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# A periodic epidemic model in a patchy environment ☆

Fang Zhang *, Xiao-Qiang Zhao

*Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, NL A1C 5S7, Canada*

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**Abstract**

An epidemic model in a patchy environment with periodic coefficients is investigated in this paper. By employing the persistence theory, we establish a threshold between the extinction and the uniform persistence of the disease. Further, we obtain the conditions under which the positive periodic solution is globally asymptotically stable. At last, we present two examples and numerical simulations.


*Keywords:* Epidemic model; Population dispersal; Persistence and extinction of disease; Periodic solution

## 1. Introduction

It has been observed that population dispersal affects the spread of many infectious diseases. In 1976, Hethcote [5] put forth an epidemic model with population dispersal between two patches. After him, Brauer and van den Driessche [2] proposed a model with immigration of infectives. In [10], Wang and Zhao presented a disease transmission model with population dispersal among $n$ patches

$$\begin{cases} S_i' = B_i(N_i)N_i - \mu_i S_i - \beta_i S_i I_i + \gamma_i I_i + \sum_{j=1}^{n} a_{ij} S_j, & 1 \leqslant i \leqslant n, \\ I_i' = \beta_i S_i I_i - (\mu_i + \gamma_i) I_i + \sum_{j=1}^{n} b_{ij} I_j, & 1 \leqslant i \leqslant n, \end{cases} \tag{1.1}$$

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* Corresponding author.

*E-mail addresses:* fzhang@math.mun.ca (F. Zhang), xzhao@math.mun.ca (X.-Q. Zhao).

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with the properties

$$\sum_{j=1}^{n} a_{ji} = 0, \quad \sum_{j=1}^{n} b_{ji} = 0, \quad \forall 1 \leqslant i \leqslant n, \tag{1.2}$$

and established a threshold between the extinction and the uniform persistence of the disease for this model. They also considered the global attractivity of the disease free equilibrium under the condition that the dispersal rates of susceptible and infective individuals are the same in each patch. Recently, the uniqueness and the global attractivity of the endemic equilibrium of this model has been studied by Jin and Wang [8]. However, these authors only considered the constant coefficients in model (1.1). Since the periodicity has been observed in the incidence of many infectious diseases, such as measles, chickenpox, mumps, rubella, poliomyelitis, diphtheria, pertussis and influenza(see, e.g., [6]), it is more realistic to assume that all the coefficients depend on time periodically. As mentioned in [4], the seasonality is an important factor for the spread of infectious diseases, such as the marked change of the contact rate caused by the school system or the weather changes (e.g., measles), the emergence of the insects caused by the seasonal variation (e.g., temperature, humidity, etc.). We will assume that these coefficients are periodic with a common period due to the seasonal effects.

In this paper, we consider the following periodic system:

$$\begin{cases} S_i' = B_i(t, N_i)N_i - \mu_i(t)S_i - \beta_i(t)S_i I_i + \gamma_i(t)I_i + \displaystyle\sum_{j=1}^{n} a_{ij}(t)S_j, & 1 \leqslant i \leqslant n, \\ I_i' = \beta_i(t)S_i I_i - \big(\mu_i(t) + \gamma_i(t)\big)I_i + \displaystyle\sum_{j=1}^{n} b_{ij}(t)I_j, & 1 \leqslant i \leqslant n, \end{cases} \tag{1.3}$$

with all functions being continuous, $\omega$-periodic in $t$. Here $S_i$, $I_i$ are the numbers of susceptible and infectious individuals in patch $i$, respectively. $N_i = S_i + I_i$ is the number of the population in patch $i$, $B_i(t, N_i)$ is the birth rate of the population in the $i$th patch, $\mu_i(t)$ is the death rate of the population in the $i$th patch, and $\gamma_i(t)$ is the recovery rate of infectious individuals in the $i$th patch. $-a_{ii}(t), -b_{ii}(t) \geqslant 0$ represent the emigration rates of susceptible and infectious individuals in the $i$th patch, respectively. $a_{ij}(t)$, $b_{ij}(t)$, $j \neq i$, represent the immigration rates of susceptible and infectious individuals from $j$th patch to $i$th patch. Since the death rates and birth rates of the individuals during the dispersal process are ignored in this model, we have

$$\sum_{j=1}^{n} a_{ji}(t) = 0, \quad \sum_{j=1}^{n} b_{ji}(t) = 0, \quad \forall 1 \leqslant i \leqslant n, \ \forall t \in [0, \omega]. \tag{1.4}$$

We further assume that

(A1) $a_{ij}(t) \geqslant 0$, $b_{ij}(t) \geqslant 0$, $a_{ii}(t) \leqslant 0$, $b_{ii}(t) \leqslant 0$, $\forall 1 \leqslant i \neq j \leqslant n$, $t \in [0, \omega]$, and two $n \times n$ matrices $(a_{ij}(t))$ and $(b_{ij}(t))$ are irreducible.

(A2) $B_i(t, N_i) > 0$, $\forall (t, N_i) \in \mathbb{R}_+ \times (0, \infty)$, $1 \leqslant i \leqslant n$.

(A3) $B_i(t, N_i)$ is continuously differentiable with $\frac{\partial B_i(t, N_i)}{\partial N_i} < 0$, $\forall (t, N_i) \in \mathbb{R}_+ \times (0, \infty)$, $1 \leqslant i \leqslant n$.

(A4) $B_i^u(\infty) := \lim_{N_i \to \infty} B_i^u(N_i) < \mu_i^l$, $1 \leqslant i \leqslant n$, where $B_i^u(N_i) := \max_{t \in [0,\omega]} B_i(t, N_i)$, $\mu_i^l := \min_{t \in [0,\omega]} \mu_i(t)$.

Biologically, (A1) implies that these $n$ patches cannot be separated into two groups such that there is no immigration of susceptible and infective individuals from first group to second group (see the definition of irreducibility in Section 2); (A2) and (A3) mean that each birth rate function is positive and decreasing; and (A4) represents the case where each birth rate cannot exceed the death rate when the population number is sufficiently large.

This paper is organized as follows. In Section 2, a threshold between the extinction and persistence of the disease is established. In Section 3, we prove the uniqueness and the global asymptotic stability of the positive periodic solution when susceptible and infectious individuals have the same dispersal rates, and the global attractivity of the positive periodic solution when the dispersal rates of susceptible and infectious individuals are very close. Finally, we present the numerical simulations for the model with two patches.

## 2. Threshold dynamics

Let $(\mathbb{R}^k, \mathbb{R}^k_+)$ be the standard ordered $k$-dimensional Euclidean space with a norm $\|\cdot\|$. For $u, v \in \mathbb{R}^k$, we write $u \geqslant v$ provided $u - v \in \mathbb{R}^k_+$, $u > v$ provided $u - v \in \mathbb{R}^k_+ \setminus \{0\}$, and $u \gg v$ provided $u - v \in \mathrm{Int}(\mathbb{R}^k_+)$.

Recall that a $k \times k$ matrix $(a_{ij})$ is said to be cooperative if all of its off-diagonal entries are nonnegative; irreducible if its index set $\{1, 2, \ldots, k\}$ cannot be split into two complementary sets (without common indices) $\{m_1, m_2, \ldots, m_\mu\}$ and $\{n_1, n_2, \ldots, n_\nu\}$ $(\mu + \nu = k)$ such that $a_{m_p n_q} = 0$, $\forall 1 \leqslant p \leqslant \mu$, $1 \leqslant q \leqslant \nu$.

Let $A(t)$ be a continuous, cooperative, irreducible, and $\omega$-periodic $k \times k$ matrix function, $\Phi_{A(\cdot)}(t)$ be the fundamental solution matrix of the linear ordinary differential system $x' = A(t)x$, and $r(\Phi_{A(\cdot)}(\omega))$ be the spectral radius of $\Phi_{A(\cdot)}(\omega)$. It then follows from [1, Lemma 2] (see also [7, Theorem 1.1]) that $\Phi_{A(\cdot)}(t)$ is a matrix with all entries positive for each $t > 0$. By the Perron–Frobenius theorem, $r(\Phi_{A(\cdot)}(\omega))$ is the principal eigenvalue of $\Phi_{A(\cdot)}(\omega)$ in the sense that it is simple and admits an eigenvector $v^* \gg 0$. The following result is useful for our subsequent comparison arguments.

**Lemma 2.1.** *Let $\mu = \frac{1}{\omega} \ln r(\Phi_{A(\cdot)}(\omega))$. Then there exists a positive, $\omega$-periodic function $v(t)$ such that $e^{\mu t} v(t)$ is a solution of $x' = A(t)x$.*

**Proof.** Let $v^* \gg 0$ be an eigenvector associated with the principal eigenvalue $r(\Phi_{A(\cdot)}(\omega))$. By the change of variable $x(t) = e^{\mu t} v(t)$, we reduce the linear system $x' = A(t)x$ to

$$v' = A(t)v - \mu v = \big(A(t) - \mu I\big)v. \tag{2.1}$$

Thus, $v(t) := \Phi_{(A(\cdot)-\mu I)}(t)v^*$ is a positive solution of (2.1). It is easy to see that

$$e^{\mu t}\Phi_{(A(\cdot)-\mu I)}(t) = \Phi_{A(\cdot)}(t).$$

Moreover,

$$v(\omega) = \Phi_{(A(\cdot)-\mu I)}(\omega)v^* = e^{-\mu\omega}\Phi_{A(\cdot)}(\omega)v^* = e^{-\mu\omega} r\big(\Phi_{A(\cdot)}(\omega)\big)v^* = v^* = v(0).$$

Thus, $v(t)$ is a positive $\omega$-periodic solution of (2.1), and hence, $x(t) = e^{\mu t} v(t)$ is a solution of $x' = A(t)x$. $\square$

Let $P : \mathbb{R}^{2n}_+ \to \mathbb{R}^{2n}_+$ be the Poincaré map associated with (1.3), that is,

$$P\big(x^0\big) = u\big(\omega, x^0\big), \quad \forall x^0 \in \mathbb{R}^{2n}_+,$$

where $u(t,x^0)$ is the unique solution of (1.3) with $u(0,x^0)=x^0$. In order to find the disease free periodic solutions of (1.3), we consider

$$S_i' = B_i(t,S_i)S_i - \mu_i(t)S_i + \sum_{j=1}^{n} a_{ij}(t)S_j, \quad 1 \leqslant i \leqslant n. \tag{2.2}$$

Let $P_1 : \mathbb{R}_+^n \to \mathbb{R}_+^n$ be the Poincaré map associated with (2.2), that is,

$$P_1(S^0) = u_1(\omega, S^0), \quad \forall S^0 \in \mathbb{R}_+^n,$$

where $u_1(t,S^0)$ is the solution of (2.2) with $u_1(0,S^0)=S^0$.

If $z$ is a nonnegative constant, we define an auxiliary matrix

$$M(t,z) = \begin{bmatrix} B_1(t,z)-\mu_1(t)+a_{11}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & \cdots & a_{2n}(t) \\ \vdots & \ddots & \vdots \\ a_{n1}(t) & \cdots & B_n(t,z)-\mu_n(t)+a_{nn}(t) \end{bmatrix}.$$

This matrix will be used to prove the existence and the uniqueness of a positive fixed point of $P_1$ and is different from the standard Jacobian matrix.

Let $F : \mathbb{R}_+^1 \times \mathbb{R}_+^n \to \mathbb{R}^n$ be defined by the right-hand side of (2.2). It is easy to see that $F$ has the following properties:

(B1) $F_i(t,S) \geqslant 0$ for every $S \geqslant 0$ with $S_i = 0$, $t \in \mathbb{R}_+^1$, $1 \leqslant i \leqslant n$;

(B2) $\frac{\partial F_i}{\partial S_j} \geqslant 0$, $i \neq j$, $\forall (t,S) \in \mathbb{R}_+^1 \times \mathbb{R}_+^n$, and $D_S F(t,0)$ is irreducible for each $t \in \mathbb{R}_+^1$, $S \in \mathbb{R}_+^n$;

(B3) for each $t \geqslant 0$, $F(t,\cdot)$ is strictly subhomogeneous on $\mathbb{R}_+^n$ in the sense that $F(t,\alpha S) > \alpha F(t,S)$, $\forall S \gg 0$, $\alpha \in (0,1)$;

(B4) $F(t,0) \equiv 0$, and $F(t,S) < D_S F(t,0)S$, $\forall t \geqslant 0$, $S \gg 0$.

Note that the nonlinear system (2.2) is dominated by the linear system $S' = D_S F(t,0)S$. It then follows that for any $S^0 \in \mathbb{R}_+^n$, the unique solution $u_1(t,S^0)$ of (2.2) satisfying $u_1(0,S^0)=S^0$ exists globally on $[0,\infty)$ and $u_1(t,S^0) \geqslant 0$, $\forall t \geqslant 0$. We claim that (2.2) admits a bounded positive solution. Indeed, in view of (A4), we can choose a sufficient large real number $K$ such that $\int_0^\omega (\mu_i(t) - B_i(t,K))\,dt > 0$, $i = 1,\ldots,n$. Then by Lemma 2.1, there is a positive, $\omega$-periodic function $v(t) = (v_1(t), v_2(t), \ldots, v_n(t))$ such that $V(t) = e^{\bar{\mu}t}v(t)$ is a solution of $V' = M(t,K)V$, where $\bar{\mu} = \frac{1}{\omega}\ln r(\Phi_{M(\cdot,K)}(\omega))$. Let $\Sigma(t) = \sum_{i=1}^n V_i(t) = e^{\bar{\mu}t}\sum_{i=1}^n v_i(t)$. By the first equation in (1.4), it easily follows that $\Sigma'(t) \leqslant a(t)\Sigma(t)$, $\forall t \geqslant 0$, where $a(t) = \max\{B_i(t,K) - \mu_i(t){:}\ 1 \leqslant i \leqslant n\}$. Thus, $\lim_{t\to\infty}\Sigma(t) = 0$, and hence $\bar{\mu} < 0$, i.e., $r(\Phi_{M(\cdot,K)}(\omega)) < 1$. Choose $l > 0$ large enough such that $lv_i(t) > K$, $1 \leqslant i \leqslant n$, $\forall t \in [0,\omega]$. Set $H(t) \equiv lv(t)$. If we rewrite (2.2) as $S' = F(t,S)$, it is easy to see that

$$F(t,H(t)) < M(t,K)H(t), \quad \forall t \geqslant 0, \tag{2.3}$$

where (A3) is used. By the standard comparison theorem (see, e.g., [9, Theorem B.1]), it follows that

$$\begin{aligned} 0 < u_1(m\omega, lv(0)) &\leqslant \Phi_{M(\cdot,K)}(m\omega)lv(0) = r(\Phi_{M(\cdot,K)}(m\omega))lv(0) \\ &= r(\Phi_{M(\cdot,K)}(\omega))^m lv(0) < lv(0), \quad \forall m \geqslant 0, \end{aligned}$$

that is, $P_1^m(lv(0)) < lv(0)$, $\forall m \geqslant 0$. Consequently, $P_1^m(lv(0))$ is bounded. In order for $P_1$ to admit a positive fixed point, we need to assume that

(A5) $r(\Phi_{M(\cdot,0)}(\omega)) > 1$.

By [11, Theorem 2.1.2], it then follows that the Poincaré map $P_1$ has a unique positive fixed point $S^*(0) = (S_1^*(0), S_2^*(0), \ldots, S_n^*(0))$ which is globally attractive for $S^0 \in \mathbb{R}_+^n \setminus \{0\}$. Thus, $E_0 = (S_1^*(0), S_2^*(0), \ldots, S_n^*(0), 0, \ldots, 0)$ is the unique disease free fixed point of the Poincaré map $P$.

To investigate the global dynamics of (1.3), we first show that (1.3) admits a family of compact, positively invariant sets. For convenience, we denote the positive solution $(S_1(t), \ldots, S_n(t), I_1(t), \ldots, I_n(t))$ of (1.3) by $(S(t), I(t))$.

**Lemma 2.2.** *Let* (A1)–(A5) *hold. Then there is $N^* > 0$ such that every forward solution in $\mathbb{R}_+^{2n}$ of* (1.3) *eventually enters into $G_{N^*} := \{(S, I) \in \mathbb{R}_+^{2n}\colon \sum_{i=1}^n (S_i + I_i) \leqslant N^*\}$, and for each $N \geqslant N^*$, $G_N$ is positively invariant for* (1.3).

**Proof.** Let $N = \sum_{i=1}^n N_i$, $N_i = S_i + I_i$. By (1.3) and (1.4), we have

$$N' = \sum_{i=1}^n \big(B_i(t, N_i) - \mu_i(t)\big)N_i \leqslant \sum_{i=1}^n \big(B_i^u(N_i) - \mu_i^l\big)N_i. \tag{2.4}$$

If $B_i^u(0+) := \lim_{N_i \to 0+} B_i^u(N_i) < \mu_i^l$, $i = 1, 2, \ldots, n$, then there exists $\alpha > 0$ such that $N'(t) \leqslant -\alpha N(t)$, $\forall t \geqslant 0$, and hence, Lemma 2.2 holds for any positive number $N^*$. Otherwise, we partition $\{1, 2, \ldots, n\}$ into two sets $P_1$ and $P_2$ such that

$$B_i^u(0+) > \mu_i^l, \quad \forall i \in P_1,$$
$$B_i^u(0+) \leqslant \mu_i^l, \quad \forall i \in P_2.$$

Without loss of generality, we suppose that $P_1 = \{1, \ldots, m\}$ and $P_2 = \{m+1, \ldots, n\}$. For $i \in P_1$, since $B_i^u(0+) > \mu_i^l$ and $B_i^u(\infty) < \mu_i^l$, (A3) implies that there is a unique $k_i > 0$ such that $B_i^u(k_i) - \mu_i^l = 0$. It follows from (A4) that there is $N^0 > 0$ such that

$$\big(B_i^u(N) - \mu_i^l\big)N < -\sum_{j=1}^m k_j B_j^u(0+) - 1, \quad \forall N \geqslant N^0,\ 1 \leqslant i \leqslant n.$$

Let $N^* = nN^0$. By the definition of $N$, it is easy to see that $N \geqslant N^*$ implies $N_{i_0} \geqslant N^0$ for some $1 \leqslant i_0 \leqslant n$. It then follows from (2.4) that

$$N'(t) \leqslant \sum_{j=1}^m B_j^u(0+)k_j + \big(B_{i_0}^u(N_{i_0}) - \mu_{i_0}^l\big)N_{i_0} < -1, \quad \text{if } N(t) \geqslant N^*,$$

which implies that $G_N$, $N \geqslant N^*$, is positively invariant and every forward orbit enters into $G_{N^*}$ eventually. $\square$

Define

$$M_1(t)=\begin{bmatrix}\bar{b}_{11}(t) & \cdots & b_{1n}(t)\\ b_{21}(t) & \cdots & b_{2n}(t)\\ \vdots & \ddots & \vdots\\ b_{n1}(t) & \cdots & \bar{b}_{nn}(t)\end{bmatrix},$$

where $\bar{b}_{ii}(t)=\beta_i(t)S_i^*(t)-\mu_i(t)-\gamma_i(t)+b_{ii}(t)$, $1\leqslant i\leqslant n$. Clearly, $M_1(t)$ is irreducible and has nonnegative off-diagonal elements.

In the case where the susceptible and infective individuals in each patch have the same dispersal rate, we have the following result on the global attractivity of the $\omega$-periodic solution $(S^*(t),0)$.

**Theorem 2.1.** *Let* (A1)–(A5) *hold and* $r(\Phi_{M_1(\cdot)}(\omega))<1$. *If* $a_{ij}(t)=b_{ij}(t)$ *for* $1\leqslant i,j\leqslant n$, $\forall t\in[0,\omega]$, *then* $\lim_{t\to\infty}(S(t)-S^*(t))=0$, $\lim_{t\to\infty}I(t)=0$ *for all* $(S^0,I^0)\in(\mathbb{R}_+^n\setminus\{0\})\times\mathbb{R}_+^n$.

**Proof.** Let us consider a nonnegative solution $(S(t),I(t))$ of (1.3). We want to show that

$$\lim_{t\to\infty}I(t)=0. \tag{2.5}$$

By (1.3), we have

$$N_i'=B_i(t,N_i)N_i-\mu_i(t)N_i+\sum_{j=1}^n a_{ij}(t)N_j,\quad 1\leqslant i\leqslant n. \tag{2.6}$$

By the afore-mentioned conclusion for (2.2), the Poincaré map associated with (2.6) has a unique positive fixed point $S^*(0)$ which is globally attractive in $\mathbb{R}_+^n\setminus\{0\}$. It then follows that for any $\epsilon>0$, there holds $N(t)=S(t)+I(t)<S^*(t)+\bar{\epsilon}$, where $\bar{\epsilon}=(\epsilon,\dots,\epsilon)\in\mathrm{Int}(\mathbb{R}_+^n)$, when $t$ is sufficiently large. Thus, if $t$ is sufficiently large, we have

$$I_i'<\beta_i(t)\big(S_i^*(t)+\epsilon\big)I_i-\big(\mu_i(t)+\gamma_i(t)\big)I_i+\sum_{j=1}^n b_{ij}(t)I_j,\quad 1\leqslant i\leqslant n. \tag{2.7}$$

It then suffices to show that positive solutions of the following auxiliary system

$$\check{I}_i'=\beta_i(t)\big(S_i^*(t)+\epsilon\big)\check{I}_i-\big(\mu_i(t)+\gamma_i(t)\big)\check{I}_i+\sum_{j=1}^n b_{ij}(t)\check{I}_j,\quad 1\leqslant i\leqslant n, \tag{2.8}$$

tend to zero as $t$ goes to infinity. Let $M_2(t)$ be the matrix defined by

$$M_2(t)=\mathrm{diag}\big(\beta_1(t),\beta_2(t),\dots,\beta_n(t)\big).$$

Since $r(\Phi_{M_1(\cdot)}(\omega))<1$ and $r(\Phi_{M_1(\cdot)+\epsilon M_2(\cdot)}(\omega))$ is continuous for small $\epsilon$, we can fix $\epsilon>0$ small enough such that $r(\Phi_{M_1(\cdot)+\epsilon M_2(\cdot)}(\omega))<1$. By Lemma 2.1, there is a positive, $\omega$-periodic function $\bar{v}(t)=(\bar{v}_1(t),\bar{v}_2(t),\dots,\bar{v}_n(t))$ such that $\rho e^{\tilde{\mu}t}\bar{v}(t)$ is a solution of (2.8) for any constant $\rho$, where $\tilde{\mu}=\frac{1}{\omega}\ln r(\Phi_{M_1(\cdot)+\epsilon M_2(\cdot)}(\omega))$. $\forall I^0\in\mathbb{R}_+^n$, we can choose a real number $\rho^0>0$ such that $I^0\leqslant\rho^0\bar{v}(0)$. By the standard comparison theorem (see, e.g., [9, Theorem A.4]), we then get (2.5).

For any $(S^0,I^0)\in(\mathbb{R}_+^n\setminus\{0\})\times\mathbb{R}_+^n$, we have $N^0=S^0+I^0\in\mathbb{R}_+^n\setminus\{0\}$. By the global attractivity of $S^*(0)$ for $P_1$, it then follows that

$$\lim_{t\to\infty}\big(S(t)-S^*(t)\big)=\lim_{t\to\infty}\big(N(t)-I(t)-S^*(t)\big)=0.\qquad\square$$

If the susceptible and infective individuals in each patch have different dispersal rate, and the initial value $I^0$ is small, we still have the result on the attractivity of the $\omega$-periodic solution $(S^*(t), 0)$.

**Theorem 2.2.** *Let* (A1)–(A5) *hold and* $r(\Phi_{M_1(\cdot)}(\omega)) < 1$. *Then there exists* $\delta > 0$ *such that for every* $(S^0, I^0) \in G_{N^*}$ *with* $S^0 \neq 0$ *and* $I_i^0 < \delta$, $1 \leqslant i \leqslant n$, *the solution* $(S(t), I(t))$ *of* (1.3) *satisfies* $\lim_{t\to\infty}(S(t) - S^*(t)) = 0$, $\lim_{t\to\infty} I(t) = 0$.

**Proof.** Let us consider an auxiliary system

$$\tilde{S}_i' = B_i(t, \tilde{S}_i)\tilde{S}_i - \mu_i(t)\tilde{S}_i + \big(B_i(t, 0+) + \gamma_i(t)\big)\epsilon + \sum_{j=1}^{n} a_{ij}(t)\tilde{S}_j, \quad 1 \leqslant i \leqslant n, \tag{2.9}$$

where $\epsilon > 0$ is a small constant to be determined. By (A5) and the previous analysis of system (2.2), we can restrict $\epsilon$ small enough such that (2.9) admits a globally attractive and positive $\omega$-periodic solution $S^*(t, \epsilon)$. Let $S^\epsilon(t, N^*)$ be the solution of (2.9) through $(N^*, \dots, N^*)$ at $t = 0$. We choose an integer $n_1 > 0$ such that

$$S^\epsilon(t, N^*) < S^*(t, \epsilon) + \bar{\epsilon}, \quad \forall t \geqslant n_1\omega.$$

Define a matrix $M_1(t, \epsilon)$ by

$$\begin{bmatrix} \beta_1(t)S_1^*(t,\epsilon) - \mu_1(t) - \gamma_1(t) + b_{11}(t) & \cdots & b_{1n}(t) \\ b_{21}(t) & \cdots & b_{2n}(t) \\ \vdots & \ddots & \vdots \\ b_{n1}(t) & \cdots & \beta_n(t)S_n^*(t,\epsilon) - \mu_n(t) - \gamma_n(t) + b_{nn}(t) \end{bmatrix}.$$

Since $M_1(t, 0) = M_1(t)$ and $r(\Phi_{M_1(\cdot,\epsilon)+\epsilon M_2(t)}(\omega))$ is continuous for small $\epsilon$, we can now restrict $\epsilon$ small enough such that $r(\Phi_{M_1(\cdot,\epsilon)+\epsilon M_2(\cdot)}(\omega)) < 1$. By Lemma 2.1, there is a positive $\omega$-periodic function $v(t) = (v_1(t), \dots, v_n(t))$ such that $\check{I}(t) = e^{\mu_3 t}v(t)$ is a solution of $\check{I}' = (M_1(t, \epsilon) + \epsilon M_2(t))\check{I}$, where $\mu_3 = \frac{1}{\omega}\ln r(\Phi_{M_1(\cdot,\epsilon)+\epsilon M_2(\cdot)}(\omega))$. Choose $k > 0$ small enough such that $kv(t) < \bar{\epsilon}$ for all $t \in [0, \omega]$.

Now we define another auxiliary system

$$\hat{I}_i' = \beta_i(t)N^*\hat{I}_i - \big(\mu_i(t) + \gamma_i(t)\big)\hat{I}_i + \sum_{j=1}^{n} b_{ij}(t)\hat{I}_j, \quad 1 \leqslant i \leqslant n. \tag{2.10}$$

Let $\hat{I}(t, \delta)$ be the solution of (2.10) through $(\delta, \dots, \delta) \in \mathbb{R}^n$ at $t = 0$. We restrict $\delta > 0$ small enough such that

$$\hat{I}(t, \delta) < ke^{\mu_3 t}v(t) \leqslant kv(t) < \bar{\epsilon}, \quad \forall t \in [0, n_1\omega]. \tag{2.11}$$

Let $(S(t), I(t))$ be a nonnegative solution of (1.3) with $(S^0, I^0) \in G_{N^*}$, $S^0 \neq 0$ and $I_i^0 < \delta$, $1 \leqslant i \leqslant n$. It then follows that $S(t) \gg 0$, $\forall t > 0$. We further claim that $I(t) \leqslant ke^{\mu_3 t}v(t)$, $\forall t \geqslant 0$. Suppose not. By the comparison principle and (2.11), there exist $q$, $1 \leqslant q \leqslant n$, and $T_1 > n_1\omega$ such that

$$\begin{aligned} &I(t) \leqslant ke^{\mu_3 t}v(t), \quad \text{for } 0 \leqslant t \leqslant T_1, \\ &I_q(T_1) = k\big(e^{\mu_3 T_1}v(T_1)\big)_q, \\ &I_q(t) > k\big(e^{\mu_3 T_1}v(T_1)\big)_q, \quad \text{for } 0 < t - T_1 \ll 1. \end{aligned} \tag{2.12}$$

Note that for $0 \leqslant t \leqslant T_1$, we have

$$S_i' < B_i(t, S_i)S_i - \mu_i(t)S_i + \big(B_i(t, 0+) + \gamma_i(t)\big)\epsilon + \sum_{j=1}^{n} a_{ij}(t)S_j, \quad 1 \leqslant i \leqslant n. \tag{2.13}$$

It follows from the comparison principle that $S(T_1) < S^*(T_1, \epsilon) + \bar{\epsilon}$. Then, for $0 \leqslant t - T_1 \ll 1$, we have $S(t) < S^*(t, \epsilon) + \bar{\epsilon}$, and hence

$$I_i' < \beta_i(t)\big(S_i^*(t, \epsilon) + \epsilon\big)I_i - \big(\mu_i(t) + \gamma_i(t)\big)I_i + \sum_{j=1}^{n} b_{ij}(t)I_j, \quad 1 \leqslant i \leqslant n.$$

Since $I(T_1) \leqslant ke^{\mu_3 T_1}v(T_1)$, the comparison principle implies that

$$I(t) < ke^{\mu_3 t}v(t), \quad \text{for } 0 \leqslant t - T_1 \ll 1,$$

and hence,

$$I_q(t) < k\big(e^{\mu_3 t}v(t)\big)_q, \quad \text{for } 0 < t - T_1 \ll 1,$$

which contradicts to (2.12). This proves the claim.

By the claim above, (2.13) holds for all $t \geqslant 0$. Thus, the comparison principle implies that $S(t) < S^*(t, \epsilon) + \bar{\epsilon}$, $\forall t \geqslant n_1\omega$. By a similar argument as above, it then follows that

$$I(t) < ke^{\mu_3 t}v(t), \quad \forall t > T_1.$$

Consequently, $I(t) \to 0$ as $t \to \infty$.

Since $P^m(x^0) = u(m\omega, x^0)$, $\forall x^0 \in \mathbb{R}_+^{2n}$, we have

$$P^m\big(S^0, I^0\big) = u\big(m\omega, \big(S^0, I^0\big)\big) = \big(S(m\omega), I(m\omega)\big), \quad \forall \big(S^0, I^0\big) \in \mathbb{R}_+^n \times \mathbb{R}_+^n.$$

Given $(S^0, I^0) \in G_{N^*}$ with $S^0 \neq 0$ and $I_i^0 < \delta$, $1 \leqslant i \leqslant n$, it easily follows that $S(t) \in \mathrm{Int}(\mathbb{R}_+^n)$, $\forall t > 0$. Let

$$\omega = \omega\big(S^0, I^0\big) := \Big\{(S_*, I_*)\colon \exists \{m_k\} \to \infty \text{ such that } \lim_{k \to \infty} P^{m_k}\big(S^0, I^0\big) = (S_*, I_*)\Big\}$$

be the omega limit set of $(S^0, I^0)$ for $P$. Since $\lim_{t\to\infty} I(t) = 0$, there holds $\omega = \bar{\omega} \times \{0\}$. We claim that $\bar{\omega} \neq \{0\}$. Assume that, by contradiction, $\bar{\omega} = \{0\}$. $\lim_{n\to\infty} P^n(S^0, I^0) = (0, 0)$, then $\lim_{t\to\infty} S(t) = 0$. By assumption (A5), we can choose a small $\eta > 0$ such that $r(\Phi_{M(\cdot,0)-\eta E}(\omega)) > 1$, where $E = \mathrm{diag}(1, \ldots, 1)$. It follows that there exists $\bar{t} > 0$ such that

$$B_i\big(t, N_i(t)\big) - \beta_i(t)I_i(t) \geqslant B_i(t, 0+) - \eta, \quad \forall t \geqslant \bar{t},\ 1 \leqslant i \leqslant n.$$

Then $S(t) = (S_1(t), \ldots, S_n(t))$ satisfies

$$S_i'(t) > \big(B_i(t, 0+) - \eta\big)S_i - \mu_i(t)S_i + \sum_{j=1}^{n} a_{ij}(t)S_j, \quad \forall t \geqslant \bar{t},\ 1 \leqslant i \leqslant n. \tag{2.14}$$

Let $p(t) = (p_1(t), \ldots, p_n(t))$ be the positive $\omega$-periodic function such that $e^{\mu_4 t}p(t)$ is a solution of the linear system

$$\hat{S}_i' = \big(B_i(t, 0+) - \eta\big)\hat{S}_i - \mu_i(t)\hat{S}_i + \sum_{j=1}^{n} a_{ij}(t)\hat{S}_j, \quad 1 \leqslant i \leqslant n, \tag{2.15}$$

where $\mu_4 = \frac{1}{\omega}\ln(r(\Phi_{M(\cdot,0)-\eta E}(\omega))) > 0$. Since $S(\bar{t}) \in \mathrm{Int}(\mathbb{R}^n_+)$, we can choose a small number $\alpha > 0$ such that $S(\bar{t}) \geqslant \alpha p(0)$. Then the comparison theorem implies that

$$S(t) \geqslant \alpha e^{\mu_4(t-\bar{t})} p(t-\bar{t}) \geqslant \alpha e^{\mu_4(t-\bar{t})} \min_{t-\bar{t}\geqslant 0} p(t-\bar{t}), \quad \forall t \geqslant \bar{t},$$

and hence $\lim_{t\to\infty} S_i(t) = \infty$, $1 \leqslant i \leqslant n$, a contradiction. Note that for any $S^0 \in \mathbb{R}^n_+$, we have $u(t, (S^0, 0)) = (u_1(t, S^0), 0)$, $\forall t \geqslant 0$. It then follows that

$$P^m(S^0, 0) = (P_1^m(S^0), 0), \quad \forall S^0 \in \mathbb{R}^n_+, m \geqslant 0.$$

Since $\omega$ is an internal chain transitive set for $P$, and hence, $\bar{\omega}$ is an internal chain transitive set for $P_1$. Let

$$W^s(S^*(0)) := \{S^0 : P_1^m(S^0) \to S^*(0): m \to \infty\}.$$

Since $\bar{\omega} \neq \{0\}$ and $S^*(0)$ is globally attractive for $P_1$ in $\mathbb{R}^n_+ \setminus \{0\}$, we have $\bar{\omega} \cap W^s(S^*(0)) \neq \emptyset$. By [11, Theorem 1.2.1], we then get $\bar{\omega} = \{S^*(0)\}$, and hence $\omega = \{(S^*(0), 0)\}$. Thus, $\lim_{t\to\infty}(S(t) - S^*(t)) = 0$ and $\lim_{t\to\infty} I(t) = 0$. □

The following result shows that actually $r(\Phi_{M_1(\cdot)}(\omega))$ is a threshold parameter for the extinction and the uniform persistence of the disease. When $r(\Phi_{M_1(\cdot)}(\omega)) > 1$, the model (1.3) admits at least one positive periodic solution, and the disease is uniformly persistent.

**Theorem 2.3.** *Let* (A1)–(A5) *hold and* $r(\Phi_{M_1(\cdot)}(\omega)) > 1$. *Then system* (1.3) *admits at least one positive periodic solution, and there is a positive constant* $\epsilon$ *such that for all* $(S^0, I^0) \in \mathbb{R}^n_+ \times \mathrm{Int}(\mathbb{R}^n_+)$, *the solution* $(S(t), I(t))$ *of* (1.3) *satisfies*

$$\liminf_{t\to\infty} I_i(t) \geqslant \epsilon, \quad 1 \leqslant i \leqslant n.$$

**Proof.** Define

$$X = \mathbb{R}^{2n}_+, \qquad X_0 = \mathbb{R}^n_+ \times \mathrm{Int}(\mathbb{R}^n_+), \qquad \partial X_0 = X \setminus X_0.$$

We first prove that $P$ is uniformly persistent with respect to $(X_0, \partial X_0)$. By the form of (1.3), it is easy to see that both $X$ and $X_0$ are positively invariant. Clearly, $\partial X_0$ is relatively closed in $X$. Furthermore, system (1.3) is point dissipative (see Lemma 2.2). Set

$$M_\partial = \{(S^0, I^0) \in \partial X_0: P^m(S^0, I^0) \in \partial X_0, \ \forall m \geqslant 0\}.$$

We now show that

$$M_\partial = \{(S, 0): S \geqslant 0\}. \tag{2.16}$$

Obviously, $\{(S, 0): S \geqslant 0\} \subseteq M_\partial$.

For any $(S^0, I^0) \in \partial X_0 \setminus \{(S, 0): S \geqslant 0\}$, we partition $\{1, 2, \ldots, n\}$ into two sets $Q_1$ and $Q_2$ such that

$$I_j^0 = 0, \quad \forall j \in Q_1,$$
$$I_i^0 > 0, \quad \forall i \in Q_2,$$

where $Q_1$ and $Q_2$ are nonempty. $\forall j \in Q_1, i \in Q_2$, we have

$$I_j'(0) \geqslant b_{ji} I_i(0) > 0.$$

It follows that $(S(t), I(t)) \notin \partial X_0$ for $0 < t \ll 1$. Thus, the positive invariance of $X_0$ implies (2.16). It is clear that there are two fixed points of $P$ in $M_\partial$, which are $M_0 = (0, 0)$ and $M_1 = (S^*(0), 0)$.

Choose $\eta > 0$ small enough such that $r(\Phi_{M_1(\cdot)-\eta M_2(\cdot)}(\omega)) > 1$. Let us consider a perturbed system of (2.2)

$$\hat{S}_i' = B_i(t, \hat{S}_i + \delta)\hat{S}_i - \big(\mu_i(t) + \beta_i(t)\delta\big)\hat{S}_i + \sum_{j=1}^{n} a_{ij}(t)\hat{S}_j, \quad 1 \leqslant i \leqslant n. \tag{2.17}$$

As in our previous analysis of system (2.2), we can choose $\delta > 0$ small enough such that the Poincaré map associated with (2.17) admits a unique positive fixed point $S^*(0, \delta)$ which is globally attractive in $\mathbb{R}^n_+ \setminus \{0\}$. By the implicit function theorem, it follows that $S^*(0, \delta)$ is continuous in $\delta$. Thus, we can fix a small number $\delta > 0$ such that $S^*(t, \delta) > S^*(t) - \bar{\eta}$, $\forall t \geqslant 0$, where $\bar{\eta} = (\eta, \dots, \eta)$. By the continuity of solutions with respect to the initial values, there exists $\delta_0^* > 0$ such that for all $(S^0, I^0) \in X_0$ with $\|(S^0, I^0) - M_i\| \leqslant \delta_0^*$, we have $\|u(t, (S^0, I^0)) - u(t, M_i)\| < \delta$, $\forall t \in [0, \omega]$, $i = 0, 1$. We now claim that

$$\limsup_{m\to\infty} d\big(P^m\big(S^0, I^0\big), M_i\big) \geqslant \delta_0^*, \quad i = 0, 1.$$

Suppose, by contradiction, that $\limsup_{n\to\infty} d(P^m(S^0, I^0), M_i) < \delta_0^*$ for some $(S^0, I^0) \in X_0$ and $i$. Without loss of generality, we can assume that $d(P^m(S^0, I^0), M_i) < \delta_0^*$, $\forall m \geqslant 0$. Then, we have $\|u(t, P^m(S^0, I^0)) - u(t, M_i)\| < \delta$, $\forall m \geqslant 0$, $\forall t \in [0, \omega]$. For any $t \geqslant 0$, let $t = m\omega + t'$, where $t' \in [0, \omega)$ and $m = [\frac{t}{\omega}]$ is the greatest integer less than or equal to $\frac{t}{\omega}$. Thus, we get

$$\big\|u\big(t, \big(S^0, I^0\big)\big) - u(t, M_i)\big\| = \big\|u\big(t', P^m\big(S^0, I^0\big)\big) - u(t', M_i)\big\| < \delta, \quad \forall t \geqslant 0.$$

Let $(S(t), I(t)) = u(t, (S^0, I^0))$. It then follows that $0 \leqslant I_i(t) \leqslant \delta$, $\forall t \geqslant 0$, $\forall 1 \leqslant i \leqslant n$. Then for $t \geqslant 0$, we have

$$S_i' \geqslant B_i(t, S_i + \delta)S_i - \big(\mu_i(t) + \beta_i(t)\delta\big)S_i + \sum_{j=1}^{n} a_{ij}(t)S_j, \quad 1 \leqslant i \leqslant n. \tag{2.18}$$

Since the fixed point $S^*(0, \delta)$ of the Poincaré map associated with (2.17) is globally attractive and $S^*(t, \delta) > S^*(t) - \bar{\eta}$, there is $T > 0$ such that $S(t) \geqslant S^*(t) - \bar{\eta}$ for $t \geqslant T$. As a consequence, for $t \geqslant T$, there holds

$$I_i' \geqslant \beta_i(t)\big(S_i^*(t) - \eta\big)I_i - \big(\mu_i(t) + \gamma_i(t)\big)I_i + \sum_{j=1}^{n} b_{ij}(t)I_j, \quad 1 \leqslant i \leqslant n. \tag{2.19}$$

Since $r(\Phi_{M_1(\cdot)-\eta M_2(\cdot)}(\omega)) > 1$, it is easy to see that $\lim_{t\to\infty} I_i(t) = \infty$, $i = 1, 2, \dots, n$, which leads to a contradiction.

Note that $S^*(0)$ is globally attractive in $\mathbb{R}^n_+ \setminus \{0\}$ for $P_1$. By the afore-mentioned claim, it then follows that $M_0$ and $M_1$ are isolated invariant sets in $X$, $W^s(M_0) \cap X_0 = \emptyset$, and $W^s(M_1) \cap X_0 = \emptyset$. Clearly, every orbit in $M_\partial$ converges to either $M_0$ or $M_1$, and $M_0$ and $M_1$ are acyclic in $M_\partial$. By [11, Theorem 1.3.1] for a stronger repelling property of $\partial X_0$, we conclude that $P$ is uniformly persistent with respect to $(X_0, \partial X_0)$. Thus, [11, Theorem 3.1.1] implies the uniform persistence of the solutions of system (1.3) with respect to $(X_0, \partial X_0)$. By [11, Theorem 1.3.6], $P$ has a fixed point $(\bar{S}(0), \bar{I}(0)) \in X_0$. Then $\bar{S}(0) \in \mathbb{R}^n_+$ and $\bar{I}(0) \in \mathrm{Int}(\mathbb{R}^n_+)$. We further claim that $\bar{S}(0) \in \mathbb{R}^n_+ \setminus \{0\}$. Suppose that $\bar{S}(0) = 0$. By the second equation in (1.4), we then get $0 = -\sum_{i=1}^{n}(\mu_i(t)+\gamma_i(t))\bar{I}_i(0)$, and hence $\bar{I}_i(0)=0,\ i=1,2,\dots,n$, a contradiction. By the first equation in (1.3) and the irreducibility of the cooperative matrix $(a_{ij}(t))$, it follows that $u(t,(\bar{S}(0),\bar{I}(0))) \in \mathrm{Int}(\mathbb{R}_+^n)$, $\forall t>0$. Then $(\bar{S}(0),\bar{I}(0))$ is a componentwise positive fixed point of $P$. Thus, $(\bar{S}(t),\bar{I}(t))$ is a positive $\omega$-periodic solution of (1.3). □

## 3. The positive periodic solutions

In the case where the dispersal rate of susceptible individuals and infective individuals are equal, we are able to prove the uniqueness and global asymptotic stability of the positive $\omega$-periodic solution under the condition that $r(\Phi_{M_1(\cdot)}(\omega))>1$.

**Theorem 3.1.** *Let* (A1)–(A5) *hold and* $r(\Phi_{M_1(\cdot)}(\omega))>1$. *If* $a_{ij}(t)=b_{ij}(t)$ *for* $1\leqslant i,j\leqslant n$, $\forall t\in[0,\omega]$, *then system* (1.3) *admits a unique positive ω-periodic solution which is globally asymptotically stable in* $\mathbb{R}_+^n\times(\mathbb{R}_+^n\setminus\{0\})$.

**Proof.** By (1.3), when $a_{ij}(t)=b_{ij}(t)$, we have

$$\begin{cases} S_i' = B_i(t,N_i)N_i - \mu_i(t)S_i - \beta_i(t)S_iI_i + \gamma_i(t)I_i + \sum_{j=1}^{n} a_{ij}(t)S_j, & 1\leqslant i\leqslant n, \\ I_i' = \beta_i(t)S_iI_i - \big(\mu_i(t)+\gamma_i(t)\big)I_i + \sum_{j=1}^{n} a_{ij}(t)I_j, & 1\leqslant i\leqslant n. \end{cases} \tag{3.1}$$

Then we obtain

$$N_i' = B_i(t,N_i)N_i - \mu_i(t)N_i + \sum_{j=1}^{n} a_{ij}(t)N_j, \quad 1\leqslant i\leqslant n. \tag{3.2}$$

By the afore-mentioned conclusion for (2.2), the Poincaré map associated with (3.2) has a unique positive fixed point $S^*(0)$ which is globally attractive for $N\in\mathbb{R}_+^n\setminus\{0\}$. Then (3.1) is equivalent to the following system:

$$\begin{cases} N_i' = B_i(t,N_i)N_i - \mu_i(t)N_i + \sum_{j=1}^{n} a_{ij}(t)N_j, & 1\leqslant i\leqslant n, \\ I_i' = \beta_i(t)(N_i-I_i)I_i - \big(\mu_i(t)+\gamma_i(t)\big)I_i + \sum_{j=1}^{n} a_{ij}(t)I_j, & 1\leqslant i\leqslant n. \end{cases} \tag{3.3}$$

Since $\lim_{t\to\infty}(N(t)-S^*(t))=0$, the second equation of (3.3) has the following limiting system:

$$\tilde{I}_i' = \beta_i(t)\big(S_i^*(t)-\tilde{I}_i\big)\tilde{I}_i - \big(\mu_i(t)+\gamma_i(t)\big)\tilde{I}_i + \sum_{j=1}^{n} a_{ij}(t)\tilde{I}_j, \quad 1\leqslant i\leqslant n. \tag{3.4}$$

Let $F_1:\mathbb{R}_+^1\times\mathbb{R}_+^n\to\mathbb{R}^n$ be defined by the right-hand side of (3.4). Clearly, $F_1$ satisfies (B1)–(B4). Let $P_2:\mathbb{R}_+^n\to\mathbb{R}_+^n$ be the Poincaré map associated with (3.4), that is,

$$P_2\big(I^0\big) = u_2\big(\omega, I^0\big), \quad \forall I^0\in\mathbb{R}_+^n,$$

where $u_2(t,I^0)$ is the solution of (3.4) with $u_2(0,I^0)=I^0$. We claim that (3.4) admits a bounded positive solution.

Define

$$M_2(t,Z)=\begin{bmatrix}\bar a_{11}(t) & \cdots & a_{1n}(t)\\ a_{21}(t) & \cdots & a_{2n}(t)\\ \vdots & \ddots & \vdots\\ a_{n1}(t) & \cdots & \bar a_{nn}(t)\end{bmatrix},$$

where $\bar a_{ii}(t)=\beta_i(t)(S_i^*(t)-Z)-\mu_i(t)-\gamma_i(t)+a_{ii}(t)$, $1\leqslant i\leqslant n$. We can choose a sufficiently large real number $Z>0$ such that $\int_0^\omega(\beta_i(t)(S_i^*(t)-Z)-\mu_i(t)-\gamma_i(t))\,dt<0$, $1\leqslant i\leqslant n$. By Lemma 2.1, there is a positive, $\omega$-periodic function $v(t)=(v_1(t),v_2(t),\dots,v_n(t))$ such that $V(t)=e^{\mu_5 t}v(t)$ is a solution of $V'=M_2(t,Z)V$, where $\mu_5=\frac{1}{\omega}\ln r(\Phi_{M_2(\cdot,Z)}(\omega))$. Let $\Sigma(t)=\sum_{i=1}^n V_i(t)=e^{\mu_5 t}\sum_{i=1}^n v_i(t)$. By the first equation in (1.4), it easily follows that $\Sigma'(t)\leqslant b(t)\Sigma(t), \forall t\geqslant 0$, where $b(t)=\max\{\beta_i(t)(S_i^*(t)-Z)-\mu_i(t)-\gamma_i(t)\colon 1\leqslant i\leqslant n\}$. Thus, $\lim_{t\to\infty}\Sigma(t)=0$, and hence $\mu_5<0$, i.e., $r(\Phi_{M_2(\cdot,Z)}(\omega))<1$. Choose $l>0$ large enough such that $lv_i(t)>Z, \forall t\in[0,\omega]$, $i=1,2,\dots,n$. Set $H(t)\equiv lv(t)$. If we rewrite (3.4) as $\tilde I'=F_1(t,\tilde I)$, it is easy to see that

$$F_1\big(t,H(t)\big)<M_2(t,Z)H(t),\quad \forall t\geqslant 0. \tag{3.5}$$

By the standard comparison theorem (see, e.g., [9, Theorem B.1]), it follows that

$$\begin{aligned}0&<u_2\big(m\omega,lv(0)\big)\leqslant \Phi_{M_2(\cdot,Z)}(m\omega)lv(0)=r\big(\Phi_{M_2(\cdot,Z)}(m\omega)\big)lv(0)\\ &=r\big(\Phi_{M_2(\cdot,Z)}(\omega)\big)^m lv(0)<lv(0),\quad \forall m\geqslant 0.\end{aligned}$$

That is, $P_2^m(lv(0))<lv(0), \forall m\geqslant 0$. Consequently, $P_2^m(lv(0))$ is bounded. Since $r(\Phi_{M_2(\cdot,0)}(\omega))=r(\Phi_{M_1(\cdot)}(\omega))>1$. By [11, Theorem 2.1.2], it then follows that the Poincaré map $P_2$ has a unique positive fixed point $\bar I(0)$ which is globally attractive for $I^0\in\mathbb{R}_+^n\setminus\{0\}$. Thus, the Poincaré map $P$ associated with (3.1) admits a unique fixed point $(S^*(0)-\bar I(0),\bar I(0))$. It then follows from Theorem 2.3 that the unique fixed point is positive. We denote it by $(\bar S(0),\bar I(0))$.

Let $P_3 : X:=\mathbb{R}_+^{2n}\to\mathbb{R}_+^{2n}$ be the Poincaré map associated with (3.3), that is,

$$P_3\big(x^0\big)=u_3\big(\omega,x^0\big),\quad \forall x^0\in\mathbb{R}_+^{2n},$$

where $u_3(t,x^0)$ is the solution of (3.3) with $u_3(0,x^0)=x^0$. Thus, we have

$$P_3^m\big(N^0,I^0\big)=u_3\big(m\omega,\big(N^0,I^0\big)\big),\quad \forall\big(N^0,I^0\big)\in\mathbb{R}_+^n\times\mathbb{R}_+^n.$$

Let $(N^0,I^0)\in(\mathbb{R}_+^n\setminus\{0\})\times(\mathbb{R}_+^n\setminus\{0\})$ be fixed. It then follows that

$$\big(N(t),I(t)\big)=u_3\big(t,\big(N^0,I^0\big)\big)\in\operatorname{Int}\big(\mathbb{R}_+^n\big)\times\operatorname{Int}\big(\mathbb{R}_+^n\big),\quad \forall t>0.$$

Let $\omega=\omega(N^0,I^0)$ be the omega limit set of $(N^0,I^0)$ for $P_3$. Since $\lim_{t\to\infty}(N(t)-S^*(t))=0$, there holds $\omega=\{S^*(0)\}\times\tilde\omega$. We claim that $\tilde\omega\neq\{0\}$. Assume that, by contradiction, $\tilde\omega=\{0\}$. Then $\lim_{m\to\infty}P_3^m(N^0,I^0)=(S^*(0),0)$, that is, $\lim_{t\to\infty}(N(t)-S^*(t))=0, \lim_{t\to\infty}I(t)=0$. Since $r(\Phi_{M_1(\cdot)}(\omega))>1$ , we can choose a small $\eta>0$ such that $r(\Phi_{M_1(\cdot)-\eta E}(\omega))>1$, where $E=\operatorname{diag}(1,\dots,1)$. It follows that there exists $\bar t>0$ such that

$$\beta_i(t)\big(N_i(t)-I_i(t)\big)\geqslant\beta_i(t)S_i^*(t)-\eta,\quad \forall t\geqslant\bar t,\ 1\leqslant i\leqslant n.$$

Then $I(t)=(I_1(t),\dots,I_n(t))$ satisfies

$$I_i'>\big(\beta_i(t)S_i^*(t)-\eta\big)I_i-\big(\mu_i(t)+\gamma_i(t)\big)I_i+\sum_{j=1}^n a_{ij}(t)I_j,\quad \forall t\geqslant\bar t,\ 1\leqslant i\leqslant n. \tag{3.6}$$

Let $q(t) = (q_1(t), \dots, q_n(t))$ be the positive $\omega$-periodic function such that $e^{\mu_6 t} q(t)$ is a solution of the linear system

$$\hat{I}_i' = \big(\beta_i(t) S_i^*(t) - \eta\big)\hat{I}_i - \big(\mu_i(t) + \gamma_i(t)\big)\hat{I}_i + \sum_{j=1}^n a_{ij}(t)\hat{I}_j, \quad 1 \leqslant i \leqslant n, \tag{3.7}$$

where $\mu_6 = \frac{1}{\omega}\ln(r(\Phi_{M_1(\cdot)-\eta E}(\omega))) > 0$. Since $I(\bar{t}) \in \mathrm{Int}(\mathbb{R}_n^+)$, we can choose a small number $\alpha > 0$ such that $I(\bar{t}) \geqslant \alpha q(0)$. Then the comparison theorem implies that

$$I(t) \geqslant \alpha e^{\mu_6(t-\bar{t})} q(t-\bar{t}) \geqslant \alpha e^{\mu_6(t-\bar{t})} \min_{t-\bar{t}\geqslant 0} q(t-\bar{t}), \quad \forall t \geqslant \bar{t},$$

and hence $\lim_{t\to\infty} I_i(t) = \infty$, $1 \leqslant i \leqslant n$, a contradiction. Note that for any $I^0 \in \mathbb{R}_+^n$, we have $u_3(t, (S^*(0), I^0)) = (S^*(t), u_2(t, I^0))$, $\forall t \geqslant 0$. It then follows that

$$P_3^m\big(S^*(0), I^0\big) = \big(S^*(0), P_2^m\big(I^0\big)\big), \quad \forall I^0 \in \mathbb{R}_+^n, m \geqslant 0.$$

Since $\omega$ is an internal chain transitive set for $P_3$, $\tilde{\omega}$ is an internal chain transitive set for $P_2$. Let

$$W^s\big(\bar{I}(0)\big) := \Big\{I^0\colon \lim_{m\to\infty}\big(P_2^m\big(I^0\big)\big) = \bar{I}(0)\Big\}.$$

Since $\tilde{\omega} \neq \{0\}$ and $\bar{I}(0)$ is globally attractive for $P_2$ in $\mathbb{R}_+^n \setminus \{0\}$, we have $\tilde{\omega} \cap W^s(\bar{I}(0)) \neq \emptyset$. By [11, Theorem 1.2.1], we then get $\tilde{\omega} = \{\bar{I}(0)\}$, and hence $\omega = \{(S^*(0), \bar{I}(0))\}$, which implies that the positive fixed point $(\bar{S}(0), \bar{I}(0))$ of $P$ is globally attractive in $\mathbb{R}_+^n \times (\mathbb{R}_+^n \setminus \{0\})$. It follows that system (1.3) admits a unique positive $\omega$-periodic solution $(\bar{S}(t), \bar{I}(t))$ such that $\lim_{t\to\infty}(S(t) - \bar{S}(t)) = 0$ and $\lim_{t\to\infty}(I(t) - \bar{I}(t)) = 0$, $\forall (S^0, I^0) \in \mathbb{R}_+^n \times (\mathbb{R}_+^n \setminus \{0\})$.

It remains to prove the stability of $(\bar{S}(t), \bar{I}(t))$ for (1.3), which is equivalent to the stability of $(\bar{N}(t), \bar{I}(t)) := (\bar{S}(t) + \bar{I}(t), \bar{I}(t))$ for (3.3). The associated Jacobian matrix is

$$A(t) = \begin{bmatrix} A_1(t) & 0 \\ A_2(t) & A_3(t) \end{bmatrix},$$

where

$$A_1(t) = \begin{bmatrix} a_{11}^*(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}^*(t) & \cdots & b_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}^*(t) \end{bmatrix}$$

with $a_{ii}^*(t) = \frac{\partial B_i(t,N_i)}{\partial N_i}|_{N_i=\bar{N}_i}\bar{N}_i + B_i(t,\bar{N}_i) - \mu_i(t) + a_{ii}(t)$,

$$A_2(t) = \mathrm{diag}\big(\beta_1(t)\bar{I}_1, \beta_2(t)\bar{I}_2, \dots, \beta_n(t)\bar{I}_n\big),$$

and

$$A_3(t) = \begin{bmatrix} b_{11}^*(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & b_{22}^*(t) & \cdots & b_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & b_{nn}^*(t) \end{bmatrix}$$

with $b_{ii}^*(t) = \beta_i(t)\bar{N}_i - 2\beta_i(t)\bar{I}_i - \mu_i(t) - \gamma_i(t) + a_{ii}(t)$.

Obviously,

$$A_1(t) < \begin{bmatrix} \bar{a}_{11}^*(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & \bar{a}_{22}^*(t) & \cdots & b_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & \bar{a}_{nn}^*(t) \end{bmatrix} := C_1(t)$$

with $\bar{a}_{ii}^*(t) = B_i(t, \bar{N}_i) - \mu_i(t) + a_{ii}(t)$, and

$$A_3(t) < \begin{bmatrix} \bar{b}_{11}^*(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & \bar{b}_{22}^*(t) & \cdots & b_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & \bar{b}_{nn}^*(t) \end{bmatrix} := C_3(t)$$

with $\bar{b}_{ii}^*(t) = \beta_i(t)\bar{N}_i - \beta_i(t)\bar{I}_i - \mu_i(t) - \gamma_i(t) + a_{ii}(t)$. The comparison principle implies that $\Phi_{A_1(\cdot)}(t) \leqslant \Phi_{C_1(\cdot)}(t)$, $\Phi_{A_3(\cdot)}(t) \leqslant \Phi_{C_3(\cdot)}(t)$, and hence $\Phi_{A_1(\cdot)}(\omega) \leqslant \Phi_{C_1(\cdot)}(\omega)$, $\Phi_{A_3(\cdot)}(\omega) \leqslant \Phi_{C_3(\cdot)}(\omega)$. By [9, Theorem A.4], we have $\mu(\Phi_{A_1(\cdot)}(\omega)) < \mu(\Phi_{C_1(\cdot)}(\omega))$ and $\mu(\Phi_{A_3(\cdot)}(\omega)) < \mu(\Phi_{C_3(\cdot)}(\omega))$. Notice that $(\bar{N}_1(t), \ldots, \bar{N}_n(t))$ is a positive $\omega$-periodic solution of the system $N' = C_1(t)N$. Then we have

$$\Phi_{C_1(\cdot)}(t) \begin{pmatrix} \bar{N}_1(0) \\ \vdots \\ \bar{N}_n(0) \end{pmatrix} = \begin{pmatrix} \bar{N}_1(t) \\ \vdots \\ \bar{N}_n(t) \end{pmatrix}.$$

It follows that

$$\Phi_{C_1(\cdot)}(\omega) \begin{pmatrix} \bar{N}_1(0) \\ \vdots \\ \bar{N}_n(0) \end{pmatrix} = \begin{pmatrix} \bar{N}_1(\omega) \\ \vdots \\ \bar{N}_n(\omega) \end{pmatrix} = \begin{pmatrix} \bar{N}_1(0) \\ \vdots \\ \bar{N}_n(0) \end{pmatrix},$$

and hence $\mu(\Phi_{C_1(\cdot)}(\omega)) = 1$. On the other hand, $(\bar{I}_1(t), \ldots, \bar{I}_n(t))$ is a positive $\omega$-periodic solution of the system $I' = C_3(t)I$. Thus, we obtain

$$\Phi_{C_3(\cdot)}(t) \begin{pmatrix} \bar{I}_1(0) \\ \vdots \\ \bar{I}_n(0) \end{pmatrix} = \begin{pmatrix} \bar{I}_1(t) \\ \vdots \\ \bar{I}_n(t) \end{pmatrix}.$$

It follows that

$$\Phi_{C_3(\cdot)}(\omega) \begin{pmatrix} \bar{I}_1(0) \\ \vdots \\ \bar{I}_n(0) \end{pmatrix} = \begin{pmatrix} \bar{I}_1(\omega) \\ \vdots \\ \bar{I}_n(\omega) \end{pmatrix} = \begin{pmatrix} \bar{I}_1(0) \\ \vdots \\ \bar{I}_n(0) \end{pmatrix},$$

and hence $\mu(\Phi_{C_3(\cdot)}(\omega)) = 1$. Consequently, we have

$$\mu(\Phi_A(\cdot)(\omega)) = \max\{\mu(\Phi_{A_1(\cdot)}(\omega)), \mu(\Phi_{A_3(\cdot)}(\omega))\} < 1,$$

which implies the stability of $(\bar{N}(t), \bar{I}(t))$. □

At last, we prove the global attractivity of positive periodic solution in the case where $\{b_{ij}(t)\}$ is very close to $\{a_{ij}(t)\}$. Let $\Lambda_0$ be the set of all continuous and $\omega$-periodic $n \times n$ matrix functions satisfying $a_{ij}(t) > 0$, $i \neq j$, $a_{ii}(t) < 0$ and $\sum_{j=1}^n a_{ji}(t) = 0$.

**Theorem 3.2.** *Assume* (A1)–(A5) *hold. Let* $\lambda_0 = \{a_{ij}(t)\colon\ 1 \leqslant i, j \leqslant n\} \in \Lambda_0$ *be fixed,* $\lambda = \{b_{ij}(t)\colon\ 1 \leqslant i, j \leqslant n\} \in \Lambda_0$, $M_{1\lambda}(t)$ *be the matrix* $M_1(t)$ *with parameter* $\lambda$, *and* $M_{1\lambda_0}(t)$ *be the matrix* $M_{1\lambda}(t)$ *with* $\lambda = \lambda_0$. *If* $r(\Phi_{M_{1\lambda_0}(\cdot)}(\omega)) > 1$, *then there exists* $\epsilon > 0$ *such that for any* $\lambda$ *with* $\|\lambda - \lambda_0\| \leqslant \epsilon$, *system* (1.3) *admits a unique positive* $\omega$*-periodic solution* $(\bar{S}_{\lambda 1}(t), \bar{S}_{\lambda 2}(t), \ldots, \bar{S}_{\lambda n}(t), \bar{I}_{\lambda 1}(t), \ldots, \bar{I}_{\lambda n}(t))$ *such that* $\lim_{t\to\infty}(S_i(t) - \bar{S}_{\lambda i}(t)) = 0$ *and* $\lim_{t\to\infty}(I_i(t) - \bar{I}_{\lambda i}(t)) = 0$ *for every* $(S^0, I^0) \in \mathbb{R}^n_+ \times \mathrm{Int}(\mathbb{R}^n_+)$.

**Proof.** There exist $\epsilon_0 > 0$, $\eta > 0$, such that $r(\Phi_{M_{1\lambda}(\cdot)}(\omega)) > 1$ and $r(\Phi_{M_{1\lambda-\eta M_2}(\cdot)}(\omega)) > 1$ whenever $\|\lambda - \lambda_0\| \leqslant \epsilon_0$. We fix a sufficiently small $\delta > 0$ such that the Poincaré map associated with (2.17) admits a unique positive fixed point $S^*(0, \delta)$, which is globally attractive in $\mathbb{R}^n_+ \setminus \{0\}$ and $S^*(t, \delta) > S^*(t) - \bar{\eta}$. Let $u(t, (S^0, I^0), \lambda)$ be the solution of (1.3) with parameter $\lambda$ and initial value $(S^0, I^0) \in X$. By the continuity of solutions with respect to initial values and parameter $\lambda$, there exist positive numbers $\delta_0^*$ and $\epsilon_0^*$ such that $\|u(t, (S^0, I^0), \lambda) - u(t, M_i, \lambda_0)\| < \delta$, $\forall t \in [0, \omega]$, $\|(S^0, I^0) - M_i\| \leqslant \delta_0^*$ and $\|\lambda - \lambda_0\| \leqslant \epsilon_0^*$, $i = 0, 1$. Let $\epsilon^* = \min\{\epsilon_0, \epsilon_0^*\}$. By the argument similar to that of the claim in the proof of Theorem 2.3, it follows that for any $\lambda$ with $\|\lambda - \lambda_0\| \leqslant \epsilon^*$, and all $(S^0, I^0) \in \mathbb{R}^n_+ \times \mathrm{Int}(\mathbb{R}^n_+)$, there holds

$$\limsup_{m\to\infty} d\big(P_\lambda^m\big(S^0, I^0\big), M_i\big) \geqslant \delta_0^*, \quad i = 0, 1,$$

where $P_\lambda$ is the Poincaré map associated with (1.3) with parameter $\lambda$. Moreover, Lemma 2.2 implies that solutions of (1.3) in $X$ are uniformly bounded and ultimately bounded for each $\lambda \in \Lambda_0$. It follows that $P$ has a global attractor $A_\lambda \subset X_0$ for each $\lambda \in \Lambda_0$. Let $\Lambda_1 = \Lambda_0 \cap \overline{\{\lambda\colon\ \|\lambda - \lambda_0\| \leqslant \epsilon^*\}}$. Then there exists a bounded and closed set $G^*$ in $\mathbb{R}^n_+ \times \mathbb{R}^n_+$, such that $\cup_{\lambda\in\Lambda_1} A_\lambda \subset G^*$. Hence, by [11, Theorem 1.4.2], there exists a $\delta_0 > 0$ such that for any $\lambda \in \Lambda_1$,

$$\liminf_{m\to\infty} d\big(P_\lambda^m\big(S^0, I^0\big), \partial X_0\big) \geqslant \delta_0.$$

Since $\overline{\bigcup_{\lambda\in\Lambda_1} P(A_\lambda)} = \overline{\bigcup_{\lambda\in\Lambda_1} A_\lambda} \subset \overline{G^*} = G^* \subset X_0$, $\overline{\bigcup_{\lambda\in\Lambda_1} P(A_\lambda)}$ is compact. By applying [11, Theorem 1.4.1] on the perturbation of a globally stable fixed point, we complete the proof. □

## 4. Numerical simulations

In order to simulate the periodic solutions, we consider the case that the patch number is 2. For simplicity, we assume that the contact rate $\beta_i(t)$, $i = 1, 2$, is $\omega$-periodic with the expression $\beta_1(t) = \beta_2(t) = m\sin(pt) + q$, and other parameters are independent of time $t$. Then $\omega = \frac{2\pi}{p}$, and assumption (1.4) is equivalent to that $a_{12} = -a_{22}$, $a_{21} = -a_{11}$, $b_{12} = -b_{22}$, $b_{21} = -b_{11}$. Thus, (1.3) reduces to

$$\begin{cases} S_1' = B_1(N_1)N_1 - (\mu_1 - a_{11})S_1 - \beta(t)S_1I_1 + \gamma_1 I_1 - a_{22}S_2, \\ S_2' = B_2(N_2)N_2 - (\mu_2 - a_{22})S_2 - \beta(t)S_2I_2 + \gamma_2 I_2 - a_{11}S_1, \\ I_1' = \beta(t)S_1I_1 - (\mu_1 + \gamma_1 - b_{11})I_1 - b_{22}I_2, \\ I_2' = \beta(t)S_2I_2 - (\mu_2 + \gamma_2 - b_{22})I_2 - b_{11}I_1. \end{cases} \tag{4.1}$$

As mentioned in [3], we choose $B_i(N_i) = \frac{r_i}{N_i} + c_i$, where $c_i < \mu_i$, $i = 1, 2$. Suppose that $r_1 = r_2 = r$, $c_1 = c_2 = c$, $\mu_1 = \mu_2 = \mu$, $\gamma_1 = \gamma_2 = \gamma$, $a_{11} = a_{22} = b_{11} = b_{22} = -\theta < 0$. Then (4.1) reduces to

$$\begin{cases} S_1' = r - (\mu + \theta - c)S_1 - \big(m\sin(pt) + q\big)S_1 I_1 + (c+\gamma)I_1 + \theta S_2, \\ S_2' = r - (\mu + \theta - c)S_2 - \big(m\sin(pt) + q\big)S_2 I_2 + (c+\gamma)I_2 + \theta S_1, \\ I_1' = \big(m\sin(pt) + q\big)S_1 I_1 - (\mu + \gamma + \theta)I_1 + \theta I_2, \\ I_2' = \big(m\sin(pt) + q\big)S_2 I_2 - (\mu + \gamma + \theta)I_2 + \theta I_1. \end{cases} \tag{4.2}$$

It is easy to verify that conditions (A1)–(A5) are satisfied. In this case, $(S_1^*(0), S_2^*(0))$ can be obtained explicitly as

$$S_1^*(0) = S_2^*(0) = \frac{r}{\mu - c}.$$

Under all assumptions above, we get

$$\begin{aligned} M_1(t) &= \left[\begin{pmatrix} \beta(t)\frac{r}{\mu-c} - \mu - \gamma - \theta & \theta \\ \theta & \beta(t)\frac{r}{\mu-c} - \mu - \gamma - \theta \end{pmatrix}\right] \\ &= \left[\begin{pmatrix} -\mu - \gamma - \theta & \theta \\ \theta & -\mu - \gamma - \theta \end{pmatrix}\right] + \left[\begin{pmatrix} \beta(t)\frac{r}{\mu-c} & 0 \\ 0 & \beta(t)\frac{r}{\mu-c} \end{pmatrix}\right]. \end{aligned}$$

Let $A$ be a $2 \times 2$ constant matrix, and $\alpha(t)$ be a continuous $\omega$-periodic function. Note that if $x(t)$ is a solution of $x' = (A + \alpha(t)I)x$, then $y(t) = e^{\int_0^t -\alpha(s)\,ds}x(t)$ satisfies

$$y'(t) = e^{\int_0^t -\alpha(s)\,ds}\big(x' - \alpha(t)x\big) = e^{\int_0^t -\alpha(s)\,ds}Ax(t) = Ae^{\int_0^t -\alpha(s)\,ds}x(t) = Ay(t).$$

Thus, we have $\phi_{A+\alpha(\cdot)I}(t) = e^{\int_0^t \alpha(s)\,ds}e^{At}$.

By the above observation, it follows that

$$r\big(\Phi_{M_1(\cdot)}(\omega)\big) = e^{\frac{r}{\mu-c}\int_0^\omega \beta(t)\,dt}e^{-(\mu+\gamma)\omega}.$$

Fix $\mu = 0.2$, $c = 0.1$, $\theta = 1$, $\gamma = 4$, $m = 1$, $p = 2\pi$, $q = 0.1$, $r = 1$. Since $\omega = 1$, we have

$$r\big(\Phi_{M_1(\cdot)}(1)\big) < 1.$$

By Theorem 2.1, system (1.3) has a positive $\omega$-periodic solution such that $\lim_{t\to\infty}(S(t) - S^*(t)) = 0$ and $\lim_{t\to\infty} I(t) = 0$ for all $(S^0, I^0) \in \mathbb{R}_+^n \times \mathbb{R}_+^n$. Our numerical simulations in Fig. 1 confirm this result.

Fix $\mu = 2$, $c = 1$, $\theta = 1$, $\gamma = 0.1$, $m = 1$, $p = 2\pi$, $q = 1$, $r = 10$. We then have $\omega = 1$ and $r(\Phi_{M_1(\cdot)}(1)) > 1$. By Theorem 3.1, system (1.3) has a unique positive $\omega$-periodic solution such that $\lim_{t\to\infty}(S(t) - \bar{S}(t)) = 0$ and $\lim_{t\to\infty}(I(t) - \bar{I}(t)) = 0$ for all $(S^0, I^0) \in \mathbb{R}_+^n \times (\mathbb{R}_+^n \setminus \{0\})$. Our numerical simulations in Fig. 2 confirm this result.

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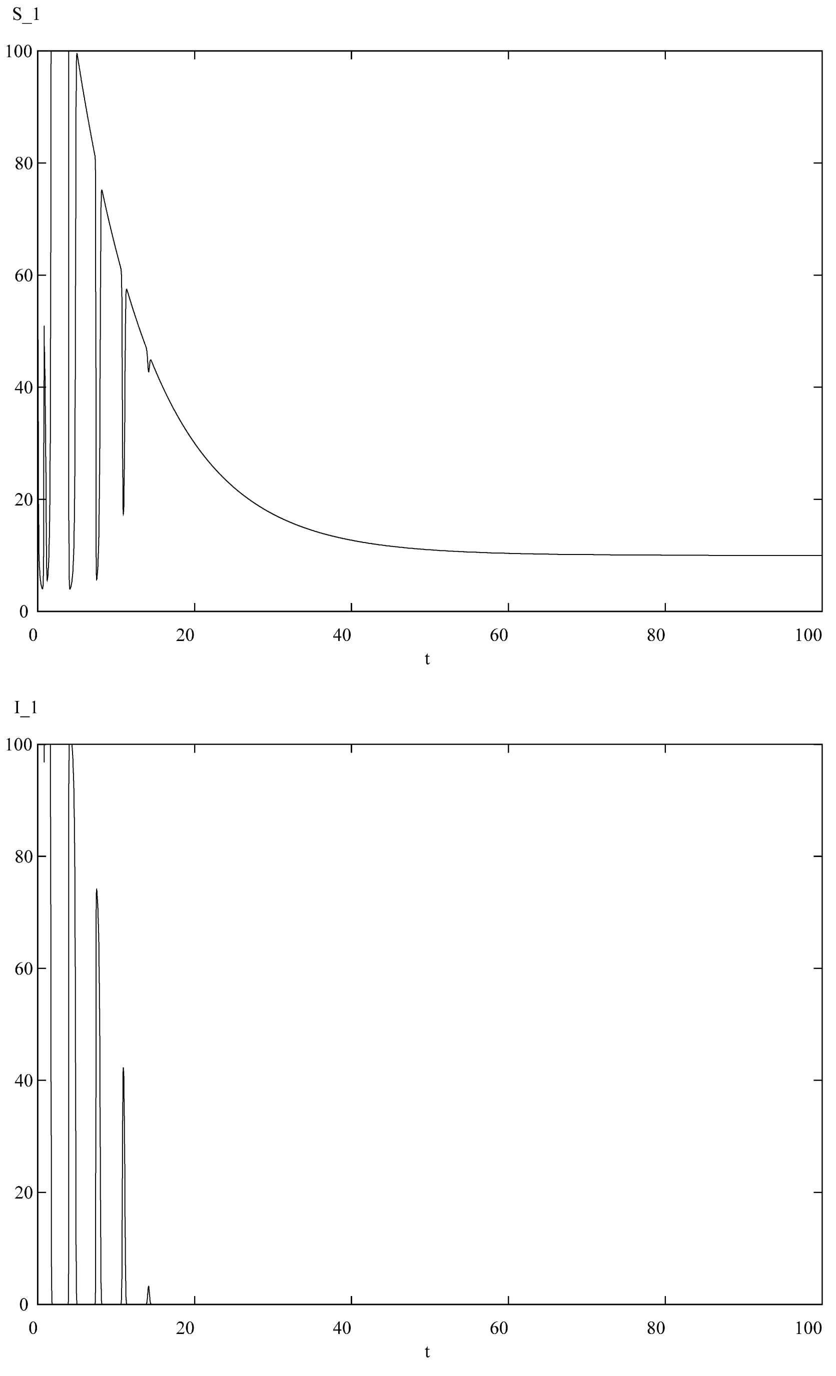
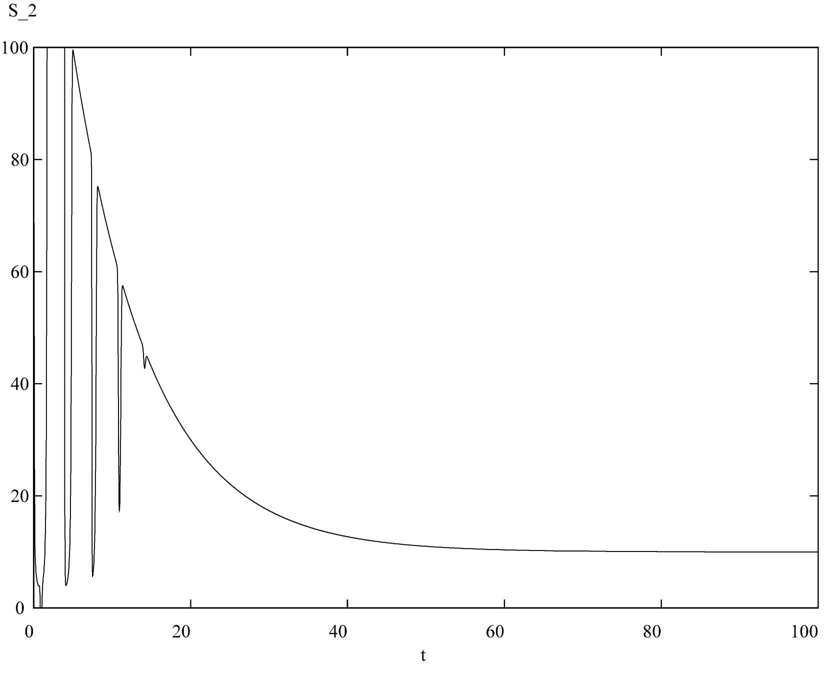



Fig. 1. The extinction of the disease.



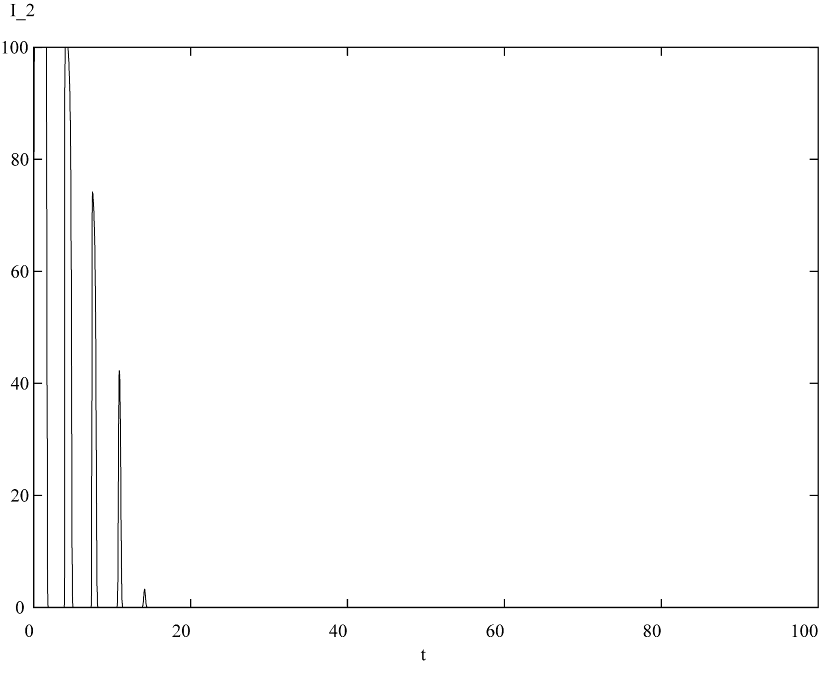




Fig. 1. (*continued*)

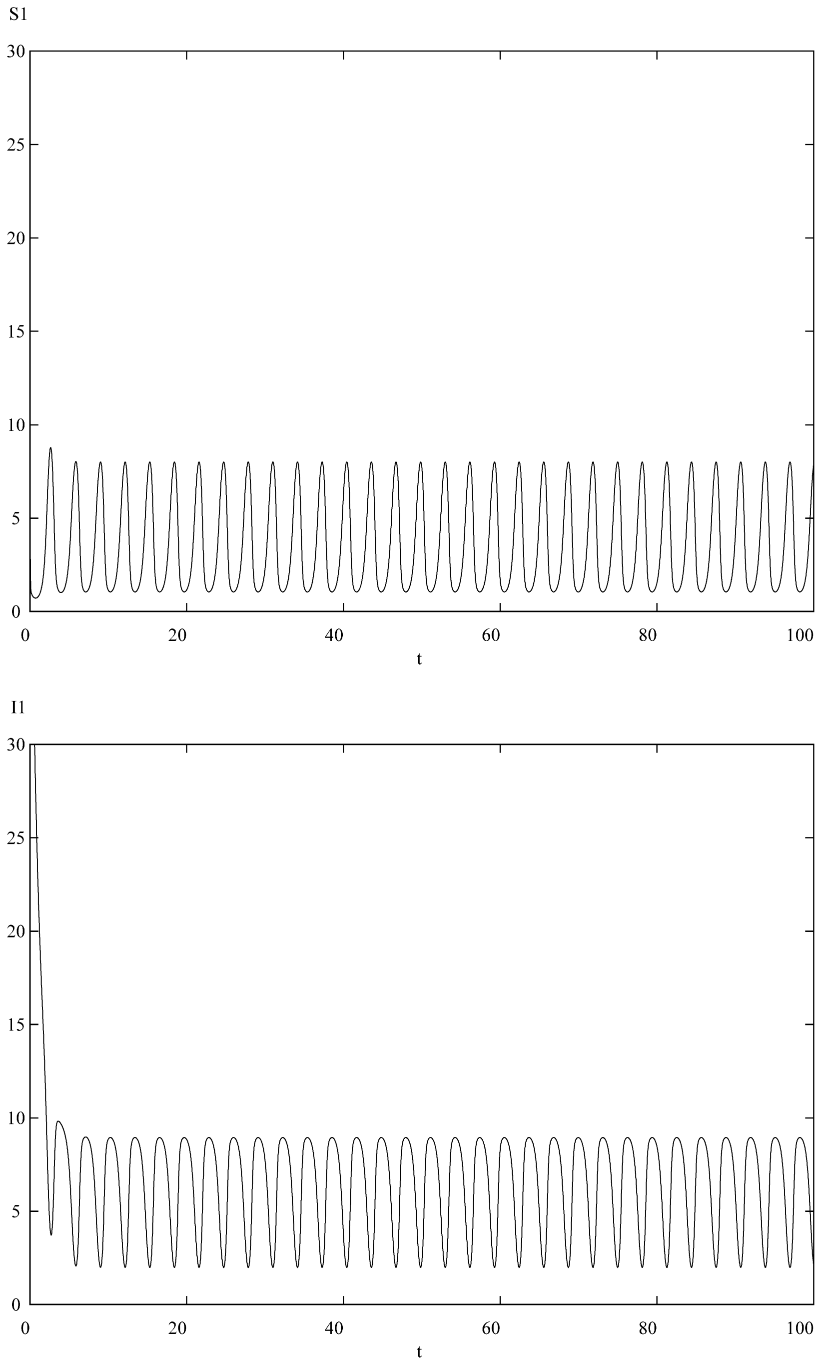
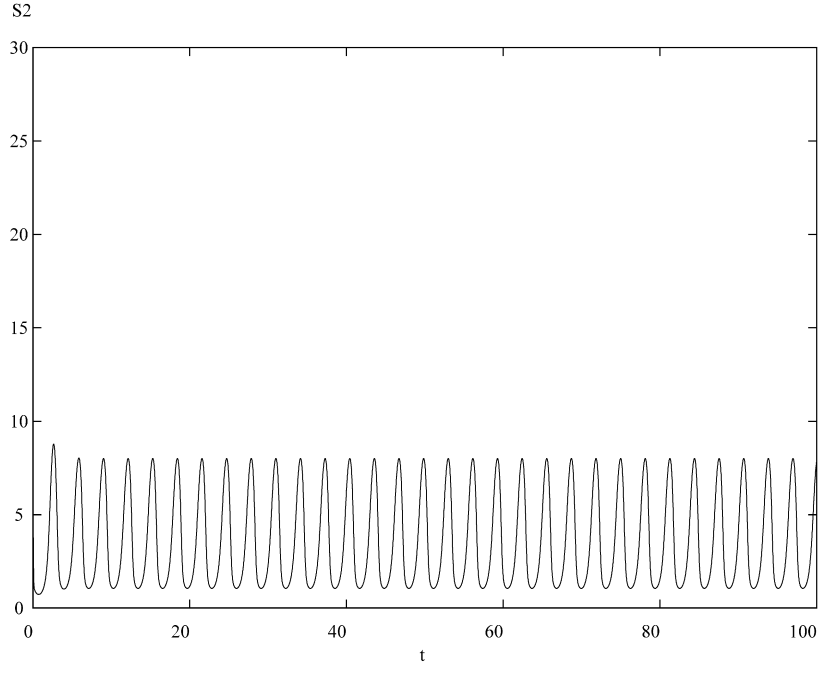



Fig. 2. The uniform persistence of the disease.



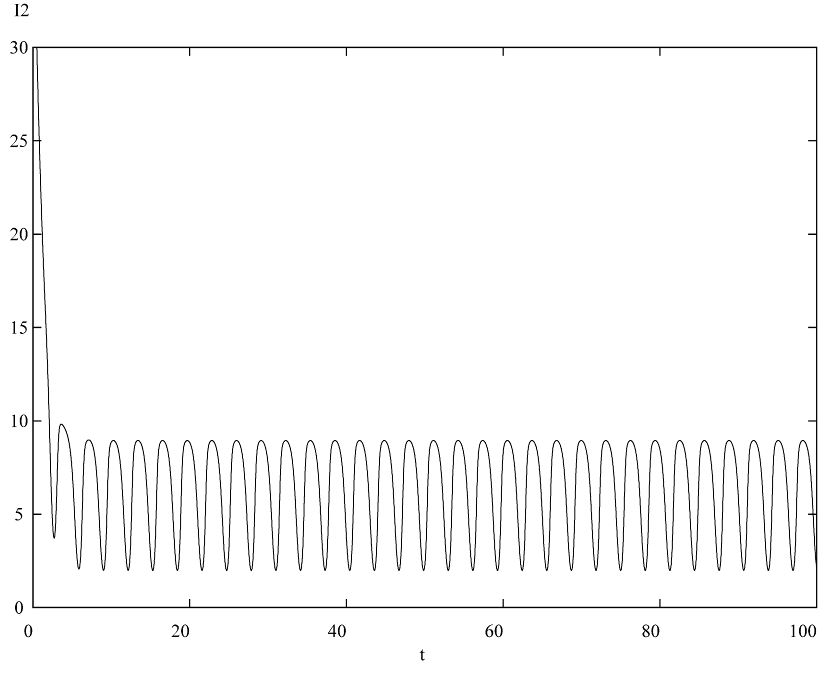




Fig. 2. (*continued*)

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