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Central limit theorems for arrays of decimated linear processes

F. Roueff[a,*], M.S. Taqqu[b]

[a] *Telecom Paristech, Institut Télécom, CNRS LTCI, 46, rue Barrault, 75634 Paris Cédex 13, France*
[b] *Department of Mathematics and Statistics, Boston University, Boston, MA 02215, USA*

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Abstract

Linear processes are defined as a discrete-time convolution between a kernel and an infinite sequence of i.i.d. random variables. We modify this convolution by introducing decimation, that is, by stretching time accordingly. We then establish central limit theorems for arrays of squares of such decimated processes. These theorems are used to obtain the asymptotic behavior of estimators of the spectral density at specific frequencies. Another application, treated elsewhere, concerns the estimation of the long-memory parameter in time series, using wavelets.

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1. Introduction

Consider a linear process, that is, a weakly stationary sequence

$$\sum_{t\in\mathbb{Z}} v(k-t)\xi_k, \quad k\in\mathbb{Z}, \quad \text{where } \sum_{t\in\mathbb{Z}} v^2(t) < \infty$$

and $\{\xi_t, t\in\mathbb{Z}\}$ is a centered white noise sequence, that is an uncorrelated sequence with mean zero. We shall sometimes make the following additional assumptions on $\{\xi_t, t\in\mathbb{Z}\}$.

* Corresponding author. Tel.: +33 1 45 81 78 53; fax: +33 1 45 81 71 44.
E-mail addresses: roueff@tsi.enst.fr, roueff@telecom-paristech.fr (F. Roueff), murad@math.bu.edu (M.S. Taqqu).

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Assumptions A

(A-1) $\{\xi_t, t \in \mathbb{Z}\}$ is a sequence of independent and identically distributed real-valued random variables such that $\mathbb{E}[\xi_0] = 0$, $\mathbb{E}[\xi_0^2] = 1$.

(A-2) $\{\xi_t, t \in \mathbb{Z}\}$ satisfies (A-1) and $\kappa_4 \stackrel{\text{def}}{=} \mathbb{E}[\xi_0^4] - 3$ is finite.

We will allow decimation and consider, moreover, not one but N linear sequences, all using the same $\{\xi_t, t \in \mathbb{Z}\}$.

Definition 1. An array of N-dimensional decimated linear processes is a process admitting the following linear representation:

$$Z_{i,j,k} = \sum_{t \in \mathbb{Z}} v_{i,j}(\gamma_j k - t)\xi_t, \quad i = 1, \ldots, N, k \in \mathbb{Z}, j \geq 0, \tag{1}$$

where $\{\xi_t, t \in \mathbb{Z}\}$ is a centered weak white noise, $(\gamma_j)_{j \geq 0}$ is a diverging sequence of positive integers and, for all $i = 1, \ldots, N$ and $j \geq 0$, $\{v_{i,j}(t), t \in \mathbb{Z}\}$ is real valued and satisfies $\sum_{t \in \mathbb{Z}} v_{i,j}^2(t) < \infty$.

Remark 1. $Z_{i,j,k}$ involves three indices. The index $i = 1, \ldots, N$ is used to define an N-variate version, the index j labels the decimation factor γ_j, and the index k corresponds to time. Because of the presence of the factor γ_j in (1), $Z_{i,j,k}$ is not a usual convolution. It can be viewed as a decimated convolution of a white noise in the sense that, after convolution, one keeps only values spaced by γ_j. A typical choice of decimation is $\gamma_j = 2^j$, $j \geq 0$.

Our goal is to study the asymptotic behavior of the sample mean square of $Z_{i,j,k}$, namely to find conditions on the kernels $v_{i,j}$, the decimation factor γ_j and normalization n_j, so that the normalized vector

$$\left\{ n_j^{-1/2} \sum_{k=0}^{n_j - 1} \left(Z_{i,j,k}^2 - \mathbb{E}\left[Z_{i,j,k}^2 \right] \right), i = 1, \ldots, N \right\}$$

converges to a multivariate normal $\mathcal{N}(0, \Gamma)$ distribution. We want also to characterize the limiting covariance matrix Γ. Thus we are interested in the sum of squares of the $Z_{i,j,k}$.

Such results are useful in estimation. In Section 4, for example, we apply our result to obtain a central limit theorem for the estimator $\widehat{f_n}(0)$ of the spectral density at the origin $f(0)$ of a linear process. This CLT is compared to [1], Eq. 3.9, as discussed in Remark 13. We also include in Section 4 a central limit theorem for estimators of the spectral density at several frequencies.

Another, more involved application, which involves wavelets, can be found in [2], where we consider linear processes, not necessarily Gaussian, with long, short or negative memory. The memory parameter is estimated semiparametrically using wavelets from a sample $X_1, \ldots, X_n$ of the process. We show that both the log-regression wavelet estimator and the wavelet Whittle estimator of the memory parameter are asymptotically normal as the sample size $n \to \infty$ and we obtain an explicit expression for the limit variance. To do so, we use a general result on the asymptotic normality of the empirical scalogram for linear processes. The scalogram is an array of quadratic forms involving the wavelet coefficients of the observed sample. In contrast to quadratic forms computed on the Fourier coefficients such as the periodogram, the scalogram involves correlations which do not vanish as the sample size $n \to \infty$. To establish the results mentioned here, we use Theorem 2 below. For a general review of this type of application see [3].

The paper is structured as follows. In Section 2, we indicate the main assumptions. The central limit theorems (Theorems 1 and 2) for decimated sequences are stated in Section 3. Section 4 contains an application to the estimation of the spectral density at various frequencies (Theorems 3 and 4). Theorems 1 and 2 are proved in Sections 5 and 6 respectively. Theorems 3 and 4 are proved in Section 7. Section 8 contains technical lemmas.

2. Main assumptions

Our assumptions will be expressed in terms of the Fourier series of the ℓ^2 sequences $\{v_{i,j}(t), t \in \mathbb{Z}\}$, namely

$$v^*_{i,j}(\lambda) = (2\pi)^{-1/2} \sum_{t \in \mathbb{Z}} v_{i,j}(t) \mathrm{e}^{-\mathrm{i}\lambda t}, \quad \lambda \in \mathbb{R}. \tag{2}$$

We suppose that for any $i = 1, \ldots, N$, as $j \to \infty$, the Fourier series $v^*_{i,j}$ concentrates around some frequency $\lambda_{i,\infty} \in [0, \pi)$. By "concentrate", we mean that when adequately normalized, translated and rescaled around these frequencies, the series $v^*_{i,j}$ converges as $j \to \infty$ to some limit functions $v^*_{i,\infty}$, with a uniform polynomial control (see Eqs. (7) and (8) below). Because of the particular structure of the γ_j-decimation in (1), however, in order to derive the asymptotic behavior for the processes, we need to introduce sequences of frequencies $(\lambda_{i,j})_{j \geq 0}$ that satisfy some special conditions and converge to $\lambda_{i,\infty}$ for all $i = 1, \ldots, N$. We shall first specify the conditions on the Fourier series $v^*_{i,j}$, the frequencies $\lambda_{i,\infty}$ and the limit functions $v^*_{i,\infty}$, and then comment on these conditions.

Condition B

(i) There exist an N-dimensional array of frequencies $(\lambda_{i,j})_{i \in 1,\ldots,N, j \geq 0}$ valued in $[0, \pi)$ such that, for all $i = 1, \ldots, N$,

$$\gamma_j \lambda_{i,j} \in 2\pi \mathbb{Z}_+, \quad \text{for } j \text{ large enough}, \tag{3}$$

$$\lambda_{i,j} \to \lambda_{i,\infty}, \quad \text{as } j \to \infty, \tag{4}$$

$$\text{if } \lambda_{i,\infty} = 0, \quad \text{then } \lambda_{i,j} = 0, \quad \text{for } j \text{ large enough}, \tag{5}$$

and, for all $1 \leq i < i' \leq N$,

$$\text{if } \lambda_{i,\infty} = \lambda_{i',\infty}, \quad \text{then } \lambda_{i,j} = \lambda_{i',j}, \quad \text{for } j \text{ large enough}. \tag{6}$$

(ii) Moreover there exist $\delta > 1/2$ and an array of $[-\pi, \pi)$-valued functions $\Phi_{i,j}(\lambda)$ defined on $\lambda \in \mathbb{R}$ such that, for all $i = 1, \ldots, N$,

$$\sup_{j \geq 0} \sup_{\lambda \in [0,\pi)} \gamma_j^{-1/2} |v^*_{i,j}(\lambda)| (1 + \gamma_j |\lambda - \lambda_{i,j}|)^{\delta} < \infty, \tag{7}$$

$$\lim_{j \to \infty} \gamma_j^{-1/2} v^*_{i,j}(\gamma_j^{-1} \lambda + \lambda_{i,j}) \mathrm{e}^{\mathrm{i} \Phi_{i,j}(\lambda)} = v^*_{i,\infty}(\lambda) \quad \text{for all } \lambda \in \mathbb{R}, \tag{8}$$

and, for all $1 \leq i < i' \leq N$,

$$\text{if } \lambda_{i,\infty} = \lambda_{i',\infty}, \quad \text{then } \Phi_{i,j} \equiv \Phi_{i',j}, \quad \text{for } j \text{ large enough}. \tag{9}$$

The following remarks provide some insight into these conditions.

Remark 2. Eqs. (4) and (8) imply that the spectral density $\lambda \mapsto |v^*_{i,j}|^2(\lambda)$ of the *undecimated* stationary process

$$\tilde{Z}_{i,j,k} = \sum_{t \in \mathbb{Z}} v_{i,j}(k-t)\xi_t, \quad k \in \mathbb{Z},$$

concentrates, as $j \to \infty$, around the frequency $\lambda_{i,\infty}$. In practical applications of the theorem, the limiting frequencies $\{\lambda_{i,\infty}, i \in 1, \ldots, N\}$ are given. However, one can often easily find sequences $(\gamma_j)_{j\geq 0}$ and $(\lambda_{i,j})_{j\geq 0}$ that satisfy Conditions (3) and (4). In the particular case where the $\lambda_{i,\infty}$ are such that $q\lambda_{i,\infty} \in 2\pi\mathbb{Z}$ for all $i = 1, \ldots, N$ and some positive integer q, one may take $\lambda_{i,j} = \lambda_{i,\infty}$ and γ_j as a multiple of q. This happens for instance when the limiting frequencies are all at the origin, that is, $\lambda_{1,\infty} = \cdots = \lambda_{N,\infty} = 0$ and $\gamma_j = 2^j$.

Remark 3. The presence of the phase function $\Phi_{i,j}$ in (8) offers flexibility and implies that $\gamma_j^{-1/2} v_{i,j}^*(\gamma_j^{-1}\lambda + \lambda_0)$ converges to $v_{i,\infty}^*(\lambda)$ *up to a change of phase*. Observe, however, that Condition (9) requires that $\Phi_{i,j}$ and $\Phi_{i',j}$ be equal if the asymptotic frequencies $\lambda_{i,\infty}$ and $\lambda_{i',\infty}$ are the same. If $\lambda_{1,\infty} < \cdots < \lambda_{N,\infty}$, Condition (8) is equivalent to requiring that $\gamma_j^{-1/2}|v_{i,j}^*(\gamma_j^{-1}\lambda + \lambda_{i,j})|$ converges to $|v_{i,\infty}^*(\lambda)|$ for all i. However, because of Condition (9), if some of the frequencies $\lambda_{1,\infty}, \ldots, \lambda_{N,\infty}$ are equal, the latter condition does not imply Conditions (8) and (9). The presence of the phase $\Phi_{i,j}$ is consistent with the fact that the asymptotic covariance matrix Γ defined in (25) is invariant through a phase translation of the functions $v_{i,\infty}^*$ for all $i = 1, \ldots, N$, provided that these phase translations are identical for any pair i, i' such that $\mathrm{C}_{i,i'} \neq 0$, that is, $\lambda_{i,\infty} = \lambda_{i',\infty}$; see (18).

Remark 4. Condition (6) states that if two limits $\lambda_{i,\infty}$ and $\lambda_{i',\infty}$ are equal, then the $\lambda_{i,j}$ and $\lambda_{i',j}$ which converge to them must coincide for large enough j. Condition (5) has a similar interpretation.

Remark 5. Conditions (6), (8) and (9) imply that, for all $1 \leq i \leq i' \leq N$ such that $\lambda_{i,\infty} = \lambda_{i',\infty}$,

$$\lim_{j\to\infty} \gamma_j^{-1}[v_{i,j}^* \overline{v_{i',j}^*}](\gamma_j^{-1}\lambda + \lambda_{i,j}) = [v_{i,\infty}^* \overline{v_{i',\infty}^*}](\lambda) \quad \text{for all } \lambda \in \mathbb{R}. \tag{10}$$

Here $\overline{z}$ denotes the conjugate of the complex z.

Remark 6. Since $v_{i,j}(t)$ is real valued, we have

$$v_{i,j}^*(-\lambda) = \overline{v_{i,j}^*(\lambda)}. \tag{11}$$

Thus, Conditions (7) and (10) imply that

$$\sup_{j\geq 0} \sup_{\lambda\in(-\pi,\pi)} \gamma_j^{-1/2}|v_{i,j}^*(\lambda)| \left(1 + \gamma_j \left||\lambda| - \lambda_{i,j}\right|\right)^\delta < \infty, \tag{12}$$

$$\lim_{j\to\infty} \gamma_j^{-1}[v_{i,j}^* \overline{v_{i',j}^*}](\gamma_j^{-1}\lambda - \lambda_{i,j}) = [\overline{v_{i,\infty}^*} v_{i',\infty}^*](-\lambda) \quad \text{for all } \lambda \in \mathbb{R}. \tag{13}$$

In particular, if $\lambda_{i,\infty} = \lambda_{i',\infty} = 0$, by (5), (10) and (13), we have

$$[v_{i,\infty}^* \overline{v_{i',\infty}^*}](\lambda) = [\overline{v_{i,\infty}^*} v_{i',\infty}^*](-\lambda). \tag{14}$$

Remark 7. Since (γ_j) is a diverging sequence and $\lambda_{i,j} \to \lambda_{i,\infty} \in [0, \pi)$, for any $\lambda \in \mathbb{R}$, for j large enough, we have $\gamma_j^{-1}\lambda + \lambda_{i,j} \in [0, \pi)$. Hence Conditions (7) and (8) imply that, for all

$i = 1, \ldots, N$,

$$\sup_{\lambda \in \mathbb{R}} \left| v^*_{i,\infty}(\lambda) \right| (1 + |\lambda|)^{\delta} < \infty. \tag{15}$$

To better understand these assumptions, we start with a result on the asymptotic behavior of the cross-covariance function for the array (1). In this proposition, we set, without loss of generality, $N = 2$.

Proposition 1. *Let $\{Z_{i,j,k}, i = 1, 2, j \geq 0, k \in \mathbb{Z}\}$ be an array of two-dimensional decimated linear processes as defined by (1). Assume that Condition* (B) *holds for some $\lambda_{i,\infty} \in [0, \pi)$ and functions $v^*_{i,\infty}$, $i = 1, 2$, from $\mathbb{R} \to \mathbb{Z}$. Then, for all $k, k' \in \mathbb{Z}$, as $j \to \infty$,*

$$\mathrm{Cov}\left(Z_{1,j,k}, Z_{2,j,k'}\right) \to \mathrm{C}_{1,2} \int_{-\infty}^{\infty} w^*_{1,2}(\lambda) \mathrm{e}^{\mathrm{i}\lambda(k'-k)} \mathrm{d}\lambda, \tag{16}$$

where, for any $i, i' \in \{1, 2\}$,

$$w^*_{i,i'}(\lambda) = \frac{1}{2} \left[\overline{v^*_{i,\infty}(-\lambda)} v^*_{i',\infty}(-\lambda) + v^*_{i,\infty}(\lambda) \overline{v^*_{i',\infty}(\lambda)} \right], \quad \lambda \in \mathbb{R}, \tag{17}$$

and

$$\mathrm{C}_{i,i'} = \begin{cases} 0 & \text{if } \lambda_{i,\infty} \neq \lambda_{i',\infty} \\ 1 & \text{if } \lambda_{i,\infty} = \lambda_{i',\infty} = 0 \\ 2 & \text{if } \lambda_{i,\infty} = \lambda_{i',\infty} > 0. \end{cases} \tag{18}$$

Proof. Using (1) and Parseval's theorem, we have

$$\mathrm{Cov}\left(Z_{1,j,k}, Z_{2,j,k'}\right) = \sum_{t \in \mathbb{Z}} v_{1,j}(\gamma_j k - t) v_{2,j}(\gamma_j k' - t) \tag{19}$$

$$= \int_{-\pi}^{\pi} [v^*_{1,j} \overline{v^*_{2,j}}](\lambda) \mathrm{e}^{\mathrm{i}\gamma_j \lambda(k'-k)} \mathrm{d}\lambda. \tag{20}$$

We now consider separately the three cases $\lambda_{1,\infty} \neq \lambda_{2,\infty}$, $\lambda_{1,\infty} = \lambda_{2,\infty} > 0$ and $\lambda_{1,\infty} = \lambda_{2,\infty} = 0$.

(1) Suppose $\lambda_{1,\infty} \neq \lambda_{2,\infty}$. Then by (7), there is a constant $C > 0$ such that

$$\left|\mathrm{Cov}\left(Z_{1,j,k}, Z_{2,j,k'}\right)\right| \leq C\gamma_j \int_0^{\pi} (1 + \gamma_j |\lambda - \lambda_{1,j}|)^{-\delta} (1 + \gamma_j |\lambda - \lambda_{2,j}|)^{-\delta} \mathrm{d}\lambda$$
$$\to 0 \quad \text{as } j \to \infty, \tag{21}$$

since $\gamma_j \to \infty$, $\delta > 1/2$ and $|\lambda_{1,j} - \lambda_{2,j}|$ has a positive limit.

(2) Suppose $\lambda_{1,\infty} = \lambda_{2,\infty} > 0$. Setting $\lambda = \gamma_j^{-1}\xi + \lambda_{1,j}$ and using (3), we have

$$\int_0^{\pi} [v^*_{1,j} \overline{v^*_{2,j}}](\lambda) \mathrm{e}^{\mathrm{i}\gamma_j \lambda(k'-k)} \mathrm{d}\lambda = \int_{-\gamma_j \lambda_{1,j}}^{\gamma_j(\pi - \lambda_{1,j})} \gamma_j^{-1} [v^*_{1,j} \overline{v^*_{2,j}}](\gamma_j^{-1}\xi + \lambda_{1,j}) \mathrm{e}^{\mathrm{i}\xi(k'-k)} \mathrm{d}\xi$$
$$\to \int_{-\infty}^{\infty} [v^*_{1,\infty} \overline{v^*_{2,\infty}}](\xi) \mathrm{e}^{\mathrm{i}\xi(k'-k)} \mathrm{d}\xi \quad \text{as } j \to \infty, \tag{22}$$

where the limit follows from Conditions (6), (4), (7), (10) and dominated convergence. Similarly we have

$$\int_{-\pi}^{0} [v^*_{1,j}\overline{v^*_{2,j}}](\lambda)\mathrm{e}^{\mathrm{i}\gamma_j\lambda(k'-k)}\mathrm{d}\lambda \to \int_{-\infty}^{\infty} [\overline{v^*_{1,\infty}}v^*_{2,\infty}](-\xi)\mathrm{e}^{\mathrm{i}\xi(k'-k)}\mathrm{d}\xi \quad \text{as } j\to\infty,$$

by using (13) instead of (10). The last display, (20) and (22) yield

$$\mathrm{Cov}\left(Z_{1,j,k}, Z_{2,j,k'}\right) \to 2\int_{-\infty}^{\infty} w^*_{1,2}(\xi)\mathrm{e}^{\mathrm{i}\xi(k'-k)}\mathrm{d}\xi \quad \text{as } j\to\infty. \tag{23}$$

(3) Suppose finally $\lambda_{1,\infty} = \lambda_{2,\infty} = 0$. Setting $\lambda = \gamma_j^{-1}\xi$ gives

$$\begin{aligned}\int_{-\pi}^{\pi} [v^*_{1,j}\overline{v^*_{2,j}}](\lambda)\mathrm{e}^{\mathrm{i}\gamma_j\lambda(k'-k)}\mathrm{d}\lambda &= \int_{-\gamma_j\pi}^{\gamma_j\pi} \gamma_j^{-1}[v^*_{1,j}\overline{v^*_{2,j}}](\gamma_j^{-1}\xi)\mathrm{e}^{\mathrm{i}\xi(k'-k)}\mathrm{d}\xi \\ &\to \int_{-\infty}^{\infty} [v^*_{1,\infty}\overline{v^*_{2,\infty}}](\xi)\mathrm{e}^{\mathrm{i}\xi(k'-k)}\mathrm{d}\xi\end{aligned}$$

by using Conditions (5), (7) and (10) and dominated convergence. The last display, (21) and (22) yield (16). □

3. Main results

We let $\xrightarrow{\mathcal{L}}$ denote the convergence in law. Our first result provides the asymptotic behavior of the sample mean square of an array of a decimated linear sequence under a global assumption on the behavior of the spectral density (the bound (7)). A local version of this assumption is considered in Theorem 2.

Theorem 1. *Let $\{Z_{i,j,k}, i = 1, 2, j \geq 0, k \in \mathbb{Z}\}$ be an array of N-dimensional decimated linear processes as defined by (1). Assume* (A-2) *and that γ_j is even for j large enough. For each $i = 1, \ldots, N$, we let $\lambda_{i,\infty}$ denote a frequency in $[0, \pi)$ and $v^*_{i,\infty}$ a continuous $\mathbb{R} \to \mathbb{Z}$ function such that Condition* (B) *holds. Then, for any diverging sequence (n_j),*

$$n_j^{-1/2}\sum_{k=0}^{n_j-1}\begin{bmatrix} Z^2_{1,j,k} - \mathbb{E}[Z^2_{1,j,k}] \\ \vdots \\ Z^2_{N,j,k} - \mathbb{E}[Z^2_{N,j,k}]\end{bmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Gamma), \tag{24}$$

where Γ is the covariance matrix defined by

$$\Gamma_{i,i'} = 4\pi \mathrm{C}_{i,i'}\int_{-\pi}^{\pi}\left|\sum_{p\in\mathbb{Z}} w^*_{i,i'}(\lambda + 2p\pi)\right|^2 \mathrm{d}\lambda, \quad 1 \leq i, i' \leq N, \tag{25}$$

where $\mathrm{C}_{i,i'}$ and $w^*_{i,i'}$ are defined in (18) and (17).

Remark 8. From (15), it follows that the doubly infinite sum in (25) is well defined and bounded on $\lambda \in \mathbb{R}$ and hence Γ is well defined.

Remark 9. The index k appears n_j times in the centered sum (24) and asymptotic normalization occurs as j and n_j tend to ∞.

Remark 10. The presence of the factor γ_j in (1), and hence of decimation, is essential for the central limit theorem to hold in this generality because it ensures that the dependence of the

$Z_{i,j,k}$'s decreases sufficiently fast as $j \to \infty$. Decimation of this type is typically encountered in settings involving wavelets, or more generally filter banks; see [4].

Remark 11. In applications, the expectations in (24), which depend on j, will be approximated by quantities that are independent of j. To see why this is possible, observe that, applying the relation (16) in Proposition 1 with $k = k' = 0$ and $i = i' = 1, \ldots, N$, we get

$$\lim_{j\to\infty} \mathbb{E}\left[Z_{i,j,0}^2\right] = \int_{-\infty}^{\infty} \left|v_{i,\infty}^*(\lambda)\right|^2 \mathrm{d}\lambda < \infty. \tag{26}$$

Thus, when the rate of convergence to this limit is fast enough, the expectations in (24) can be replaced by $\int_{-\infty}^{\infty} \left|v_{i,\infty}^*(\lambda)\right|^2 \mathrm{d}\lambda$, $i = 1, \ldots, N$, which does not depend on j.

We have assumed in (7) a bound for $v_{i,j}^*(\lambda)$ for $\lambda \in (-\pi, \pi)$. This bound implies that the spectral density of the process $Z_{i,j,\cdot}$ defined in (1) is bounded on $(-\pi, \pi)$. We shall now weaken this assumption by only assuming a local bound around the frequency $\lambda_{i,j}$ as follows.

Theorem 2. *Assume that all the conditions of Theorem 1 hold except that (7) is replaced by*

$$\sup_{j\geq 0} \sup_{|\lambda-\lambda_{i,\infty}|\leq\varepsilon} \gamma_j^{-1/2} |v_{i,j}^*(\lambda)| \left(1 + \gamma_j \left|\lambda - \lambda_{i,j}\right|\right)^{\delta} < \infty, \tag{27}$$

where $\varepsilon > 0$ is arbitrarily small. Suppose in addition that

$$n_j^{1/2} \int_0^{\pi} \mathbb{1}(|\lambda - \lambda_{i,\infty}| > \varepsilon) |v_{i,j}^*(\lambda)|^2 \mathrm{d}\lambda \to 0 \quad \text{as } j \to \infty. \tag{28}$$

Then the conclusion of Theorem 1, that is, the CLT (24), still holds.

Remark 12. Since (7) is replaced by the local condition (27), we impose the additional condition (28) involving λ's not considered in (27). Condition (28), which is basically a condition on the growth of n_j, does not appear in the conditions of Theorem 1, where it was only required that $n_j \to \infty$.

4. Application to spectral density estimation

Let $\{X_u, u \in \mathbb{Z}\}$ be a standard linear process,

$$X_u = \sum_{t\in\mathbb{Z}} a(u - t)\xi_t, \tag{29}$$

where $\{\xi_t, t \in \mathbb{Z}\}$ is a centered weak white noise with unit variance satisfying (A-2) and where $\{a(t), t \in \mathbb{Z}\}$ is a real-valued sequence such that $\sum_k a_k^2 < \infty$ with Fourier series

$$a^*(\lambda) = (2\pi)^{-1/2} \sum_{t\in\mathbb{Z}} a(t)\mathrm{e}^{-\mathrm{i}\lambda t}. \tag{30}$$

Then $\{X_k, k \in \mathbb{Z}\}$ admits the following spectral density:

$$f(\lambda) = \left|a^*(\lambda)\right|^2, \quad \lambda \in (-\pi, \pi).$$

We first consider the problem of estimating $f(0)$ from T observations $X_1, \ldots, X_T$.

Let us denote by W a bounded $\mathbb{R} \to \mathbb{R}$ function with compact support and by $\widehat{W}$ its Fourier transform,

$$\widehat{W}(\xi) = \int_{-\infty}^{\infty} W(t) \mathrm{e}^{-\mathrm{i}\xi t} \mathrm{d}t.$$

Let (γ_j) be any diverging sequence of even integers.

We let $N = 1$, and $\lambda_{1,j} = \lambda_{1,\infty} = 0$ for all $j \geq 0$, which yields (3)–(6) in Condition (B).

Define

$$Z_{1,j,k} = \gamma_j^{-1/2} \sum_{u \in \mathbb{Z}} W(k - \gamma_j^{-1} u) X_u. \tag{31}$$

We assume that:

(C-1) As $\lambda \to 0$,

$$f(\lambda) = f(0) + O(|\lambda|^2). \tag{32}$$

(C-2) The support of W is included in $[-1, 0]$, $\sup_{\xi \in \mathbb{R}} |\widehat{W}(\xi)|(1 + |\xi|)^\beta < \infty$ with $\beta > 1$ and

$$\int_{-\infty}^{\infty} \left|\widehat{W}(\lambda)\right|^2 \mathrm{d}\lambda = 1. \tag{33}$$

Assumptions (C-1) and (C-2) are related to the standard bias control of kernel estimates of the spectral density.

Define

$$n_j = [\gamma_j^{-1}(T + 1)]. \tag{34}$$

Observe that (C-2) implies that, for all $k = 0, \ldots, n_j - 1$, $W(k - \gamma_j^{-1} u)$ vanishes for $u \leq 0$ and $u \geq T + 1$. Hence the infinite sum in (31) can be written as

$$Z_{1,j,k} = \gamma_T^{-1/2} \sum_{u=1}^{T} W(k - \gamma_j^{-1} u) X_u. \tag{35}$$

In other words, $\{Z_{1,j,k}, k = 0, \ldots, n_j - 1\}$ can be computed from the T observations $X_1, \ldots, X_T$. Thus

$$\widehat{f_T}(0) \stackrel{\text{def}}{=} n_T^{-1} \sum_{k=0}^{n_T - 1} Z_{1,T,k}^2 \tag{36}$$

can be used as an estimator of $f(0)$. Here we put $j = T$ so that the estimator only depends on the number of observations T, the sequence (γ_T) and the kernel W. The following theorem provides a central limit result for $\widehat{f_T}(0)$.

Theorem 3. *Assume* (C-1) *and* (C-2) *with $\beta > 2$. Let (γ_T) be a diverging sequence of even integers such that $\gamma_T^{-1} T \to \infty$ and*

$$T^{1/2} \gamma_T^{1/2 - 2\beta} \to 0. \tag{37}$$

Then, as $T \to \infty$,

$$\mathbb{E}\left[\widehat{f_T}(0)\right] = \mathbb{E}\left[Z_{1,T,0}^2\right] = f(0) + O(\gamma_T^{-2}) \tag{38}$$

and

$$(\gamma_T^{-1}T)^{1/2}\left\{\widehat{f}_T(0)-\mathbb{E}\left[Z_{1,T,0}^2\right]\right\}\xrightarrow{\mathcal{L}}\mathcal{N}(0,\sigma^2), \tag{39}$$

where

$$\sigma^2=4\pi f(0)^2\int_{-\pi}^{\pi}\left(\sum_{p\in\mathbb{Z}}\left|\widehat{W}(\lambda+2p\pi)\right|^2\right)^2 \mathrm{d}\lambda.$$

Remark 13. Our CLT (39) can be compared with [1, Eq. 3.9], although the estimators are different since ours involve a decimation and the one in [1] is expressed as a weighted integral of the standard periodogram. In (39), our γ_T has a role similar to that of the $q = q_T$ for their estimator. Our bias estimate (38) has a faster decrease than the corresponding one $O(q^{-1})$ in [1]; see the last display in their Section 3. Our conditions also differ from those of [1]. Our condition on the weight sequence $a(t)$ is much more general, since we assume a polynomial decrease neither of this sequence nor of $a(t) - a(t + 1)$, as assumed for the corresponding (one-sided) sequence $(\psi_j)_{j\geq 0}$ in Assumption 2.1 of [1]. Standard results on spectral estimation (see e.g. [5, Theorem 9.4.1]) usually assume the even stronger condition $\sum_t |a(t)| < \infty$. On the other hand we do assume that the noise sequence $\{\xi_t\}$ has a fourth finite moment (see (A-2)) while only a finite $2 + \epsilon$ moment (with $\epsilon > 0$ arbitrarily small) is assumed in [1]. It is an open question whether a similar moment condition can be used for our estimator $\widehat{f}_T(0)$.

Remark 14. The sequence (γ_T) must satisfy both $\gamma_T^{-1}T \to \infty$ and (37). The former is equivalent to $n_T \to \infty$, see (34). The decimation $\gamma_T = 2^T$ will not satisfy $n_T \to \infty$, nor will $\gamma_T = 1$ satisfy (37). An intermediate decimation, for example, $\gamma_T = [T^a]$, will work for $0 < a < 1$ as long as the Fourier decay index β in (C-2) satisfies $\beta > (1 + 1/a)/4$. We also assume in Theorem 3 that $\beta > 2$. Thus, if $a \leq 1/7$, then we require $\beta > (1 + 1/a)/4$ and when $a \geq 1/7$, we require $\beta > 2$. Since $\gamma_T = [T^a]$ the normalization coefficient $(\gamma_T^{-1}T)^{1/2}$ in (39) is asymptotically equivalent to $T^{(1-a)/2}$.

We now give an extension of Theorem 3 for estimating the spectral density f at several frequencies $0 \leq \lambda_{1,\infty} < \cdots < \lambda_{N,\infty} < \pi$ from observations $X_1, \ldots, X_T$. Given a sequence of even integers (γ_T), for each $i = 1, \ldots, N$, we consider the point $\lambda_{i,T}$, closest to $\lambda_{i,\infty}$, that satisfies $\gamma_T\lambda_{i,T} \in 2\pi\mathbb{Z}_+$. For sake of brevity, we will not investigate the bias $f(\lambda_{i,\infty}) - f(\lambda_{i,T})$ as it depends both on the smoothness f and on the distance between $\lambda_{i,\infty}$ and $2\pi\gamma_T^{-1}\mathbb{Z}_+$. It may happen for instance that $\lambda_{i,\infty} \in 2\pi\gamma_T^{-1}\mathbb{Z}_+$ for T large enough, and so $\lambda_{i,T} = \lambda_{\infty,T}$, in which case this bias vanishes.

We define an array of N-dimensional decimated linear process by

$$Z_{i,j,k}=\gamma_j^{-1/2}\sum_{u\in\mathbb{Z}}W_{i,j}(k-\gamma_j^{-1}u)X_u,\quad i=1,\ldots,N,\ j\geq 0, \tag{40}$$

where X_u is defined in (29) and

$$W_{i,j}(t)=\mathrm{C}_{i,i}^{1/2}W(t)\cos(\gamma_j\lambda_{i,j}t),\quad t\in\mathbb{R}, \tag{41}$$

with $\mathrm{C}_{i,i}$ defined in (18). The estimator (36) is then extended at frequencies $\lambda_{i,T}$, $i = 1, \ldots, N$, as follows:

$$\widehat{f}_T(\lambda_{i,T})\overset{\text{def}}{=}n_T^{-1}\sum_{k=0}^{n_T-1}Z_{i,T,k}^2,\quad i=1,\ldots,N,$$

with n_T defined in (34). Observe that, as in (35), (C-2) implies that, for all $i = 1, \dots, N$, $j \geq 0$ and $k \in \{0, \dots, n_T - 1\}$,

$$Z_{i,j,k} = \gamma_j^{-1/2} \sum_{u=1}^{T} W_{i,j}(k - \gamma_j^{-1} u) X_u,$$

and thus $\widehat{f}_T(\lambda_{i,T})$ is an estimator based on the observations $X_1, \dots, X_T$.
Instead of Condition (C-1), we use now

(C$'$-1) For each $i = 1, \dots, N$, there exists a bounded real-valued sequence $(\zeta_{i,j})_{j\geq 1}$ such that, as $\lambda \to 0$,

$$\sup_{j\geq 1} \left| f(\lambda_{i,j} + \lambda) - f(\lambda_{i,j}) - \lambda \zeta_{i,j} \right| = O(|\lambda|^2). \tag{42}$$

This condition is satisfied for instance by $\zeta_{i,j} = \dot{f}(\lambda_{i,j})$ if f has a bounded second derivative in a neighborhood of $\lambda_{i,\infty}$ or if $\lambda_{i,j} = \lambda_{i,\infty}$ for j large enough and f has a finite second derivative at $\lambda_{i,\infty}$.

Theorem 4. *Assume* (C$'$-1) *and* (C-2) *with $\beta > 2$. Let (γ_T) be a diverging sequence of even integers such that $\gamma_T^{-1} T \to \infty$ and (37) holds. Let $(\lambda_{i,j})_{i=1,\dots,N,j=1,\dots,\infty}$ be an array satisfying Condition* (B)(i). *Then, as $T \to \infty$, for $i = 1, \dots, N$,*

$$\mathbb{E}\left[\widehat{f}_T(\lambda_{i,T})\right] = \mathbb{E}\left[Z_{i,T,0}^2\right] = f(\lambda_{i,T}) + O(\gamma_T^{-2}) \tag{43}$$

and

$$(\gamma_T^{-1} T)^{1/2} \begin{bmatrix} \widehat{f}_T(\lambda_{1,T}) - \mathbb{E}\left[Z_{1,T,0}^2\right] \\ \vdots \\ \widehat{f}_T(\lambda_{N,T}) - \mathbb{E}\left[Z_{N,T,0}^2\right] \end{bmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Gamma), \tag{44}$$

where

$$\Gamma_{i,i'} = 4\pi f(\lambda_{i,\infty}) f(\lambda_{i',\infty}) \frac{\mathrm{C}_{i,i'}}{\mathrm{C}_{i,i}\mathrm{C}_{i',i'}} \int_{-\pi}^{\pi} \left(\sum_{p\in\mathbb{Z}} \left| \widehat{W}(\lambda + 2p\pi) \right|^2 \right)^2 \mathrm{d}\lambda, \tag{45}$$

where $\mathrm{C}_{i,i'}$ is defined in (18) and where $\widehat{W}$ is the Fourier transform of the function W in (C-2).

5. Proof of Theorem 1

We first establish two central limit theorems which will be used in the proof of Theorem 1. The first involves a sequence of linear filters of the sequence $\{\xi_t, t \in \mathbb{Z}\}$.

Proposition 2. *Define, for all $i = 1, \dots, N$, $j \geq 0$, $k \in \mathbb{Z}$,*

$$Z_{i,j} = \sum_{t\in\mathbb{Z}} v_{i,j}(t) \xi_t, \tag{46}$$

where for all $i = 1, \dots, N$ and $j \geq 0$, $\{v_{i,j}(t), t \in \mathbb{Z}\}$ is real valued and satisfies $\sum_{t\in\mathbb{Z}} v_{i,j}^2(t) < \infty$ and $\{\xi_t, t \in \mathbb{Z}\}$ satisfies (A-1). *Assume that*

$$\lim_{j\to\infty} \sup_{t\in\mathbb{Z}} |v_{i,j}(t)| = 0 \quad \textit{for all } i = 1, \dots, N, \tag{47}$$

$$\lim_{j\to\infty} \sum_{t\in\mathbb{Z}} \left(v_{i,j}(t) v_{i',j}(t)\right) = \Sigma_{i,i'} \quad \textit{for all } i, i' = 1, \dots, N, \tag{48}$$

where Σ is a $N \times N$ given matrix. Then, as $j \to \infty$,

$$\left[Z_{1,j} \dots Z_{N,j}\right]^{\mathrm{T}} \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Sigma). \tag{49}$$

Remark 15. A study on the weak convergence of such sequence without assuming Assumption (A-1) can be found in [6].

Proof. This is a standard application of the Lindeberg–Feller central limit theorem. Using the Cramér–Wold device for the multivariate central limit theorem, it is sufficient by (47) and (48) to prove the result for $N = 1$, in which case we simply denote $v_{1,j}(t)$ by $v_j(t)$ and Σ by σ^2. Let (m_j) be a sequence of integers tending to infinity with j, such that $\sum_{|t|\geq m_j} v_j^2(t) \leq 2^{-j}$. We now show that the Lindeberg conditions hold for the sequence $\sum_{|t|\leq m_j} v_j(t)\xi_t$. The first holds because, by (48),

$$\lim_{j\to\infty} \sum_{|t|\leq m_j} \mathrm{Var}(v_j(t)\xi_t) = \lim_{j\to\infty} \sum_{|t|\leq m_j} v_j^2(t) \stackrel{\text{def}}{=} \sigma^2.$$

The second holds because, for all $\varepsilon > 0$,

$$\sum_{|t|\leq m_j} \mathbb{E}\left[v_j^2(t)\xi_t^2 \mathbb{1}(|v_j(t)\xi_t| \geq \varepsilon)\right] \leq \left(\sum_{t\in\mathbb{Z}} v_j^2(t)\right) \mathbb{E}\left[\xi_0^2 \mathbb{1}(|\xi_0| \geq \varepsilon/S_j)\right],$$

where $S_j = \sup_{t\in\mathbb{Z}} |v_j(t)|$, and, by (47), the right-hand side of the last display tends to 0 as $j \to \infty$. This concludes the proof. □

The second central limit theorem deals with m-dependent arrays. Recall that $\{Y_k\}$ is said to be m-dependent if, for all $p \geq 1$ and all $k_1, \dots, k_p$ such that $k_1 + m \leq k_2, \dots, k_{p-1} + m \leq k_p$, $Y_{k_1}, \dots, Y_{k_p}$ are independent.

Proposition 3. *Let m be a fixed integer and (n_j) a sequence of integers such that $n_j \to \infty$ as $j \to \infty$. Let $\{Y_{j,k}, k = 0, \dots, n_j - 1, j \geq 1\}$ be an array of $\mathbb{R}^d$-valued random vectors, such that, for each $j \geq 0$, $\{Y_{j,k}, k = 0, \dots, n_j - 1\}$ has zero mean, and is strictly stationary and m-dependent. Assume that there exists a $d \times d$ matrix Γ and a centered stationary $\mathbb{R}^d$-valued process $\{\mathbf{Y}_k, k \geq 0\}$ with finite variance such that*

$$Y_{j,\cdot} \xrightarrow{\mathcal{L}} \mathbf{Y}_\cdot \quad \textit{as } j \to \infty, \tag{50}$$

$$\lim_{j\to\infty} \mathrm{Cov}\left(Y_{j,k}, Y_{j,k'}\right) = \mathrm{Cov}\left(\mathbf{Y}_k, \mathbf{Y}_{k'}\right) \quad \textit{for all } k, k' \geq 0 \tag{51}$$

$$\lim_{l\to\infty} \lim_{j\to\infty} \mathrm{Cov}\left(l^{-1/2} \sum_{k=0}^{l-1} Y_{j,k}\right) = \Gamma. \tag{52}$$

Then we have, as $j \to \infty$,

$$n_j^{-1/2} \sum_{k=0}^{n_j-1} Y_{j,k} \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Gamma). \tag{53}$$

Proof. We may suppose that $d = 1$ since the vector case follows by the Cramé–Wold device. For convenience, we set $\Gamma = \sigma^2$. Let s be a positive integer larger than m. We decompose $\sum_{k=0}^{n_j-1} Y_{j,k}$ in sums of random variables spaced by m, as follows:

$$\sum_{k=0}^{n_j-1} Y_{j,k} = \sum_{k=0}^{p_j} S_{j,k}^{(s)} + \sum_{k=0}^{p_j} T_{j,k}^{(s)} + R_j^{(s)}, \tag{54}$$

where

$$S_{j,k}^{(s)} = \sum_{i=0}^{s-1} Y_{j,k(m+s)+i}, \quad T_{j,k}^{(s)} = \sum_{i=0}^{m-1} Y_{j,k(m+s)+s+i} \quad \text{and} \quad R_j^{(s)} = \sum_{i=0}^{q_j-1} Y_{j,p_j(m+s)+i},$$

and where p_j and q_j are the non-negative integers defined by the Euclidean division

$$n_j = p_j(m+s) + q_j \quad \text{with } q_j \in \{0, \ldots, m+s-1\}. \tag{55}$$

The m-dependent and the strict stationarity of the sequences $Y_{j,\cdot}$ ensure that for all $j \geq 0$ and all $s \geq m$, $S_{j,\cdot}^{(s)}$ and $T_{j,\cdot}^{(s)}$ are sequences of centered independent and identically distributed random variables. Hence, by (51), we have

$$\lim_{j\to\infty} \sum_{k=0}^{p_j} \mathrm{Var}\left(p_j^{-1/2} S_{j,k}^{(s)}\right) = \lim_{j\to\infty} \mathrm{Var}\left(S_{j,0}^{(s)}\right) = \mathrm{Var}\left(\mathbf{S}^{(s)}\right), \tag{56}$$

where

$$\mathbf{S}^{(s)} \stackrel{\text{def}}{=} \sum_{i=0}^{s-1} \mathbf{Y}_i,$$

$$\lim_{j\to\infty} \mathrm{Var}\left(p_j^{-1/2} \sum_{k=0}^{p_j} T_{j,k}^{(s)}\right) = \lim_{j\to\infty} \mathrm{Var}\left(T_{j,0}^{(s)}\right) = \mathrm{Var}\left(\mathbf{S}^{(m)}\right) \tag{57}$$

and

$$\limsup_{j\to\infty} \mathrm{Var}\left(R_j^{(s)}\right) \leq \max\left\{\mathrm{Var}\left(\mathbf{S}^{(t)}\right) : t = 0, 1, \ldots, m+s-1\right\} < \infty. \tag{58}$$

In addition, for any $\varepsilon > 0$,

$$\sum_{k=0}^{p_j} \mathbb{E}\left[(p_j^{-1/2} S_{j,k}^{(s)})^2 \mathbb{1}(p_j^{-1/2} |S_{j,k}^{(s)}| > \varepsilon)\right] = \mathbb{E}\left[(S_{j,0}^{(s)})^2 \mathbb{1}(p_j^{-1/2} |S_{j,k}^{(s)}| > \varepsilon)\right]$$

and hence, since $p_j \to \infty$,

$$\limsup_{j\to\infty} \sum_{k=0}^{p_j} \mathbb{E}\left[(p_j^{-1/2} S_{j,k}^{(s)})^2 \mathbb{1}(p_j^{-1/2} |S_{j,k}^{(s)}| > \varepsilon)\right]$$
$$\leq \inf_{M>0} \limsup_{j\to\infty} \mathbb{E}\left[(S_{j,0}^{(s)})^2 \mathbb{1}(|S_{j,k}^{(s)}| > M)\right]. \tag{59}$$

But using (50), we have

$$S_{j,0}^{(s)} \xrightarrow{\mathcal{L}} \mathbf{S}^{(s)} \quad \text{as } j \to \infty.$$

Hence, denoting by ϕ_M some continuous $\mathbb{R}_+ \to [0,1]$ function satisfying $\mathbb{1}(x \leq M/2) \leq \phi_M(x) \leq \mathbb{1}(x \leq M)$, so that $x^2\phi_M(x)$ is continuous and bounded, we have

$$\begin{aligned}\mathbb{E}\left[(S_{j,0}^{(s)})^2\mathbb{1}(|S_{j,k}^{(s)}| \leq M)\right] \geq\ & \mathbb{E}\left[(S_{j,0}^{(s)})^2\phi_M(S_{j,0}^{(s)})\right]\\ &\to \mathbb{E}\left[(\mathbf{S}^{(s)})^2\phi_M(\mathbf{S}^{(s)})\right] \quad \text{as } j \to \infty\\ &\to \mathbb{E}\left[(\mathbf{S}^{(s)})^2\right] \quad \text{as } M \to \infty.\end{aligned}$$

Using (51), we have $\mathbb{E}\left[(S_{j,0}^{(s)})^2\right] \to \mathbb{E}\left[(\mathbf{S}^{(s)})^2\right]$ as $j \to \infty$ and hence, for any $M > 0$,

$$\limsup_{j\to\infty} \mathbb{E}\left[(S_{j,0}^{(s)})^2\mathbb{1}(|S_{j,k}^{(s)}| > M)\right] = \mathbb{E}\left[(\mathbf{S}^{(s)})^2\right] - \liminf_{j\to\infty}\mathbb{E}\left[(S_{j,0}^{(s)})^2\mathbb{1}(|S_{j,k}^{(s)}| \leq M)\right].$$

The two last displays and (59) imply the second Lindeberg condition, namely,

$$\limsup_{j\to\infty} \sum_{k=0}^{p_j} \mathbb{E}\left[(p_j^{-1/2} S_{j,k}^{(s)})^2 \mathbb{1}(p_j^{-1/2}|S_{j,k}^{(s)}| > \varepsilon)\right] = 0 \quad \text{for any } \varepsilon > 0.$$

Using this and (56), we may apply the Lindeberg–Feller CLT for arrays of independent r.v.'s and we obtain

$$p_j^{-1/2} \sum_{k=0}^{p_j} S_{j,k}^{(s)} \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \mathrm{Var}\left(\mathbf{S}^{(s)}\right)\right). \tag{60}$$

By (52), we have

$$\lim_{s\to\infty} s^{-1}\mathrm{Var}\left(\mathbf{S}^{(s)}\right) = \sigma^2. \tag{61}$$

Using (60) and (55) with $j \to \infty$ and then (61) with $s \to \infty$ yields

$$n_j^{-1/2} \sum_{k=0}^{p_j} S_{j,k}^{(s)} \xrightarrow[j\to\infty]{\mathcal{L}} \mathcal{N}\left(0, (m+s)^{-1}\mathrm{Var}\left(\mathbf{S}^{(s)}\right)\right) \xrightarrow[s\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \sigma^2\right).$$

On the other hand, by (54), (57) and (58), we have

$$\limsup_{j\to\infty} \mathbb{E}\left[\left(n_j^{-1/2}\sum_{k=0}^{n_j-1} Y_{j,k} - n_j^{-1/2}\sum_{k=0}^{p_j} S_{j,k}^{(s)}\right)^2\right] \leq (m+s)^{-1}\mathrm{Var}\left(\mathbf{S}^{(m)}\right) \to 0$$

as $s \to \infty$. Using the last two displays and [7, Theorem 3.2], we obtain (53), which concludes the proof. □

Proof of Theorem 1. The proof is in three steps. We show in a first step the convergence of the process $[Z_{i,j,k}, i = 1, \ldots, N, k = 0, \ldots, n_j]$ as $j \to \infty$ towards a Gaussian limit. In the second step we prove Theorem 1 under the additional assumption that the sequence

$$Y_{j,k} \stackrel{\text{def}}{=} \left[\{Z_{1,j,k}^2 - \mathbb{E}[Z_{1,j,k}^2]\} \ldots \{Z_{N,j,k}^2 - \mathbb{E}[Z_{N,j,k}^2]\}\right]^{\mathrm{T}}, \quad k = 0, \ldots, n_j - 1$$

is m-dependent. The third step exhibits an m-dependent approximation and extends the m-dependent case to the general case. The proof uses a number of auxiliary results proved in Section 8.

Step 1. We shall apply Proposition 2. By the relation (16) in Proposition 1, we get, for all $i, i' = 1, \dots, N$ and all $k, k' \in \mathbb{Z}$, as $j \to \infty$,

$$\mathrm{Cov}\left(Z_{i,j,k}, Z_{i',j,k'}\right) \to \mathrm{C}_{i,i'} \int_{-\infty}^{\infty} w^*_{i,i'}(\lambda) \mathrm{e}^{\mathrm{i}\lambda(k'-k)} \mathrm{d}\lambda. \tag{62}$$

Moreover, by (7), one has, for all $i = 1, \dots, N$,

$$\begin{aligned} \sup_{t \in \mathbb{Z}} \left| v_{i,j}(t) \right| &\le 2(2\pi)^{-1/2} \int_0^{\pi} \left| v^*_{i,j}(\lambda) \right| \mathrm{d}\lambda \\ &\le 2(2\pi)^{-1/2} \gamma_j^{-1/2} \int_{-\gamma_j \lambda_{i,j}}^{\gamma_j(\pi - \lambda_{i,j})} (1 + |\lambda|)^{-\delta} \mathrm{d}\lambda, \end{aligned}$$

which, by (4), tends to 0 as $j \to \infty$ for any $\delta > 1/2$. Hence, by Proposition 2, for any $p \ge 1$, any $i_1, \dots, i_p \in \{1, \dots, N\}$ and any $k_1, \dots, k_p \in \mathbb{Z}$, we have, as $j \to \infty$,

$$\left[Z_{i_1,j,k_1} \dots Z_{i_p,j,k_p} \right]^{\mathrm{T}} \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Sigma), \tag{63}$$

where Σ is the covariance matrix with entries given for all $1 \le n, n' \le p$ by

$$\Sigma_{n,n'} = \mathrm{C}_{i_n,i_{n'}} \int_{-\infty}^{\infty} w^*_{i_n,i_{n'}}(\lambda) \mathrm{e}^{\mathrm{i}\lambda(k_{n'}-k_n)} \mathrm{d}\lambda.$$

Expressing this integral as $\sum_p \int_{-\pi+2p\pi}^{\pi+2p\pi}$, we get

$$\Sigma_{n,n'} = \mathrm{C}_{i_n,i_{n'}} \int_{-\pi}^{\pi} \left(\sum_{p \in \mathbb{Z}} w^*_{i_n,i_{n'}}(\lambda + 2p\pi) \right) \mathrm{e}^{\mathrm{i}\lambda(k_{n'}-k_n)} \mathrm{d}\lambda.$$

The convergence (63) can be written equivalently as

$$\left[Z_{1,j,\cdot} \dots Z_{N,j,\cdot} \right]^{\mathrm{T}} \xrightarrow{\mathcal{L}} \mathbf{Z}_{\cdot}, \tag{64}$$

as $j \to \infty$, where $\{\mathbf{Z}_k = [Z_{1,\infty,k} \dots Z_{N,\infty,k}]^{\mathrm{T}}, k \ge 0\}$ is a stationary Gaussian $\mathbb{R}^N$-valued process with spectral density matrix function D with entries

$$D_{i,i'}(\lambda) = \mathrm{C}_{i,i'} \sum_{p \in \mathbb{Z}} w^*_{i_n,i_{n'}}(\lambda + 2p\pi), \quad 1 \le i, i' \le N.$$

Step 2. In this step, we prove (24), assuming that for each $j \ge 1$, $\{Y_{j,k}, k = 0, \dots, n_j - 1\}$ is m-dependent. We shall apply Proposition 3 under this additional assumption. We thus need to show that (50)–(52) hold. Relations (26), (64) and the continuous mapping theorem imply (50) with

$$\mathbf{Y}_k = \left[\{Z^2_{1,\infty,k} - \mathbb{E}[Z^2_{1,\infty,k}]\} \dots \{Z^2_{N,\infty,k} - \mathbb{E}[Z^2_{N,\infty,k}]\} \right]^{\mathrm{T}}.$$

Since $\mathbf{Z}_{\cdot}$ is Gaussian, we have, for all $k, k' \ge 0$ and all $i, i' = 1, \dots, N$,

$$\begin{aligned} \mathrm{Cov}\left(\mathbf{Y}_{k,i}, \mathbf{Y}_{k',i'}\right) &= 2\mathrm{Cov}^2\left(Z_{i,\infty,k}, Z_{i',\infty,k'}\right) \\ &= 2\mathrm{C}^2_{i,i'} \left(\int_{-\infty}^{\infty} w^*_{i,i'}(\lambda) \mathrm{e}^{\mathrm{i}\lambda(k'-k)} \mathrm{d}\lambda \right)^2. \end{aligned}$$

Hence the relation (158) in Corollary 1 and the previous display yield (51). The final condition (52) follows from the relation (159) of Corollary 1 with a covariance matrix Γ with entries (25). Applying Proposition 3 then yields (24), with Γ given by (25).

Step 3. Let $K(t)$ be a non-negative infinitely differentiable function defined on $t \in \mathbb{R}$ whose support is included in $[-1/2, 1/2]$ and such that $K(0) = 1$. We will denote by $\widehat{K}$ its Fourier transform,

$$\widehat{K}(\xi) = \int_{-\infty}^{\infty} K(t)\mathrm{e}^{-\mathrm{i}\xi t}\mathrm{d}t.$$

Observe that, by the assumptions on K, $\widehat{K}(\xi)$ decreases faster than any polynomial as $|\xi| \to \infty$. In particular $\widehat{K}(\xi)$ is integrable on $\xi \in \mathbb{R}$ and, for all $t \in \mathbb{R}$,

$$K(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{K}(\xi)\mathrm{e}^{\mathrm{i}\xi t}\mathrm{d}\xi. \tag{65}$$

The function K will be used to approximate the $v_{i,j}^{(m)}$ sequence by a sequence whose dependence structure can be controlled. We thus define, for any $i = 1, \ldots, N$, $j \geq 0$ and $m \geq 1$,

$$v_{i,j}^{(m)}(t) = v_{i,j}(t)K(t/(m\gamma_j)),$$

which vanishes for all $|t| \geq m\gamma_j/2$, and we define $Z_{i,j,k}^{(m)}$ and $v_{i,j}^{*(m)}(\lambda)$ accordingly.

Let m be a fixed integer. Then, for all $j \geq 0$,

$$\{[Z_{1,j,k}^{(m)} \ldots Z_{N,j,k}^{(m)}]^{\mathrm{T}}, k \in \mathbb{Z}\}$$

is an m-dependent sequence of vectors. We shall now show that $\{v_{i,j}^{*(m)}, j \geq 0\}$ satisfy conditions similar to (7) and (8) and then apply Step 2. Using (65) and (2) in the equation

$$v_{i,j}^{*(m)}(\lambda) = (2\pi)^{-1/2} \sum_{t \in \mathbb{Z}} v_{i,j}(t)K(t/(m\gamma_j))\mathrm{e}^{-\mathrm{i}\lambda t},$$

we get that

$$v_{i,j}^{*(m)}(\lambda) = \frac{m}{2\pi} \int_{-\infty}^{\infty} \widehat{K}(m\xi)v_{i,j}^{*}(\lambda - \gamma_j^{-1}\xi)\mathrm{d}\xi. \tag{66}$$

It follows from Condition (7) that there exists a constant $C > 0$ such that for all $j \geq 0$ and $\lambda \in [0, \pi)$,

$$\left|v_{i,j}^{*}(\lambda)\right| \leq C\gamma_j^{1/2}(1 + |\gamma_j\lambda - \gamma_j\lambda_{i,j}|)^{-\delta}. \tag{67}$$

Using (11) and the (2π)-periodicity of $v_{i,j}^{*}$, we can express (67) using the symmetric $(2\pi\gamma_j)$-periodic function $h_{2\pi\gamma_j,\gamma_j\lambda_{i,j}}(\xi)$ defined in Lemma 5 and equal to $(1 + |\xi - \gamma_j\lambda_{i,j}|)^{-\delta}$ for $0 \leq \xi \leq \pi\gamma_j$. With $\xi = \gamma_j\lambda$, one gets

$$\left|v_{i,j}^{*}(\lambda)\right| \leq C\gamma_j^{1/2}h_{2\pi\gamma_j,\gamma_j\lambda_{i,j}}(\gamma_j\lambda), \quad j \geq 0, \lambda \in \mathbb{R}.$$

Let $g(t) = m|\widehat{K}(mt)|$ and observe that $\|g\|_1 = \|\widehat{K}\|_1 < \infty$ and

$$g(t) \leq c_0\, m(m|t|)^{-\delta-1} \leq c_0|t|^{-\delta-1} \quad \text{for all } |t| \geq 1 \text{ and } m \geq 1,$$

where c_0 is a positive constant such that $|\widehat{K}(u)| \leq c_0|u|^{-\delta-1}$ for $|u| \geq 1$. Applying these bounds to (66) gives

$$\left|v_{i,j}^{*(m)}(\lambda)\right| \leq C\gamma_j^{1/2} \int_{-\infty}^{\infty} g(\xi) h_{2\pi\gamma_j,\gamma_j\lambda_{i,j}}(\gamma_j\lambda - \xi)\mathrm{d}\xi.$$

Applying Lemma 5 to this convolution, we get

$$\left|v_{i,j}^{*(m)}(\lambda)\right| \leq C\gamma_j^{1/2}(1 + \gamma_j|\lambda - \lambda_{i,j}|)^{-\delta},$$

for different constants C depending neither on $m \geq 1$, $j \geq 0$ nor on $\lambda \in [0, \pi)$. One has therefore the following version of (7) for $v_{i,j}^{*(m)}$, uniform in $m \geq 1$:

$$\sup_{j\geq 0}\sup_{m\geq 1}\sup_{\lambda\in[0,\pi)} \gamma_j^{-1/2}|v_{i,j}^{*(m)}(\lambda)|(1 + \gamma_j|\lambda - \lambda_{i,j}|)^{\delta} < \infty. \tag{68}$$

To get a version of (8) for $v_{i,j}^{*(m)}$, observe that, by (66), we have

$$\gamma_j^{-1/2} v_{i,j}^{*(m)}(\gamma_j^{-1}\lambda + \lambda_{i,j}) = \frac{m}{2\pi}\int_{-\infty}^{\infty} \widehat{K}(m\xi)\left[\gamma_j^{-1/2} v_{i,j}^{*}(\gamma_j^{-1}(\lambda - \xi) + \lambda_{i,j})\right]\mathrm{d}\xi.$$

Condition (7) implies that the term in brackets is bounded independently of ξ and j and hence by (8) and dominated convergence, one has

$$\lim_{j\to\infty} \gamma_j^{-1/2} v_{i,j}^{*(m)}(\gamma_j^{-1}\lambda + \lambda_{i,j})\mathrm{e}^{\mathrm{i}\Phi_{i,j}(\lambda)} = v_{i,\infty}^{*(m)}(\lambda) \quad \text{for all } \lambda \in \mathbb{R}, \tag{69}$$

where

$$v_{i,\infty}^{*(m)}(\lambda) \stackrel{\text{def}}{=} \frac{m}{2\pi}\int_{-\infty}^{\infty} \widehat{K}(m\xi) v_{i,\infty}^{*}(\lambda - \xi)\mathrm{d}\xi. \tag{70}$$

Note that $v_{i,\infty}^{*(m)}(\lambda)$ is an approximating sequence of $v_{i,\infty}^{*}(\lambda)$ in the sense that, since $v_{i,\infty}^{*}$ is bounded (by (15)) and continuous (by hypothesis), and since $(2\pi)^{-1}\int_{-\infty}^{\infty}\widehat{K}(u)\mathrm{d}u = K(0) = 1$, for all $\lambda \in \mathbb{R}$,

$$v_{i,\infty}^{*(m)}(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{K}(u) v_{i,\infty}^{*}(\lambda - u/m)\mathrm{d}u \to v_{i,\infty}^{*}(\lambda) \quad \text{as } m \to \infty. \tag{71}$$

Relations (68) and (69) are the corresponding versions of Conditions (7) and (8) for $v_{i,j}^{*(m)}$ and, since we are in the m-dependent case, we may apply the result proved in Step 2, and obtain, as $j \to \infty$,

$$n_j^{-1/2}\begin{bmatrix} \sum_{k=0}^{n_j-1}\{Z_{1,j,k}^{(m)2} - \mathbb{E}[Z_{1,j,k}^{(m)2}]\} \\ \vdots \\ \sum_{k=0}^{n_j-1}\{Z_{N,j,k}^{(m)2} - \mathbb{E}[Z_{N,j,k}^{(m)2}]\}\end{bmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Gamma^{(m)}), \tag{72}$$

where $\Gamma^{(m)}$ is the covariance matrix with entries

$$\Gamma^{(m)}_{i,i'} \stackrel{\text{def}}{=} 2\pi \mathrm{C}_{i,i'} \int_{-\pi}^{\pi} \left| \sum_{p\in\mathbb{Z}} w^{*(m)}_{i,i'}(\lambda + 2p\pi) \right|^2 \mathrm{d}\lambda, \quad 1 \le i, i' \le N, \tag{73}$$

where $w^{*(m)}_{i,i'}$ is the equivalent of $w^*_{i,i'}$ in (17),

$$w^{*(m)}_{i,i'}(\lambda) = \frac{1}{2}\left[\overline{v^{*(m)}_{i,\infty}(-\lambda)}v^{*(m)}_{i',\infty}(-\lambda) + v^{*(m)}_{i,\infty}(\lambda)\overline{v^{*(m)}_{i',\infty}(\lambda)}\right], \quad \lambda \in \mathbb{R}.$$

To obtain the corresponding result (24) for the $\{Z_{i,j,k}\}$ sequence, we apply [7, Theorem 3.2] as follows. We show that

$$\lim_{m\to\infty} \Gamma^{(m)}_{i,i'} = \Gamma_{i,i'} \quad 1 \le i, i' \le N \tag{74}$$

and, for all $i = 1, \ldots, N$,

$$\lim_{m\to\infty} \lim_{j\to\infty} \mathrm{Var}\left(n_j^{-1/2} \sum_{k=0}^{n_j-1} Z^{(m)2}_{i,j,k} - n_j^{-1/2} \sum_{k=0}^{n_j-1} Z^2_{i,j,k} \right) = 0. \tag{75}$$

Relation (74) says that the RHS of (72) converges in distribution to the RHS of (24) and the relation (75) says that the LHS of (72) is a good approximation to the LHS (24) on choosing m arbitrarily large.

To prove (75), it is sufficient to establish the following equalities for all $i = 1, \ldots, N$:

$$\lim_{m\to\infty} \lim_{j\to\infty} \mathrm{Cov}\left(n_j^{-1/2} \sum_{k=0}^{n_j-1} Z^{(m)2}_{i,j,k}, n_j^{-1/2} \sum_{k=0}^{n_j-1} Z^2_{i,j,k} \right)$$
$$= \lim_{m\to\infty} \lim_{j\to\infty} \mathrm{Var}\left(n_j^{-1/2} \sum_{k=0}^{n_j-1} Z^{(m)2}_{i,j,k} \right) = \lim_{j\to\infty} \mathrm{Var}\left(n_j^{-1/2} \sum_{k=0}^{n_j-1} Z^2_{i,j,k} \right).$$

Using the relation (159) of Corollary 1, the limits as $j \to \infty$ (and hence $n_j \to \infty$) in the previous display are, respectively,

$$\Gamma^{(m,\infty)}_{i,i} \stackrel{\text{def}}{=} 2\pi \mathrm{C}_{i,i'} \int_{-\pi}^{\pi} \left| \sum_{p\in\mathbb{Z}} w^{*(m,\infty)}_{i,i'}(\lambda + 2p\pi) \right|^2 \mathrm{d}\lambda, \quad \Gamma^{(m)}_{i,i} \quad \text{and} \quad \Gamma_{i,i},$$

where $\Gamma^{(m)}_{i,i}$ is defined in (73) and $\Gamma_{i,i}$ in (25) and

$$w^{*(m,\infty)}_{i,i'}(\lambda) = \frac{1}{2}\left[\overline{v^{*(m)}_{i,\infty}(-\lambda)}v^{*}_{i',\infty}(-\lambda) + v^{*(m)}_{i,\infty}(\lambda)\overline{v^{*}_{i',\infty}(\lambda)}\right], \quad \lambda \in \mathbb{R}.$$

Hence to prove (74) and (75), it is sufficient to show that

$$\lim_{m\to\infty} \Gamma^{(m,\infty)}_{i,i'} = \lim_{m\to\infty} \Gamma^{(m)}_{i,i'} = \Gamma_{i,i'}, \quad i, i' = 1, \ldots, N. \tag{76}$$

Observe first that the relations (68) and (69) imply

$$\sup_{\lambda\in\mathbb{R}} \sup_{m\ge 1} \left| v^{*(m)}_{i,\infty}(\lambda) \right| (1 + |\lambda|)^{\delta} < \infty, \tag{77}$$

which is the uniform version of (15). Eq. (76) now follows from (71), (68) and (77), and dominated convergence. □

6. Proof of Theorem 2

The following proposition is the key point for proving Theorem 2 since it shows how Condition (7) in Theorem 1 can be recovered for an approximation of the sample mean square, when using the alternative Condition (27). Condition (28) in Theorem 2 can then be used to control the sharpness of the approximation.

Proposition 4. *Let $\{Z_{1,j,k}, i = 1, 2, j \geq 0, k \in \mathbb{Z}\}$ be an array of one-dimensional decimated linear processes as defined by (1). Assume that $\{v_{1,j}(t), j \geq 0, t \in \mathbb{Z}\}$ satisfies (27) for $\delta > 1/2$, a sequence $(\lambda_{1,j})$ taking its values in $[0, \pi)$ and some $\varepsilon > 0$. Then there exists an array $\{\widehat{v_{1,j}}(t), j \geq 0, t \in \mathbb{Z}\}$ whose Fourier series coincide with those of $\{v_{1,j}(t), j \geq 0, t \in \mathbb{Z}\}$ in ε-neighborhoods of the frequencies $\{\lambda_{1,j}, j \geq 0\}$ and satisfying (7), that is, such that*

$$\widehat{v^*_{1,j}}(\lambda) = v^*_{1,j}(\lambda) \quad \text{for all } \lambda \in (-\pi, \pi) \text{ such that } |\lambda - \lambda_{1,j}| \leq \varepsilon, \tag{78}$$

$$\sup_{j \geq 0} \sup_{\lambda \in [0,\pi)} \gamma_j^{-1/2} |\widehat{v^*_{1,j}}(\lambda)| (1 + \gamma_j |\lambda - \lambda_{i,j}|)^\delta < \infty, \tag{79}$$

and the following approximation holds:

$$n_j^{-1/2} \left(\sum_{k=0}^{n_j-1} \{Z^2_{1,j,k} - \mathbb{E}[Z^2_{1,j,k}]\} \right) = n_j^{-1/2} \sum_{k=0}^{n_j-1} \{\widehat{Z}^2_{1,j,k} - \mathbb{E}[\widehat{Z}^2_{1,j,k}]\} + R_j, \tag{80}$$

where

$$\widehat{Z}_{1,j,k} = \sum_{t \in \mathbb{Z}} \widehat{v_{1,j}}(\gamma_j k - t) \xi_t, \tag{81}$$

and, for some positive constant C not depending on j,

$$\mathbb{E}\left[|R_j|\right] \leq C \left[n_j^{1/2} I_j + I_j^{1/2} \right], \tag{82}$$

where

$$I_j \stackrel{\text{def}}{=} \int_0^\pi \mathbb{1}(|\lambda - \lambda_{1,\infty}| > \varepsilon) \left| v^*_{1,j}(\lambda) \right|^2 \mathrm{d}\lambda. \tag{83}$$

Proof. Let $\mathrm{L}_0 = [-\lambda_{1,\infty} - \varepsilon, -\lambda_{1,\infty} + \varepsilon] \cup [\lambda_{1,\infty} - \varepsilon, \lambda_{1,\infty} + \varepsilon]$. We write

$$v^*_{1,j}(\lambda) = \widehat{v^*_{1,j}}(\lambda) + \widetilde{v^*_{1,j}}(\lambda), \quad \lambda \in (-\pi, \pi),$$

where $\widehat{v^*_{1,j}}(\lambda) = \mathbb{1}_{\mathrm{L}_0}(\lambda) v^*_{1,j}(\lambda)$ so that (78) holds. We define $\widehat{v_{1,j}}$, $\widetilde{v_{1,j}}$ accordingly, so that $v_{1,j}(t) = \widehat{v_{1,j}}(t) + \widetilde{v_{1,j}}(t)$ and, since $\widehat{v^*_{1,j}}$ and $\widetilde{v^*_{1,j}}$ are in $L^2(-\pi, \pi)$, $\widehat{v_{1,j}}$ and $\widetilde{v_{1,j}}$ are in $l^2(\mathbb{Z})$. Hence $Z_{1,j,k} = \widehat{Z}_{1,j,k} + \widetilde{Z}_{1,j,k}$ with $\widehat{Z}_{1,j,k}$ defined by (81) and

$$\widetilde{Z}_{1,j,k} = \sum_{t \in \mathbb{Z}} \widetilde{v}_{1,j}(\gamma_j k - t) \xi_t. \tag{84}$$

Moreover, by (27) and the definition of $\widehat{v^*_{1,j}}$, Condition (79) holds.

We now show that the remainder R_j defined by (80) satisfies (82). Observe that $\widehat{Z}_{1,j,k}$ and $\widetilde{Z}_{1,j,k}$ are centered and, since $\mathbb{E}[\widehat{Z}_{1,j,k}\widetilde{Z}_{1,j,k}] = \int_{-\pi}^{\pi} \widehat{v^*_{1,j}}(\lambda)\widetilde{v^*_{1,j}}(\lambda)\mathrm{d}\lambda = 0$, uncorrelated. Thus we get $\mathbb{E}[Z^2_{1,j,k}] = \mathbb{E}[\widehat{Z}^2_{1,j,k}] + \mathbb{E}[\widetilde{Z}^2_{1,j,k}]$ and hence the remainder R_j defined by (80) is

$$R_j = P_j + 2Q_j \quad \text{with} \tag{85}$$

$$P_j = n_j^{-1/2} \sum_{k=0}^{n_j-1} \{\widetilde{Z}^2_{1,j,k} - \mathbb{E}[\widetilde{Z}^2_{1,j,k}]\} \quad \text{and} \quad Q_j = n_j^{-1/2} \sum_{k=0}^{n_j-1} \widetilde{Z}_{1,j,k}\widehat{Z}_{1,j,k}. \tag{86}$$

We have

$$\mathbb{E}[|P_j|] \le 2n_j^{1/2}\mathbb{E}[\widetilde{Z}^2_{1,j,0}] = 2n_j^{1/2} \sum_{t\in\mathbb{Z}} \widetilde{v}^2_{1,j}(t) = 4n_j^{1/2} I_j \tag{87}$$

by the Parseval theorem and the definitions of $\widetilde{v^*_{1,j}}$ and I_j.

Using that $\widehat{Z}_{1,j,k}$ and $\widetilde{Z}_{1,j,k'}$ are centered and uncorrelated, we have, using a standard formula for cumulants of products, for all $k, k' \in \mathbb{Z}$,

$$\begin{aligned}\mathrm{Cov}\left(\widetilde{Z}_{1,j,k}\widehat{Z}_{1,j,k}, \widetilde{Z}_{1,j,k'}\widehat{Z}_{1,j,k'}\right) &= \mathrm{Cov}\left(\widetilde{Z}_{1,j,k}, \widetilde{Z}_{1,j,k'}\right)\mathrm{Cov}\left(\widehat{Z}_{1,j,k}, \widehat{Z}_{1,j,k'}\right) \\ &\quad + \mathrm{cum}\left(\widetilde{Z}_{1,j,k}, \widehat{Z}_{1,j,k}, \widetilde{Z}_{1,j,k'}, \widehat{Z}_{1,j,k'}\right).\end{aligned}$$

Hence, $\mathrm{Var}\left(Q_j\right) = A_j + B_j$ where

$$A_j = n_j^{-1} \sum_{k=0}^{n_j-1}\sum_{k'=0}^{n_j-1} \mathrm{Cov}\left(\widetilde{Z}_{1,j,k}, \widetilde{Z}_{1,j,k'}\right)\mathrm{Cov}\left(\widehat{Z}_{1,j,k}, \widehat{Z}_{1,j,k'}\right)$$

$$\text{and} \quad B_j = n_j^{-1} \sum_{k=0}^{n_j-1}\sum_{k'=0}^{n_j-1} \mathrm{cum}\left(\widetilde{Z}_{1,j,k}, \widehat{Z}_{1,j,k}, \widetilde{Z}_{1,j,k'}, \widehat{Z}_{1,j,k'}\right).$$

Denote by $\widehat{f}_j$ and $\widetilde{f}_j$ the respective spectral densities of the weakly stationary processes $\widehat{Z}_{1,j,\cdot}$ and $\widetilde{Z}_{1,j,\cdot}$. Replacing the covariances in the definition of A_j by their respective expressions as Fourier coefficients of the spectral density, e.g. $\mathrm{Cov}\left(\widetilde{Z}_{1,j,k}, \widetilde{Z}_{1,j,k'}\right) = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(k-k')\lambda}\widetilde{f}_j(\lambda)\mathrm{d}\lambda$, we get

$$A_j = n_j^{-1} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \widehat{f}_j(\lambda)\widetilde{f}_j(\lambda') \left|\sum_{k=0}^{n_j-1} \mathrm{e}^{\mathrm{i}k(\lambda+\lambda')}\right|^2 \mathrm{d}\lambda\mathrm{d}\lambda',$$

which implies that

$$0 \le A_j \le 2\pi \sup_{\lambda\in(-\pi,\pi)} \widehat{f}_j(\lambda) \times \int_{-\pi}^{\pi} \widetilde{f}_j(\lambda')\mathrm{d}\lambda', \tag{88}$$

where, in the last inequality, we used that, for any λ', $\int_{-\pi}^{\pi} \left|\sum_{k=0}^{n_j-1} \mathrm{e}^{\mathrm{i}k(\lambda+\lambda')}\right|^2 \mathrm{d}\lambda = 2\pi n_j$. Observe that, by definition of $\widehat{Z}_{1,j,k}$ in (81),

$$\mathrm{Cov}\left(\widehat{Z}_{1,j,0}, \widehat{Z}_{1,j,k}\right) = (2\pi)\int_{-\pi}^{\pi} \left|\widehat{v^*_{1,j}}(\lambda)\right|^2 \mathrm{e}^{\mathrm{i}\gamma_j k\lambda}\mathrm{d}\lambda.$$

Using Lemma 4 with the (2π)-periodic function $g(\lambda) = \left|\widehat{v^*_{1,j}}(\lambda)\right|^2 \mathrm{e}^{\mathrm{i}\gamma_j k\lambda}$, we get

$$\mathrm{Cov}\left(\widehat{Z}_{1,j,0}, \widehat{Z}_{1,j,k}\right) = (2\pi)\gamma_j^{-1} \int_{-\pi}^{\pi} \left(\sum_{p=0}^{\gamma_j-1} \left|\widehat{v^*_{1,j}}(\gamma_j^{-1}(\lambda + 2p\pi))\right|^2 \right) \mathrm{e}^{\mathrm{i}k\lambda}\mathrm{d}\lambda.$$

Hence we have $\widehat{f}_j(\lambda) = (2\pi)^{-1}\gamma_j^{-1} \sum_{p=0}^{\gamma_j-1} \left|\widehat{v^*_{1,j}}(\gamma_j^{-1}(\lambda + 2p\pi))\right|^2$. Using (79), since $|\gamma_j^{-1}(\lambda + 2p\pi)| < \pi$ for $0 \le p \le \gamma_j - 1$ and $-\gamma_j\pi < \lambda < -\gamma_j\pi + 2\pi$, we get

$$\widehat{f}_j(\lambda) \le C \sum_{p=0}^{\gamma_j-1} \left(1 + \left||\lambda + 2p\pi| - \gamma_j\lambda_{1,j}\right|\right)^{-2\delta}, \quad \lambda \in (-\gamma_j\pi, -\gamma_j\pi + 2\pi).$$

Using (123) in Lemma 6 and that $\widehat{f}_j$ is (2π)-periodic, we obtain

$$\sup_{j\ge 0} \sup_{\lambda\in(-\pi,\pi)} \widehat{f}_j(\lambda) < \infty. \tag{89}$$

Moreover, we have

$$\int_{-\pi}^{\pi} \widetilde{f}_j(\lambda')\mathrm{d}\lambda' = \mathrm{Var}\left(\widetilde{Z}_{1,j,0}\right) = \sum_{t\in\mathbb{Z}} \widetilde{v}^2_{1,j}(t) = \int_{-\pi}^{\pi} \left|\widetilde{v^*_{1,j}}(\lambda)\right|^2 \mathrm{d}\lambda, \tag{90}$$

by the Parseval theorem. Hence by (88), there is a positive constant C such that

$$\left|A_j\right| \le C \int_{-\pi}^{\pi} \left|\widetilde{v^*_{1,j}}(\lambda)\right|^2 \mathrm{d}\lambda. \tag{91}$$

We now consider B_j. Using (A-2) and the definitions of $\widetilde{Z}_{1,j,k}$ and $\widehat{Z}_{1,j,k}$ in (81) and (84),

$$\begin{aligned}&\mathrm{cum}\left(\widetilde{Z}_{1,j,k}, \widehat{Z}_{1,j,k}, \widetilde{Z}_{1,j,k'}, \widehat{Z}_{1,j,k'}\right)\\ &\quad = \kappa_4 \sum_{t\in\mathbb{Z}} \widetilde{v}_{1,j}(\gamma_j k - t)\widehat{v}_{1,j}(\gamma_j k - t)\widetilde{v}_{1,j}(\gamma_j k' - t)\widehat{v}_{1,j}(\gamma_j k' - t).\end{aligned}$$

Hence

$$\begin{aligned}|B_j| &\le \kappa_4 \sum_{t,\tau\in\mathbb{Z}} \left|\widetilde{v}_{1,j}(t)\widehat{v}_{1,j}(t)\widetilde{v}_{1,j}(t+\gamma_j\tau)\widehat{v}_{1,j}(t+\gamma_j\tau)\right|\\ &\le \kappa_4 \left(\sum_{t\in\mathbb{Z}} \left|\widetilde{v}_{1,j}(t)\widehat{v}_{1,j}(t)\right|\right)^2\\ &\le \kappa_4 \sum_{t\in\mathbb{Z}} \left|\widetilde{v}_{1,j}(t)\right|^2 \times \sum_{t\in\mathbb{Z}} \left|\widehat{v}_{1,j}(t)\right|^2\\ &= \kappa_4 \int_{-\pi}^{\pi} \left|\widetilde{v^*_{1,j}}(\lambda)\right|^2 \mathrm{d}\lambda \times \int_{-\pi}^{\pi} \left|\widehat{v^*_{1,j}}(\lambda)\right|^2 \mathrm{d}\lambda.\end{aligned} \tag{92}$$

By definition of $\widehat{v^*_{1,j}}$ and (89), we have $\int_{-\pi}^{\pi} \left|\widehat{v^*_{1,j}}(\lambda)\right|^2 \mathrm{d}\lambda = \mathrm{Var}\left(\widehat{Z}_{1,j,0}\right) = \int_{-\pi}^{\pi} \widehat{f}_j(\lambda)\mathrm{d}\lambda \le C$, and hence

$$B_j \le C \int_{-\pi}^{\pi} \left|\widetilde{v^*_{1,j}}(\lambda)\right|^2 \mathrm{d}\lambda, \tag{93}$$

where C denotes a positive constant not depending on j. The bounds (91) and (93) and the definition of $\widetilde{v^*_{1,j}}$ yield

$$\mathbb{E}\left[Q_j^2\right] = \mathrm{Var}(Q_j) = A_j + B_j \le 2C \int_{-\pi}^{\pi} \left|\widetilde{v^*_{1,j}}(\lambda)\right|^2 \mathrm{d}\lambda$$
$$\le 4C \int_0^{\pi} \mathbb{1}\left(|\lambda - \lambda_{1,\infty}| > \varepsilon\right) \left|v^*_{1,j}(\lambda)\right|^2 \mathrm{d}\lambda = 4C I_j$$

by the definitions of $\widetilde{v^*_{1,j}}$ and I_j. This, with (85), (87) and Jensen's inequality yields (82), which concludes the proof. □

Proof of Theorem 2. The general case can easily be adapted from the case $N = 1$, which we assume here. We apply Proposition 4. It follows from (78) and (79) that the assumptions of Theorem 1 are verified for $\widehat{Z}_{1,j,k}$ and we obtain

$$n_j^{-1/2}\left(\sum_{k=0}^{n_j-1}\{\widehat{Z}^2_{1,j,k} - \mathbb{E}[\widehat{Z}^2_{1,j,k}]\}\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Gamma).$$

It follows from (28) that $R_j \xrightarrow{\mathbb{P}} 0$ as $j \to \infty$. Hence (80) yields the CLT (24). □

7. Proof of Theorems 3 and 4

By setting $N = 1$ and $\lambda_{1,j} = 0$ for $j = 1, 2, \ldots, \infty$ in Theorem 4 and observing that (C-1) implies (C$'$-1) with $\zeta_{1,j} = 0$, we obtain Theorem 3. Hence we only prove Theorem 4. We shall use the following lemmas.

Lemma 1. *Assume* (C$'$-1) *and* (C-2). *Let* (γ_j) *and* $(\lambda_{i,j})_{1\le i\le N, j=0,\ldots,\infty}$ *satisfy Condition* (B)(i) *in Section 2. For* $\varepsilon > 0$ *small enough, if* $\beta > 2$, *then for all* $i = 1, \ldots, N$,

$$\left|2\int_0^{\pi} \mathbb{1}(|\lambda - \lambda_{i,j}| \le \varepsilon)\gamma_j \left|\widehat{W}_{i,j}(\gamma_j\lambda)\right|^2 f(\lambda)\mathrm{d}\lambda - f(\lambda_{i,j})\right| = O\left(\gamma_j^{-2}\right), \tag{94}$$

where $\widehat{W}_{i,j}$ *is the Fourier transform of* $W_{i,j}$ *defined in* (41).

Proof. We first consider the case $\lambda_{i,\infty} = 0$. By Condition (B)(i), $\lambda_{i,j} = 0$ and $W_{i,j} = W$ for j large enough and (94) can be written as

$$\left|\int_{-\varepsilon}^{\varepsilon} \gamma_j \left|\widehat{W}(\gamma_j\lambda)\right|^2 f(\lambda)\mathrm{d}\lambda - f(0)\right| = O\left(\gamma_j^{-2}\right). \tag{95}$$

Using (32), for $\varepsilon > 0$ small enough, the left-hand side of (95) is at most

$$\left|f(0)\int_{-\varepsilon}^{\varepsilon} \gamma_j \left|\widehat{W}(\gamma_j\lambda)\right|^2 \mathrm{d}\lambda - f(0)\right| + C\left|\int_{-\varepsilon}^{\varepsilon} \gamma_j \left|\widehat{W}(\gamma_j\lambda)\right|^2 \lambda^2\mathrm{d}\lambda\right|, \tag{96}$$

where C is a positive constant. To evaluate the first integral in (96) we write $\int_{-\varepsilon}^{\varepsilon} = \int_{-\infty}^{\infty} - \int_{-\infty}^{-\varepsilon} - \int_{\varepsilon}^{\infty}$. Using (C-2), one gets

$$\int_{-\varepsilon}^{\varepsilon} \gamma_j \left|\widehat{W}(\gamma_j\lambda)\right|^2 \mathrm{d}\lambda = 1 + O\left(\gamma_j^{1-2\beta}\right). \tag{97}$$

Defining $C' = \int_{-\infty}^{\infty} \left|\widehat{W}(\lambda)\right|^2 \lambda^2 \mathrm{d}\lambda$, the second integral in (96) is bounded by

$$\int_{-\infty}^{\infty} \gamma_j \left|\widehat{W}(\gamma_j\lambda)\right|^2 \lambda^2 \mathrm{d}\lambda = C'\gamma_j^{-2}. \tag{98}$$

Since $C' < \infty$ for $\beta > 3/2$ by (C-2), we obtain (95) for $\beta > 3/2$.
The stronger condition $\beta > 2$ will be useful for the case $\lambda_{i,\infty} > 0$, which we now consider. By (C$'$-1), we may choose $\varepsilon > 0$ small enough so that, for all $\lambda \in [\lambda_{i,j} - \varepsilon, \lambda_{i,j} + \varepsilon]$ and $j \geq 0$,

$$\left|f(\lambda) - f(\lambda_{i,j}) - (\lambda - \lambda_{i,j})\zeta_{i,j}\right| \leq C|\lambda - \lambda_{i,j}|^2, \tag{99}$$

where C is a positive constant. In particular, since $(\zeta_{i,j})_j$ is assumed bounded in (C$'$-1), f is bounded in $\cup_j[\lambda_{i,j} - \varepsilon, \lambda_{i,j} + \varepsilon]$. We also impose $\varepsilon < \lambda_{i,\infty}/2$, so that, since $\lambda_{i,j} \to \lambda_{i,\infty}$, for j large enough,

$$[\lambda_{i,j} - \varepsilon, \lambda_{i,j} + \varepsilon] \subset [\lambda_{i,\infty} - 2\varepsilon, \lambda_{i,\infty} + 2\varepsilon] \subset (0, \pi). \tag{100}$$

Hence in the following we may replace $\int_0^{\pi} \mathbb{1}(|\lambda - \lambda_{i,j}| \leq \varepsilon)(\ldots)\mathrm{d}\lambda$ by $\int_{\lambda_{i,j}-\varepsilon}^{\lambda_{i,j}+\varepsilon}(\ldots)\mathrm{d}\lambda$. In view of (41), the Fourier transform of $W_{i,j}$ is given by

$$\widehat{W}_{i,j}(\xi) = \frac{\mathrm{C}_{i,i}^{1/2}}{2}\left[\widehat{W}(\xi - \gamma_j\lambda_{i,j}) + \widehat{W}(\xi + \gamma_j\lambda_{i,j})\right], \quad \xi \in \mathbb{R}. \tag{101}$$

By (100) and (C-1), we have, for j large enough,

$$\sup_{|\lambda - \lambda_{i,j}| \leq \varepsilon} \left|\widehat{W}(\gamma_j(\lambda + \lambda_{i,j}))\right| = O(\gamma_j^{-\beta}). \tag{102}$$

It follows that, using $\int_0^{\pi} f(\lambda)\mathrm{d}\lambda < \infty$,

$$\int_{\lambda_{i,j}-\varepsilon}^{\lambda_{i,j}+\varepsilon} \gamma_j \left|\widehat{W}(\gamma_j\lambda + \gamma_j\lambda_{i,j})\right|^2 f(\lambda)\mathrm{d}\lambda = O(\gamma_j^{1-2\beta}),$$

and, using that f is bounded in $\cup_j[\lambda_{i,j} - \varepsilon, \lambda_{i,j} + \varepsilon]$ and $\int_{-\infty}^{\infty} \gamma_j \left|\widehat{W}(\gamma_j\xi)\right| \mathrm{d}\xi = \|\widehat{W}\|_1 < \infty$,

$$\int_{\lambda_{i,j}-\varepsilon}^{\lambda_{i,j}+\varepsilon} \gamma_j \left|\widehat{W}(\gamma_j(\lambda - \lambda_{i,j}))\right| \left|\widehat{W}(\gamma_j(\lambda + \lambda_{i,j}))\right| f(\lambda)\mathrm{d}\lambda = O(\gamma_j^{-\beta}).$$

Since $\lambda_{i,\infty} > 0$, we have $\mathrm{C}_{i,i} = 2$ and it follows from (101) that, for all $\xi \in \mathbb{R}$,

$$\begin{aligned}
&\left|2\left|\widehat{W}_{i,j}(\xi)\right|^2 - \left|\widehat{W}(\xi - \gamma_j\lambda_{i,j})\right|^2\right| \\
&\quad \leq \left|\widehat{W}(\xi + \gamma_j\lambda_{i,j})\right|^2 + 2\left|\widehat{W}(\xi - \gamma_j\lambda_{i,j})\right| \left|\widehat{W}(\xi + \gamma_j\lambda_{i,j})\right|.
\end{aligned}$$

The last three displays give that

$$\begin{aligned}
&2\int_{\lambda_{i,j}-\varepsilon}^{\lambda_{i,j}+\varepsilon} \gamma_j \left|\widehat{W}_{i,j}(\gamma_j\lambda)\right|^2 f(\lambda)\mathrm{d}\lambda \\
&\quad = \int_{\lambda_{i,j}-\varepsilon}^{\lambda_{i,j}+\varepsilon} \gamma_j \left|\widehat{W}(\gamma_j(\lambda - \lambda_{i,j}))\right|^2 f(\lambda)\mathrm{d}\lambda + O(\gamma_j^{1-2\beta} + \gamma_j^{-\beta}).
\end{aligned}$$

Moreover, by symmetry, $\int_{\lambda_{i,j}-\varepsilon}^{\lambda_{i,j}+\varepsilon} \gamma_j \left|\widehat{W}(\gamma_j(\lambda - \lambda_{i,j}))\right|^2 (\lambda - \lambda_{i,j})\mathrm{d}\lambda = 0$. Using (99) and the last two relations, the left-hand side of (94) is at most

$$\left| f(\lambda_{i,j}) \int_{-\varepsilon}^{\varepsilon} \gamma_j \left|\widehat{W}(\gamma_j\lambda)\right|^2 \mathrm{d}\lambda - f(\lambda_{i,j}) \right|$$
$$+ C \left| \int_{-\varepsilon}^{\varepsilon} \gamma_j \left|\widehat{W}(\gamma_j\lambda)\right|^2 \lambda^2 \mathrm{d}\lambda \right| + O(\gamma_j^{1-2\beta} + \gamma_j^{-\beta}),$$

where C is a positive constant. Using (97) and (98), we get that the left-hand side of (94) is at most $O(\gamma_j^{-2} + \gamma_j^{1-2\beta} + \gamma_j^{-\beta})$; hence (94) since we assumed $\beta > 2$. □

Lemma 2. *Assume* (C-2). *Let* (γ_j) *and* $(\lambda_{i,j})_{1\le i\le N, j=0,\ldots,\infty}$ *satisfy Condition* (B)(i). *Define for all* $i = 1, \ldots, N$ *and* $j \ge 0$,

$$B_{i,j}(\lambda) \stackrel{\text{def}}{=} \sum_{p\in\mathbb{Z}} \gamma_j^{1/2} \widehat{W}_{i,j}(\gamma_j(\lambda + 2p\pi)), \tag{103}$$

where $\widehat{W}_{i,j}$ *is the Fourier transform of* $W_{i,j}$ *defined in* (41). *Then the following assertions hold.*

(i) *There is a positive constant C such that, for all* $j \ge 0$ *and* $\lambda \in (-\pi, \pi)$,

$$\left| \gamma_j^{-1/2} B_{i,j}(\lambda) - \widehat{W}_{i,j}(\gamma_j\lambda) \right| \le C\gamma_j^{-\beta}, \tag{104}$$

$$\left| \left|B_{i,j}(\lambda)\right|^2 - \gamma_j \left|\widehat{W}_{i,j}(\gamma_j\lambda)\right|^2 \right| \le C \left[\gamma_j^{1-\beta} \left|\widehat{W}_{i,j}(\gamma_j\lambda)\right| + \gamma_j^{1-2\beta} \right], \tag{105}$$

(ii) *For any positive* ε,

$$\sup_{\lambda\in[0,\pi)} \mathbb{1}(|\lambda - \lambda_{i,\infty}| > \varepsilon) \left|B_{i,j}(\lambda)\right| = O(\gamma_j^{1/2-\beta}). \tag{106}$$

(iii) *For an arbitrarily small* $\varepsilon > 0$,

$$\sup_{j\ge 0} \sup_{|\lambda-\lambda_{i,\infty}|\le\varepsilon} \gamma_j^{-1/2} \left|B_{i,j}(\lambda)\right| (1 + \gamma_j|\lambda - \lambda_{i,j}|)^{\beta} < \infty. \tag{107}$$

Proof. By (103) and (101), we have $B_{i,j}(\lambda) = \gamma_j^{1/2}\widehat{W}_{i,j}(\gamma_j\lambda) + R_{i,j}(\lambda)$, where

$$R_{i,j}(\lambda) = \sum_{p\ne 0} \gamma_j^{1/2} \widehat{W}_{i,j}(\gamma_j(\lambda + 2p\pi))$$
$$= \frac{\mathrm{C}_{i,i}^{1/2}}{2} \sum_{s=-1,1} \sum_{p\ne 0} \gamma_j^{1/2} \widehat{W}(\gamma_j(\lambda + s\lambda_{i,j} + 2p\pi)).$$

Since $\lambda_{i,j} \to \lambda_{i,\infty} \in [0, \pi)$, there exists $\eta \in (0, 2)$ such that, for j large enough, $s = -1, 1$ and all $\lambda \in (-\pi, \pi)$, $|\lambda + s\lambda_{i,j}| < \eta\pi$. Using (C-2), since $\beta > 1$, we thus have for j large enough,

$$\sup_{\lambda\in(-\pi,\pi)} \left|R_{i,j}(\lambda)\right| \le C\gamma_j^{1/2} \sum_{p\ne 0} (1 + (2|p| - \eta)\gamma_j\pi)^{-\beta} = O(\gamma_j^{1/2-\beta}).$$

This bound implies (104).

The bound (105) follows from (104) and

$$\left| |z_1|^2 - |z_2|^2 \right| \le 2|z_2| \times |z_1 - z_2| + |z_1 - z_2|^2$$

applied with $z_1 = B_{i,j}(\lambda)$ and $z_2 = \gamma_j^{1/2}\widehat{W}_{i,j}(\gamma_j\lambda)$.

By (5) and (4) in Condition (B)(i), we have $|\lambda_{i,j} - \lambda_{i,\infty}| < \varepsilon/2$ for j large enough, which implies $|\lambda-\lambda_{i,j}| \geq |\lambda-\lambda_{i,\infty}|-\varepsilon/2$. Thus, $|\lambda-\lambda_{i,\infty}| > \varepsilon$ implies $|\lambda-\lambda_{i,j}| > \varepsilon/2$ and by (C-2),

$$\sup_{\lambda\in[0,\pi)} \mathbb{1}(|\lambda - \lambda_{i,\infty}| > \varepsilon) \left|\widehat{W}(\gamma_j\lambda - \gamma_j\lambda_{i,j})\right| = O(\gamma_j^{-\beta}). \tag{108}$$

But we also have

$$\sup_{\lambda\in[0,\pi)} \mathbb{1}(|\lambda - \lambda_{i,\infty}| > \varepsilon) \left|\widehat{W}(\gamma_j\lambda + \gamma_j\lambda_{i,j})\right| = O(\gamma_j^{-\beta}). \tag{109}$$

Indeed, if $\lambda_{i,\infty} = 0$, the left-hand sides of (109) and (108) are the same for j large enough by (5) in Condition (B)(i) and, if $\lambda_{i,\infty} > 0$, $\lambda \in [0, \pi)$ implies that $\lambda + \lambda_{i,j} \geq \lambda_{i,j} \geq \lambda_{i,\infty}/2 > 0$ for j large enough by (4) in Condition (B)(i).

With (101), the bounds (108) and (109) imply

$$\sup_{\lambda\in[0,\pi)} \mathbb{1}(|\lambda - \lambda_{i,\infty}| > \varepsilon) \left|\widehat{W}_{i,j}(\gamma_j\lambda)\right| = O(\gamma_j^{-\beta}).$$

This bound and the relation (104) yield (106).

We now prove (107). By (C-2), we have

$$\sup_{j\geq 0}\sup_{\lambda\in\mathbb{R}} \left|\widehat{W}\left(\gamma_j\lambda - \gamma_j\lambda_{i,j}\right)\right| \left(1 + \gamma_j|\lambda - \lambda_{i,j}|\right)^{\beta} < \infty.$$

If $\lambda_{i,\infty} > 0$, we may apply (102) for ε small enough, and, by (101), we get that, for all λ such that $|\lambda - \lambda_{i,\infty}| \leq \varepsilon$,

$$|\widehat{W}_{i,j}(\gamma_j\lambda)| \leq C\left(1 + \gamma_j|\lambda - \lambda_{i,j}|\right)^{-\beta} + O\left(\gamma_j^{-\beta}\right) \tag{110}$$

where C and the O-term do not depend on λ. If $\lambda_{i,\infty} = 0$, $W_{i,j} = W$ for j large enough and the last display holds as a direct consequence of the previous one. The bound (110) and the relation (104) yield (107). □

Lemma 3. *Assume* (C$'$-1) *and* (C-2). *Let* (γ_j) *and* $(\lambda_{i,j})_{1\leq i\leq N, j=0,\ldots,\infty}$ *satisfy Condition* (B)(i) *and define, for all* $i = 1, \ldots, N$ *and* $j \geq 0$,

$$v_{i,j}(s) \stackrel{\text{def}}{=} \gamma_j^{-1/2} \sum_{v\in\mathbb{Z}} W_{i,j}(\gamma_j^{-1}(s - v))a(v), \quad s \in \mathbb{Z} \tag{111}$$

$$v^*_{i,\infty}(\lambda) \stackrel{\text{def}}{=} \mathrm{C}_{i,i}^{-1/2} a^*(\lambda_{i,\infty})\widehat{W}(\lambda), \quad \lambda \in \mathbb{R}, \tag{112}$$

where $\mathrm{C}_{i,i}$ *is defined in* (18), $a(v)$ *is the kernel in* (29) *and* $a^*(\lambda)$ *its Fourier series* (30). *Then for some arbitrarily small* $\varepsilon > 0$, *for all* $i = 1, \ldots, N$,

$$\sup_{j\geq 0}\sup_{|\lambda-\lambda_{i,\infty}|\leq\varepsilon} \gamma_j^{-1/2}\left|v^*_{i,j}(\lambda)\right|(1 + \gamma_j|\lambda - \lambda_{i,j}|)^{\beta} < \infty, \tag{113}$$

$$\lim_{j\to\infty} \gamma_j^{-1/2} v^*_{i,j}(\gamma_j^{-1}\lambda + \lambda_{i,j})\mathrm{e}^{\mathrm{i}\Phi_{i,j}(\lambda)} = v^*_{i,\infty}(\lambda) \quad \textit{for all } \lambda \in \mathbb{R}, \tag{114}$$

where $\Phi_{i,j}(\lambda)$ *is an array of* $[-\pi, \pi)$*-valued functions defined on* $\lambda \in \mathbb{R}$ *satisfying Condition* (9). *Moreover, as* $j \to \infty$,

$$\int_0^{\pi} \mathbb{1}(|\lambda - \lambda_{i,j}| > \varepsilon)|v^*_{i,j}(\lambda)|^2 \mathrm{d}\lambda = O\left(\gamma_j^{1-2\beta}\right). \tag{115}$$

Proof. By (30), the Fourier series (2) of $v_{i,j}(s)$ is given by

$$v^*_{i,j}(\lambda) = \gamma_j^{-1/2} a^*(\lambda) \sum_{u\in\mathbb{Z}} W_{i,j}(\gamma_j^{-1}u)\mathrm{e}^{-\mathrm{i}\lambda u}, \quad \lambda \in (-\pi, \pi). \tag{116}$$

Moreover we have, for all $u \in \mathbb{Z}$,

$$\begin{aligned} W_{i,j}(\gamma_j^{-1}u) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \gamma_j \widehat{W}_{i,j}(\gamma_j\xi)\mathrm{e}^{\mathrm{i}u\xi}\mathrm{d}\xi \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{p\in\mathbb{Z}} \gamma_j \widehat{W}_{i,j}(\gamma_j(\lambda+2p\pi))\right]\mathrm{e}^{\mathrm{i}u\lambda}\mathrm{d}\lambda, \end{aligned}$$

and hence the term in brackets is the Fourier series of $\{W_{i,j}(\gamma_j^{-1}u), u \in \mathbb{Z}\}$ and thus

$$\sum_{u\in\mathbb{Z}} W_{i,j}(\gamma_j^{-1}u)\mathrm{e}^{-\mathrm{i}\lambda u} = \sum_{p\in\mathbb{Z}} \gamma_j \widehat{W}_{i,j}(\gamma_j(\lambda+2p\pi)),$$

which is some times called the Poisson formula. Inserting this in (116), we obtain

$$v^*_{i,j}(\lambda) = a^*(\lambda) B_{i,j}(\lambda), \quad \lambda \in (-\pi, \pi), \tag{117}$$

where $B_{i,j}$ is the (2π)-periodic function defined by (103).

Applying (117) and (107) and that $|a^*(\lambda)| = \sqrt{f(\lambda)}$ is bounded in a neighborhood of the origin by (C$'$-1), we get (113).

Applying (117) and (104), we have, as $j \to \infty$,

$$\gamma_j^{-1/2} v^*_{i,j}(\gamma_j^{-1}\lambda + \lambda_{i,j}) = a^*(\gamma_j^{-1}\lambda + \lambda_{i,j})\widehat{W}_{i,j}(\lambda + \gamma_j\lambda_{i,j}) + O(\gamma_j^{-\beta}).$$

By (C$'$-1), since $|a^*| = \sqrt{f}$, $\lambda_{i,j} \to \lambda_{i,\infty}$ and $\gamma_j \to \infty$, we have, for all $\lambda \in \mathbb{R}$, $|a^*(\gamma_j^{-1}\lambda + \lambda_{i,j})| \to |a^*(\lambda_{i,\infty})|$. Hence defining $\Phi_{i,j}(\lambda)$ as the phase of $a^*(\gamma_j^{-1}\lambda + \lambda_{i,j})$ minus the phase of $a^*(\lambda_{i,\infty})$, we have, for all $\lambda \in \mathbb{R}$,

$$\lim_{j\to\infty} a^*(\gamma_j^{-1}\lambda + \lambda_{i,j})\mathrm{e}^{\mathrm{i}\Phi_{i,j}(\lambda)} = a^*(\lambda_{i,\infty}).$$

Moreover, in view of (6), the array $\Phi_{i,j}(\lambda)$ satisfies (9). By (101) we have

$$\widehat{W}_{i,j}(\lambda + \gamma_j\lambda_{i,j}) = \frac{\mathrm{C}_{i,i}^{1/2}}{2}\left[\widehat{W}(\lambda) + \widehat{W}(\lambda + 2\gamma_j\lambda_{i,j})\right], \quad \lambda \in \mathbb{R}.$$

Hence if $\lambda_{i,\infty} = 0$, we get

$$\widehat{W}_{i,j}(\lambda + \gamma_j\lambda_{i,j}) = \widehat{W}(\lambda) = \mathrm{C}_{i,i}^{-1/2}\widehat{W}(\lambda), \quad \lambda \in \mathbb{R},$$

for j large enough, and if $\lambda_{i,\infty} > 0$, $\lambda + 2\gamma_j\lambda_{i,j} \to \infty$ and by (C-1), (18), we get

$$\lim_{j\to\infty} \widehat{W}_{i,j}(\lambda + \gamma_j\lambda_{i,j}) = \frac{\mathrm{C}_{i,i}^{1/2}}{2}\widehat{W}(\lambda) = \mathrm{C}_{i,i}^{-1/2}\widehat{W}(\lambda), \quad \lambda \in \mathbb{R}.$$

The last five displays give (114).

Applying (117) and (106) and $\int_{-\pi}^{\pi} |a^*(\lambda)|^2\mathrm{d}\lambda < \infty$, we obtain (115). □

Proof of Theorem 4. By (29) and (40), we have for all $i = 1, \dots, N$,

$$
\begin{aligned}
Z_{i,j,k} &= \sum_{t\in\mathbb{Z}} \gamma_j^{-1/2} \sum_{u\in\mathbb{Z}} W_{i,j}(k-\gamma_j^{-1}u)a(u-t)\xi_t \\
&= \sum_{t\in\mathbb{Z}} \gamma_j^{-1/2} \sum_{v\in\mathbb{Z}} W_{i,j}(k-\gamma_j^{-1}t-\gamma_j^{-1}v)a(v)\xi_t \\
&= \sum_{t\in\mathbb{Z}} v_{i,j}(\gamma_j k - t)\xi_t,
\end{aligned}
$$

where $v_{i,j}$ is defined in (111). Thus $\{Z_{i,j,t},\ j \geq 0, t \in \mathbb{Z}\}$ is an array of N-dimensional decimated linear processes as in Definition 1.

We are now in a position to show first (43) then (44). Applying (20), we have

$$
\mathbb{E}\left[Z_{i,j,0}^2\right] = \mathrm{Var}\left(Z_{i,j,0}\right) = \int_{-\pi}^{\pi} \left|v_{i,j}^*(\lambda)\right|^2 \mathrm{d}\lambda = 2\int_0^{\pi} \left|v_{i,j}^*(\lambda)\right|^2 \mathrm{d}\lambda.
$$

Using (117) and (115) and then (105), this gives for any $\varepsilon > 0$ small enough,

$$
\begin{aligned}
\mathbb{E}\left[Z_{i,j,0}^2\right] &= 2\int_0^{\pi} \mathbb{1}(|\lambda-\lambda_{i,j}|\leq\varepsilon)\left|a^*(\lambda)B_{i,j}(\lambda)\right|^2 \mathrm{d}\lambda + O\left(\gamma_j^{1-2\beta}\right) \\
&= 2\int_0^{\pi} \mathbb{1}(|\lambda-\lambda_{i,j}|\leq\varepsilon)\gamma_j \left|a^*(\lambda)\widehat{W}_{i,j}(\gamma_j\lambda)\right|^2 \mathrm{d}\lambda \\
&\quad + O\left(\gamma_j^{1-\beta}\int_{-\infty}^{\infty}\left|\widehat{W}_{i,j}(\gamma_j\lambda)\right| \mathrm{d}\lambda\right) + O\left(\gamma_j^{1-2\beta}\right).
\end{aligned}
$$

In the last line, since $|a^*(\lambda)|^2 = f(\lambda)$, by Lemma 1, the first term is $f(\lambda_{i,j}) + O\left(\gamma_j^{-2}\right)$ and, by a change of variable, the second term is less than $O(\gamma_j^{-\beta}\|\widehat{W}_{i,j}\|_1) = O(\gamma_j^{-\beta})$ by (101) since $\|\widehat{W}\|_1 < \infty$ for $\beta > 1$ by (C-2). Hence (43) follows since $\beta > 2$.

Condition (B)(i) is satisfied by assumption. In view of Lemma 3, the relations (8) and (9) in Condition (B) hold, as well as the relation (27) of Theorem 2 (recall that in that theorem, the relation (27) replaces the relation (7) in Condition (B)). The bound (115) yields (28) under the condition $\gamma_j^{1-2\beta} = o(n_j^{-1/2})$, where n_j is given by (34). The assumption (37) implies that condition for $j = T$. We have also $n_T \sim \gamma_T^{-1}T \to \infty$. Finally, since (A-2) is assumed in the definition (29) of $\{X_u\}$, we may apply Theorem 2 with $j = T$ and obtain (44). In view of (112) and (25), the limiting covariance $\Gamma_{i,i'}$ is given by (45). □

8. Technical lemmas

The following lemma will be used several times.

Lemma 4. *Let g be a (2π)-periodic locally integrable function. Then for all positive integer γ, the function defined by*

$$
g_\gamma(\lambda) = \sum_{p=0}^{\gamma-1} g(\gamma^{-1}(\lambda + 2p\pi))
$$

is (2π)-periodic. Moreover, one has

$$
\int_{-\pi}^{\pi} g(\lambda)\mathrm{d}\lambda = \gamma^{-1}\int_{-\pi}^{\pi} g_\gamma(\lambda)\mathrm{d}\lambda. \tag{118}
$$

Proof. Observe that, for all $\lambda \in \mathbb{R}$,

$$g_\gamma(\lambda + 2\pi) = \sum_{p=0}^{\gamma-1} g\left(\gamma^{-1}(\lambda + (p+1)2\pi)\right)$$
$$= \sum_{p=1}^{\gamma-1} g\left(\gamma^{-1}(\lambda + 2p\pi)\right) + g\left(\gamma^{-1}(\lambda + \gamma 2\pi)\right) = g_\gamma(\lambda),$$

since $g(\gamma^{-1}(\lambda + \gamma 2\pi)) = g(\gamma^{-1}\lambda)$ by the (2π)-periodicity of g. Hence $g_\gamma(\lambda)$ is (2π)-periodic.

With a change of variable, one gets

$$\gamma^{-1}\int_0^{2\pi} g_\gamma(\lambda)\mathrm{d}\lambda = \sum_{p=0}^{\gamma-1}\int_0^{2\pi\gamma^{-1}} g(\xi + 2p\pi\gamma^{-1})\mathrm{d}\xi = \int_0^{2\pi} g(\xi)\mathrm{d}\xi.$$

Relation (118) follows by (2π)-periodicity of the integrands. □

The next lemma relates the rates of decrease of two functions with the rate of decrease of their convolution.

Lemma 5. *Let $\delta > 0$. For all $T > 0$ and $t_0 \in [0, T/2)$, we let $h_{T,t_0}(t)$, $t \in \mathbb{R}$ be the even and T-periodic function such that*

$$h_{T,t_0}(t) = (1 + |t - t_0|)^{-\delta} \quad \textit{for all } t \in [0, T/2].$$

Let g be an integrable non-negative function on $\mathbb{R}$ such that

$$g(t) \le c_0 |t|^{-\delta-1} \quad \textit{for } |t| \ge 1. \tag{119}$$

Then there exists a positive constant c, depending only on δ, $\|g\|_1 = \int_{-\infty}^{\infty} g(s)\mathrm{d}s$ and c_0 such that, for all $T > 0$, $t_0 \in [0, T/2)$, and $t \in [0, T/2]$,

$$g * h_{T,t_0}(t) = \int_{-\infty}^{\infty} g(t-u) h_{T,t_0}(u)\mathrm{d}u \le c(1 + |t - t_0|)^{-\delta}. \tag{120}$$

Proof. Let $t_0 \in [0, T/2)$. We shall use the bound, valid for all $t \in \mathbb{R}$,

$$g * h_{T,t_0}(t) \le \|g\|_1. \tag{121}$$

This bound yields (120) only for t close enough to t_0. We shall derive a different bound valid only for $t \in [0, T/2]$ with $|t - t_0| \ge 2$, namely

$$g * h_{T,t_0}(t) \le 2^{1-\delta} c_0 \delta^{-1} |t - t_0|^{-\delta} + \|g\|_1 (1 + |t - t_0|/2)^{-\delta}. \tag{122}$$

Applying (121) for $|t - t_0| \le 2$ and (122) for $|t - t_0| \ge 2$ yields (120).

Hence it only remains to establish (122) for $t \in [0, T/2]$ with $|t - t_0| \ge 2$. We shall suppose that $t \in [t_0 + 2, T/2]$ (the case $t \in [0, t_0 - 2]$ is obtained similarly). Let u be such that $|t - u| \le |t - t_0|/2$. Then $t - (t - t_0)/2 \le u \le t + (t - t_0)/2$ and thus, using that $t \le T/2$ implies $t + t/2 \le T - t/2$, we get $(t + t_0)/2 \le u \le T - (t + t_0)/2$. Observe that the middle point between $(t + t_0)/2$ and $T - (t + t_0)/2$ is $T/2$. Since $h_{T,t_0}(u)$ is decreasing on $[t_0, T/2]$, and symmetric around $T/2$, we get $h_{T,t_0}(u) \le h_{T,t_0}((t + t_0)/2) = (1 + |t - t_0|/2)^{-\delta}$. Hence we may bound $h_{T,t_0}(u)$ by 1 for $|t - u| > |t - t_0|/2$ and by $(1 + |t - t_0|/2)^{-\delta}$ otherwise, which gives

$$\int_{-\infty}^{\infty} g(t-u) h_{T,t_0}(u)\mathrm{d}u \le \int_{|s| > |t - t_0|/2} g(s)\mathrm{d}s + \|g\|_1 (1 + |t - t_0|/2)^{-\delta}.$$

Since $|t - t_0| \geq 2$ we may apply the bound (119) in the integral of the RHS of the previous display. Hence we get (122), which concludes the proof. □

The following lemma is used, in particular, to bound $\widehat{f_j}$ in the proof of Theorem 2. It will be used again in the proof of Lemma 9 below. Applying it, one can bound $g_{j,\gamma_j}(\lambda)$ independently of j and λ, where g_{j,γ_j} is defined as in Lemma 4 with g replaced by g_j, and the sequence g_j satisfies a uniform bound of the form (7), namely

$$\sup_{j\geq 0} \sup_{\lambda\in[-\pi,\pi)} |g_j(\lambda)| \left(1 + \gamma \left||\lambda| - \lambda_j\right|\right)^{\delta} < \infty,$$

with $\lambda_j \to \lambda_\infty \in [0, \pi)$ as $j \to \infty$.

Lemma 6. *Let $\delta > 1/2$. Then*

$$\sup_{t\in\mathbb{R}} \sup_{t'\in\mathbb{R}} \sum_{p\in\mathbb{Z}} \left(1 + \left||t + 2p\pi| - t'\right|\right)^{-2\delta} < \infty. \tag{123}$$

Moreover, as $u \to \infty$,

$$\sup_{t\in\mathbb{R}} \sup_{t'\in\mathbb{R}} \sum_{p\in\mathbb{Z}} \left(1 + \left||t + 2p\pi| - t'\right|\right)^{-\delta} \left(1 + \left||t + 2p\pi| - t' - u\right|\right)^{-\delta} \to 0. \tag{124}$$

Proof. Let $S(t, t') = \sum_{p\in\mathbb{Z}}(3\pi + \left||t + 2p\pi| - t'\right|)^{-2\delta}$. Since, for any t', $t \mapsto S(t, t')$ is (2π)-periodic we have

$$\sup_{t\in\mathbb{R}} \sup_{t'\in\mathbb{R}} S(t, t') = \sup_{t'\in\mathbb{R}} \sup_{t:|t-t'|\leq\pi} S(t, t'). \tag{125}$$

Suppose that $t, t' \in \mathbb{R}$ are such that $|t - t'| \leq \pi$. Then for any $a, b \in \mathbb{R}$, we have, if $t + a \geq 0$,

$$\left||t + a| - t' - b\right| \geq |a - b| - \pi, \tag{126}$$

and, if $t + a \leq 0$,

$$\left||t + a| - t' - b\right| = \left|t - t' + 2t' + a + b\right| \geq |2t' + a + b| - \pi. \tag{127}$$

Adding 3π to each of the last two displays with $a = 2p\pi$ and $b = 0$, we get that, for all $|t - t'| \leq \pi$ and $p \in \mathbb{Z}$,

$$\left(3\pi + \left||t + 2p\pi| - t'\right|\right)^{-2\delta} \leq (2|p|\pi + 2\pi)^{-2\delta} + \left(|2t' + 2p\pi| + 2\pi\right)^{-2\delta}. \tag{128}$$

Since $\sum_{p\in\mathbb{Z}}(2|p|\pi + \pi)^{-2\delta} < \infty$ and $\sup_{t'\in\mathbb{R}} \sum_{p\in\mathbb{Z}}(|2t' + 2p\pi| + \pi)^{-2\delta} < \infty$, the relation (125) gives that $\sup_{t\in\mathbb{R}} \sup_{t'\in\mathbb{R}} S(t, t') < \infty$ and (123) follows.

We now prove (124). Let

$$S(t, t', u) = \sum_{p\in\mathbb{Z}} \left(3\pi + \left||t + 2p\pi| - t'\right|\right)^{-\delta} \left(3\pi + \left||t + 2p\pi| - t' - u\right|\right)^{-\delta}.$$

As above, we have

$$\sup_{t\in\mathbb{R}} \sup_{t'\in\mathbb{R}} S_j(t, t') = \sup_{t'\in\mathbb{R}} \sup_{t:|t-t'|\leq\pi} S_j(t, t'). \tag{129}$$

Suppose that $t, t' \in \mathbb{R}$ are such that $|t - t'| \leq \pi$. Adding 3π to (126) and (127) with $a = 2p\pi$ and $b = u$, we have

$$\left(3\pi + \left||t + 2p\pi| - t' - u\right|\right)^{-\delta} \leq (|2p\pi - u| + 2\pi)^{-\delta} + (|2t' + 2p\pi + u| + 2\pi)^{-\delta}.$$

Using (129), (128) and the previous display, we obtain

$$\begin{aligned}\sup_{t\in\mathbb{R}} \sup_{t'\in\mathbb{R}} S(t, t', u) \leq & \sum_{p\in\mathbb{Z}} (|2p\pi| + 2\pi)^{-\delta} (|2p\pi - u| + 2\pi)^{-\delta} \\ & + \sup_{t'\in\mathbb{R}} \sum_{p\in\mathbb{Z}} (|2t' + 2p\pi| + 2\pi)^{-\delta} (|2t' + 2p\pi + u| + 2\pi)^{-\delta} \\ & + \sup_{t'\in\mathbb{R}} \sum_{p\in\mathbb{Z}} (|2p\pi| + 2\pi)^{-\delta} (|2t' + 2p\pi + u| + 2\pi)^{-\delta} \\ & + \sup_{t'\in\mathbb{R}} \sum_{p\in\mathbb{Z}} (|2t' + 2p\pi| + 2\pi)^{-\delta} (|2p\pi - u| + 2\pi)^{-\delta}.\end{aligned}$$

Since the three functions in t' appearing in the right-hand side of the last display are π-periodic the $\sup_{t'\in\mathbb{R}}$ can be replaced by $\sup_{|t'|\leq\pi/2}$. Since $|2t' + 2p\pi| \geq |2p\pi| - \pi$ and $|2t' + 2p\pi + u| \geq |2p\pi + u| - \pi$ for $|t'| \leq \pi/2$, we thus obtain

$$\sup_{t\in\mathbb{R}} \sup_{t'\in\mathbb{R}} S(t, t', u) \leq 4 \sum_{p\in\mathbb{Z}} (|2p\pi| + \pi)^{-\delta} (|2p\pi + u| + \pi)^{-\delta} \to 0 \quad \text{as } u \to \infty,$$

which conclude the proof. □

The following lemma will be used in the proof of Lemma 9.

Lemma 7. *Let p be a positive integer. For all $\mathbb{C}^p$-valued functions $\mathbf{g} \in L^2(-\pi, \pi)$ and $n \geq 1$, define*

$$M_n(\mathbf{g}) \stackrel{\text{def}}{=} \left\{ \sum_{k\in\mathbb{Z}} \left(1 - \frac{|k|}{n}\right)_+ |\mathbf{c}_k|^2 \right\}^{1/2} = \left\{ \sum_{k=-n+1}^{n-1} \left(1 - \frac{|k|}{n}\right) |\mathbf{c}_k|^2 \right\}^{1/2}, \tag{130}$$

where $\mathbf{c}_k = (2\pi)^{-1/2} \int_{-\pi}^{\pi} \mathbf{g}(\lambda) \mathrm{e}^{\mathrm{i}k\lambda} \mathrm{d}\lambda$ and $|\cdot|$ denotes the Euclidean norm in any dimension. Then, for all $\mathbf{g}_1$ and $\mathbf{g}_2$ in $L^2(-\pi, \pi)$,

$$|M_n(\mathbf{g}_1) - M_n(\mathbf{g}_2)| \leq \left(\int_{-\pi}^{\pi} |\mathbf{g}_1(\lambda) - \mathbf{g}_2(\lambda)|^2 \mathrm{d}\lambda \right)^{1/2}. \tag{131}$$

Moreover, for all $\mathbf{g}$ in $L^2(-\pi, \pi)$, as $n \to \infty$,

$$M_n(\mathbf{g}) \to \left(\int_{-\pi}^{\pi} |\mathbf{g}(\lambda)|^2 \mathrm{d}\lambda \right)^{1/2}. \tag{132}$$

Proof. See [8, Lemma 1 (Appendix B)]. □

The following lemmas are used to compute the limiting covariances (73) and (25).

Lemma 8. *Let $\{Z_{i,j,k}, i = 1, 2, j \geq 0, k \in \mathbb{Z}\}$ be an array of two-dimensional decimated linear processes as defined by (1). Assume* (A-2). *Then for all $j \geq 0$ and all $n \geq 1$, one has*

$$\frac{1}{n}\mathrm{Cov}\left(\sum_{k=0}^{n-1} Z_{1,j,k}^2, \sum_{k=0}^{n-1} Z_{2,j,k}^2\right) = 2A_j(n) + \kappa_4 B_j(n), \tag{133}$$

where

$$A_j(n) = \sum_{\tau=-n+1}^{n-1} (1 - |\tau|/n) \left(\sum_{u\in\mathbb{Z}} v_{1,j}(u) v_{2,j}(\gamma_j \tau + u)\right)^2 \tag{134}$$

and

$$B_j(n) = \sum_{u\in\mathbb{Z}} v_{1,j}^2(u) \sum_{\tau=-n+1}^{n-1} (1 - |\tau|/n) v_{2,j}^2(\gamma_j \tau + u). \tag{135}$$

Proof. Using a standard formula for cumulants of products, we have

$$\begin{aligned}
\mathrm{Cov}\left(\sum_{k=0}^{n-1} Z_{1,j,k}^2, \sum_{k=0}^{n-1} Z_{2,j,k}^2\right) &= \sum_{k=0}^{n-1}\sum_{k'=0}^{n-1} \mathrm{Cov}\left(Z_{1,j,k}^2, Z_{2,j,k'}^2\right) \\
&= 2\sum_{k=0}^{n-1}\sum_{k'=0}^{n-1} \mathrm{Cov}^2\left(Z_{1,j,k}, Z_{2,j,k'}\right) \\
&\quad + \sum_{k=0}^{n-1}\sum_{k'=0}^{n-1} \mathrm{cum}\left(Z_{1,j,k}, Z_{1,j,k}, Z_{2,j,k'}, Z_{2,j,k'}\right).
\end{aligned}$$

By definition of $\{Z_{i,j,k}, i = 1, 2, k \in \mathbb{Z}\}$ the covariance and the fourth-order cumulant in the previous display read respectively

$$\begin{aligned}
&\mathrm{Cov}\left(Z_{1,j,k}, Z_{2,j,k'}\right) = \sum_{t\in\mathbb{Z}} v_{1,j}(\gamma_j k - t) v_{2,j}(\gamma_j k' - t) \\
&\text{and} \quad \mathrm{cum}\left(Z_{1,j,k}, Z_{1,j,k}, Z_{2,j,k'}, Z_{2,j,k'}\right) = \kappa_4 \sum_{t\in\mathbb{Z}} v_{1,j}^2(\gamma_j k - t) v_{2,j}^2(\gamma_j k' - t).
\end{aligned}$$

Hence we obtain (133). □

Lemma 9. *Let $A_j(n)$ and $B_j(n)$ be defined by (134) and (135), respectively, and $v_{i,j}^*$ by (2). Then the following inequalities hold for all $j \geq 0$ and all $n \geq 1$:*

$$A_j(n) \leq 2\pi \int_{-\pi}^{\pi} \left| \gamma_j^{-1} \sum_{p=0}^{\gamma_j - 1} [v_{1,j}^* \overline{v_{2,j}^*}](\gamma_j^{-1}(\lambda + 2\pi p)) \right|^2 \mathrm{d}\lambda \tag{136}$$

$$B_j(n) \leq \int_{-\pi}^{\pi} \left| v_{1,j}^*(\lambda) \right|^2 \mathrm{d}\lambda \int_{-\pi}^{\pi} \left(\gamma_j^{-1} \sum_{p=0}^{\gamma_j - 1} \left| v_{2,j}^*(\gamma_j^{-1}(\lambda + 2\pi p)) \right| \right)^2 \mathrm{d}\lambda. \tag{137}$$

Suppose moreover that $\gamma_j \to \infty$ as $j \to \infty$, γ_j is an even integer for j large enough and that Condition (B) *holds for some $\lambda_{i,\infty} \in [0, \pi)$ and $\mathbb{R} \to \mathbb{Z}$ functions $v_{i,\infty}^*$, $i = 1, 2$. Then, as*

(n, j) jointly tends to (∞, ∞),

$$A_j(n) \to 2\pi \mathrm{C}_{1,2} \int_{-\pi}^{\pi} \left| \sum_{p\in\mathbb{Z}} w^*_{1,2}(\lambda + 2\pi p) \right|^2 \mathrm{d}\lambda, \tag{138}$$

where $\mathrm{C}_{1,2}$ and $w^*_{1,2}$ are defined in (18) and (17) respectively. Moreover,

$$\lim_{j\to\infty} \sup_{n\geq 1} \left|B_j(n)\right| = 0. \tag{139}$$

Proof. Step 1. Using properties of the convolution of square summable sequences, we have, for all $t \in \mathbb{Z}$,

$$\sum_{u\in\mathbb{Z}} v_{1,j}(u) v_{2,j}(t+u) = \int_{-\pi}^{\pi} v^*_{1,j}(\lambda) \overline{v^*_{2,j}(\lambda)} \mathrm{e}^{\mathrm{i}t\lambda} \mathrm{d}\lambda.$$

For any $\tau \in \mathbb{Z}$, applying Lemma 4 to the (2π)-periodic function $\lambda \mapsto v^*_{1,j}(\lambda)\overline{v^*_{2,j}(\lambda)}\mathrm{e}^{\mathrm{i}\gamma_j\tau\lambda}$, one gets

$$\sum_{u\in\mathbb{Z}} v_{1,j}(u) v_{2,j}(\gamma_j\tau+u) = (2\pi)^{-1/2} \int_{-\pi}^{\pi} \left((2\pi)^{1/2} \gamma_j^{-1} \sum_{p=0}^{\gamma_j-1} v^*_{1,j}(\xi_{j,p}(\lambda)) \overline{v^*_{2,j}(\xi_{j,p}(\lambda))} \right) \mathrm{e}^{\mathrm{i}\tau\lambda} \mathrm{d}\lambda,$$

$$\text{where } \xi_{j,p}(\lambda) \stackrel{\text{def}}{=} \gamma_j^{-1}(\lambda + 2\pi p). \tag{140}$$

Using the notation of Lemma 7, we can express $A_j(n)$ defined in (134) as

$$A_j(n) = \left(M_n(g_j)\right)^2 \tag{141}$$

$$\text{where} \quad g_j(\lambda) = (2\pi)^{1/2} \gamma_j^{-1} \sum_{p=0}^{\gamma_j-1} v^*_{1,j}(\xi_{j,p}(\lambda)) \overline{v^*_{2,j}(\xi_{j,p}(\lambda))}. \tag{142}$$

The bound (131) in Lemma 7 with $\mathbf{g}_1 = g_j$ and $\mathbf{g}_2 = 0$ thus gives (136).

Step 2. Let us now show (137). Since

$$v_{2,j}(\gamma_j\tau+u) = (2\pi)^{-1/2} \int_{-\pi}^{\pi} v^*_{2,j}(\lambda) \mathrm{e}^{\mathrm{i}\lambda(\gamma_j\tau+u)} \mathrm{d}\lambda,$$

we can apply Lemma 4 to the (2π)-periodic function $\lambda \mapsto v^*_{2,j}(\lambda)\mathrm{e}^{\mathrm{i}\lambda(\gamma_j\tau+u)}$ for all u and τ in $\mathbb{Z}$ and get

$$v_{2,j}(\gamma_j\tau+u) = (2\pi)^{-1/2} \int_{-\pi}^{\pi} \left(\gamma_j^{-1} \sum_{p=0}^{\gamma_j-1} v^*_{2,j}(\xi_{j,p}(\lambda)) \mathrm{e}^{\mathrm{i}u\xi_{j,p}(\lambda)} \right) \mathrm{e}^{\mathrm{i}\tau\lambda} \mathrm{d}\lambda.$$

Using the Parseval formula, we get, for all $u \in \mathbb{Z}$,

$$\sum_{\tau=-n+1}^{n-1} (1 - |\tau|/n) v^2_{2,j}(\gamma_j\tau+u) \leq \sum_{\tau\in\mathbb{Z}} v^2_{2,j}(\gamma_j\tau+u)$$

$$= \int_{-\pi}^{\pi} \left| \gamma_j^{-1} \sum_{p=0}^{\gamma_j - 1} v_{2,j}^*(\xi_{j,p}(\lambda)) e^{iu\xi_{j,p}(\lambda)} \right|^2 d\lambda$$

$$\leq \int_{-\pi}^{\pi} \left(\gamma_j^{-1} \sum_{p=0}^{\gamma_j - 1} \left| v_{2,j}^*(\xi_{j,p}(\lambda)) \right| \right)^2 d\lambda.$$

Observing that the resulting bound is independent of $u \in \mathbb{Z}$ and using the Parseval formula $\sum_{u\in\mathbb{Z}} v_{1,j}^2(u) = \int_{-\pi}^{\pi} \left| v_{1,j}^*(\lambda) \right|^2 d\lambda$, we obtain the bound (137) for $B_j(n)$ defined in (135).

Step 3. We now establish the limit (138) successively in the cases $\lambda_{1,\infty} \neq \lambda_{2,\infty}$ and $\lambda_{1,\infty} = \lambda_{2,\infty}$. The (2π)-periodicity of $v_{1,j}^* \overline{v_{2,j}^*}$ and Lemma 4 entail that g_j (defined in (142)) is (2π)-periodic. By definition of M_n in Lemma 7, it follows that, for any $j \geq 0$ and any $\tau \in \mathbb{R}$,

$$M_n(g_j) = M_n(g_j^{(\tau)}) \quad \text{with } g_j^{(\tau)}(\lambda) = g_j(\lambda - \tau), \lambda \in \mathbb{R}, \tag{143}$$

since the modulus of the Fourier coefficients of g_j and $g_j^{(\tau)}$ are equal. In the following we will take $\tau = \pi\gamma_j$. Observe that, for all $p \in \{0, \ldots, \gamma_j - 1\}$, $\lambda \in (0, 2\pi)$ and $j \geq 0$,

$$\xi_{j,p}(\lambda - \pi\gamma_j) \in (-\pi, \pi). \tag{144}$$

Consider the case where $\lambda_{1,\infty} \neq \lambda_{2,\infty}$, which, by (4), implies

$$\gamma_j |\lambda_{1,j} - \lambda_{2,j}| \to \infty \quad \text{as } j \to \infty. \tag{145}$$

Using (142)–(144) and (12), we have, for some constant $C > 0$, for all $j \geq 0$,

$$\begin{aligned} \sup_{\lambda\in(0,2\pi)} \left| g_j^{(\pi\gamma_j)}(\lambda) \right| &= \sup_{\lambda\in(0,2\pi)} \left| g_j(\lambda - \pi\gamma_j) \right| \\ &\leq C \sum_{p=0}^{\gamma_j - 1} \prod_{i=1,2} \left(1 + \gamma_j \left| |\xi_{j,p}(\lambda - \pi\gamma_j)| - \lambda_{i,j} \right| \right)^{-\delta} \\ &\leq C \sup_{t,t'\in\mathbb{R}} \sum_{p\in\mathbb{Z}} \left(1 + \left| |t + 2\pi p| - t' \right| \right)^{-\delta} \\ &\quad \times \left(1 + \left| |t + 2\pi p| - t' - \gamma_j |\lambda_{1,j} - \lambda_{2,j}| \right| \right)^{-\delta} \\ &\to 0 \quad \text{as } j \to \infty, \end{aligned}$$

by (145) and (124) in Lemma 6. Applying (141), (143) and the bound (131) in Lemma 7 with $\mathbf{g}_1 = g_j$ and $\mathbf{g}_2 = 0$ yields

$$A_j(n) = \left(M_n(g_j^{(\pi\gamma_j)}) \right)^2 \leq \int_{-\pi}^{\pi} \left| g_j^{(\pi\gamma_j)} \right|^2 d\lambda.$$

The last two displays and the (2π)-periodicity of g_j imply $A_j(n) \to 0$ as $j \to \infty$. This proves (138) since by (18), $C_{1,2} = 0$ when $\lambda_{1,\infty} \neq \lambda_{2,\infty}$.

We now consider the case $\lambda_{1,\infty} = \lambda_{2,\infty}$. By Conditions (6) and (9), we have $\lambda_{1,j} = \lambda_{2,j}$ and $\Phi_{1,j} \equiv \Phi_{2,j}$ for j large enough. Let $p_j = \gamma_j \lambda_{1,j}/(2\pi) + \gamma_j/2$ so that

$$\lambda_{1,j} = \lambda_{2,j} = 2\pi\gamma_j^{-1} p_j - \pi. \tag{146}$$

By Condition (3) and since γ_j is even for j large enough by assumption, we get that p_j is an integer for j large enough. Writing

$$\sum_{p=0}^{\gamma_j-1} = \sum_{p=0}^{[\gamma_j/2]-1} + \sum_{p=[\gamma_j/2]}^{\gamma_j-1} = \sum_{q=-(\gamma_j-p_j)}^{[\gamma_j/2]-(\gamma_j-p_j)-1} + \sum_{r=[\gamma_j/2]-p_j}^{\gamma_j-p_j-1},$$

where $q = p - (\gamma_j - p_j)$ and $r = p - p_j$ and observe that, with these definitions, (140) and (146), we have $\xi_{j,q}(\lambda) = \gamma_j^{-1}(\lambda + 2\pi q) = \gamma_j^{-1}[(\lambda - \pi\gamma_j) + 2\pi p] + 2\pi\gamma_j^{-1}p_j - \pi = \xi_{j,p}(\lambda - \pi\gamma_j) + \lambda_{1,j}$, and, similarly, $\xi_{j,q}(\lambda) = \xi_{j,p}(\lambda - \pi\gamma_j) - \lambda_{1,j}$, so $g_j^{(\pi\gamma_j)}(\lambda) = g_j(\lambda - \pi\gamma_j)$ defined in (143) and (142) can be expressed as

$$g_j^{(\pi\gamma_j)}(\lambda) = (2\pi)^{1/2}\left[\sum_{q=-(\gamma_j-p_j)}^{[\gamma_j/2]-(\gamma_j-p_j)-1} \frac{v_{1,j}^*\overline{v_{2,j}^*}}{\gamma_j}(\xi_{j,q}(\lambda) - \lambda_{1,j}) + \sum_{r=[\gamma_j/2]-p_j}^{\gamma_j-p_j-1} \frac{v_{1,j}^*\overline{v_{2,j}^*}}{\gamma_j}(\xi_{j,r}(\lambda) + \lambda_{1,j})\right]. \tag{147}$$

Since $\lim_{j\to\infty}\gamma_j = \infty$ and, by Condition (4), $\lim_{j\to\infty}\lambda_{i,j} = \lambda_{i,\infty} \in [0,\pi)$, we have $\lim_{j\to\infty}\gamma_j^{-1}p_j \in [1/2, 1)$ and thus

$$-(\gamma_j - p_j) \to -\infty \quad \text{and} \quad \gamma_j - p_j - 1 \to \infty, \tag{148}$$

namely, in (147), the upper limit of the first sum tends to ∞ and the bottom limit of the second sum tends to $-\infty$. We now consider the remaining limits. If $\lambda_{1,\infty} = \lambda_{2,\infty} > 0$, then $\lim_{j\to\infty}\gamma_j^{-1}p_j$ falls in the open interval $(1/2, 1)$ and thus

$$[\gamma_j/2] - (\gamma_j - p_j) - 1 \to \infty \quad \text{and} \quad [\gamma_j/2] - p_j \to -\infty. \tag{149}$$

If $\lambda_{1,\infty} = \lambda_{2,\infty} = 0$, using (5), (146) implies $p_j = \gamma_j/2$ and thus, for j large enough so that p_j is integer valued and γ_j even,

$$[\gamma_j/2] - (\gamma_j - p_j) - 1 = -1 \quad \text{and} \quad [\gamma_j/2] - p_j = 0. \tag{150}$$

In view of (140), Conditions (6)–(9) (which imply (10), (12) and (13)), (148)–(150) and dominated convergence yield, for all $\lambda \in (0, 2\pi)$,

$$g_j^{(\pi\gamma_j)}(\lambda) \to g_\infty(\lambda) \quad \text{as } j \to \infty, \tag{151}$$

where

$$g_\infty(\lambda) = (2\pi)^{1/2}\left[\sum_{q\in\mathbb{Z}}[\overline{v_{1,\infty}^*}v_{2,\infty}^*](-\lambda - 2\pi q) + \sum_{r\in\mathbb{Z}}[v_{1,\infty}^*\overline{v_{2,\infty}^*}](\lambda + 2\pi r)\right]$$

if $\lambda_{1,\infty} = \lambda_{2,\infty} > 0$, and

$$g_\infty(\lambda) = (2\pi)^{1/2}\left[\sum_{q=-\infty}^{-1}[\overline{v_{1,\infty}^*}v_{2,\infty}^*](-\lambda - 2\pi q) + \sum_{r=0}^{\infty}[v_{1,\infty}^*\overline{v_{2,\infty}^*}](\lambda + 2\pi r)\right]$$

if $\lambda_{1,\infty} = \lambda_{2,\infty} = 0$. By definition of w^* in (17) and using (14), one has $[\overline{v^*_{1,\infty}} v^*_{2,\infty}](-\lambda) = [v^*_{1,\infty} \overline{v^*_{2,\infty}}](\lambda) = w^*_{1,2}(\lambda)$; the two previous displays read

$$g_\infty(\lambda) = (2\pi)^{1/2} \mathrm{C}_{1,2} \sum_{p\in\mathbb{Z}} w^*_{1,2}(\lambda + 2\pi p). \tag{152}$$

Conditions (7) and (8) imply (15), and thus that $g_\infty(\lambda)$ is bounded, and hence square integrable on $\lambda \in (-\pi, \pi)$. Moreover, applying the same dominated convergence argument as above, one has

$$\lim_{j\to 0} \int_{-\pi}^{\pi} \left| g_j^{(\pi\gamma_j)}(\lambda) - g_\infty(\lambda) \right|^2 \mathrm{d}\lambda = 0. \tag{153}$$

One gets by (143), (131) and (153)

$$\left| M_n(g_j) - M_n(g_\infty) \right|^2 \le \int_{-\pi}^{\pi} \left| g_j(\lambda) - g_\infty(\lambda) \right|^2 \mathrm{d}\lambda \to 0 \quad \text{as } j \to \infty. \tag{154}$$

By applying the limit (132) with $\mathbf{g} = g_\infty$, one gets

$$M_n(g_\infty)^2 \to \int_{-\pi}^{\pi} |g_\infty(\lambda)|^2 \mathrm{d}\lambda \tag{155}$$

as $n \to \infty$. Hence, setting $M_n(g_j) = \left(M_n(g_j) - M_n(g_\infty)\right) + M_n(g_\infty)$, the limit (138) follows from (141), (152), (154) and (155).

Step 4. We now establish the limit (139). By Condition (7), we have

$$\sup_{j\ge 0} \int_{-\pi}^{\pi} \left| v^*_{1,j}(\lambda) \right|^2 \mathrm{d}\lambda < \infty. \tag{156}$$

Using arguments similar to those above and Condition (7), we have

$$\begin{aligned}
&\int_{-\pi}^{\pi} \left(\gamma_j^{-1} \sum_{p=0}^{\gamma_j - 1} \left| v^*_{2,j}(\xi_{j,p}(\lambda)) \right| \right)^2 \mathrm{d}\lambda \\
&= \gamma_j^{-1} \int_0^{2\pi} \left(\sum_{p=0}^{\gamma_j - 1} \gamma_j^{-1/2} \left| v^*_{2,j}(\xi_{j,p}(\lambda - \pi\gamma_j)) \right| \right)^2 \mathrm{d}\lambda \\
&\le C\gamma_j^{-1} \int_0^{2\pi} \left(\sum_{p=0}^{\gamma_j - 1} \left(1 + \left| |\lambda + 2\pi(p - \gamma_j/2)| - \gamma_j \lambda_{2,j} \right| \right)^{-\delta} \right)^2 \mathrm{d}\lambda.
\end{aligned} \tag{157}$$

Using that $||a + b| - c| \ge ||b| - c| - |a|$ and $\lambda \in [0, 2\pi]$, we have

$$\left| |\lambda + 2\pi(p - \gamma_j/2)| - \gamma_j \lambda_{2,j} \right| \ge \left| 2\pi |p - \gamma_j/2| - \gamma_j \lambda_{2,j} \right| - 2\pi.$$

Take j large enough so that γ_j is even. Since $\lambda_{2,j} \in [0, \pi)$, as $p \in \{0, \dots, \gamma_j - 1\}$, $2\pi|p - \gamma_j/2| - \gamma_j \lambda_{2,j}$ is a sequence of numbers with lag 2π and belonging to $[-\gamma_j \pi, \gamma_j \pi]$ and can thus be written as a sequence $2\pi q + c$, where q belongs to $\{-\gamma_j/2, \dots, \gamma_j/2\}$ and c to $[-\pi, \pi]$ so that

$$\left| 2\pi |p - \gamma_j/2| - \gamma_j \lambda_{2,j} \right| - 2\pi \ge 2\pi |q| - 3\pi.$$

From the last two displays, we have

$$\left(5\pi + \left||\lambda + 2\pi(p - \gamma_j/2)| - \gamma_j\lambda_{2,j}\right|\right)^{-\delta} \le (2\pi)^{-\delta}(1 + |q|)^{-\delta},$$

with q describing $\{-\gamma_j/2, \dots, \gamma_j/2\}$ as p describes $\{0, \dots, \gamma_j - 1\}$. Inserting this bound in (157), we get

$$\int_{-\pi}^{\pi} \left(\gamma_j^{-1} \sum_{p=0}^{\gamma_j - 1} \left|v_{2,j}^*(\xi_{j,p}(\lambda))\right|\right)^2 \mathrm{d}\lambda \le C\gamma_j^{-1} \left(\sum_{q=0}^{\gamma_j/2} (1+q)^{-\delta}\right)^2$$

for some constant C not depending on $j \ge 0$. Since the last right-hand side of the previous display tends to 0 as $j \to \infty$ for any $\delta > 1/2$, with (156) and (137), we obtain (139). □

Remark 16. The factor $4\pi = 2 \times 2\pi$ in (25) is due to the factor 2 in the right-hand side of (133) and the presence of 2π in the right-hand side of (138).

Corollary 1. *Let $\{Z_{i,j,k}, i = 1, 2, j \ge 0, k \in \mathbb{Z}\}$ be an array of two-dimensional decimated linear processes as defined by (1). Assume (A-2), that γ_j is even for j large enough and that Condition (B) holds for some $\lambda_{i,\infty} \in [0, \pi)$ and $\mathbb{R} \to \mathbb{Z}$ functions $v_{i,\infty}^*$, $i = 1, 2$. Then, for all $k, k' \in \mathbb{Z}$, as $j \to \infty$,*

$$\mathrm{Cov}\left(Z_{1,j,k}^2, Z_{2,j,k'}^2\right) \to 2\mathrm{C}_{1,2}^2 \left(\int_{-\infty}^{\infty} w_{1,2}^*(\lambda)\mathrm{e}^{\mathrm{i}\lambda(k'-k)}\mathrm{d}\lambda\right)^2, \tag{158}$$

where $\mathrm{C}_{1,2}$ and $w_{1,2}^*$ are defined in (18) and (17), and, as (n, j) jointly tends to (∞, ∞),

$$\frac{1}{n}\mathrm{Cov}\left(\sum_{k=0}^{n-1} Z_{1,j,k}^2, \sum_{k=0}^{n-1} Z_{2,j,k}^2\right) \to 4\pi\mathrm{C}_{1,2} \int_{-\pi}^{\pi} \left|\sum_{p\in\mathbb{Z}} w_{1,2}^*(\lambda + 2\pi p)\right|^2 \mathrm{d}\lambda. \tag{159}$$

Proof. Setting $n = 1$ in (133) and (134) and replacing $v_{1,j}(-t)$ by $v_{1,j}(\gamma_j k - t)$ and $v_{2,j}(-t)$ by $v_{2,j}(\gamma_j k' - t)$ so that $Z_{1,j,0}$ is replaced by $Z_{1,j,k}$ and $Z_{2,j,0}$ by $Z_{2,j,k'}$, we get

$$\begin{aligned}\mathrm{Cov}\left(Z_{1,j,k}^2, Z_{2,j,k'}^2\right) &= 2A_j(1) + \kappa_4 B_j(1)\\ &= 2\left(\sum_{t\in\mathbb{Z}} v_{1,j}(\gamma_j k - t)v_{2,j}(\gamma_j k' - t)\right)^2 + \kappa_4 B_j(1),\end{aligned}$$

and thus (158) follows from (139), (19) and (16).

Relation (159) is obtained by applying Lemmas 8 and 9. □

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References

[1] R.J. Bhansali, L. Giraitis, P.S. Kokoszka, Approximations and limit theory for quadratic forms of linear processes, Stochastic Process. Appl. 117 (1) (2007) 71–95.

[2] F. Roueff, M.S. Taqqu, Asymptotic normality of wavelet estimators of the memory parameter for linear processes, Technical report, HAL/arXiv, 2008.

[3] G. Faÿ, E. Moulines, F. Roueff, M.S. Taqqu, Estimators of long-memory: Fourier versus wavelets, J. Econometrics (2009), in press (doi:10.1016/j.jeconom.2009.03.005).

[4] S. Mallat, A Wavelet Tour of Signal Processing, Academic Press Inc., San Diego, CA, 1998.

[5] T.W. Anderson, The Statistical Analysis of Time Series, John Wiley & Sons Inc., New York, 1994.

[6] G. Lang, P. Soulier, Convergence de mesures spectrales aléatoires et applications à des principes d'invariance, in: 19th Rencontres Franco-Belges de Statisticiens (Marseille, 1998), Stat. Inference Stoch. Process. 3 (1–2) (2000) 41–51.

[7] P. Billingsley, Convergence of Probability Measures, second edition, in: Wiley Series in Probability and Statistics: Probability and Statistics, John Wiley & Sons Inc., New York, 1999. A Wiley-Interscience Publication.

[8] E. Moulines, F. Roueff, M. Taqqu, On the spectral density of the wavelet coefficients of long memory time series with application to the log-regression estimation of the memory parameter, J. Time Ser. Anal. 28 (2) (2007) 155–187.