1}^{\epsilon_1} \cdot w_{r_2}^{\epsilon_2} \cdots w_{r_m}^{\epsilon_m}$$

for $\epsilon_i = \pm 1$ for $1 \leqslant r_i \leqslant q-1$, the action of $\mathcal{W}$ is specified by the formula

$$\mathcal{W}(x_t) = \begin{cases} x_t & \textit{if } t \neq q, \\ (x_{r_1}^{\epsilon_1} \cdot x_{r_2}^{\epsilon_2} \cdots x_{r_m}^{\epsilon_m}) \cdot x_q \cdot (x_{r_1}^{\epsilon_1} \cdot x_{r_2}^{\epsilon_2} \cdots x_{r_m}^{\epsilon_m})^{-1} & \textit{if } t = q. \end{cases}$$

Thus for example, the action of the commutator

$$[w_{r_1}, w_{r_2}] = w_{r_1}^{-1} w_{r_2}^{-1} w_{r_1} w_{r_2}$$

on x_t is specified by

$$[w_{r_1}, w_{r_2}](x_t) = \begin{cases} x_t & \text{if } t \neq q, \\ [x_{r_1}, x_{r_2}] \cdot x_q \cdot [x_{r_1}, x_{r_2}]^{-1} & \text{if } t = q. \end{cases}$$

The formula for the action of the commutator

$$\Lambda = \big[\cdots\big[[w_{r_1}, w_{r_2}]\cdots\big]w_{r_m}\big] \in \mathrm{IA}_n$$

on x_t for $1 \leqslant r_i \leqslant q-1$ is thus given by the formula

$$\Lambda(x_t) = \begin{cases} x_t & \text{if } t \neq q, \\ \Lambda_x \cdot x_q \cdot \Lambda_x^{-1} & \text{if } t = q \end{cases}$$

where

$$\Lambda_x = \big[\cdots\big[[x_{r_1}, x_{r_2}]\cdots\big]x_{r_m}\big],$$

the commutator formally obtained by replacing each w_{r_i} by x_{r_i} in the commutator Λ.

Values resulting from applying the Johnson homomorphism are recorded next.

Proposition 6.2. *Consider the commutator $\Lambda = [\cdots[[w_{r_1}, w_{r_2}], \ldots], w_{r_t}]$. If $r_1, r_2, \ldots, r_t < q$, then*

$$\tau_s(\Lambda)(x_t) = \begin{cases} x_t & \textit{if } t \neq q, \\ \Lambda_x \cdot x_q \cdot \Lambda_x^{-1} \cdot x_q^{-1} = [\Lambda_x^{-1}, x_q{}^{-1}] & \textit{if } t = q. \end{cases} \qquad \square$$

The next statement records implications of these formulae on the level of Lie algebras.

Corollary 6.3. *The composite morphism of Lie algebras denoted*

$$J : \bigoplus_{s\geqslant 1} gr_s^{DCS}\big(\mathrm{P}\Sigma_n^+\big) \to Der\big(L[V_n]\big)$$

given by

$$\bigoplus_{s\geqslant 1} gr_s^{DCS}\big(\mathrm{P}\Sigma_n^+\big) \to \bigoplus_{s\geqslant 1} gr_s^{J}(\mathrm{IA}_n) \to Der\big(L[V_n]\big)$$

which is induced by the Johnson homomorphisms is injective, and is split injective as abelian groups (*but not split as Lie algebras*).

7. The last step

By Corollary 6.3, the composite morphism of Lie algebras

$$J : gr_*^{DCS}\big(\mathrm{P}\Sigma_n^+\big) \to Der\big(L[V_n]\big)$$

given by

$$\bigoplus_{s\geqslant 1} gr_s^{DCS}\big(\mathrm{P}\Sigma_n^+\big) \to \bigoplus_{s\geqslant 1} gr_s^{J}(\mathrm{IA}_n) \to Der\big(L[V_n]\big)$$

is injective, and is additively split. By 5.6, the morphism of Lie algebras

$$gr_*^{DCS}\big(H(n,k)\big) \xrightarrow{gr_*(\Theta(n,k))} gr_*^{DCS}\big(\mathrm{P}\Sigma_n^+\big)$$

is a monomorphism which is additively split in case $2 \leqslant k \leqslant n$. The next theorem follows at once.

Theorem 7.1. *If $n \geqslant 3$, and $2 \leqslant k \leqslant n$, the composite homomorphism*

$$H(n,k) \to \mathrm{P}\Sigma_n^+ \to \mathrm{IA}_n$$

induces a morphism of Lie algebras

$$gr_*^{DCS}\big(H(n,k)\big) \to gr_*^{DCS}\big(\mathrm{P}\Sigma_n^+\big) \to gr_*^{J}(\mathrm{IA}_n) \to Der\big(L[V_n]\big).$$

This composite is a monomorphism of Lie algebras and is a split monomorphism of abelian groups.

Since the composite map of Theorem 7.1

$$\gamma : gr_s^{DCS}\big(H(n,k)\big) \to Hom\big(V_n, L_{s+1}[V_n]\big)$$

is a split monomorphism of finitely generated, free abelian groups, the map γ induces a split epimorphism in integer cohomology

$$\gamma^* : H^*\big(Hom\big(V_n, L_{s+1}[V_n]\big)\big) \to H^*\big(gr_s^{DCS}\big(H(n,k)\big)\big).$$

Observe that the cohomology ring of $gr_s^{DCS}(H(n,k))$ is isomorphic to that of the product

$$\Gamma^s\big(G(n,k,k-1)\big) \times \Gamma^s\big(G(n,k+1,k-1)\big) \times \cdots \times \Gamma^s\big(G(n,n,k-1)\big)$$

by Lemma 5.4. Thus, the cohomology of $\Gamma^s H(n,k)$ contains

$$\bigotimes_{n-k+1} \big(\mathbb{Z} \oplus L_s[V_{k-1}]^*\big)$$

by Lemma 5.7. On the other hand, the natural quotient map

$$\Gamma^s H(n,k) \xrightarrow{gr_s(\Theta(n,k))} gr_s^{DCS}\big(H(n,k)\big)$$

induces a surjection onto its image in cohomology given in Corollary 5.9 by

$$\bigotimes_{n-k+1} \big(\mathbb{Z} \oplus L_s[V_{k-1}]^*\big).$$

The next statement as well as the main Theorem 1.1 follows by setting $q = k-1$: the case $k = 2$ is deleted as $L_s[V_1]^* = \{0\}$ for $s > 1$.

Theorem 7.2. *If $n \geqslant 3$, and $3 \leqslant k \leqslant n$, the integral cohomology ring $H^*(J_n^s)$ contains a direct summand which is additively isomorphic to*

$$\bigotimes_{n-k+1} \big(\mathbb{Z} \oplus L_s[V_{k-1}]^*\big).$$

Furthermore, this summand is in the image of the map induced by the Johnson homomorphism on integral cohomology groups

$$(\tau_s)^* : H^*\big(Hom\big(V_n, L_{s+1}[V_n]\big)\big) \to H^*\big(J_n^s\big).$$

Since the composite

$$J_n^{s+1} \to J_n^s \xrightarrow{\tau_s} Hom\big(V_n, L_{s+1}[V_n]\big)$$

is constant by definition of the Johnson homomorphism (as given in Section 2), the next result, the ‘fragility’ of these cohomology classes, follows at once.

Corollary 7.3. *If $n \geqslant 3$, and $2 \leqslant k \leqslant n$, the composite*

$$J_n^{s+1} \to J_n^s \xrightarrow{\tau_s} Hom\big(V_n, L_{s+1}[V_n]\big)$$

gives the trivial map in cohomology when restricted to

$$\bigotimes_{n-k+1} L_s[V_{k-1}]^*,$$

and these classes are in the kernel of the map

$$H^*\big(J_n^s\big) \to H^*\big(J_n^{s+1}\big).$$

To estimate the ranks of the free abelian groups $L_s[V_q]^*$ for fixed $1 \leqslant q \leqslant n-1$ where $V_q = \bigoplus_q \mathbb{Z} = H_1(F_q)$, classical work of Witt is recalled next [24,25]. For fixed filtration degree s, write $d_s(V)$ for the rank of the free abelian group $L_s[V]$ occurring above and consider the power series

$$\sum_{s \geqslant 0} d_s(V) t^s$$

where by convention

$$d_0(V) = 1.$$

Thus

$$d_1(V) = q.$$

Following Witt's application of the Poincaré–Birkhoff–Witt theorem,

$$1/(1-qt) = \prod_{s \geqslant 1} 1/\big(1-t^s\big)^{d_s(V)}.$$

To find an inductive formula for the coefficients $d_s(V)$, take formal logarithms of both sides of this equation to obtain the formula

$$q^s = \sum_{m|s} m d_m(V).$$

An elegant exposition for this information is in Serre's book [24]. Observe that

$$s d_s(V) = q^s - \sum_{m|s,\, m<s} m d_m(V).$$

Next, specialize to the case of filtration degree s for the summand $L_s[V]$ with

$$s = p^r, \quad p \text{ is assumed to be prime.}$$

The formula $q^s = \sum_{m|p^r} m d_m(V)$ then simplifies to

$$q^{p^r} = \sum_{0 \leqslant i \leqslant r} p^i d_{p^i}(V) = d_1(V) + p d_p(V) + \cdots + p^{r-1} d_{p^{r-1}}(V) + p^r d_{p^r}(V).$$

To illustrate this computation, some values are listed next:

$$d_{p^r}(V) = \begin{cases} q, & r = 0, \\ (q^{p^r} - q^{p^{r-1}})/p^r, & r > 0. \end{cases}$$

Thus in case $q > p$ for a fixed prime p, the previous formula illustrates the rapid growth of the values $d_{p^r}(V)$.

8. Further comparison with earlier work

This section consists of a remark concerning work of M. Bestvina, K. Bux, and D. Margalit [2]. They exhibit an abelian subgroup of IA_n determined by the automorphisms

$$\rho(p_j, q_j) : H_n \to F_n$$

defined by

$$x_1 \to x_1,$$
$$x_2 \to x_2,$$
$$x_j \to w^{p_j} x_j w^{q_j},$$

where $w = [x_1, x_2]$, $j > 2$.

Depending on the choices of p_j and q_j, these elements live in various stages of the Johnson filtrations. For example, if

$$p_j = 1 \quad \text{and} \quad q_j = -1,$$

then the elements $\rho(p_j, q_j)$ are in $\Gamma^2 \mathrm{P}\Sigma_n^+$. The groups $H(n,k)$ of Section 5 give non-trivial abelian subgroups in J_n^s for large s. It is natural to ask whether the above methods imply that $H^i(J_n^s)$ fails to be finitely generated as long as $n > 2$, $s > 2$, and $2 \leqslant i \leqslant n - 2$.

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Appendix A

The purpose of this section is to list the natural Euler–Poincaré series associated to the Lie algebra

$$Der\big(L[V_n]\big) = \bigoplus_{1 \leqslant s} Hom\big(V_n, L_{s+1}[V_n]\big)$$

where each module $Hom(V_n, L_{s+1}[V_n])$ is formally assigned gradation s.

The reason for doing so is that these modules are the natural images of the Johnson homomorphism, which is injective. Thus these maps are split rationally, and so the computation given next may provide a setting for enumerating the cokernel of the Johnson homomorphisms in a 'global' way.

Recall that the rank of V_n is n, and the rank of $L_s[V_n]$ is

$$d_s(V_n)$$

subject to the relations discovered by Witt as described in Section 7. Thus the natural Euler–Poincaré series associated to the Lie algebra $Der(L[V_n])$ is

$$\chi\big(Der\big(L[V_n]\big)\big) = \sum_{1\leqslant s} n \cdot d_{s+1}(V_n) \cdot t^s.$$

It seems likely that the analogous series for the Johnson Lie algebra should admit an analogous description in terms of the $d_{s+1}(V_n)$.

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