Quasineutral limit of a time-dependent drift–diffusion–Poisson model for p-n junction semiconductor devices

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Abstract

In this paper the vanishing Debye length limit of the bipolar time-dependent drift–diffusion–Poisson equations modelling insulated semiconductor devices with p-n junctions (i.e., with a fixed bipolar background charge) is studied. For sign-changing and smooth doping profile with ‘good’ boundary conditions the quasineutral limit (zero-Debye-length limit) is performed rigorously by using the multiple scaling asymptotic expansions of a singular perturbation analysis and the carefully performed classical energy methods. The key point in the proof is to introduce a ‘density’ transform and two $\lambda$-weighted Liapunov-type functionals first, and then to establish the entropy production integration inequality, which yields the uniform estimate with respect to the scaled Debye length. The basic point of the idea involved here is to control strong nonlinear oscillation by the interaction between the entropy and the entropy dissipation.

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1. Introduction and formal asymptotics

In this paper we study the quasineutral limit of a singularly perturbed mixed system of parabolic and elliptic equations modelling p-n junction devices. We consider an insulated semiconductor with the generally physical doping profile of p-n junctions including a semiconductor (for example, germanium) which is doped with donor atoms (positive ions) in one side and with acceptor atoms (negative ions) in another side of the device. The physics of the p-n junctions are explained by Sze in [33] and Smith in [32]. The (scaled) equations governing the potential (or the electric field) distribution, the electron and hole densities are given in the case of one space dimension (see Van Roosbroeck [27]) as follows:

$$\begin{align*}
n_\lambda^t &= (n_\lambda^x + n_\lambda^E^\lambda^E^\lambda)_x, \quad 0 < x < 1, \quad t > 0, \\
p_\lambda^t &= (p_\lambda^x - p_\lambda^E^\lambda^E^\lambda)_x, \quad 0 < x < 1, \quad t > 0, \\
-\lambda^2 E_\lambda^x &= n_\lambda^x - p_\lambda^E^\lambda^E^\lambda - D, \quad 0 < x < 1, \quad t > 0, \\
E_\lambda^x &= -\Phi_\lambda^x.
\end{align*}$$

System (1)–(3) is subject to the boundary and initial conditions:

$$\begin{align*}
n_\lambda^x \big|_{x=0} &= n_\lambda^0(x), \quad p_\lambda^x \big|_{x=0} = p_\lambda^0(x), \quad 0 \leq x \leq 1. 
\end{align*}$$

The unknowns $n_\lambda^x$, $p_\lambda^x$, $E_\lambda^x$ or $\Phi_\lambda^x$ are the electron density, the hole density, the electric field or the electric potential, respectively. The characteristic length of the device is scaled to be unit. The parameter $\lambda$ is the scaled Debye length. $D = D(x)$ is the given function of space and models the doping profile (i.e., the preconcentration of electrons and holes). Because of the physical background of realistic p-n junction in semiconductor device, the physical doping profile $D(x)$ has the property that $D(x)$ changes sign.

In this paper, we assume that $D(x)$ is a smooth $C^4$ function satisfying $D_x(x = 0, 1) = D_{xxx}(x = 0, 1) = 0$.

A necessary solvability condition for the Poisson equation (3) subject to the Neumann boundary conditions for the field in (4) is global space charge neutrality,

$$\int_0^1 (n_\lambda^x - p_\lambda^E^\lambda^E^\lambda - D) \, dx = 0.$$
Since the total numbers of electrons and holes are conserved, it is sufficient to require the corresponding condition for the initial data:

$$\int_0^1 (n_0^\lambda - p_0^\lambda - D) \, dx = 0.$$  \hspace{1cm} (6)

Usually semiconductor physics are concerned with large scales structure with respect to the Debye length $\lambda$ (for $\lambda$ takes small values, typically $\lambda^2 \approx 10^{-7}$). For such scales, the semiconductor is electrically neutral, i.e., there is no space charge separation or electric field. This is so-called quasineutrality assumption to semiconductors or plasma physics, which had been applied by Shockley [30] in the first theoretical studies of semiconductor devices in 1949, but also in other contexts such as the modelling of plasmas [31] and ionic membranes [28]. Under the assumption of space charge neutrality, i.e., $\lambda = 0$, we formally arrive at the following quasineutral drift–diffusion model

$$n_t = (n_x + nE)_x, \hspace{1cm} (7)$$

$$p_t = (p_x - pE)_x, \hspace{1cm} (8)$$

$$0 = n - p - D, \hspace{1cm} (9)$$

$$E = -\Phi_x.$$

This formal limit was obtained by Roosbroeck [27] in 1950. For further formal asymptotic analysis, see [19,24,26].

Generally speaking, it should be expected at least formally that $(n^\lambda, p^\lambda, E^\lambda) \to (n, p, E)$ as $\lambda \to 0$ in the interior of interval $[0, 1]$ while it cannot be a priori expected that all boundary and initial value conditions hold for the limit problem because of the singular perturbation character of the problem (the Poisson equation becomes an algebraic equation in the limit). However, by the conservation form of the continuity equations the property of zero flux through the boundary will prevail in the limit:

$$(n_x + nE)(x = 0, 1) = 0, \hspace{1cm} (p_x - pE)(x = 0, 1) = 0$$ \hspace{1cm} (10)

while the boundary condition for the electric field $E^\lambda$ does not except for some special cases.

Similarly, we can a priori expect that quasineutral drift–diffusion models (7)–(9) is supplemented by the following initial data:

$$n(t = 0) = n_0, \hspace{1cm} p(t = 0) = p_0$$ \hspace{1cm} (11)

satisfying space charge neutrality locally

$$n_0 - p_0 - D = 0.$$ \hspace{1cm} (12)

The aim of this paper is to justify rigorously the above formal limit to sufficiently smooth solutions for small time as well as large time.

It is important to mention that the quasineutral limit is a well-known challenging and physically very complex modelling problem for the (bipolar) fluid-dynamic models and for the kinetic
models of semiconductors and plasmas. In both cases there exist only partial results. In particular, for time-dependent transport models, the limit $\lambda \to 0$ has been performed in Vlasov–Poisson system by Brenier [2], Grenier [12,13] and Masmoudi [20], in Schrödinger–Poisson system by Puel [25] and Jüngel and Wang [16], in drift–diffusion–Poisson system by Gasser et al. [10,11], Jüngel and Peng [15] and Schmeiser and Wang [29], and in Euler–Poisson system by Cordier and Grenier [5,6], Cordier et al. [4] and Wang [34], respectively. However, all these results are restricted to the special case of the non-physical doping profile, i.e., either assuming that $D(x)$ is constant, or assuming that $D(x)$ does not change the sign. For physically interesting doping profile, i.e., for the case where the doping profile can change its sign, there is no rigorous result available for time-dependent semiconductor models both to fluid-dynamic models and to the kinetic models up to now. Therefore, it is natural to study the quasineutral limit on the level of the drift–diffusion–Poisson models. For the stationary drift–diffusion–Poisson models, rigorous convergence results for p-n junction devices with contacts can be found in Markowich [18] and recent extensions [18] were given by Carfarelli et al. [3] and Dolbeault et al. [7].

In this paper we consider quasineutral limit of time-dependent drift–diffusion–Poisson models (1)–(6) for semiconductors with the physically interesting doping profile. To provide some insight into the rigorous justification of the quasineutrality assumptions, here we only consider the sign-changing smooth doping profile with ‘good’ boundary conditions. More general case (formal asymptotic requires the matching of the two time scales (slow and fast spatial) at the boundary and hence further complicated techniques is required) will be discussed in another coming paper.

Let us summary up our results of this paper as follows: The convergence of drift–diffusion–Poisson models (1)–(6) to (7)–(12) is rigorously proven under some restrictive assumptions:

(i) Jump discontinuities in the doping profile $D(x)$ are excluded by our smoothness assumption;
(ii) $D_x(x = 0, 1) = D_{xxx}(x = 0, 1) = 0$;
(iii) An insulated piece of semiconductor without contacts is considered.

Although assumption (i) still excludes some cases with physically practical interest, but it does contain some p-n junction devices for semiconductor with bipolar space charge background. Hence our result is the first rigorous quasineutrality result for time-dependent semiconductor model with physically bipolar fixed charge background.

Assumptions (ii) and (iii) are the technical ones, but physically this is feasible. The two conditions are assumed to avoid the complicated structure of the solution near the boundary, i.e., the occurrence of the boundary layer. But, we allow the presence to another kind of layers—initial time layers, where the solutions varying on the fast dielectric relaxation time scale are responsible to the connection between initial conditions and the quasineutral inner problem. This is because we do not assume the locally space charge neutral initial condition any more. Even so, here we only treat with an isolated and one-dimensional semiconductor model problem. However we believe that the tools developed in this paper are also able to be applied to the analysis for more complicated and realistic small Debye length limit problems.

We mention that one of the main difficulties in dealing with quasineutral limit is the oscillatory behavior of the electric field. Usually it is difficult to obtain the uniform estimates on the electric field with respect to the Debye length $\lambda$.

For the stationary case, the system (1)–(6) can be reduced to pure elliptic system [18] or single elliptic equation [4,7] and then the variation structure of elliptic equations gives the desired
uniform estimates with respect to $\lambda$. However, for time-dependent case, this is completely different. Usually, strongly connected with quasineutrality is the presence of the two different time scales—the slow diffusion time scale and the fast dielectric relaxation time scale, see formal analysis results [26]. The existence of the fast time scale yields the oscillation of the solution, in particular, the electric field, in the time direction. In this case, it is hard to establish the a priori estimates uniformly with respect to $\lambda$. Recently, entropy method is successively used by Gasser, Levermore, Markowich and Schmeiser in [11] to obtain uniform estimates of the electric field. Their essential assumption is that sign does not change in the doping profile and, thus, $p$-n junctions are excluded, which guarantees the validity of the entropy method. But, this eliminates the cases of great practical interest. In this paper, we remove this ‘pure’ mathematical assumption and obtain rigorous convergence results. The key point in establishing uniform estimates with respect to the scaled Debye length here is to introduce a ‘density’ transform and two $\lambda$-weighted Liapunov-type functionals, and then to derive a new entropy production integration inequality (containing an integration of the entropy and entropy-dissipation’s production, see below (56)).

Our basic idea in the proof is to control the strong nonlinear oscillation caused by the small Debye length $\lambda \to 0$ by the interaction of the physically motivated entropy and the entropy dissipation.

Finally, we should mention that for the related classical drift–diffusion–Poisson models there have been many results about existence, uniqueness, large time asymptotic behaviors, stability of stationary state and regularities of weak solutions, etc., for example, see [1, 8, 9, 14, 19, 21–23].

The paper is organized as follows. In Section 2 we give the main results of this paper. In Section 3 we discuss the existence and regularity of the solutions to quasineutral drift–diffusion models. Sections 4 and 5 are devoted to the proofs of the main theorems of this paper.

2. Reformulation of the equations and main results

Introduce the new variables $(z^\lambda, E^\lambda)$ by the following transform

$$E^\lambda = -\Phi^\lambda_x, \quad n^\lambda = \frac{z^\lambda + D - \lambda^2 E^\lambda_x}{2}, \quad p^\lambda = \frac{z^\lambda - D + \lambda^2 E^\lambda_x}{2} \quad (z^\lambda = n^\lambda + p^\lambda). \quad (13)$$

Then we can reduce the initial boundary value problem (1)–(6) to the following equivalent system

$$z^\lambda_t = (z^\lambda_x + D E^\lambda_x)_x - \lambda^2 (E^\lambda_x E^\lambda_x)_x, \quad 0 \leq x \leq 1, \quad t > 0, \quad (14)$$

$$\lambda^2 (E^\lambda_t - E^\lambda_x) = -(D_x + z^\lambda E^\lambda), \quad 0 \leq x \leq 1, \quad t > 0, \quad (15)$$

$$z^\lambda_x = E^\lambda = 0, \quad x = 0, 1, \quad t > 0, \quad (16)$$

$$z^\lambda(t = 0) = z^\lambda_0, \quad E^\lambda(t = 0) = E^\lambda_0, \quad 0 \leq x \leq 1. \quad (17)$$

Note that the equivalence between system (1)–(6) and system (14)–(17) is easy to be verified for classical solutions. Thus, we have:

**Proposition 1 (Existence and uniqueness).** Assume that $(z^\lambda_0, E^\lambda_0) \in (C^2)^2$ satisfying compatibility conditions

$$z^\lambda_{0,x} = E^\lambda_0 = E^\lambda_{0,xx} = 0, \quad \text{at} \ x = 0, 1.$$

Then system (14)–(17) has a unique classical solution $(z^\lambda, E^\lambda) \in C^{2,1}([0, 1] \times [0, T]).$
Remark 1. The existence in Proposition 1 is obtained by known existence results for (1)–(6), see, for example, [9,23], and by transform (13). The uniqueness in Proposition 1 is easily be proven for any $H^1$-solution to (14)–(17).

Similarly, by the transform
\[
\begin{align*}
n & = \frac{Z + D}{2}, \quad p = \frac{Z - D}{2}, \quad \xi = -\Phi_x,
\end{align*}
\] the formal limit problem (7)–(12) is reduced to the following equivalent systems:
\[
\begin{align*}
Z_t &= (Z_x + D\xi)_x, \quad 0 < x < 1, \quad t > 0, \\
0 &= -(D_x + Z\xi), \quad 0 < x < 1, \quad t > 0, \\
Z_x + D\xi &= 0, \quad x = 0, 1, \quad t > 0, \\
Z(t = 0) &= Z_0, \quad 0 \leq x \leq 1.
\end{align*}
\] For this limit systems, we have the following existence and regularity results:

**Proposition 2 (Existence and regularity).** Let $E(x, t = 0) = \frac{D_x(x)}{Z_0(x)}$ and $Z_0(x) \geq \delta_0 > 0$ for some constant $\delta_0$. Assume that $D(x) \in C^4$ satisfying $D_x(x = 0, 1) = 0$ and that $Z_0 \in C^3$ satisfying the compatibility conditions
\[
\begin{align*}
Z_{0x}(x) &= 0, \\
(Z_{0x}(x) + D(x)\xi(x, t = 0))_{xx}(x) &= 0, \quad \text{at } x = 0, 1.
\end{align*}
\] Then the limit problem (19)–(22) has a unique solution $(Z, \xi) \in C^{3,\frac{3}{2}}([0, 1] \times [0, T])$, well-defined in $[0, T]$ for some $T > 0$, satisfying $Z \geq \delta > 0$ for some constant $\delta$. Moreover, if $Z_0$ is suitably large, then $T = +\infty$.

The equivalent or more general form of Proposition 2 will be given in Section 3.

Now we give our main results in this paper.

**Theorem 3 (The case of well-prepared initial data).** Let $(z^\lambda, E^\lambda)$ and $(Z, \xi)$ be the classical solutions of the problem (14)–(17) and the problem (19)–(22), respectively. Let all assumptions of Proposition 2 hold and $T$ be given by Proposition 2. Also, assume that $D_x(x = 0, 1) = D_{xxx}(x = 0, 1) = 0$ and that $Z_0(x) \geq \delta_0 > 0$ and $E_0(x) = -\frac{D_x(x)}{Z_0(x)}$, and that there exist a constant $M$ and an $\alpha > 0$ such that, for any $\lambda > 0$
\[
\begin{align*}
\|z_0^\lambda - Z_0(\cdot)\|_{H^1([0, 1])}^2 + \lambda^2 \|(E_0^\lambda - E_0(\cdot))\|_{H^1([0, 1])}^2 & \leq M\lambda^{\min\{\alpha, 4\}}.
\end{align*}
\] Then there exists an $\lambda_0$: $0 < \lambda_0 \ll 1$, depending upon $T$, such that, for any $\lambda$: $0 < \lambda \leq \lambda_0$,
\[
\sup_{0 \leq t \leq T} \|(z^\lambda - Z, \lambda(E^\lambda - \xi))(\cdot, t)\|_{H^1([0, 1])}^2 \leq M\lambda^{\min\{\alpha, 4\} - \delta}
\] for any $\delta \in (0, \min\{\alpha, 4\})$. 

Theorem 4 (The case of ill-prepared initial data). Let \((z^\delta, E^\delta)\) and \((\mathcal{Z}, \mathcal{E})\) be the classical solutions of the problem (14)–(17) and the problem (19)–(22), respectively. Let all assumptions of Proposition 2 hold and \(T\) be given by Proposition 2. Assume that \(D_x(x = 0, 1) = D_{xxx}(x = 0, 1) = 0\). Let \(E_0(x) \in C^3\) be any given function satisfying the compatibility conditions for (14)–(17), i.e., \(E_0(x = 0, 1) = E_{0,xx}(x = 0, 1) = 0\). Assume that there exist a constant \(M\) and an \(\alpha > 0\) such that, for any \(\lambda > 0\),
\[
\|z_0^\delta - Z_0(\cdot)\|_{H^1([0,1])}^2 + \lambda^2 \|E_0^\delta - E_0(\cdot)\|_{H^1([0,1])}^2 \leq M\lambda^{\min[\alpha,4]}.
\]
Then there exists an \(\lambda_0 > 0\) such that, for any \(\lambda : 0 < \lambda < \lambda_0\),
\[
\sup_{0 \leq t \leq T} \|\left( z^\delta - \mathcal{Z} - \lambda^2 z^\delta, \lambda\left( E^\delta - \mathcal{E} - E_1^\delta \right) \right)(\cdot, t) \|_{H^1([0,1])}^2 \leq M\lambda^{\min[\alpha,4]-\delta}
\]
for any \(\delta \in (0, \min[\alpha,4])\), where \((z^\delta_t(x, s), E^\delta_t(x, s))\) solves the problem
\[
\begin{align*}
(z^\delta_t)_s &= (D(x)E^\delta_t)_x, \quad (25) \\
E^\delta_t|_{s=0} &= -Z_0(x)E^\delta, \quad (26) \\
z^\delta|_{s=0} &= 0, \quad (27) \\
E^\delta|_{s=0} &= E_0 - \mathcal{E}(t = 0). \quad (28)
\end{align*}
\]
Here we had used the fast scaling time \(s\) to denote \(\frac{t}{\lambda^2}\).

If one comes back to the original problem (1)–(5), we have the following equivalent statements of Theorems 3 and 4.

Theorem 5 (The case of well-prepared initial data). Let \((n^\lambda, p^\lambda, E^\lambda)\) and \((n, p, \mathcal{E})\) be the classical solutions of the problem (1)–(6) and the problem (7)–(12) with \(n_0, p_0 \in C^3\), respectively. Let \(T\) be given by Proposition 2 with \(Z_0(x) = n_0(x) + p_0(x)\). Also, assume that \(D_x(x = 0, 1) = D_{xxx}(x = 0, 1) = 0\) and that \(E_0(x) = -\frac{D_x(x)}{(n_0 + p_0)(x)}\), and that there exist a constant \(M\) and an \(\alpha > 0\) such that, for any \(\lambda > 0\),
\[
\|\left( n_0^\lambda - n_0, p_0^\lambda - p_0 \right)(\cdot) \|_{H^1([0,1])}^2 + \lambda^2 \|E^\delta(t = 0) - E_0(\cdot)\|_{H^1([0,1])}^2 \leq M\lambda^{\min[\alpha,4]}.
\]
Then there exists an \(\lambda_0 > 0\) depending upon \(T\), such that, for any \(\lambda : 0 < \lambda < \lambda_0\),
\[
\sup_{0 \leq t \leq T} \left( \|\left( n^\lambda - n, p^\lambda - p \right)(\cdot, t) \|_{L^2([0,1])}^2 + \lambda^2 \|E^\lambda - \mathcal{E}(\cdot, t)\|_{H^1([0,1])}^2 \right) \leq \tilde{M}\lambda^{\min[\alpha,4]-\delta}
\]
for any \(\delta \in (0, \min[\alpha,4])\) and some constant \(\tilde{M}\) independent of \(\lambda\).

Theorem 6 (The case of ill-prepared initial data). Let \((n^\lambda, p^\lambda, E^\lambda)\) and \((n, p, \mathcal{E})\) be the classical solutions of the problem (1)–(6) and the problem (7)–(12) with \(n_0, p_0 \in C^3\), respectively. Let \(T\) be given by Proposition 2 with \(Z_0(x) = n_0(x) + p_0(x)\). Assume that \(D_x(x = 0, 1) = D_{xxx}(x = 0, 1) = 0\), and that \(E_0(x) = -\frac{D_x(x)}{(n_0 + p_0)(x)}\), and that there exist a constant \(M\) and an \(\alpha > 0\) such that, for any \(\lambda > 0\),
\[
\|\left( n_0^\lambda - n_0, p_0^\lambda - p_0 \right)(\cdot) \|_{H^1([0,1])}^2 + \lambda^2 \|E^\delta(t = 0) - E_0(\cdot)\|_{H^1([0,1])}^2 \leq M\lambda^{\min[\alpha,4]}.
\]
Then there exists an \(\lambda_0 > 0\) depending upon \(T\), such that, for any \(\lambda : 0 < \lambda < \lambda_0\),
\[
\sup_{0 \leq t \leq T} \left( \|\left( n^\lambda - n, p^\lambda - p \right)(\cdot, t) \|_{L^2([0,1])}^2 + \lambda^2 \|E^\lambda - \mathcal{E}(\cdot, t)\|_{H^1([0,1])}^2 \right) \leq \tilde{M}\lambda^{\min[\alpha,4]-\delta}
\]
for any \(\delta \in (0, \min[\alpha,4])\) and some constant \(\tilde{M}\) independent of \(\lambda\).
$D_{xxx}(x = 0, 1) = 0$. Also, assume that there exist a constant $M$ and an $\alpha > 0$ such that, for any $\lambda > 0$,
\[
\left\| (n_0^\lambda - n_0, p_0^\lambda - p_0)(\cdot) \right\|_{H^1([0,1])}^2 + \lambda^2 \left\| (E^\lambda(t = 0) - \mathcal{E}_0)(\cdot) \right\|_{H^1([0,1])}^2 \leq M\lambda^{\min[\alpha, 4]},
\]
for some function $\mathcal{E}_0(x) \in C^3$ satisfying $\mathcal{E}_0(x = 0, 1) = \mathcal{E}_{0,xx}(x = 0, 1) = 0$. Then there exists an $\lambda_0: 0 < \lambda_0 \ll 1$ such that, for any $\lambda: 0 < \lambda \leq \lambda_0$,
\[
\sup_{0 \leq t \leq T} \left( \left\| (n_1^\lambda - n - \lambda^2 z_1^\lambda, p_1^\lambda - p - \lambda^2 z_1^\lambda) \right\|_{L^2([0,1])}^2 + \lambda^2 \left\| (E_1^\lambda - \mathcal{E} - E_1^\lambda)(\cdot, t) \right\|_{H^1([0,1])}^2 \right) \leq \tilde{M}\lambda^{\min[\alpha, 4] - \delta}
\]
for any $\delta \in (0, \min[\alpha, 4])$ and some constant $\tilde{M} > 0$, where $(z_1^\lambda, E_1^\lambda)$ satisfies the problem (25)–(28).

3. The existence and regularity of solutions to quasineutral drift–diffusion models

In this section we investigate the existence and regularity of classical solutions to the limit problem.

We should point out that there are very little results about quasineutral drift–diffusion model in the literature although this is the simplest model in the modelling sets for semiconductors and plasma physics. The main difficulty is due to the fact that the third equation is an algebraic equation, which is very different from the classical drift–diffusion model for semiconductors. In this case, if the doping profile can change the sign, some physical phenomena such as the vacuum phenomena of the density or the singularity of the electric field at the vacuum set of the density may occur. This causes essential difficulties in mathematical analysis on this model.

Now we give the main results of this section.

**Theorem 7** (Local existence). Assume that $D \in C^2$, $D_x(x = 0, 1) = 0$ and $n_0, p_0 \in C^2$ satisfying that $n_0, p_0 \geq \delta_0 > 0$. Then the initial-boundary value problem (7)–(12) has a classical solution $(n, p, \mathcal{E}) \in (C^{2, 1}([0, 1] \times [0, T_{\max}]))^2 \times C^{1, 1}([0, 1] \times [0, T_{\max}])$ defined on $[0, T_{\max})$, $0 < T_{\max} \leq \infty$ is the maximal existence time. Moreover, if $T_{\max} < \infty$, then
\[
\lim_{t \to T_{\max}^-} \min_{0 \leq x \leq 1} (n + p)(x, t) = 0.
\]

Also, if $D$ and $D_x$ never vanish together, then $(n + p)(x, t) > 0$ for all $(x, t) \in [0, 1] \times [0, T_{\max})$ and $\lim_{t \to T_{\max}^-} \max_{0 \leq x \leq 1} \mathcal{E} = \infty$. In particular, if either $D \equiv \text{constant}$ or $D \neq 0$ for any $x \in [0, 1]$, then $T_{\max} = \infty$. In this case, we say that the solution exists globally in time.

**Theorem 8** (Uniqueness). The classical solution of the initial boundary-value problem (19)–(22) with $(n + p)(x, t) > 0$ is unique. Moreover, if $D \in C^4$ and $Z_0 \in C^3$ satisfying the compatibility conditions
\[
n_{0,x} = p_{0,x} = (n_0 + n_0 \mathcal{E}(t = 0))_{xx} = (p_0 - p_0 \mathcal{E}(t = 0))_{xx} = 0, \quad \text{at} \ x = 0, 1,
\]
where $\mathcal{E}(t = 0) = -\frac{D_x(x)}{n_0(x) + p_0(x)}$. Then $(n, p, \mathcal{E}) \in C^{3, 2}([0, 1] \times [0, T_{\max}])$. 
Theorem 9 (Global existence). Under the assumptions of Theorem 7, for suitably large initial data the initial-boundary value problem (7)–(12) has a unique global classical solution \((n, p, E)\) satisfying that \(n + p\) has a strictly positive lower bound.

Before giving the proofs of these theorems, we first discuss some properties of system (7)–(12) and system (19)–(22).

Property 1. If \((n, p, E)\) solves problem (7)–(12), then \((n, p) \geq (0, 0)\) if \((n_0, p_0) \geq (0, 0)\).

In fact, we rewrite (7), (8) and (10) as

\[
\begin{align*}
nt &= n_{xx} + En_x + E_n, \\
p_t &= p_{xx} - Ep_x - E_p, \\
n_x &= -En, \\
p_x &= E_p, \\
x &= 0, 1.
\end{align*}
\]

By minimal principle, it is easy to prove that \(n \geq 0\) and \(p \geq 0\) if \((n_0, p_0) \geq (0, 0)\).

Remark 2. Since \(E\) can change sign, we are not able to obtain the uniformly positive lower bound of the densities \(n\) and \(p\) even though the initial data \((n_0, p_0)\) has positive lower bound.

Property 2. If \((Z, E)\) solves problem (19)–(22) with \(Z_0 \geq |D|\), then \(Z \geq |D|\).

In fact, if \((Z, E)\) solves problem (19)–(22) with \(Z_0 \geq |D|\), then \((n, p, E)\), where \((n, p)\) is defined by (18), solves (7)–(9), (10) and (11). By Property 1, one gets \((n, p) \geq (0, 0)\). By the transform (18), we get Property 3.

By Property 2, we have:

Property 3. Assume that \((Z, E)\) solves problem (19)–(22) with \(Z_0 \geq |D|\). Then if \(Z(x_0, t) = 0\) for some \(x_0 \in [0, 1]\), then \(D(x_0) = 0\).

Now we prove the main theorems of this section.

Proof of Theorem 7. Take a \(C^\infty([0, \infty))\) nondecreasing smooth cut-off function satisfying

\[
f(z) = \begin{cases} 
2M_2, & z \geq 2M_2, \\
\ldots & \ldots \\
M_1, & M_1 \leq z \leq M_2, \\
\ldots & \ldots \\
\frac{M_1}{2}, & 0 \leq z \leq \frac{M_1}{2},
\end{cases}
\]

(29)

where \(M_1, M_2\) are positive constants to be determined later.

First, consider the following system

\[
\begin{align*}
0 &= -(D_x + f(\tilde{z})\tilde{E}), & 0 < x < 1, \ t > 0, \\
\tilde{z}_t &= (\tilde{z}_x + D\tilde{E})_x, & 0 < x < 1, \ t > 0, \\
(\tilde{z}_x + D\tilde{E})(x = 0, 1) &= 0, & t > 0, \\
\tilde{z}(t = 0) &= n_0 + p_0, & 0 \leq x \leq 1.
\end{align*}
\]

(30) (31) (32) (33)
Using (29) and $D_x(x = 0, 1) = 0$, we know that (30)–(33) is equivalent to the system

$$
\begin{align*}
\tilde{z}_t &= \left(\tilde{z}_x - \frac{DD_x}{f(\tilde{z})}\right)_x, \quad 0 < x < 1, \ t > 0, \\
\tilde{z}_x(x = 0, 1) &= 0, \quad t > 0, \\
\tilde{z}(t = 0) &= \tilde{Z}_0 = n_0 + p_0 > \delta_0, \quad 0 < x < 1.
\end{align*}
$$

(34)

This is a standard quasi-linear parabolic system. By the known parabolic theory, see, for example, [17], we know that (34)–(36) has a unique global classical solution $z \in C^{2,1}([0, 1] \times [0, \infty))$.

Next, we claim that there exist positive constants $M_1, M_2$ and a time $T > 0$ such that

$$
M_1 \leq z(x, t) \leq M_2, \quad x \in [0, 1], \ 0 \leq t \leq T.
$$

(37)

Take the function $\tilde{z}(x, t) = e^{\kappa_1 t} \max_{x \in [0, 1]} Z_0(x)$, where the positive constant $\kappa_1$ is chosen to satisfy

$$
\kappa_1 \geq \frac{2 \max_{x \in [0, 1]} |(DD)_x|}{M_1 \max_{x \in [0, 1]} Z_0(x)}.
$$

It is easy to verify that $\tilde{z}(x, t)$ is a upper solution of system (34)–(36). By the comparison principle one gets

$$
\tilde{z}(x, t) \leq \tilde{z}(x, t), \quad x \in [0, 1], \ t \in [0, \infty).
$$

(38)

On the other hand, take the function $\bar{z}(x, t) = \delta_0 e^{-\kappa_2 t}$, where $\kappa_2$ is a positive constant to be determined later.

By the direct calculation, one gets that $\bar{z}(x, t)$ is a lower solution of system (34)–(36) if

$$
-\delta_0 \kappa_2 e^{-\kappa_2 t} \leq -\frac{(DD_x)_x}{f(\bar{z})},
$$

(39)

which can be guaranteed by

$$
\delta_0 \kappa_2 M_1 \geq 2e^{\kappa_2 T} \max_{x \in [0, 1]} |(DD_x)_x|.
$$

(40)

Noting that if $(DD_x)_x \leq 0$, which implies that $D \equiv constant$ according to $D_x(x = 0, 1) = 0$, then (39) holds for any $\kappa_2 > 0$ and any $t > 0$. In this case, for any given $T > 0$, there exist positive constants $M_1, M_2$, depending on $T$, such that (37) holds.

For general case, we take $\kappa_2, T$ and $M_1$ such that

$$
\kappa_2 = \frac{\delta_0^2}{4 \max_{x \in [0, 1]} |(DD_x)_x|},
$$

$T$ satisfies $e^{2\kappa_2 T} = 2$ and $M_1 = \delta_0 e^{-\kappa_2 T}$. Then it is easy to verify that (40) holds for $x \in [0, 1], \ t \in [0, T]$. Hence, by comparison principle again, we have

$$
\tilde{z}(x, t) \geq \bar{z}(x, t) \geq M_1, \quad x \in [0, 1], \ t \in [0, T].
$$

(41)

Combining (38) and (41), we get (37) with $M_1 = \delta_0 e^{-\kappa_2 T}$ and $M_2 = e^{\kappa_1 T} \max_{0 \leq x \leq 1} Z_0(x)$. 

Thus (34)–(36) has a classical solution $\tilde{z}(x, t)$ satisfying $M_1 \leq \tilde{z}(x, t) \leq M_2$ for $x \in [0, 1]$, $t \in [0, T]$, where $M_1, M_2, T$ are positive constants depending only upon initial data and the doping profile. Moreover, when $D(x) = constant$, then $T$ is arbitrary. Hence (30)–(33) has a classical solution $(\tilde{z}(x, t), \tilde{E})$, where

$$\tilde{E} = -\frac{D_x}{f(\tilde{z})} = -\partial_x \int_0^x \frac{D_x(x)}{f(\tilde{z})(x, t)} \, dx,$$

satisfying $M_1 \leq \tilde{z}(x, t) \leq M_2$ for $x \in [0, 1]$, $t \in [0, T]$.

Finally, by the definition of $f(z)$, (19)–(22) has a local classical solution $(Z, E) \in C^{2,1}([0, 1] \times [0, T]) \times C^{1,1}([0, 1] \times [0, T])$. Moreover, when $D(x) = constant$, $T$ is arbitrary.

Now denote the maximal existence time of the solution to (19)–(22) by $T_{\text{max}}$. Then we claim that if $T_{\text{max}} < \infty$, then \( \lim_{t \to T_{\text{max}}} \min_{x \in [0, 1]} Z(x, t) = 0 \). Otherwise, there exist $\delta_1, \delta_2 > 0$ such that, for any $t \in [T_{\text{max}} - \epsilon, T_{\text{max}})$ and for any $\epsilon > 0$ small enough, $\delta_1 \leq Z(x, t) \leq \delta_2$. Consider the following problem

$$0 = -(D_x + \tilde{z}\tilde{E}), \quad 0 < x < 1, \quad t > T_{\text{max}} - \epsilon, \quad (42)$$

$$\tilde{z}_t = (\tilde{z}_x + D\tilde{E})_x, \quad 0 < x < 1, \quad t > T_{\text{max}} - \epsilon, \quad (43)$$

$$(\tilde{z}_x + D\tilde{E})_x(x = 0, 1) = 0, \quad t > T_{\text{max}} - \epsilon, \quad (44)$$

$$\tilde{z}(t = T_{\text{max}} - \epsilon) = Z(T_{\text{max}} - \epsilon), \quad 0 \leq x \leq 1. \quad (45)$$

Repeating the above procedure, we can prove that (42)–(45) has a classical solution defined on $[T_{\text{max}} - \epsilon, T_{\text{max}} - \epsilon + \ln 2/\delta_3)$, where $\delta_3$ depends only upon $\delta_1$ and $\delta_2$. Thus, taking $\epsilon$ small enough, we can extend the solution of (19)–(22) beyond $T_{\text{max}}$. This is a contradiction, which gives our claim. By the Property 2, there exists $x_0 \in [0, 1]$ such that $Z(x_0, T_{\text{max}}) = 0$. Hence $D(x_0) = 0$. By assumption, $D_x(x_0) \neq 0$. Thus we have $\lim_{t \to T_{\text{max}} - \epsilon} \max_{0 \leq x \leq 1} E = \infty$ by using Eq. (20).

By Property 3, when $D(x) \neq 0$ for any $x \in [0, 1]$, $Z(x, t) \geq |D| > 0$. Hence, $T_{\text{max}} = \infty$.

The rest is to prove that if $D(x)$ and $D_x(x)$ never vanish together in the interval $[0, 1]$, then $Z(x, t) > 0$ for all $(x, t) \in [0, 1] \times [0, T_{\text{max}})$.

In fact, if $D(x)$ never vanishes in the interval $[0, 1]$, then, by Property 2, $Z(x, t) > |D(x)| > 0$.

If $D(x)$ vanishes in the interval $[0, 1]$, denote the zero root set of $D(x)$ by $X$. Assume that the result does not hold. Then $Z(x_0, t_0) = 0$ for some $(x_0, t_0) \in [0, 1] \times [0, T_{\text{max}})$. By Property 3, $x_0 \in X$. Since $D_x(x_0) \neq 0$, by Eq. (20), we have $E(x_0, t_0) = \infty$, which contradicts the regularity of the electric field $E$. Hence $Z(x, t) > 0$ for all $(x, t) \in [0, 1] \times [0, T_{\text{max}})$. By transform (18), we easily get Theorem 7.

This completes the proof of Theorem 7. \( \square \)

**Proof of Theorem 8. Uniqueness.** Since systems (7)–(11) and (19)–(22) are equivalent, it suffices to prove the uniqueness of the solution to system (7)–(11). Assume that $(Z_i, E_i)$, $i = 1, 2$, are any two classical solutions satisfying $Z_i > 0$. Denote $z = Z_1 - Z_2$, $E = E_1 - E_2$. Then $(z, E)$ satisfies

$$0 = -(zE_1 + Z_2 E), \quad (46)$$

$$z_t = (z_x + DE)_x, \quad (47)$$
\[ z_x + DE = 0, \quad x = 0, 1, \quad z(t = 0) = 0. \]

(48)

(49)

Multiplying (46) by \( \Phi \) defined by \( \Phi x = E \) and integrating by parts, one gets

\[ 0 = - \int_0^1 (z E_1 + Z_2 E) E \, dx. \]

(50)

Using the positivity of \( Z_2 \) and the Cauchy–Schwarz inequality, it follows from (50) that there exists a positive constant \( M = M(Z_2, E_1) \) such that

\[ \int_0^1 E^2 \, dx \leq M \int_0^1 z^2 \, dx. \]

(51)

Multiplying (47) by \( z \) and integrating by parts, one gets

\[ \frac{1}{2} \frac{d}{dt} \int_0^1 z^2 \, dx = - \int_0^1 |z_x|^2 \, dx - \int_0^1 DE z_x \, dx. \]

(52)

Using the Cauchy–Schwarz inequality, we have

\[ \frac{1}{2} \frac{d}{dt} \int_0^1 z^2 \, dx \leq - \frac{1}{2} \int_0^1 |z_x|^2 \, dx + M \int_0^1 E^2 \, dx. \]

(53)

By (51) and (53), one gets

\[ \frac{1}{2} \frac{d}{dt} \int_0^1 z^2 \, dx \leq - \frac{1}{2} \int_0^1 |z_x|^2 \, dx + M \int_0^1 z^2 \, dx. \]

Gronwall’s lemma and (49) give \( z = 0 \) and hence \( E = 0 \) a.e.

Regularities. The standard elliptic and parabolic regularity theory gives the desired regularity results, see [17].

This completes the proof of Theorem 8. ∎

Proof of Theorem 9. According to the proof of Theorem 7, we just prove that (30)–(33) has a solution \( (\bar{z}, \bar{E}) \) satisfying that, for any \( T > 0 \), there exist positive constants \( M_1(T), M_2(T) \) such that

\[ M_1(T) \leq \bar{z} \leq M_2(T) \] for \((x, t) \in [0, 1] \times [0, T] \).

As in the proof of Theorem 7, it is easy to verify that \( \bar{z} \) defined by the above in Theorem 7 is a global upper solution of (34)–(36). This gives the existence of \( M_2(T) \) for any \( T > 0 \).

The rest is to prove there is a global lower solution \( \bar{z} \geq M_1(T) \) for some \( M_1(T) \). In fact, take the function \( \bar{z}(x, t) = z^*(x) \), \( z^*(x) \) is defined by
z^*(x) = \int_0^x f(s) ds = \frac{D^2(x)}{2} + \delta_2^2, \quad x \in [0, 1],

where \delta_2 is any given positive constant and \( f(s) \) is given by (29). Then a simple calculation gives that \( z^*(x) \geq \frac{\delta_2^2}{2M_2(T)} \) is a positive lower solution of (34)–(36). Hence, by comparison principle, \( \tilde{z}(x, t) \geq z^*(x, t) \geq \frac{\delta_2^2}{2M_2(T)} = M_1 \) for \( (x, t) \in [0, 1] \times [0, T] \). This gives the existence of \( M_1(T) \).

Thus, we have proven that (30)–(33) has a global classical solution \( (\tilde{z}, \tilde{E}) \) satisfying

\[ M_1 \leq \tilde{z} \leq M_2 \]

for some \( M_1, M_2 > 0 \). By the definition of \( f(z) \), we easily get our result.

The proof of Theorem 9 is completed. \( \square \)

Proof of Proposition 2. By Theorems 7–9 and the transform (18), we easily get our results. \( \square \)

4. The case of well-prepared initial data

In this section we investigate the asymptotic behavior of the solution of the problem (1)–(5) for well-prepared initial data as \( \lambda \to 0 \). We will prove Theorems 3 and 5. The basic ideas of the proof are to introduce two entropies, one of which is called as the entropy-dissipation because it is derived from the dissipation term of the error equations. And then to establish an entropy production integration inequality by the careful energy methods. The key point in the Sobolev energy estimates is to control the strong nonlinear oscillation coming from the nonlinear terms in the ‘error’ equations by the interaction between the entropy and the entropy dissipation, more precisely, by the integration term whose integral function is the production of the entropy and the entropy-dissipation. This is completely different from the ideas used by the other authors listed in the references. We will perform the energy estimates carefully step by step in the following to control the strong nonlinear terms.

Proof of Theorem 3. Replacing \((z^\lambda, E^\lambda)\) by \((Z + z^\lambda_R, E + E^\lambda_R)\) in (14), (15) and subtracting (19), (20), one gets

\[
(z^\lambda_R)_t = (z^\lambda_x + DE^\lambda_R)_x - \lambda^2 (E^\lambda_R E^\lambda_{R,x})_x - \lambda^2 (E E^\lambda R)_x, \\
\lambda^2 (E^\lambda_R - E^\lambda_{R,xx}) + Z E^\lambda_R = -E z^\lambda_R - z^\lambda_R E^\lambda_R - \lambda^2 (E_l - E_{lx}).
\] (54) \hspace{1cm} (55)

To show that the error functions are small enough, we use the following technical lemma.

Lemma 10. Let \( \Gamma^\lambda(t), G^\lambda(t) \) be nonnegative functions well-defined in \([0, T]\) satisfying

\[
\Gamma^\lambda(t) + c_0 \int_0^t G^\lambda(s) ds \leq M \Gamma^\lambda(t = 0) + M \int_0^t (\Gamma^\lambda(s) + (\Gamma^\lambda(s))^\prime) ds \\
+ M \lambda^\prime \int_0^t \Gamma^\lambda(s) G^\lambda(s) ds + M \lambda^\prime, \quad 0 \leq t \leq T,
\] (56)
where \( t > 1, q > 0, \tau \geq 0 \) and \( c_0, M \) are some positive constants independent of \( \lambda \). Then \( \Gamma^\lambda(t = 0) \leq \tilde{M} \lambda^{\min\{\alpha, q\}} \), for some constant \( \tilde{M} \) independent of \( \lambda \) and some \( \alpha > 0 \), implies that there exists an \( \lambda_0 \ll 1 \) such that, for any \( \lambda : 0 < \lambda \leq \lambda_0 \),

\[
\Gamma^\lambda(t) \leq \tilde{M} \lambda^{\min\{\alpha, q\}} - \delta, \quad 0 \leq t \leq T
\]

holds for any \( \delta \in (0, \min\{\alpha, q\}) \).

Moreover, if \( \tau > 0 \), then \( \Gamma^\lambda(t = 0) \leq \tilde{M} \) implies that there exists a \( T_0 : 0 < T_0 \leq T \) and an \( \lambda_0 \ll 1 \) such that, for any \( \lambda : 0 < \lambda \leq \lambda_0 \),

\[
\Gamma^\lambda(t) \leq 3\tilde{M}(M + 1), \quad 0 \leq t \leq T_0.
\]

**Remark 3.** Inequality (56) is one of the Gronwall’s type with an extra integration term in which the integral function is the production of the entropy and the entropy-dissipation. Hence (56) is called as the entropy production integration inequality.

**Proof of Lemma 10.** We just prove the first part of this lemma. For the second part, the proof is similar, which we omit.

Assume that the result does not hold, then, for any \( 0 < \lambda_0 \ll 1 \), there exists some \( 0 < \lambda \leq \lambda_0 \) such that

\[
\Gamma^\lambda(t^*_0) > \tilde{M} \lambda^{\min\{\alpha, q\}} - \delta
\]

holds for some \( \delta \in (0, \min\{\alpha, q\}) \) and for some \( 0 < t^*_0 \leq T \).

Denote the first root of \( \Gamma^\lambda(t) = \tilde{M} \lambda^{\min\{\alpha, q\}} - \delta \) in \([0, t^*_0]\) by \( t^*_1 \). Then we have

\[
\Gamma^\lambda(t) \leq \tilde{M} \lambda^{\min\{\alpha, q\}} - \delta, \quad 0 \leq t \leq t^*_1 \leq t^*_0 \leq T, \quad \Gamma^\lambda(t^*_1) = \tilde{M} \lambda^{\min\{\alpha, q\}} - \delta. \quad (57)
\]

It follows from (56) and (57) that

\[
\Gamma^\lambda(t) + c_0 \int_0^t G^\lambda(s) \, ds \leq M \tilde{M} \lambda^{\min\{\alpha, q\}} + M(1 + (\tilde{M} \lambda^{\min\{\alpha, q\}} - \delta)^{-1}) \int_0^t \Gamma^\lambda(s) \, ds
\]

\[
+ M \tilde{M} \lambda^{\tau + \min\{\alpha, q\}} - \delta \int_0^t G^\lambda(s) \, ds + M \lambda^q,
\]

which gives, with the help of the smallness of \( \lambda_0 \) and the fact \( \lambda \leq \lambda_0 \), that

\[
\Gamma^\lambda(t) \leq M(\tilde{M} + 1) \lambda^{\min\{\alpha, q\}} + 2M \int_0^t \Gamma^\lambda(s) \, ds.
\]

Gronwall’s lemma and the smallness of \( \lambda_0 \) give, for \( 0 \leq t \leq t^*_1 \), that
\[ \Gamma_{\lambda}(t) \leq M(\tilde{M} + 1)(2Me^{2MT} + 1)\lambda^{\min\{\alpha, q\}} \]
\[ \leq M(\tilde{M} + 1)(2Me^{2MT} + 1)\lambda^\delta \lambda^{\min\{\alpha, q\} - \delta} \leq \frac{\tilde{M}}{2} \lambda^{\min\{\alpha, q\} - \delta}, \]

which contradicts with (57).

The proof of Lemma 10 is completed. \(\square\)

To perform the energy estimates, we derive some boundary conditions for the error functions \(z_{\lambda}^R, E_{\lambda}^R\).

Since \(D_x(x = 0, 1) = 0\), one gets from (20) that
\[ \mathcal{E}(x = 0, 1; t) = 0 \tag{58} \]
and hence from (21)
\[ \mathcal{Z}_x(x = 0, 1; t) = 0. \tag{59} \]

Then it follows from the boundary condition (16), (58) and (59) that
\[ z_{\lambda, x}^R = E_{\lambda, x}^R = 0, \quad x = 0, 1, \ t > 0. \tag{60} \]

Also, using \(D_{xxx}(x = 0, 1) = 0\), from (58), (59) and the fact that \(\mathcal{Z} \geq \delta\), one gets
\[ \mathcal{E}_{xx}(x = 0, 1; t) = 0. \tag{61} \]

Hence from (55), (60), (58), (61) it follows
\[ E_{\lambda, xx}^R(x = 0, 1; t) = 0. \tag{62} \]

Now we make the energy estimates. In the following, we use \(c_i, \delta_i, \epsilon\) and \(M(\epsilon)\) or \(M\) to denote the constants which are independent of \(\lambda\), but can be different from line to line.

First, we have the following basic energy estimates.

**Lemma 11.** Under the assumptions of Theorem 3, we have
\[ \frac{d}{dt}(\delta_4 \| z_{\lambda}^R(t) \|_{L_2^x}^2 + \lambda^2 \| E_{\lambda}^R(t) \|_{L_2^x}^2) + c_1 \delta_4 \| z_{\lambda, x}^R \|_{L_2^x}^2 + 2\lambda^2 \| E_{\lambda, x}^R \|_{L_2^x}^2 + c_2 \| E_{\lambda}^R \|_{L_2^x}^2 \]
\[ \leq M \lambda^4 \|(E_{\lambda}^R, E_{\lambda, x}^R)\|_{L_2^x}^2 + M \lambda^4 \|(E_{\lambda, x}^R, z_{\lambda}^R, z_{\lambda, x}^R, z_{\lambda, xx}^R)\|_{L_2^x}^2 \]
\[ + M \|z_{\lambda}^R\|_{L_2^x}^2 \|E_{\lambda}^R\|_{L_2^x}^2 + M \lambda^4 \]
\[ \tag{63} \]

for some \(\delta_4\), small enough.

**Proof.** Multiplying (54) by \(z_{\lambda}^R\) and integrating the resulting equation over \([0, 1]\) with respect to \(x\), one gets, by (58), (60) and integrations by parts, that
\[
\frac{1}{2} \frac{d}{dt} \|z^\lambda_R(t)\|_{L^2_x}^2 + \|z^\lambda_{R,x}\|_{L^2_x}^2 \\
= - \int_0^1 D(x) E^\lambda_R z^\lambda_{R,x} \, dx + \lambda^2 \int_0^1 E^\lambda_R E^\lambda_{R,x} \, dx \\
+ \lambda^2 \int_0^1 \mathcal{E} E^\lambda_{R,x} z^\lambda_{R,x} \, dx + \lambda^2 \int_0^1 E_x E^\lambda_{R,x} \, dx + \lambda^2 \int_0^1 \mathcal{E} \mathcal{E}_x z^\lambda_{R,x} \, dx.
\] (64)

Now we estimate each term in the right-hand side of (64).

For the first term, by Cauchy–Schwarz’s inequality, we have
\[
- \int_0^1 D(x) E^\lambda_R z^\lambda_{R,x} \, dx \leq \epsilon \|z^\lambda_{R,x}\|_{L^2_x}^2 + M(\epsilon) \|E^\lambda_R\|_{L^2_x}^2.
\] (65)

For the second term, by Cauchy–Schwarz’s inequality and Sobolev’s lemma, one gets
\[
\lambda^2 \int_0^1 E^\lambda_R E^\lambda_{R,x} z^\lambda_{R,x} \, dx \leq \epsilon \|z^\lambda_{R,x}\|_{L^2_x}^2 + M(\epsilon)^4 \|E^\lambda_R E^\lambda_{R,x}\|_{L^2_x}^2 \\
\leq \epsilon \|z^\lambda_{R,x}\|_{L^2_x}^2 + M(\epsilon)^4 \|E^\lambda_R\|_{L^\infty_x} \|E^\lambda_{R,x}\|_{L^2_x}^2 \\
\leq \epsilon \|z^\lambda_{R,x}\|_{L^2_x}^2 + M \lambda^4 \| (E^\lambda_R, E^\lambda_{R,x}) \|_{L^2_x}^2 \|E^\lambda_{R,x}\|_{L^2_x}^2.
\] (66)

For the third term, using the regularities of \(\mathcal{E}\) and Cauchy–Schwarz’s inequality, we have
\[
\lambda^2 \int_0^1 \mathcal{E} E^\lambda_{R,x} z^\lambda_{R,x} \, dx \leq \epsilon \|z^\lambda_{R,x}\|_{L^2_x}^2 + M \lambda^4 \|E^\lambda_{R,x}\|_{L^2_x}^2.
\] (67)

The fourth and fifth terms are treated in the same way as in the third term. This yields
\[
\lambda^2 \int_0^1 E_x E^\lambda_{R,x} z^\lambda_{R,x} \, dx + \lambda^2 \int_0^1 \mathcal{E} \mathcal{E}_x z^\lambda_{R,x} \, dx \leq \epsilon \|z^\lambda_{R,x}\|_{L^2_x}^2 + M \lambda^4 \|E^\lambda_{R,x}\|_{L^2_x}^2 + M(\epsilon) \lambda^4.
\] (68)

Thus, using the above (64), (65)–(68) together and taking \(\epsilon\) small enough, one can get
\[
\frac{d}{dt} \|z^\lambda_R(t)\|_{L^2_x}^2 + c_1 \|z^\lambda_{R,x}\|_{L^2_x}^2 \leq M \|E^\lambda_R\|_{L^2_x}^2 + M \lambda^4 \| (E^\lambda_R, E^\lambda_{R,x}) \|_{L^2_x}^2 \\
+ M \lambda^4 \| (E^\lambda_R, E^\lambda_{R,x}) \|_{L^2_x}^2 \|E^\lambda_{R,x}\|_{L^2_x}^2 + M \lambda^4.
\] (69)

Multiplying (55) by \(E^\lambda_R\) and integrating the resulting equation over \([0, 1]\) with respect to \(x\), one gets, by (60) and integrations by parts, that
\[
\frac{\lambda^2}{2} \frac{d}{dt} \| E_R^\lambda \|_{L^2_x}^2 + \lambda^2 \| E_{R,x}^\lambda \|_{L^2_x}^2 + \int_0^1 Z | E_R^\lambda |^2 \, dx
\]
\[
= -\lambda^2 \int_0^1 (E_t - E_{xx}) E_R^\lambda \, dx - \int_0^1 E_{zR} E_R^\lambda \, dx - \int_0^1 z_R^\lambda E_R^\lambda E_R^\lambda \, dx. \tag{70}
\]
For the first and the second terms, by Cauchy–Schwarz’s inequality and using the regularity of \( E \), we have
\[
-\lambda^2 \int_0^1 (E_t - E_{xx}) E_R^\lambda \, dx \leq \epsilon \| E_R^\lambda \|_{L^2_x}^2 + M(\epsilon) \lambda^4 \quad \text{and} \quad \tag{71}
\]
\[
- \int_0^1 E_{zR} E_R^\lambda \, dx \leq \epsilon \| E_R^\lambda \|_{L^2_x}^2 + M(\epsilon) \| z_R^\lambda \|_{L^2_x}^2. \tag{72}
\]
For the nonlinear third term, by Cauchy–Schwarz’s inequality and Sobolev’s lemma, one gets
\[
- \int_0^1 z_R^\lambda E_R^\lambda E_R^\lambda \, dx \leq \epsilon \| E_R^\lambda \|_{L^2_x}^2 + M \left( z_R^\lambda, z_{R,x}^\lambda \right) \| E_R^\lambda \|_{L^2_x}^2 \leq \epsilon \| E_R^\lambda \|_{L^2_x}^2 + M \| z_R^\lambda \|_{L^2_x}^2 \| E_R^\lambda \|_{L^2_x}^2. \tag{73}
\]
Then, combining (70) and (71)–(73), using the positivity of \( Z \) and choosing \( \epsilon \) to be small enough, one gets
\[
\lambda^2 \frac{d}{dt} \| E_R^\lambda \|_{L^2_x}^2 + 2\lambda^2 \| E_{R,x}^\lambda \|_{L^2_x}^2 + c_2 \| E_{R,x}^\lambda \|_{L^2_x}^2 \leq M \| z_R^\lambda \|_{L^2_x}^2 + M \left( z_R^\lambda, z_{R,x}^\lambda \right) \| E_R^\lambda \|_{L^2_x}^2 + M \lambda^4. \tag{74}
\]
Combining (69) and (74), one easily get (63).
This completes the proof of Lemma 11. \( \square \)

To enclose the energy estimate, we obtain higher order Sobolev’s estimates.

**Lemma 12.** Under the assumptions of Theorem 3, we have
\[
\frac{d}{dt} \left( \delta_5 \| z_{R,x}^\lambda (t) \|_{L^2_x}^2 + \lambda^2 \| E_{R,x}^\lambda \|_{L^2_x}^2 \right) + c_3 \delta_5 \| z_{R,xx}^\lambda \|_{L^2_x}^2 + \lambda^2 \| E_{R,xx}^\lambda \|_{L^2_x}^2 + c_4 \| E_{R,x}^\lambda \|_{L^2_x}^2
\]
\[
\leq M \left( \left( z_R^\lambda, z_{R,x}^\lambda, E_R^\lambda \right) \right) \| z_{R,x}^\lambda \|_{L^2_x}^2 + M \lambda^4 \left( \| E_R^\lambda \|_{L^2_x}^2 + \| E_{R,x}^\lambda \|_{L^2_x}^2 \right) + M \left( \left( E_{R,xx}^\lambda , E_{R,xxx}^\lambda \right) \right) \| E_{R,x}^\lambda \|_{L^2_x}^2 \| E_{R,xx}^\lambda \|_{L^2_x}^2
\]
\[
+ M \left( \| z_{R,xx}^\lambda \|_{L^2_x}^2 + \| E_{R,xx}^\lambda \|_{L^2_x}^2 \right) \| z_{R,x}^\lambda \|_{L^2_x}^2 \| E_{R}^\lambda \|_{L^2_x}^2 + M \lambda^4. \tag{75}
\]
Proof. Differentiating (54) with respect to $x$, multiplying the resulting equations by $z_{R,x}^\lambda$ and then integrating it over $[0, 1]$, one gets, by (60) and integrations by parts, that

\[
\frac{1}{2} \frac{d}{dt} \|z_{R,x}^\lambda\|^2_{L_x^2} + \|z_{R,xx}^\lambda\|^2_{L_x^2} = -\int_0^1 (D(x)E_R^\lambda)_{x} z_{R,xx}^\lambda \, dx + \lambda^2 \int_0^1 (E_R^\lambda E_{R,x}^\lambda)_{x} z_{R,xx}^\lambda \, dx
\]

\[
+ \lambda^2 \int_0^1 (E_{R,x}^\lambda)_{x} z_{R,xx}^\lambda \, dx + \lambda^2 \int_0^1 (E_{x}^\lambda E_{R,x}^\lambda)_{x} z_{R,xx}^\lambda \, dx + \lambda^2 \int_0^1 (E_{x} E_{R,x}^\lambda)_{x} z_{R,xx}^\lambda \, dx. \quad (76)
\]

Now we estimate each term in the right-hand side of (76).

For the first term, by Cauchy–Schwarz’s inequality, we have

\[
-\int_0^1 (D(x)E_R^\lambda)_{x} z_{R,xx}^\lambda \, dx \leq \epsilon \|z_{R,xx}^\lambda\|^2_{L_x^2} + M(\epsilon) \|E_R^\lambda, E_{R,x}^\lambda\|^2_{L_x^2}. \quad (77)
\]

For the second term, by Cauchy–Schwarz’s inequality and Sobolev’s lemma, one gets

\[
\lambda^2 \int_0^1 (E_R^\lambda E_{R,x}^\lambda)_{x} z_{R,xx}^\lambda \, dx \leq \epsilon \|z_{R,xx}^\lambda\|^2_{L_x^2} + M(\epsilon) \lambda^4 \|E_R^\lambda, E_{R,x}^\lambda\|^2_{L_x^2}. \quad (78)
\]

For the third term, using the regularities of $E$, we have

\[
\lambda^2 \int_0^1 (E_{R,x}^\lambda)_{x} z_{R,xx}^\lambda \, dx \leq \epsilon \|z_{R,xx}^\lambda\|^2_{L_x^2} + M(\epsilon) \lambda^4 \|E_{R,x}^\lambda, E_{R,xx}^\lambda\|^2_{L_x^2}. \quad (79)
\]

The fourth and fifth terms can be treated as in the third term. Then it yields

\[
\lambda^2 \int_0^1 (E_{x}^\lambda E_R^\lambda)_{x} z_{R,xx}^\lambda \, dx + \lambda^2 \int_0^1 (E_{x} E_{R,x}^\lambda)_{x} z_{R,xx}^\lambda \, dx \leq \epsilon \|z_{R,xx}^\lambda\|^2_{L_x^2} + M(\epsilon) \lambda^4 \|E_R^\lambda, E_{R,x}^\lambda\|^2_{L_x^2} + M(\epsilon) \lambda^4. \quad (80)
\]
Thus, taking $\epsilon$ to be small enough, from (76)–(80), one gets

$$
\frac{d}{dt} \left\| z^{\lambda}_{R,x}(t) \right\|_{L^2_x}^2 + c_3 \left\| z^{\lambda}_{R,xx} \right\|_{L^2_x}^2 \\
\leq M \left\| (E^\lambda_R, E^\lambda_{R,x}) \right\|_{L^2_x}^2 + M\lambda^4 \left\| (E^\lambda_R, E^\lambda_{R,x}, E^\lambda_{R,xx}) \right\|_{L^2_x}^2 \\
+ M\lambda^4 \left( \left\| E^\lambda_R \right\|_{L^2_x}^2 \left\| E^\lambda_{R,xx} \right\|_{L^2_x}^2 + \left\| (E^\lambda_{R,x}, E^\lambda_{R,xx}) \right\|_{L^2_x}^2 \left\| E^\lambda_{R,x} \right\|_{L^2_x}^2 \right) + M\lambda^4. \quad (81)
$$

Differentiating (55) with respect to $x$, multiplying the resulting equations by $E^\lambda_{R,x}$ and then integrating it over $[0, 1]$, one gets, by (62) and integrations by parts, that

$$
\frac{\lambda^2}{2} \frac{d}{dt} \left\| E^\lambda_{R,x} \right\|_{L^2_x}^2 + \lambda^2 \left\| E^\lambda_{R,xx} \right\|_{L^2_x}^2 + \int_0^1 Z \left| E^\lambda_{R,x} \right|^2 \, dx \\
= -\lambda^2 \int_0^1 (E^\lambda_{x,t} - E^\lambda_{xxx}) E^\lambda_{R,x} \, dx - \int_0^1 (E^\lambda_{x,R})_x E^\lambda_{R,x} \, dx \\
- \int_0^1 (z^\lambda_R E^\lambda_R)_x E^\lambda_{R,x} \, dx - \int_0^1 Z_x E^\lambda_{R,x} E^\lambda_{R,x} \, dx. \quad (82)
$$

For the first term, by Cauchy–Schwarz's inequality and using the regularity of $E$, i.e., $E_{ix}, E_{xxx} \in C([0, 1] \times [0, T])$, we have

$$
-\lambda^2 \int_0^1 (E^\lambda_{x,t} - E^\lambda_{xxx}) E^\lambda_{R,x} \, dx \leq \epsilon \left\| E^\lambda_{R,x} \right\|_{L^2_x}^2 + M(\epsilon)\lambda^4. \quad (83)
$$

Similarly

$$
-\int_0^1 (E^\lambda_{x,R})_x E^\lambda_{R,x} \, dx \leq \epsilon \left\| E^\lambda_{R,x} \right\|_{L^2_x}^2 + M(\epsilon) \left( \left\| z^\lambda_R E^\lambda_R \right\|_{L^2_x}^2 \right). \quad (84)
$$

For the nonlinear third term, by Cauchy–Schwarz’s inequality and Sobolev’s lemma, one gets

$$
- \int_0^1 \left( z^\lambda_R E^\lambda_R \right)_x E^\lambda_{R,x} \, dx \\
\leq \epsilon \left\| E^\lambda_{R,x} \right\|_{L^2_x}^2 + M(\epsilon) \left( \left\| z^\lambda_R E^\lambda_R \right\|_{L^2_x}^2 \right). \quad (85)
$$
For the fourth term, by Cauchy–Schwarz’s inequality and using the regularity of $Z$, we have

$$
- \int_{0}^{1} Z_{x} E_{R}^{\lambda} E_{R,x}^{\lambda} \, dx \leq \epsilon \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}}^{2} + M(\epsilon) \left\| E_{R}^{\lambda} \right\|_{L_{x}^{2}}^{2}.
$$

Thus, combining (82) and (83)–(86), taking $\epsilon$ to be small enough and using the positivity of $Z$, one gets

$$
\lambda^{2} \frac{d}{dt} \left[ \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}}^{2} + 2 \lambda^{2} \left\| E_{R,xx}^{\lambda} \right\|_{L_{x}^{2}}^{2} + c_{4} \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}}^{2} \right]
\leq M \left( \left\| z_{R}^{\lambda} \right\|_{L_{x}^{2}} + M \left( \left\| z_{R}^{\lambda} \right\|_{L_{x}^{2}} + \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}} \left\| z_{R,x}^{\lambda} \right\|_{L_{x}^{2}} \right) + M \lambda^{4} \right).
$$

Using (81) and (87) and restricting $\lambda$ small enough, one easily gets (75).

This completes the proof of Lemma 12. □

Now the proof of Theorem 3 can be finished as follows. Due to (63) in Lemma 11 and (75) in Lemma 12, one gets

$$
\frac{d}{dt} \left[ K(\delta_{4} \left\| z_{R}^{\lambda}(t) \right\|_{L_{x}^{2}}^{2} + \lambda^{2} \left\| E_{R}^{\lambda} \right\|_{L_{x}^{2}}^{2}) + \left( \delta_{5} \left\| z_{R,x}^{\lambda}(t) \right\|_{L_{x}^{2}}^{2} + \lambda^{2} \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}}^{2} \right) \right]
\leq K \left[ M \lambda^{4} \left( \left\| E_{R}^{\lambda} \right\|_{L_{x}^{2}} + \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}} \right) \left\| z_{R}^{\lambda} \right\|_{L_{x}^{2}} + M \lambda^{4} \right]
\leq K \left[ M \lambda^{4} \left( \left\| E_{R}^{\lambda} \right\|_{L_{x}^{2}} + \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}} \right) \left\| z_{R}^{\lambda} \right\|_{L_{x}^{2}} + M \lambda^{4} \right]
\leq K \left[ M \lambda^{4} \left( \left\| E_{R}^{\lambda} \right\|_{L_{x}^{2}} + \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}} \right) \left\| z_{R}^{\lambda} \right\|_{L_{x}^{2}} + M \lambda^{4} \right]
\leq K \left[ M \lambda^{4} \left( \left\| E_{R}^{\lambda} \right\|_{L_{x}^{2}} + \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}} \right) \left\| z_{R}^{\lambda} \right\|_{L_{x}^{2}} + M \lambda^{4} \right]
\leq K \left[ M \lambda^{4} \left( \left\| E_{R}^{\lambda} \right\|_{L_{x}^{2}} + \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}} \right) \left\| z_{R}^{\lambda} \right\|_{L_{x}^{2}} + M \lambda^{4} \right]
\leq K \left( \left\| E_{R}^{\lambda} \right\|_{L_{x}^{2}} + \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}} \right)
\leq K \left( \left\| E_{R}^{\lambda} \right\|_{L_{x}^{2}} + \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}} \right)
\leq K \left( \left\| E_{R}^{\lambda} \right\|_{L_{x}^{2}} + \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}} \right)
\leq K \left( \left\| E_{R}^{\lambda} \right\|_{L_{x}^{2}} + \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}} \right)
\leq K \left( \left\| E_{R}^{\lambda} \right\|_{L_{x}^{2}} + \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}} \right)
\leq K \left( \left\| E_{R}^{\lambda} \right\|_{L_{x}^{2}} + \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}} \right)
\leq K \left( \left\| E_{R}^{\lambda} \right\|_{L_{x}^{2}} + \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}} \right)
\leq K \left( \left\| E_{R}^{\lambda} \right\|_{L_{x}^{2}} + \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}} \right)
\leq K \left( \left\| E_{R}^{\lambda} \right\|_{L_{x}^{2}} + \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}} \right)
\leq K \left( \left\| E_{R}^{\lambda} \right\|_{L_{x}^{2}} + \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}} \right)
\leq K \left( \left\| E_{R}^{\lambda} \right\|_{L_{x}^{2}} + \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}} \right)
\leq K \left( \left\| E_{R}^{\lambda} \right\|_{L_{x}^{2}} + \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}} \right)
\leq K \left( \left\| E_{R}^{\lambda} \right\|_{L_{x}^{2}} + \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}} \right)
\leq K \left( \left\| E_{R}^{\lambda} \right\|_{L_{x}^{2}} + \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}} \right)
\leq K \left( \left\| E_{R}^{\lambda} \right\|_{L_{x}^{2}} + \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}} \right)
\leq K \left( \left\| E_{R}^{\lambda} \right\|_{L_{x}^{2}} + \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}} \right)
\leq K \left( \left\| E_{R}^{\lambda} \right\|_{L_{x}^{2}} + \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}} \right)
\leq K \left( \left\| E_{R}^{\lambda} \right\|_{L_{x}^{2}} + \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}} \right)
\leq K \left( \left\| E_{R}^{\lambda} \right\|_{L_{x}^{2}} + \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}} \right)
\leq K \left( \left\| E_{R}^{\lambda} \right\|_{L_{x}^{2}} + \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}} \right)
\leq K \left( \left\| E_{R}^{\lambda} \right\|_{L_{x}^{2}} + \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}} \right)
\leq K \left( \left\| E_{R}^{\lambda} \right\|_{L_{x}^{2}} + \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}} \right)
\leq K \left( \left\| E_{R}^{\lambda} \right\|_{L_{x}^{2}} + \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}} \right)
\leq K \left( \left\| E_{R}^{\lambda} \right\|_{L_{x}^{2}} + \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}} \right)
\leq K \left( \left\| E_{R}^{\lambda} \right\|_{L_{x}^{2}} + \left\| E_{R,x}^{\lambda} \right\|_{L_{x}^{2}} \right)
\[
\leq \left[ K (\delta_4 \| z^{\lambda}_R \|^2_{L^2_x} + \lambda^2 \| E^{\lambda}_R \|^2_{L^2_x}) + (\delta_5 \| z^{\lambda}_R, x \|^2_{L^2_x} + \lambda^2 \| E^{\lambda}_R, x \|^2_{L^2_x}) (t = 0) \right] \\
+ \int_0^t \left[ M (\| z^{\lambda}_R, x \|^2_{L^2_x} + \lambda^2 \| E^{\lambda}_R, x \|^2_{L^2_x}) + \lambda^2 (\| E^{\lambda}_R \|^2_{L^2_x}) + \lambda^2 (\| E^{\lambda}_R, x \|^2_{L^2_x}) \right] (s) \mathrm{d}s \\
+ M \lambda^4 \left( \| E^{\lambda}_R \|^2_{L^2_x} + \| E^{\lambda}_R, x \|^2_{L^2_x} \right)
\]
(89)

Now introduce the \( \lambda \)-weighted Liapunov-type functional

\[
\Gamma^{\lambda}(t) = \| (z^{\lambda}_R, z^{\lambda}_R, x) (t) \|^2_{L^2_x} + \lambda^2 \| (E^{\lambda}_R, E^{\lambda}_R, x) (t) \|^2_{L^2_x}
\]
(90)

and

\[
G^{\lambda}(t) = \| (z^{\lambda}_R, z^{\lambda}_R, x, E^{\lambda}_R, E^{\lambda}_R, x) (t) \|^2_{L^2_x} + \lambda^2 \| (E^{\lambda}_R, x, E^{\lambda}_R, x, x) (t) \|^2_{L^2_x}.
\]
(91)

Then it follows from (89)–(91) that

\[
\Gamma^{\lambda}(t) + c_5 \int_0^t G^{\lambda}(s) \mathrm{d}s \leq M \Gamma^{\lambda}(t = 0) + M \int_0^t (\Gamma^{\lambda}(s) + (\Gamma^{\lambda}(s))^2) \mathrm{d}s \\
+ M \int_0^t \Gamma^{\lambda}(s) G^{\lambda}(s) \mathrm{d}s + M \lambda^4.
\]
(92)

The conditions of the Lemma 10 are then satisfied, with \( \tau = 0, \iota = 2, q = 4 \). This, together with assumption (24), yields

\[
\sup_{0 \leq t \leq T} \left( \| (z^{\lambda}_R, z^{\lambda}_R, x) (t) \|^2_{L^2_x} + \lambda^2 \| (E^{\lambda}_R, E^{\lambda}_R, x) (t) \|^2_{L^2_x} \right) \leq M \lambda^\min\{\alpha, 4\} - \delta
\]
for any \( \delta \in (0, \min\{\alpha, 4\}) \).

This completes the proof of Theorem 3. \( \Box \)

**Proof of Theorem 5.** By transforms (13) and (18) and Theorem 3, we easily obtain Theorem 5. \( \Box \)

5. The case of ill-prepared initial data

In this section we investigate the asymptotic behavior of the solution to the problem (1)–(5) for ill-prepared initial data as \( \lambda \to 0. \) It turns out that in this case there is an additional fast time scale which corresponds to the change of the displacement current \( \lambda^2 E^{\lambda}_t. \)
Proof of Theorem 4. Since the problem (14)–(17) has the initial data \((z_0^\lambda(x), E_0^\lambda(x)) = (Z_0(x) + \lambda z_{R,0}^\lambda(x), E_0(x) + \lambda E_{R,0}^\lambda(x))\) with \(E_0(x) \neq -\frac{D_x(x)}{Z_0(x)}\) while the solution of the limit or inner solution problem (19)–(22) has the value at time \(t = 0\) as
\[
(Z(x, t), E(x, t))(t = 0) = \left(Z_0(x), \frac{D_x(x)}{Z_0(x)}\right)
\]
there exists the initial time layer for the electric field in the case for generally arbitrary initial data.

Now we formally derive the equations for the initial time layer functions.

Physically there exists the scaled dielectric relaxation time scale \(s = \frac{1}{\lambda^2}\) as the reference time. Hence heuristically we take the form of the solution \((z^\lambda, E^\lambda)\) to the problem (14)–(17) as
\[
\begin{align*}
(z^\lambda(x, t) &= Z(x, t) + \lambda^2 z_I^\lambda(x, s) + z_R^\lambda(x, t), \\
E^\lambda(x, t) &= E(x, t) + E_I^\lambda(x, s) + E_R^\lambda(x, t)
\end{align*}
\]
with
\[
z_I^\lambda(x, s = 0) = 0, \quad E_I^\lambda(x, s = 0) = E_0(x) - E(x, t = 0)
\]
and
\[
z_R^\lambda(x, t = 0) = z_{R,0}^\lambda(x), \quad E_R^\lambda(x, t = 0) = E_{R,0}^\lambda(x).
\]

Replacing (93), (94) in (14), (15) and subtracting (19), (20), one gets
\[
\begin{align*}
z_{R,t}^\lambda + z_{I,s}^\lambda &= (\lambda^2 z_{I,x}^\lambda + D E_I^\lambda + z_R^\lambda + D E_R^\lambda)_x - \lambda^2 (E_R^\lambda E_{R,x})_x \\
&\quad - \lambda^2 (E_{R,i}^\lambda E_{R,x}) - \lambda^2 (E_{I,i}^\lambda E_I^\lambda)_x - \lambda^2 (E_I^\lambda E_{R,x})_x \\
&\quad - \lambda^2 (E_{I,x}^\lambda E_R^\lambda)_x - \lambda^2 (E_I^\lambda E_{I,x})_x - \lambda^2 (E_{I,x}^\lambda E_R^\lambda)_x,
\end{align*}
\]
\[
\begin{align*}
\lambda^2 ((E_{I,i}^\lambda - E_{I,x}) + (E_{R,i}^\lambda - E_{R,x}^\lambda)) + E_{I,s}^\lambda \\
&= -Z(x, 0) E_I^\lambda - Z E_R^\lambda - E z_R^\lambda - z_R^\lambda E_R^\lambda \\
&\quad - (E_I^\lambda z_R^\lambda + \lambda^2 z_I^\lambda E + \lambda^2 z_R^\lambda E_I^\lambda + \lambda^2 z_I^\lambda E_R^\lambda) - (Z(x, \lambda^2 s) - Z(x, 0)) E_I^\lambda.
\end{align*}
\]

Now we require that
\[
\begin{align*}
z_{I,s}^\lambda(x, s) &= (D(x) E_I^\lambda(x, s))_x, \\
E_{I,s}^\lambda(x, s) &= -Z(x, 0) E_I^\lambda(x, s),
\end{align*}
\]
which are so-called initial layer equations. Thus, from (97)–(100), one gets
\[
\begin{align*}
z_{R,t}^\lambda &= (z_{R,x}^\lambda + D E_R^\lambda)_x - \lambda^2 (E_R^\lambda E_{R,x})_x - \lambda^2 (E_{R,x}^\lambda E_R^\lambda)_x \\
&\quad - \lambda^2 (E_I^\lambda E_R^\lambda)_x - \lambda^2 (E_I^\lambda E_{I,x})_x - \lambda^2 (E_I^\lambda)_{xx} + \lambda^2 z_{I,xx}^\lambda,
\end{align*}
\]
\[ \lambda^2 (E_{R,t}^\lambda - E_{R,xx}^\lambda) + Z E_R^\lambda = - (E + E_I^\lambda) z_R^\lambda - z_R^\lambda E_R - \lambda^2 (\mathcal{E}_t - \mathcal{E}_{xx}) + \lambda^2 E_{I,xx}^\lambda - \lambda^2 z_I^\lambda \mathcal{E} \]
\[ - \lambda^2 z_I^\lambda E_I^\lambda - \lambda^2 z_I^\lambda E_{R}^\lambda - (Z(x, t) - Z(x, 0)) E_I^\lambda. \quad (102) \]

Noting that the system (101), (102) is similar to (54), (55) except for the additional initial layer terms, the proof of Theorem 4 is similar to that of Theorem 3 except the treatment related to the initial time layer functions \( z_I^\lambda, E_I^\lambda \). For completeness, we outline our method with special emphasis on how to control those terms.

We first give some properties of the initial time layer functions \( z_I^\lambda \) and \( E_I^\lambda \).

**Lemma 13.** (i) For any \( k_1 = 0, 1, 2, 3 \) and \( k_2 = 0, 1, 2, 3 \), there exists a positive constant \( M \) such that
\[ \| \partial^{k_1} x E_I^\lambda, \partial^{k_2} x z_I^\lambda \|_{L^\infty_x([0, 1] \times [0, T])} \leq M. \quad (103) \]

(ii) For \( x = 0, 1 \) and \( 0 \leq t \leq T \), we have
\[ E_I^\lambda(x, t) = E_{I,xx}^\lambda(x, t) = z_{I,x}^\lambda(x, t) = 0. \quad (104) \]

**Proof.** The proof is elementary. In fact, the solution of problem (99), (100) and (95) can be exactly given by
\[ z_I^\lambda(x, s) = \frac{Z_0(x) D(E_0(x) - \mathcal{E}(x, t = 0))}{Z_0(x)} - Z_0(x) s e^{-Z_0(x)s} \]
\[ + \frac{Z_0(x) D(E_0(x) - \mathcal{E}(x, t = 0)) - Z_0(x) (D(E_0(x) - \mathcal{E}(x, t = 0)))_x}{(Z_0(x))^2} \]
\[ \times (e^{-Z_0(x)s} - 1), \quad (105) \]
\[ E_I^\lambda(x, s) = (E_0(x) - \mathcal{E}(x, t = 0)) e^{-Z_0(x)s}. \quad (106) \]

Using (105), (106), \( Z_0, E_0 \in C^3 \), \( \mathcal{E}(x, t) \in C^3 \) \((0, 1] \times [0, T])\) and the fact \( D_{xx}(x = 0, 1) = 0 \), one easily get (103) and (104).

As in the previous Theorem 3, using (104) and the fact \( D_{xx}(x = 0, 1) = 0 \), we have
\[ z_{R,x}^\lambda = E_R^\lambda = E_{R,xx}^\lambda = 0, x = 0, 1, \quad t > 0. \quad (107) \]

The rest is to establish the energy estimates. We divide it into the following four steps.

**Step 1.** Multiplying (101) by \( z_R^\lambda \) and integrating the resulting equation over \([0, 1]\) with respect to \( x \), one gets, by (104), (107) and integrations by parts, that
\[ \frac{1}{2} \frac{d}{dt} \| z_R^\lambda \|_{L^2}^2 + \| z_{R,x}^\lambda \|_{L^2}^2 \]
\[ = - \int_0^1 D(x) E_R^\lambda z_{R,x}^\lambda dx + \lambda^2 \int_0^1 E_R^\lambda E_{R,x}^\lambda z_{R,x}^\lambda dx \]
\[ + \lambda^2 \int_0^1 \mathcal{E} E_{R,x}^\lambda z_{R,x}^\lambda \, dx + \lambda^2 \int_0^1 E_{R}^\lambda z_{R,x}^\lambda \, dx + \lambda^2 \int_0^1 \mathcal{E} x E_{R,x}^\lambda \, dx \]

\[ + \lambda^2 \int_0^1 (E_{I,x}^\lambda E_{R}^\lambda)_x z_{R,x}^\lambda \, dx + \lambda^2 \int_0^1 (E_{I,x}^\lambda E_{R}^\lambda + (E_{I,x}^\lambda E_{R}^\lambda)_x - z_{I,x}^\lambda) z_{R,x}^\lambda \, dx. \]  

(108)

Now we just estimate these terms related to the initial time layer functions.

By Cauchy–Schwarz’s inequality and (103), we have

\[ \lambda^2 \int_0^1 (E_{I,x}^\lambda E_{R}^\lambda)_x z_{R,x}^\lambda \, dx \leq \epsilon \| z_{R,x}^\lambda \|_{L^2_x}^2 + M(\epsilon) \lambda^4 \| (E_{R}^\lambda, E_{R,x}^\lambda) \|_{L^2_x}^2. \]  

(109)

By Cauchy–Schwarz’s inequality and using the regularity of \( \mathcal{E} \) and the properties (103) of \( E_{I,x}^\lambda \), one gets

\[ \lambda^2 \int_0^1 (E_{I,x}^\lambda E_{R}^\lambda + E_{I,x}^\lambda \mathcal{E} + \mathcal{E} E_{I,x}^\lambda - z_{I,x}^\lambda) z_{R,x}^\lambda \, dx \leq \epsilon \| z_{R,x}^\lambda \|_{L^2_x}^2 + M(\epsilon) \lambda^4. \]  

(110)

Thus, (108), together with (109), (110) and the previous estimates obtained in Theorem 3, gives, by choosing \( \epsilon \) small enough, that

\[ \frac{d}{dt} \| z_R^\lambda(t) \|_{L^2_x}^2 + c_6 \| z_{R,x}^\lambda \|_{L^2_x}^2 \leq M \| E_{R}^\lambda \|_{L^2_x}^2 + M\lambda^4 \| (E_{R}^\lambda, E_{R,x}^\lambda) \|_{L^2_x}^2 \]

\[ + M\lambda^4 \| (E_{R}^\lambda, E_{R,x}^\lambda) \|_{L^2_x}^2 \| E_{R}^\lambda \|_{L^2_x}^2 + M\lambda^4. \]  

(111)

**Step 2.** Multiplying (102) by \( E_{R}^\lambda \) and integrating the resulting equation over \([0, 1]\) with respect to \( x \), one gets, by (104), (107) and integrations by parts, that

\[ \frac{\lambda^2}{2} \frac{d}{dt} \| E_{R}^\lambda \|_{L^2_x}^2 + \lambda^2 \| E_{R,x}^\lambda \|_{L^2_x}^2 + \int_0^1 \mathcal{Z} \| E_{R}^\lambda \|_{L^2_x}^2 \, dx \]

\[ = -\lambda^2 \int_0^1 (\mathcal{E} - \mathcal{E}_{xx}) E_{R}^\lambda \, dx - \int_0^1 \mathcal{E} z_{R}^\lambda E_{R}^\lambda \, dx - \int_0^1 z_{R}^\lambda E_{R}^\lambda E_{R}^\lambda \, dx \]

\[ + \int_0^1 (\lambda^2 E_{I,xx}^\lambda - E_{I,x}^\lambda z_{R}^\lambda - \lambda^2 z_{I}^\lambda \mathcal{E} - \lambda^2 z_{I}^\lambda E_{I}^\lambda - \lambda^2 z_{I}^\lambda E_{I,x}^\lambda) E_{R}^\lambda \, dx \]

\[ - \int_0^1 (\mathcal{Z}(x, t) - \mathcal{Z}(x, 0)) E_{I,x}^\lambda E_{R}^\lambda \, dx. \]  

(112)
For the fourth term in the right-hand side of (112), caused the initial time layer, we have, by Cauchy–Schwarz’s inequality and using the regularity of $Z, E$ and the properties (103) of the initial time layer functions, that

\[
\int_0^1 \left( \lambda^2 E_{I,xx}^\lambda - E_{I}^\lambda z_R^\lambda - \lambda^2 z_I^\lambda E - \lambda^2 z_I^\lambda E_{I}^\lambda - \lambda^2 z_I^\lambda E_{R}^\lambda \right) E_{R}^\lambda \, dx \leq \epsilon \left\| E_{R}^\lambda \right\|_{L_x^2}^2 + M(\epsilon) \left( \left\| z_I^\lambda \right\|_{L_x^2}^2 + \lambda^4 \left\| E_{R}^\lambda \right\|_{L_x^2}^2 + \lambda^4 \right). \tag{113}
\]

For the fifth term in the right-hand side of (112), we have, by the mean value theorem, Cauchy–Schwarz’s inequality and using the regularity of $Z$ and the properties (103) of the initial time layer functions, that

\[
-\int_0^1 \left( Z(x, t) - Z(x, 0) \right) E_{I}^\lambda E_{R}^\lambda \, dx = - \int_0^1 \int_0^1 \partial_t Z(x, t\theta) \, d\theta E_{I}^\lambda E_{R}^\lambda \, dx \\
\leq \epsilon \left\| E_{R}^\lambda \right\|_{L_x^2}^2 + M(\epsilon) \max_{0 \leq t \leq T} \max_{0 \leq s \leq t} (t E_{I}^\lambda)^2 \\
\leq \epsilon \left\| E_{R}^\lambda \right\|_{L_x^2}^2 + M(\epsilon) \max_{0 \leq t \leq T} \left( te^{-\delta_0 \frac{t}{\lambda^2}} \right)^2 \\
\leq \epsilon \left\| E_{R}^\lambda \right\|_{L_x^2}^2 + M(\epsilon) \lambda^4 \max_{0 \leq s \leq \infty} (se^{-\delta_0 s})^2 \\
\leq \epsilon \left\| E_{R}^\lambda \right\|_{L_x^2}^2 + M(\epsilon) \lambda^4. \tag{114}
\]

Then choosing $\epsilon$ to be small enough and using the positivity of $Z$, combining (112) with (113) and (114) and the estimates obtained in the previous Theorem 3, one gets

\[
\lambda^2 \frac{d}{dt} \left\| E_{R}^\lambda \right\|_{L_x^2}^2 + 2\lambda^2 \left\| E_{R,xx}^\lambda \right\|_{L_x^2}^2 + c_2 \left\| E_{R}^\lambda \right\|_{L_x^2}^2 \\
\leq M \left\| z_{I,xx}^\lambda \right\|_{L_x^2}^2 + M \lambda^4 \left\| E_{R}^\lambda \right\|_{L_x^2}^2 + M \left\| (z_{R}^\lambda, z_{R,xx}^\lambda) \right\|_{L_x^2}^2 \left\| E_{R}^\lambda \right\|_{L_x^2}^2 + M\lambda^4. \tag{115}
\]

To enclose the energy estimate, we obtain higher order Sobolev’s estimates next. 

**Step 3.** Differentiating (101) with respect to $x$, multiplying the resulting equations by $z_{R,xx}^\lambda$ and then integrating it over $[0, 1]$, one gets, by (104), (107) and integrations by parts, that

\[
\frac{1}{2} \frac{d}{dt} \left\| z_{R,xx}^\lambda \right\|_{L_x^2}^2 + \left\| z_{R,xx}^\lambda \right\|_{L_x^2}^2 \\
= - \int_0^1 \left( D(x) E_{R,x}^\lambda \right) z_{R,xx}^\lambda \, dx + \lambda^2 \int_0^1 \left( E_{R}^\lambda E_{R,x}^\lambda \right) z_{R,xx}^\lambda \, dx
\]
\[ + \lambda^2 \int_0^1 \left( E_{R,x}^\lambda \right)_x \lambda^2 z_{R,xx}^\lambda \, dx + \lambda^2 \int_0^1 \left( E_{x} E_R^\lambda \right)_x \lambda^2 z_{R,xx}^\lambda \, dx + \lambda^2 \int_0^1 \left( \mathcal{E}_x \right)_x \lambda^2 z_{R,xx}^\lambda \, dx \]

\[ + \lambda^2 \int_0^1 \left( E_{1,x}^\lambda E_R^\lambda \right)_{xx} \lambda^2 z_{R,xx}^\lambda \, dx + \lambda^2 \int_0^1 \left( E_{I,x}^\lambda E_I^\lambda + \left( E_{I}^\lambda \mathcal{E}_x \right)_x - z_{I,xx}^\lambda \right) \lambda^2 z_{R,xx}^\lambda \, dx. \]

(116)

Now we also only estimate those terms related to the initial time layer functions.

By Cauchy–Schwarz’s inequality, we have

\[ \lambda^2 \int_0^1 \left( E_{I,x}^\lambda E_I^\lambda \right)_{xx} \lambda^2 z_{R,xx}^\lambda \, dx \leq \epsilon \left\| z_{R,xx}^\lambda \right\|^2_{L^2_x} + M(\epsilon) \lambda^4 \left\| \left( E_{R}, E_{R,x}, E_{R,xx}^\lambda \right) \right\|^2_{L^2_x}. \]

(117)

By Cauchy–Schwarz’s inequality and using the regularity of \( \mathcal{E} \) and the properties (103) of \( z_I^\lambda \) and \( E_I^\lambda \), one gets

\[ \lambda^2 \int_0^1 \left( E_{I,x}^\lambda E_{I,x}^\lambda + \left( E_{I}^\lambda \mathcal{E}_x \right)_x - z_{I,xx}^\lambda \right) \lambda^2 z_{R,xx}^\lambda \, dx \leq \epsilon \left\| z_{R,xx}^\lambda \right\|^2_{L^2_x} + M(\epsilon) \lambda^4. \]

(118)

Here we have required the regularity of order 2 for \( z_I^\lambda \).

Thus, taking \( \epsilon \) to be small enough, combining (116) with (117) and (118) and the estimates obtained in the previous Theorem 3, one gets

\[
\frac{d}{dt} \left\| z_{R,x}^\lambda \right\|^2_{L^2} + c_3 \left\| z_{R,xx}^\lambda \right\|^2_{L^2_x} \\
\leq M \left\| (E_{R}, E_{R,x}) \right\|^2_{L^2_x} + M \lambda^4 \left\| \left( E_{R}, E_{R,x}, E_{R,xx}^\lambda \right) \right\|^2_{L^2_x} + M \lambda^4 \\
+ M \lambda^4 \left( \left\| (E_{R}, E_{R,x}) \right\|^2_{L^2_x} \left\| E_{R,xx}^\lambda \right\|^2_{L^2_x} + \left\| (E_{R,x}, E_{R,xx}^\lambda) \right\|^2_{L^2_x} \left\| E_{R,xx}^\lambda \right\|^2_{L^2_x} \right). \]

(119)

Step 4. Differentiating (102) with respect to \( x \), multiplying the resulting equations by \( E_{R,x}^\lambda \) and then integrating it over \([0, 1] \), one gets, by (104), (107) and integrations by parts, that

\[
\frac{\lambda^2}{2} \frac{d}{dt} \left\| E_{R,x}^\lambda \right\|^2_{L^2} + \lambda^2 \left\| E_{R,xx}^\lambda \right\|^2_{L^2_x} + \int_0^1 \mathcal{Z} \left| E_{R,x}^\lambda \right|^2 \, dx \\
= -\lambda^2 \int_0^1 \left( \mathcal{E}_{x,t} - \mathcal{E}_{xxx} \right) E_{R,x}^\lambda \, dx - \int_0^1 \left( \mathcal{E}_{z_R}^\lambda \right)_x E_{R,x}^\lambda \, dx - \int_0^1 \left( z_R^\lambda E_{R}^\lambda \right)_x E_{R,x}^\lambda \, dx \\
- \int_0^1 \mathcal{Z}_x E_{R}^\lambda E_{R,x}^\lambda \, dx + \int_0^1 \left( \lambda^2 E_{I,xx}^\lambda - E_{I,x} z_{R} - \lambda^2 z_{R} E_{I} - \lambda^2 z_{R} E_{I}^\lambda \right)_x E_{R,x}^\lambda \, dx \\
- \int_0^1 \left( \left( \mathcal{Z}(x,t) - \mathcal{Z}(x,0) \right) E_{I}^\lambda \right)_x E_{R,x}^\lambda \, dx. \]

(120)
For the fifth term, by Cauchy–Schwarz’s inequality and using the regularity of $Z, \mathcal{E}$, the properties of the initial time layer functions, we have

$$
\int_0^1 \left( \lambda^2 E_{I,xx}^\lambda - E_I^\lambda z_R^\lambda - \lambda^2 z_I^\lambda \mathcal{E} - \lambda^2 z_I^\lambda E_I^\lambda - \lambda^2 z_I^\lambda E_R^\lambda \right) E_R, x \, dx
\leq \epsilon \| E_R, x \|_{L_x^2}^2 + M(\epsilon) \left( \| (z_R^\lambda, z_R^\lambda, E_R^\lambda, E_R, x) \|_{L_x^2}^2 + \lambda^4 \| (E_R^\lambda, E_R, x) \|_{L_x^2}^2 + \lambda^4 \right).
$$

(121)

Here we have required the regularity of order 3 for $E_I^\lambda$.

For the sixth term, we have, by the mean value theorem, Cauchy–Schwarz’s inequality and using the regularity of $Z$, i.e., $Z_{tx} \in C([0, 1] \times [0, T])$, and the properties of the initial time layer functions, that

$$
- \int_0^1 \left( (Z(x, t) - Z(x, 0)) E_I^\lambda \right) x E_R, x \, dx
= - \int_0^1 \int_0^1 \partial_t Z(x, t\theta) d\theta t E_{I,x}^\lambda E_R, x \, dx - \int_0^1 \int_0^1 \partial_x Z(x, t\theta) d\theta t E_I^\lambda E_R, x \, dx
\leq \epsilon \| E_R, x \|_{L_x^2}^2 + M(\epsilon) \max_{0 \leq t \leq T} \max_{0 \leq x \leq 1} \left( t \| (E_I^\lambda, E_{I,x}^\lambda) \|_{L_x^2}^2 \right)^2
\leq \epsilon \| E_R, x \|_{L_x^2}^2 + M(\epsilon) \max_{0 \leq t \leq T} \left( t \left( 1 + \frac{t}{\lambda^2} \right) e^{-\delta_0 \frac{t}{\lambda^2}} \right)^2
\leq \epsilon \| E_R, x \|_{L_x^2}^2 + M(\epsilon) \lambda^4 \max_{0 \leq s \leq \infty} \left( s(1 + s)e^{-\delta_0 s} \right)^2 \leq \epsilon \| E_R, x \|_{L_x^2}^2 + M\lambda^4.
$$

(122)

Then, choosing $\epsilon$ to be small enough and using the positivity of $Z$, combining (120) with (121) and (122) and the estimates obtained in the previous Theorem 3, one gets

$$
\lambda^2 \frac{d}{dt} \| E_R, x \|_{L_x^2}^2 + 2\lambda^2 \| E_{R,xx}^\lambda \|_{L_x^2}^2 + C_4 \| E_R, x \|_{L_x^2}^2
\leq M \left( \| z_R^\lambda, z_{R,x}^\lambda, E_R^\lambda \|_{L_x^2}^2 + M \left( \| z_R^\lambda \|_{L_x^2}^2 \| E_R^\lambda \|_{L_x^2}^2 + \| (E_R^\lambda, E_R, x) \|_{L_x^2}^2 \right) \right) + M\lambda^4.
$$

(123)

Thus, we have established the same energy estimates as in the previous case for well-prepared initial data. Exactly as in the proof of Theorem 3, we conclude our results.

This completes the proof of Theorem 4. □

**Proof of Theorem 6.** By transforms (13) and (18) and using Theorem 4, we easily obtain Theorem 6. □
Remark 4. Noting that the techniques to establish energy estimates used here cannot be applied to the case of the general doping profile without the boundary restriction assumption. This will be discussed in the future.

References