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On Cantor sets with shadows of prescribed dimension ☆

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ABSTRACT

We consider a question raised by John Cobb: given positive integers $n > l > k$ is there a Cantor set in $\mathbb{R}^n$ such that all whose projections onto l-dimensional planes are exactly k-dimensional? We construct in $\mathbb{R}^n$ a Cantor set such that all its shadows (projections onto hyperplanes) are k-dimensional for every $0 \leqslant k \leqslant n-1$. We also consider the extension of Cobb's question to Hilbert space.

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1. Introduction

Borsuk [2] constructed a Cantor set in $\mathbb{R}^n$ such that every projection onto a hyperplane (called a *shadow*) contains an $(n-1)$-dimensional ball, and therefore is $(n-1)$-dimensional; see Dijkstra, Goodsell, and Wright [5] for a simple proof. Cobb [4] has constructed a Cantor set in $\mathbb{R}^3$ such that all its shadows are exactly one-dimensional. He posed the following question: given integers $n > l \geqslant k \geqslant 0$, is there a Cantor set in $\mathbb{R}^n$ such that all its projections onto l-dimensional planes are exactly k-dimensional. We will refer to this question as the case (n, l, k) of Cobb's problem. So Cobb's theorem covers the case (3, 2, 1) and observe that Borsuk solved the case (n, l, l). Subsequently, Frolkina [7] used Cobb's construction as basis for her positive solution to the case $(n, l, l-1)$. Note that the case $(n, l, 0)$ is trivial, simply take a Cantor set on a line.

In the current paper we also use Cobb's construction for the case $(3, 2, 1)$ to give a positive solution for the case $(n, n-1, k)$:

Theorem 1. *Let n and k be integers with $0 \leqslant k \leqslant n-1$. Then there exists a Cantor set in $\mathbb{R}^n$ such that all its shadows are k-dimensional.*

The next two theorems concern projections of Cantor sets in ℓ^2, the Hilbert space of square summable real sequences. Theorem 2 is a positive result about projections onto finite-dimensional planes and a consequence of Theorem 1. Theorem 3 gives a negative answer concerning projections onto planes of finite codimension.

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Theorem 2. *For $m \in \mathbb{N}$ there exists a Cantor set in ℓ^2 such that its projections onto all m-planes are exactly $(m-1)$-dimensional.*

Let $\mathbb{V}$ stand for either $\mathbb{R}^n$ or ℓ^2. For $m \in \mathbb{N}$, $\mathcal{G}_m(\mathbb{V})$ denotes the Grassmann manifold consisting of all m-dimensional linear subspaces of $\mathbb{V}$. If $P \in \mathcal{G}_m(\mathbb{V})$ then ψ_P denotes the projection in $\mathbb{V}$ along P onto the orthocomplement $P^\perp$.

Theorem 3. *Let $m \in \mathbb{N}$ and let K be compact in ℓ^2. Then the restriction $\psi_P \restriction K$ is an imbedding for all P in some dense subset of $\mathcal{G}_m(\ell^2)$.*

Thus every Cantor set in ℓ^2 has many shadows that are Cantor sets.

2. Definitions and preliminaries

Let $\mathbb{V}$ stand for a separable real Hilbert space with an inner product $x \cdot y$ and origin $\mathbf{0}$. Therefore, $\mathbb{V}$ is isomorphic to either an $\mathbb{R}^n$ or ℓ^2. The norm on $\mathbb{V}$ is given by $\|u\| = \sqrt{u \cdot u}$ and the metric d is given by $d(u, v) = \|v - u\|$. A *plane* in $\mathbb{V}$ is a closed affine subspace of $\mathbb{V}$; a *k-subspace* is a k-dimensional linear subspace of $\mathbb{V}$.

Let A be a subset of $\mathbb{V}$. We have that $\operatorname{lin} A$ denotes the linear hull, $\langle A \rangle$ the convex hull of A and $\operatorname{aff} A$ the closed affine hull of A, that is, the intersection of all planes containing A. If $v, w \in \mathbb{V}$ then $[v, w] = \langle \{v, w\} \rangle$, the line segment with endpoints v and w. The *geometric interior* A° of A is the interior of A relative to $\operatorname{aff} A$. We define $A^\perp$ in the following way:

$$A^\perp = \{v \in \mathbb{V}\colon\ v \cdot x = v \cdot y \text{ for all } x, y \in A\}.$$

Also, we define $\operatorname{codim} A = \dim A^\perp \in \{0, 1, \ldots, \infty\}$. If A is a closed linear space then $A^{\perp\perp} = A$ and $A^\perp$ is called the *orthocomplement* of A. Further, let L be a plane in $\mathbb{V}$. A plane $H \subset L$ is called a *k-hyperplane* in L if $\dim(H^\perp \cap L) = k$. In other words, a k-hyperplane is a plane with codimension k in the ambient space. A *hyperplane* H of L is a plane of L of codimension 1. Given a coordinate system in $\mathbb{V}$ we denote by $\pi_i : \mathbb{V} \to \mathbb{R}$ the projection onto the ith coordinate axis.

Definition 1. Let L be a plane in $\mathbb{V}$. Then $\psi_L : \mathbb{V} \to L^\perp$ denotes the orthogonal projection along L onto $L^\perp$ defined by the conditions $\psi_L(x) - x \in L^{\perp\perp}$ and $\psi_L(x) \in L^\perp$ for each $x \in \mathbb{V}$. Note that if $\mathbf{0} \in L$ then $\{\psi_L(x)\} = L^\perp \cap (x + L)$ and that $\psi_{L^\perp}$ is the projection onto L.

Definition 2. Let $A \subset \mathbb{V}$. Any projection of A onto a hyperplane is called a *shadow* of A. Thus $\psi_\ell(A)$ is a shadow when ℓ is a line in $\mathbb{V}$.

Definition 3. Let $\mathcal{K}(\mathbb{V})$ stand for all non-empty compact subsets of a $\mathbb{V}$. Recall that the Hausdorff metric d_{H} on $\mathcal{K}(\mathbb{V})$ associated with d is defined as follows:

$$d_{\mathrm{H}}(A, B) = \sup\bigl\{d(x, A), d(y, B)\colon\ x \in B \text{ and } y \in A\bigr\}.$$

We let $\mathcal{G}_k(\mathbb{V})$ stand for the collection of all k-subspaces of $\mathbb{V}$. Consider the ball $\mathbb{B} = \{v \in \mathbb{V}\colon\ \|v\| \leqslant 1\}$. We topologize $\mathcal{G}_k(\mathbb{V})$ by defining a metric ρ on $\mathcal{G}_k(\mathbb{V})$:

$$\rho(L_1, L_2) = d_{\mathrm{H}}(L_1 \cap \mathbb{B}, L_2 \cap \mathbb{B}).$$

When $\mathbb{V}$ is finite-dimensional then $\mathcal{G}_k(\mathbb{V})$ is compact and is known as a *Grassmann manifold*.

Next, we prove a lemma that links the dimension of a subspace of $\mathbb{R}^n$ to some information about the dimension of its projections.

Lemma 4. *Let C be a subset of $\mathbb{R}^n$. If $\ell_1, \ldots, \ell_k$ are lines generated by linearly independent vectors in $\mathbb{R}^n$ such that $\psi_{\ell_i^\perp}(C)$ is zero-dimensional for each $i \in \{1, \ldots, k\}$, then C is at most $(n-k)$-dimensional.*

Proof. Set $\ell_i = \mathbb{R}u_i$ for independent unit vectors $u_1, \ldots, u_k$. Now these vectors span a k-dimensional subspace, so add $n-k$ unit vectors $u_{k+1}, \ldots, u_n$ such that the set $\{u_1, \ldots, u_n\}$ is a basis for $\mathbb{R}^n$. Then there exists a linear isomorphism of vector spaces $f : \mathbb{R}^n \to \mathbb{R}^n$ given by $(\pi_i \circ f)(x) = u_i \cdot x$. In other words, $f(x) = Ax$, where the $n \times n$-matrix A is given by putting u_i on row i, for $i = 1, \ldots, n$, that is,

$$A = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}.$$

The map f is obviously a homeomorphism. Then for $x \in \mathbb{R}^n$ we have $\psi_{\ell_i^\perp}(x) = (u_i \cdot x)u_i$. Therefore $\psi_{\ell_i^\perp}(C) \approx (\pi_i \circ f)(C)$. So in particular, $(\pi_i \circ f)(C)$ is zero-dimensional for each $i \in \{1, \ldots, k\}$. Hence, the image of C under f is contained in $(\pi_1 \circ f)(C) \times (\pi_2 \circ f)(C) \times \cdots \times (\pi_k \circ f)(C) \times \mathbb{R}^{n-k}$, which, by [6, Theorem 1.5.16], is at most $(n-k)$-dimensional. □

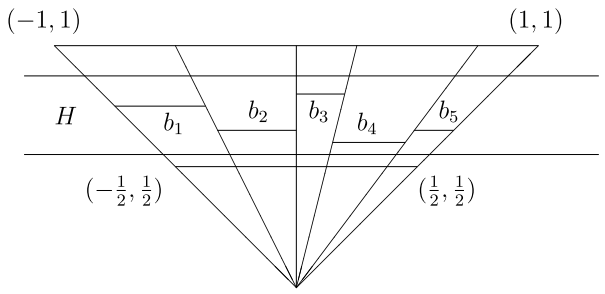

Fig. 1. Baffles b_i in the horizontal strip H.

Now, we list some background information needed for the proof of Theorem 1. Let ℓ be any line in $\mathbb{R}^2$, and let $\varepsilon > 0$. A bounded set $X \subset \mathbb{R}^2$ is said to have *projective size* (ε, δ) *with respect to* ℓ if $\psi_{\ell^\perp}(X)$ can be covered by finitely many pairwise disjoint closed intervals in ℓ, such that each interval has length less than ε, and any two intervals are at least δ apart.

A bounded set X has *absolute size* (ε, δ) (see [4]) if it is of projective size (ε, δ) with respect to *any* line $\ell \subset \mathbb{R}^2$. A bounded set X has *absolute size* ε if, for any line $\ell \subset \mathbb{R}^2$ we have that $\psi_{\ell^\perp}(X)$ contains no intervals of length at least ε. In other words, for each line ℓ, the set $\ell \setminus \psi_{\ell^\perp}(X)$ is an $\frac{\varepsilon}{2}$-net in ℓ. Obviously, if a set X has absolute size (ε, δ), for some $\delta > 0$, then it also has absolute size ε.

Lemma 5. *Let $\varepsilon > 0$. If a bounded set $X \subset \mathbb{R}^2$ has absolute size (ε, δ) and $\varepsilon' > \varepsilon$, then $\exists \eta > 0$, $\delta' > 0$ such that the closed η-neighbourhood of X has absolute size (ε', δ').*

Proof. Let $\alpha = \frac{\varepsilon' - \varepsilon}{2}$. Then choose $\eta = \min\{\alpha, \frac{\delta}{3}\}$. Let ℓ be any line in $\mathbb{R}^2$. Since the projection of the closed η-neighbourhood of X onto ℓ is equal to the closed η-neighbourhood of the projection of X we have that it can be covered by finitely many closed intervals, each of length less than ε' and such that any two intervals are at least $\frac{\delta}{3}$ apart. Therefore the closed η-neighbourhood of X has absolute size $(\varepsilon', \frac{\delta}{3})$. □

We now recall how Cobb [4] constructs sets of a particular absolute size. In $\mathbb{R}^2$, let F denote the closed horizontal segment $[(-1, 1), (1, 1)]$, and let K denote the trapezoid with vertices $(-1, 1)$; $(1, 1)$; $(-\frac{1}{2}, \frac{1}{2})$; $(\frac{1}{2}, \frac{1}{2})$. Let S be any partition of F; i.e. a finite set of points of F, including both endpoints, and let S be ordered from left to right: $S = \{s_0, s_1, \ldots, s_k\}$ with $s_0 = (-1, 1)$ and $s_k = (1, 1)$. Let H be a horizontal strip through F, which means $H = \mathbb{R} \times A$, where $A \subset (\frac{1}{2}, 1)$ is a closed interval.

A set of *baffles* $\mathcal{B}$ *for* S *in* H is a finite collection of pairwise disjoint horizontal closed segments in H, one for each segment $[s_i, s_{i+1}]$, $0 \leqslant i < k$, with endpoints on the line segments $[\mathbf{0}, s_i]$ and $[\mathbf{0}, s_{i+1}]$. Therefore, each line intersecting F and $\mathbf{0}$ passes through at least one of the baffles, and at most two of them. An example is shown in Fig. 1. We conclude this section by stating Lemma 1 from Cobb [4]:

Lemma 6. *Given a partition S of F, a finite collection $\{H_1, \ldots, H_p\}$ of pairwise disjoint horizontal strips through K, and any $\varepsilon > 0$, there exist a subdivision S' of S, a $\delta > 0$, and for each H_i a set of baffles $\mathcal{B}_i$ for S' in H_i, such that the union of all the baffles in all the $\mathcal{B}_i$'s has absolute size (ε, δ).*

3. Projections in $\mathbb{R}^n$

Our aim in this section is to prove Theorem 1. First, we do some necessary preparation. Put $m = n - k - 1$ with the k and n from Theorem 1. We are going to construct a Cantor set in $\mathbb{R}^{m+k+1}$ such that all its shadows have dimension k. In short, we prove the case $(m + k + 1, m + k, k)$ of the problem stated in the introduction.

In $\mathbb{R}^{m+1}$, let F be the m-cube $\{1\} \times [-1, 1]^m$ and let $F' = \frac{1}{2}F = \{\frac{1}{2}\} \times [-\frac{1}{2}, \frac{1}{2}]^m$. Define $K = \langle F \cup F' \rangle$. Now in $\mathbb{R}^{m+k+1}$ let

$$P = F \times [-1, 1]^k, \qquad Q = F' \times [-1, 1]^k,$$
$$W = \langle P \cup Q \rangle, \quad \text{and} \quad J = \{0\}^{m+1} \times [-1, 1]^k.$$

By Z_i, for $2 \leqslant i \leqslant m+1$, we denote the x_1x_i-plane in $\mathbb{R}^{m+k+1}$ (where we view the x_1-axis as the vertical axis and the x_i-axis as horizontal), and by Z we denote the $x_1 \ldots x_{m+1}$-plane i.e. Z can be identified with $\mathbb{R}^{m+1}$. Also let F_i be the line segment $P \cap Z_i$, for $2 \leqslant i \leqslant m + 1$, and $F'_i = Q \cap Z_i$. Let $\xi : F \to F'$ be the radial projection towards $\mathbf{0}$, that is, multiplication by $\frac{1}{2}$ in all coordinates, and let $\bar{\xi} : P \to Q$ be defined by $\bar{\xi} = \xi \times 1_k$, where 1_k is the identity on the last k coordinates.

A *regular subdivision* $\mathcal{R}$ of P is defined as follows. First we take finite partitions of the closed interval $[-1, 1]$ for each of the coordinate axes x_i, where $i = 2, \ldots, m + k + 1$, ordered from small to large: $-1 = s_{i,1}, \ldots, s_{i,n_i} = 1$. Then the sets of the $\mathcal{R}$ are obtained by taking the product of the closed intervals between successive elements of the partitions, for each

coordinate axis. In other words, each set of $\mathcal{R}$ has the form $C = \{1\} \times [a_2, b_2] \times \cdots \times [a_{m+k+1}, b_{m+k+1}]$, where a_i and b_i are consecutive elements in the partition of the x_i-axis. It is important to note that the elements of $\mathcal{R}$ form a finite collection of closed subsets of P, such that their relative interiors are non-empty and pairwise disjoint, and $\bigcup \mathcal{R} = P$. It is easy to see that if $\mathcal{R}$ is a regular subdivision of P, and $\mathcal{P}$ is obtained by subdividing each element of $\mathcal{R}$ in the way described above, then $\mathcal{P}$ is also a regular subdivision of P. In this case we will sometimes call $\mathcal{P}$ a regular subdivision of $\mathcal{R}$ instead of P. We define regular subdivisions of Q analogously.

Lemma 7. *W contains a Cantor set G such that any line through P and J will intersect G, the projection of G onto the x_1-axis is one-to-one, and the projection of G onto any line in any Z_i is zero-dimensional.*

Proof. Let $(\varepsilon_n)_n$ be a sequence of positive real numbers with $\lim_{n\to\infty} \varepsilon_n = 0$. We will recursively define regular subdivisions $\mathcal{A}^n$ of P and pairwise disjoint collections $\mathcal{C}^n$ of closed subintervals of $[\frac{1}{2}, 1]$, such that $\mathcal{A}^n$ refines $\mathcal{A}^{n-1}$ and $\mathcal{C}^n$ refines $\mathcal{C}^{n-1}$. Define

$$\mathcal{B}^n = \left\{\bar{\xi}(A)\colon\ A \in \mathcal{A}^n\right\}$$

thus it is a regular subdivision of Q, and let

$$\Psi^n = \left\{(A, B) \in \mathcal{A}^n \times \mathcal{B}^n\colon\ \xi\left(\psi_{Z^\perp}(A)\right) = \psi_{Z^\perp}(B)\right\}.$$

We will also construct functions $\varphi^n : \Psi^n \to \mathcal{C}^n$ such that the following four conditions hold:

(1) φ^n is a bijection;
(2) if $(A, B) \in \Psi^{n-1}$ and $(A', B') \in \Psi^n$ such that $A' \subset A$ and $B' \subset B$ then $\varphi^n(A', B') \subset \varphi^{n-1}(A, B)$;
(3) for any $(A, B) \in \Psi^n$ we have $\operatorname{diam}(D^n(A, B)) < \varepsilon_n$, where $D^n(A, B) = \pi_1^{-1}(\varphi^n(A, B)) \cap \langle A \cup B\rangle$;
(4) for $2 \leqslant i \leqslant m+1$, $\psi_{Z_i^\perp}(G^n)$ has absolute size ε_n, where $G^n = \bigcup\{D^n(A, B)\colon\ (A, B) \in \Psi^n\}$.

We start the induction by assuming that ε_0 is large, and then we pick $\mathcal{A}^0 = \{P\}$, $\mathcal{C}^0 = \{[\frac{1}{2}, 1]\}$. Now assume that the n-th level has been defined, i.e. we have $\mathcal{A}^n$, $\mathcal{B}^n$, Ψ^n, $\mathcal{C}^n$ and $\varphi^n : \Psi^n \to \mathcal{C}^n$. Let $\mathcal{A}_1^n$ be a regular subdivision of $\mathcal{A}^n$, such that each element has diameter at most $\varepsilon_{n+1}/2$. Let $\mathcal{B}_1^n$ and Ψ_1^n be determined by $\mathcal{A}_1^n$ in the obvious way. For any $(A, B) \in \Psi^n$, the set $\{(A_1, B_1) \in \Psi_1^n\colon\ A_1 \subset A, B_1 \subset B\}$ is finite. Corresponding to each of its elements, choose pairwise disjoint closed intervals $\varphi_1^n(A_1, B_1)$ in the interior of $\varphi^n(A, B)$. This defines $\mathcal{C}_1^n$ and φ_1^n. Then φ_1^n satisfies conditions (1) and (2).

Let S_2 be the partition of the line segment F_2 obtained by taking the endpoints of the projections of the elements of $\mathcal{A}_1^n$ onto Z_2. Now we apply Lemma 6 in the plane Z_2, using $\{C \times \mathbb{R}\colon\ C \in \mathcal{C}_1^n\}$ as horizontal strips in $W \cap Z_2$, to S_2. This gives a refinement S_2' of S_2, for each $C \in \mathcal{C}_1^n$ a collection of baffles $\mathcal{F}_C$ in the horizontal strip $C \times \mathbb{R}$, and (by Lemma 5) an $\eta > 0$ such that the η-neighbourhood of $\bigcup\bigcup\{\mathcal{F}_C\colon\ C \in \mathcal{C}_1^n\}$ in Z_2 has absolute size $\frac{1}{2}\varepsilon_{n+1}$. Without loss of generality, η can be chosen so small that the distance between any two different baffles is greater than 2η.

Define $\mathcal{A}_2^n$ by subdividing each element of $\mathcal{A}_1^n$ in the x_2-coordinate at the points of S_2'. This also defines $\mathcal{B}_2^n$ and Ψ_2^n in the same way as before. Let $(A, B) \in \Psi_1^n$. Then the set $\{(A', B') \in \Psi_2^n\colon\ A' \subset A,\ B' \subset B\}$ is finite. Since the endpoints of $\psi_{Z_2^\perp}(A')$ are in S_2', there is exactly one baffle in the set $\mathcal{F}_{\varphi_1^n(A,B)}$ with endpoints on the rays from $\mathbf{0}$ to the endpoints of $\psi_{Z_2^\perp}(A')$. Define the map

$$f : \left\{\left(A', B'\right) \in \Psi_2^n\colon\ A' \subset A,\ B' \subset B\right\} \to \mathcal{F}_{\varphi_1^n(A,B)}$$

by assigning to each pair (A', B') the baffle element in the strip $\varphi_1^n(A, B) \times \mathbb{R}$ which corresponds to the endpoints of $\psi_{Z_2^\perp}(A')$. For each pair (A', B') we create a distinct horizontal segment in the horizontal strip $\varphi_1^n(A, B) \times \mathbb{R}$ by taking the baffle $f(A', B')$ and moving it slightly in x_1-direction so that it stays within the $\frac{1}{4}\eta$-neighbourhood of $f(A', B')$. For each pair (A', B') we move the baffle to a different level, so that we end up with a finite number of horizontal segments in the strip $\varphi_1^n(A, B) \times \mathbb{R}$, one for each element of $\{(A', B') \in \Psi_2^n\colon\ A' \subset A,\ B' \subset B\}$ and each one with a different x_1-coordinate. Repeat this process for each pair $(A, B) \in \Psi_1^n$ to create a distinct horizontal segment for each pair $(A', B') \in \Psi_2^n$, such that for any two pairs in Ψ_2^n, the corresponding segments have a different x_1-coordinate. The projection of all these segments onto the x_1-axis consists of a finite number of points of $\bigcup \mathcal{C}_1^n$. Choose a collection $\mathcal{C}_2^n$ that refines $\mathcal{C}_1^n$ and consists of one interval about each of these points, such that all these intervals are pairwise disjoint, and all are of diameter less than $\min\{\frac{1}{4}\varepsilon_{n+1}, \frac{1}{4}\eta\}$. Then define φ_2^n by assigning to each pair $(A', B') \in \Psi_2^n$ the interval in $\mathcal{C}_2^n$ that contains the π_1-projection of the baffle that corresponds to (A', B'). At this point, we have constructed $\mathcal{A}_2^n$, $\mathcal{B}_2^n$, Ψ_2^n, $\mathcal{C}_2^n$, and φ_2^n. Next, we have $D_2^n(A, B)$ and G_2^n, corresponding to that construction, defined by

$$D_2^n(A, B) = \pi_1^{-1}\left(\varphi_2^n(A, B)\right) \cap \langle A \cup B\rangle \quad \text{and}$$
$$G_2^n = \bigcup\left\{D_2^n(A, B)\colon\ (A, B) \in \Psi_2^n\right\}.$$

Now φ_2^n satisfies (1) and (2), by construction. Condition (3) is satisfied because $\operatorname{diam} D_2^n(A,B) < 2\operatorname{diam}\varphi_2^n(A,B) + \operatorname{diam} A < \frac{1}{2}\varepsilon_{n+1} + \frac{1}{2}\varepsilon_{n+1} = \varepsilon_{n+1}$. Furthermore, condition (4) is also satisfied for projections onto Z_2, because $\psi_{Z_2^\perp}(G_2^n)$ is contained in the η-neighbourhood of $\bigcup\bigcup\{\mathcal{F}_C\colon C \in \mathcal{C}_1^n\}$ in Z_2.

We repeat this construction; let S_3 be the partition of F_3 obtained by taking the endpoints of $\mathcal{A}_2^n$ and projecting them onto F_3. Following the above steps and applying Lemma 6 once more we obtain $\mathcal{A}_3^n$, $\mathcal{B}_3^n$, Ψ_3^n, $\mathcal{C}_3^n$, φ_3^n, and G_2^n satisfying the corresponding conditions (1), (2), and (3), and also (4) for projections onto Z_3 and Z_2 (note that $G_3^n \subset G_2^n$).

In general, we can keep iterating this process and end up with $\mathcal{A}^{n+1} = \mathcal{A}_{m+1}^n$, $\mathcal{B}^{n+1} = \mathcal{B}_{m+1}^n$, $\Psi^{n+1} = \Psi_{m+1}^n$, $\mathcal{C}^{n+1} = \mathcal{C}_{m+1}^n$, $\varphi^{n+1} = \varphi_{m+1}^n$, and $G^{n+1} = G_{m+1}^n$ satisfying (1), (2), (3), and (4).

The set $G = \bigcap_{n=1}^\infty G^n$ is the desired Cantor set. By construction, we see that $\pi_1 \restriction G$ is one-to-one, because the map φ^n is a bijection, and because the diameters of the elements of $\mathcal{C}^n$ and those of all $D^n(A,B)$ tend to 0 as n tends to infinity. Therefore, for each point $x \in [\frac{1}{2}, 1]$, there is at most one point in G that is projected onto x. Let $n \in \mathbb{N}$ and ℓ be a line intersecting J and P. Then ℓ must intersect an $A \in \mathcal{A}^n$. Looking at the image under the projection onto Z, we see that $\psi_{Z^\perp}(\ell) \cap \psi_{Z^\perp}(A) \neq \emptyset$. Then we also know that $\psi_{Z^\perp}(\ell) \cap \xi(\psi_{Z^\perp}(A)) \neq \emptyset$ (because $\psi_{Z^\perp}(\ell)$ is a line through $\mathbf{0}$ in Z). Now consider the set $\mathcal{B}' = \{B \in \mathcal{B}^n\colon\ \psi_{Z^\perp}(B) = \xi(\psi_{Z^\perp}(A))\}$. We see that $\bigcup \mathcal{B}' = \xi(\psi_{Z^\perp}(A)) \times [-1,1]^k$. Then, by the convexity of $[-1,1]^k$, the line ℓ must intersect $\bigcup \mathcal{B}'$, hence there must be a $B \in \mathcal{B}^n$ such that $\psi_{Z^\perp}(B) = \xi(\psi_{Z^\perp}(A))$ and such that ℓ meets B. This means that the pair (A,B) is in Ψ^n, so ℓ intersects $D^n(A,B) \subset G^n$. Thus the line ℓ intersects all G^n for $n \in \mathbb{N}$, and hence G as well by compactness. Also, if ℓ is any line in Z_i for $2 \leqslant i \leqslant m+1$, then $\psi_{\ell^\perp}(G) = \psi_{\ell^\perp}(\psi_{Z_i^\perp}(G))$ is zero-dimensional because $\psi_{Z_i^\perp}(G)$ has absolute size ε_n for all n, and therefore $\psi_{\ell^\perp}(\psi_{Z_i^\perp}(G))$ does not contain non-degenerate intervals. That completes the proof. □

Proof of Theorem 1. The cases $k = 0$ and $k = n-1$ have already been discussed in the introduction, so we may assume that $0 < k < n-1$ and hence $n \geqslant 3$. Put $m = n-k-1$. Let G be the Cantor set in $\mathbb{R}^{m+k+1}$ from Lemma 7, with the property that any line through P and J intersects G. Let ℓ be a line through $\mathbf{0}$ and $P^\circ = \{1\} \times (-1,1)^{m+k}$. Define $\{p\} = \ell \cap P^\circ$ and $U = (P^\circ - p) \cap J^\circ$. Observe that $P^\circ - p$ is an open neighbourhood of $\mathbf{0}$ in $\{0\} \times \mathbb{R}^{m+k}$. Thus U is an open neighbourhood of $\mathbf{0}$ in $\operatorname{aff} J = \{0\}^{m+1} \times \mathbb{R}^k$, and hence U is k-dimensional. We have $\ell \cap \operatorname{aff} J = \{\mathbf{0}\}$, so the map $\psi_\ell \restriction J$ is an imbedding. For any $a \in U$, we have that the line $a + \ell$ intersects J in a and P in $a + p$, so it intersects G as well. Therefore, $\psi_\ell(G) \supset \psi_\ell(U)$ which is a k-dimensional set, hence $\psi_\ell(G)$ is at least k-dimensional.

Let H be any hyperplane in $\mathbb{R}^{m+k+1}$. Then $H \cap Z_i$ always contains a line ℓ_i. First of all, if H contains the x_1-axis, then $\pi_1 \restriction G = (\pi_1 \circ \psi_{H^\perp}) \restriction G$. We know that $\pi_1 \restriction G$ is one-to-one, so $\psi_{H^\perp} \restriction G$ must be one-to-one as well, and therefore $\psi_{H^\perp}(G)$ is again a Cantor set, which is zero-dimensional. On the other hand, if H does not contain the x_1-axis, then $\{\ell_i\colon\ 2 \leqslant i \leqslant m+1\}$ is a set of lines generated by linearly independent vectors such that $\psi_{\ell_i^\perp}(\psi_{H^\perp}(G))$ is zero-dimensional for $2 \leqslant i \leqslant m+1$. Therefore, by Lemma 4, we get that $\psi_{H^\perp}(G)$ is at most $(m+k) - m = k$-dimensional.

Now if ℓ is a line through P° and $\mathbf{0}$, then we know that any line ℓ' through $\mathbf{0}$ and P° has the property that the projection of G onto the hyperplane $(\ell')^\perp$ is exactly k-dimensional. Since P° is open in $\{1\} \times \mathbb{R}^{m+k}$, the set of lines through $\mathbf{0}$ and P° forms an open subset of $\mathcal{G}_1(\mathbb{R}^{m+k+1})$, the $(m+k)$-dimensional projective space. Rotating ℓ around the origin and taking such an open neighbourhood for each line, we find an open cover of $\mathcal{G}_1(\mathbb{R}^{m+k+1})$. By compactness, take a finite subcover, and take $G_1, \ldots, G_n$, a finite collection copies of G, corresponding to the elements of the subcover, such that for any hyperplane H, the projection of one of the G_i's onto H has dimension k, and the projection of all other G_i's onto H has dimension at most k. Set $C = \bigcup_{i=1}^n G_i$. Then C is a Cantor set as required. That completes the proof of the theorem. □

4. Projections in Hilbert space

Now, we extend our discussion on Cantor sets with projections of prescribed dimension to Hilbert space. First we use Theorem 1 to prove Theorem 2.

Proof of Theorem 2. Let $\{e_1, e_2, \ldots, e_n, \ldots\}$ be the standard basis for ℓ^2, where each e_j has a 1 on the jth coordinate and a 0 on every other coordinate. Let $\mathcal{P}$ be the set of all $(m+1)$-dimensional linear subspaces of ℓ^2 which are generated by $m+1$ coordinate vectors from that standard basis. There are countably many such subspaces, so let $\mathcal{P} = \{A_1, A_2, \ldots\}$. For each $A_i \in \mathcal{P}$, apply Theorem 1 to find a Cantor set $C_i \subset A_i$ such that the projection of C_i onto any m-plane in A_i is exactly $(m-1)$-dimensional, and also such that $\mathbf{0} \in C_i$ for all $i \in \mathbb{N}$, and $\lim_{i\to\infty} \operatorname{diam} C_i = 0$. Set $C = \bigcup_{i\in\mathbb{N}} C_i$ and note that C is compact and without isolated points. Moreover, being a countable union of Cantor sets C is zero-dimensional and hence a Cantor set by [3].

We are going to prove that C is as required. Indeed, let H be a linear subspace with $\dim H = m$. Let $A_i \in \mathcal{P}$ be fixed and define $L = H^\perp \cap A_i$. Since $\dim A_i > \dim H$ we have that $\dim L \geqslant 1$. Then $H \subset L^\perp$, so $\psi_{H^\perp} = (\psi_{H^\perp} \restriction L^\perp) \circ \psi_L$. Define $P = L^\perp \cap A_i$. Now we have $H^\perp \cap P = H^\perp \cap L^\perp \cap A_i = L^\perp \cap L = \{\mathbf{0}\}$, so $\psi_{H^\perp} \restriction P$ is an imbedding. Then $\psi_{H^\perp}(C_i) = \psi_{H^\perp}(\psi_L(C_i)) = \psi_{H^\perp}(\psi_{P^\perp}(C_i)) \approx \psi_{P^\perp}(C_i)$ because $C_i \subset A_i$.

We distinguish two cases, $\dim L = 1$ and $\dim L \geqslant 2$. First of all, assume $\dim L = 1$. Then P is a hyperplane in A_i. In this case, we have that $\dim \psi_H(C_i) = \dim \psi_P(C_i) = m-1$. Next, assume that $\dim L \geqslant 2$. Then $\dim P \leqslant m-1$, so we know that $\dim \psi_H(C_i) = \dim \psi_P(C_i) \leqslant m-1$.

Given H, it is always possible to find an A_i such that $\dim(H^\perp \cap A_i) = 1$. To do this, first select m elements of the standard basis, $e_{i_1}, \dots, e_{i_m}$, such that $\{\psi_{H^\perp}(e_{i_j})\colon\ j = 1, \dots, m\}$ is a basis for H. Adding any other element of the standard basis gives the set $\{e_{i_1}, \dots, e_{i_{m+1}}\}$; these basis elements determine an A_i satisfying $\dim(H^\perp \cap A_i) = 1$. Now the projection of $C_i \subset A_i$ onto H is exactly $(m-1)$-dimensional, and the projection of any other C_j onto H is at most $(m-1)$-dimensional. Therefore, $\dim \psi_{H^\perp}(C) = m - 1$ by [6, Theorem 1.5.3]. □

A slightly simpler version of this argument, but starting with the theorem of Borsuk [2] that we mentioned in the introduction, proves the following result.

Proposition 8. *For $m \in \mathbb{N}$ there exists a Cantor set in ℓ^2 such that its projections onto all m-planes are exactly m-dimensional.*

We finish with the proof of Theorem 3.

Proof of Theorem 3. First, we prove the theorem when $m = 1$, and then by an easy induction we get the rest of the theorem. Let $m = 1$ and let $P \in \mathcal{G}_1(\ell^2)$. Let us choose an orthogonal coordinate system for ℓ^2 such that the x_1-axis is P. By compactness we can set, for each $i \in \mathbb{N}$,

$$a_i = \max\bigl\{\bigl|\pi_i(v)\bigr|\colon\ v \in K\bigr\}.$$

Claim 1. $\lim_{i\to\infty} a_i = 0$.

Proof of Claim 1. Suppose that $\lim_{i\to\infty} a_i \neq 0$. Choose for each $n \in \mathbb{N}$ a $v(n) \in K$ with $|\pi_n(v(n))| = a_n$. Find a subsequence $(v(n_i))_i$ of $v(1), v(2), \dots$ and an $r > 0$ such that $\lim_{i\to\infty} v(n_i) = v \in K$ and $|\pi_{n_i}(v(n_i))| \geqslant r$ for every i. Let i be such that $\|v(n_i) - v\| < r/2$ and $|\pi_{n_i}(v)| < r/2$. Then $|\pi_{n_i}(v(n_i))| \leqslant |\pi_{n_i}(v)| + \|v(n_i) - v\| < r$, contradicting an assumption. □

We will show that given $\varepsilon > 0$ there is a $\hat{P} \in \mathcal{G}_1(\ell^2)$ such that $\rho(P, \hat{P}) < \varepsilon$ and $(w + \hat{P}) \cap \hat{K} = \{w\}$ for every $w \in \hat{K}$. Indeed, let $\varepsilon > 0$ and let $p = (1, 0, \dots, 0, \dots)$, so $\|p\| = 1$ and $p \in P$. Let $\delta > 0$ be such that whenever $v \in \ell^2$ is such that $\|v\| < \delta$ then $\rho(P, \mathbb{R}(p + v)) < \varepsilon$; see [1, Lemma 5].

Let $c_1, c_2, \dots$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} c_n^2 < \delta^2$. Since $\lim_{i\to\infty} a_i = 0$ we can find a sequence $1 < j_1 < j_2 < \cdots$ of integers such that

$$a_{j_n} < 2^{-n} c_n, \quad \text{for every } n.$$

Now, define $v \in \ell^2$ by

$$\pi_i(v) = \begin{cases} c_n, & \text{if } i = j_n \text{ for some } n \in \mathbb{N}; \\ 0, & \text{otherwise.} \end{cases}$$

If we set $\hat{p} = p + v$ and $\hat{P} = \mathbb{R}\hat{p}$ then clearly $\rho(P, \hat{P}) < \varepsilon$. Note that $\pi_{j_n}(\hat{p}) = c_n$ for every n. Now we show that $\hat{P}$ exposes every point of K:

Claim 2. *For every $w \in K$ we have $(w + \hat{P}) \cap K = \{w\}$.*

Proof of Claim 2. Let $w \in K$ and consider $w + \alpha\hat{p} \in K$. We show that $\alpha = 0$. Let $n \in \mathbb{N}$ be arbitrary. We have

$$|\alpha| c_n = \bigl|\alpha \pi_{j_n}(\hat{p})\bigr| = \bigl|\pi_{j_n}(w + \alpha\hat{p}) - \pi_{j_n}(w)\bigr| \leqslant 2a_{j_n}$$

and hence $|\alpha| \leqslant 2a_{j_n}/c_n < 2^{-n+1}$ for every n. □

Note that Claim 2 states that $\psi_{\hat{P}} \restriction K$ is one-to-one and hence by compactness it is an imbedding.

For the general case, let $m > 1$, $P \in \mathcal{G}_m(\ell^2)$ and assume that the theorem is valid for $m - 1$. Take an orthonormal basis $\{e_1, \dots, e_m\}$ for P and let $\mathcal{O}$ be a neighbourhood of P. Then, by [1, Lemma 5], there is a neighbourhood U of $\mathbf{0}$ such that the open set

$$\mathcal{L} = \bigl\{\operatorname{lin}\{v_1, v_2, \dots, v_m\}\colon\ v_i \in e_i + U,\ 1 \leqslant i \leqslant m\bigr\}$$

is contained in $\mathcal{O}$. By the case $m = 1$, we can find a vector $v \in e_1 + U$ such that for $L = \mathbb{R}v$, $\psi_L \restriction K$ is an imbedding. Consider

$$K_L = \psi_L(K),$$
$$\mathcal{L}_L = \bigl\{N \in \mathcal{G}_{m-1}\bigl(L^\perp\bigr)\colon\ N + L \in \mathcal{L}\bigr\}.$$

Then, K_L is compact, $\mathcal{L}_L$ is nonempty and open in $\mathcal{G}_{m-1}(L^\perp)$ [1, Remark 1] and we may apply the induction hypothesis in the Hilbert space $L^\perp$. So, there exists an $N \in \mathcal{L}_L$ such that $\psi_N \restriction K_L$ is an imbedding. Then $H = L + N$ is the required m-plane, namely, $H \in \mathcal{L} \subset \mathcal{O}$ and $\psi_H \restriction K = \psi_N \circ \psi_L \restriction K$ is an imbedding. That completes the proof of the theorem. □

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