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Note

Full orientability of graphs with at most one dependent arc

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ABSTRACT

Suppose that D is an acyclic orientation of a graph G. An arc of D is *dependent* if its reversal creates a directed cycle. Let $d_{\min}(G)$ ($d_{\max}(G)$) denote the minimum (maximum) of the number of dependent arcs over all acyclic orientations of G. We call G *fully orientable* if G has an acyclic orientation with exactly d dependent arcs for every d satisfying $d_{\min}(G) \leqslant d \leqslant d_{\max}(G)$. We show that a connected graph G is fully orientable if $d_{\min}(G) \leqslant 1$. This generalizes the main result in Fisher et al. [D.C. Fisher, K. Fraughnaugh, L. Langley, D.B. West, The number of dependent arcs in an acyclic orientation, J. Combin. Theory Ser. B 71 (1997) 73–78].

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1. Introduction

Graphs considered in this paper are finite, without loops, or multiple edges. For a graph $G(V, E)$ with vertex set V and edge set E, we use $|G|$ and $\|G\|$ to denote the cardinalities of V and E, respectively. The degree of a vertex v of G is denoted by $d_G(v)$. An orientation D of G is obtained by assigning a fixed direction, either $x \rightarrow y$ or $y \rightarrow x$, on every edge xy of G. The original undirected graph is called the *underlying* graph of any such orientation. *Sources* (or *sinks*) are vertices with no ingoing (or outgoing) arcs. An orientation D is called *acyclic* if there does not exist any directed cycle.

Suppose that D is an acyclic orientation of G. An arc $u \rightarrow v$ of D, or its underlying edge, is called *dependent* (in D) if there exists a directed cycle in the new orientation $D' = (D - (u \rightarrow v)) \cup (v \rightarrow u)$. Note that $u \rightarrow v$ is a dependent arc if and only if there exists a directed walk of length at least 2 from u to v. Let $d(D)$ denote the number of dependent arcs in D. Let $d_{\min}(G)$ and $d_{\max}(G)$ be, respectively, the minimum and maximum values of $d(D)$ over all acyclic orientations D of G. It is known [3] that $d_{\max}(G) = \|G\| - |G| + k$ for a graph G having k components.

If any integer d satisfying $d_{\min}(G) \leqslant d \leqslant d_{\max}(G)$ is achievable as $d(D)$ for some acyclic orientation D of G, then G is said to be *fully orientable*. Otherwise, it is called *non-fully-orientable*. West [14] showed that complete bipartite graphs are fully orientable. Let $\chi(G)$ denote the *chromatic number* of G, i.e., the least number of colors to color the vertices of G so that adjacent vertices receive distinct colors. Let $g(G)$ denote the *girth* of G, i.e., the length of a shortest cycle of G if there is any, and ∞ if G possesses no cycles. Fisher et al. [3] showed that G is fully orientable if $\chi(G) < g(G)$. They also proved that $d_{\min}(G) = 0$ when $\chi(G) < g(G)$. When viewed as an undirected graph, the Hasse diagram of a partially ordered set is called a *cover graph*. We say that G is a *cover graph* if it is the underlying graph of the Hasse diagram for some partially ordered set. Note that $d_{\min}(G) = 0$ is equivalent to G being a cover graph.

A graph G is called *k-degenerate* if each subgraph H of G contains a vertex of degree at most k in H. Lai et al. [5] established the full orientability of 2-degenerate graphs that generalizes a previous result for outerplanar graphs in Lih et al. [7].

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Lai and Lih [6] gave further examples of fully orientable graphs, such as subdivisions of Halin graphs and graphs of maximum degree at most three. There are also results about dependent arcs in Collins and Tysdal [2] as well as Rödl and Thoma [13].

The main result obtained in this paper is the following. *A connected graph G is fully orientable if* $d_{\min}(G) \leqslant 1$. In Section 2, we introduce the source-reversal operation and an old result of Mosesian [8]. We will obtain an interval of possible values for $d(D)$, and then derive our main result. We conclude our paper by posing several open questions concerning the existence of non-fully-orientable graphs.

2. Results

Let u be a source of the acyclic orientation D. A *source-reversal* operation applied to u reverses the direction of all outgoing arcs from u so that u becomes a sink afterward. This operation preserves the acyclicity of D. Note that, if there are no dependent arcs in D, neither will there be any in the new acyclic orientation.

The following result originally appeared in Mosesian [8]. It was put to good use by Pretzel in a series of papers (for example, [9–12]).

Theorem 1. *Let D be an acyclic orientation of a connected graph G. For any vertex u of G, there exists an acyclic orientation, D', of G obtained from D by a sequence of source-reversals so that u becomes the unique source of D'. The source u is connected to any vertex v by a directed path in D'.*

For an acyclic orientation D of the underlying graph G, the vertex set $V(G)$ can be partitioned as follows. Let V_1 be the set of all sources of D, and V_i the set of all sources of the acyclic directed subgraph $D - (V_1 \cup \cdots \cup V_{i-1})$. We thus obtain a decomposition $V_1 \cup \cdots \cup V_s$ of $V(G)$, called the *canonical decomposition* by Goles and Prisner [4], such that arcs only go from sets V_i toward sets V_j with $j > i$. Moreover, for $1 < i \leqslant s$, every $x \in V_i$ is the head of an arc $y \to x$ for some $y \in V_{i-1}$. If a vertex x belongs to a part V_i, then we call the index i the *level* of x.

Let G be a connected graph. Use depth-first search to construct a spanning tree T of G and index the vertices as $v_1, v_2, \ldots, v_{|G|}$ in the order of that search. Label each edge v_iv_j of $E(G) \setminus E(T)$ by the pair (i, j) with $i < j$. Then order these edges in lexicographical order by their labels. Observe that there are $\|G\| - |G| + 1$ edges in $E(G) \setminus E(T)$. We also note that each edge in $E(G) \setminus E(T)$ joins a vertex with one of its ancestors on the tree T.

Let k_T denote the least number t such that, when the first t edges of $E(G) \setminus E(T)$ in the above lexicographical order are removed, the remaining subgraph of G is a cover graph. Since T is a cover graph, $k_T \leqslant \|G\| - |G| + 1$.

Theorem 2. *If G is a connected graph and T is a spanning tree of G obtained by depth-first search, then every number d satisfying $k_T \leqslant d \leqslant \|G\| - |G| + 1$ is achievable as $d(D)$ for some acyclic orientation D of G.*

Proof. Index the vertices of T as $v_1, v_2, \ldots, v_{|G|}$ according to the order given by depth-first search. Let $k_T \leqslant d \leqslant \|G\| - |G| + 1$. We are going to construct an acyclic orientation D of G with $d(D) = d$ as follows.

Let G_0 and G' be, respectively, the subgraphs obtained from G by removing the first k_T and d edges of $E(G) \setminus E(T)$ in the lexicographical order. By definition, G_0 is a cover graph. Hence, the subgraph G' of G_0 is a cover graph. Then there exists an acyclic orientation D' of G' having no dependent arcs. By Theorem 1, we may assume that v_1 is the unique source of D'. Let $V_1 \cup \cdots \cup V_s$ be the canonical decomposition with respect to D'. Now we orient each edge xy of G into $x \to y$ if and only if the level of x is lower than that of y. Note that this new acyclic orientation D is an extension of D'.

Claim 1. *If $v_iv_j \in E(G) \setminus E(G')$ and $i < j$, then v_iv_j is a dependent edge in D and there is a directed path of length at least 2 from v_i to v_j in D'.*

Since v_i and v_j are adjacent and $i < j$, v_i is an ancestor of v_j in the spanning tree T obtained by depth-first search. The unique path P_0 from v_1 to v_j in T passes through v_i. Assume that $P_1 \neq P_0$ is another path from v_1 to v_j in G'. According to the lexicographical order, we may consider the first edge $v_pv_q \in E(P_1) \setminus E(P_0)$. If $p < i$, then v_pv_q precedes v_iv_j in the lexicographical order. By the construction of G', the edge v_pv_q should have been removed and cannot belong to $E(P_1)$. Hence, P_1 coincides with P_0 from v_1 to v_i. It follows that there is a unique path from v_1 to v_i in G' and every path from v_1 to v_j in G' passes through v_i. By Theorem 1, there is a directed path from v_1 to v_j in D'; hence there is a directed path P from v_i to v_j in D'. This implies that the level of v_i is lower than that of v_j. Since $v_iv_j \notin E(G')$, the directed path P has length at least 2. Therefore, $v_i \to v_j$ is a dependent arc in D.

Claim 2. *If $xy \in E(G')$, then xy is not a dependent edge in D.*

We may assume that the edge xy is oriented as $x \to y$. Suppose on the contrary that $x \to y$ is a dependent arc in D. Then there would be a directed walk W of length at least 2 from x to y in D. Since $x \to y$ is not a dependent arc in D', W contains an arc $v_p \to v_q$ in $E(G) \setminus E(G')$ for some $p < q$. We can find a directed path P^* from v_p to v_q in D' by Claim 1. Then the directed walk obtained from W by replacing $v_p \to v_q$ by P^* is of length at least 2 and having one fewer edge in $E(G) \setminus E(G')$. Continuing in this way, we finally obtain a directed walk $z_1(= x), z_2, \ldots, z_k(= y)$ of length at least 2 from x to y in D', and hence $x \to y$ is a dependent arc in D'. We have arrived at a contradiction.

It follows from Claims 1 and 2 that an edge e is not dependent in D if and only if $e \in E(G')$, which amounts to $d(D) = d$. ∎

Theorem 3. *If G is a connected graph with $d_{\min}(G) \leqslant 1$, then G is fully orientable.*

Proof. If $d_{\min}(G) = 0$, then $k_T = 0$ for each spanning tree T obtained by depth-first search.

If $d_{\min}(G) = 1$, then there exists an acyclic orientation of G having a unique dependent edge xy. This also shows that $G - xy$ is a cover graph.

Among all cycles containing xy, let $C : c_1(= x), c_2, \ldots, c_t(= y)$ have the shortest length. Obviously, any two vertices on C cannot be adjacent unless they are consecutive on C. We say that C is *chordless*.

Let T be the tree obtained by depth-first search rooted at x so that the edges $c_1c_2, c_2c_3, \ldots, c_{t-1}c_t$ are chosen in the first $t - 1$ steps. Since C is chordless, the first edge of $E(G) \setminus E(T)$ in the lexicographical order determined by T must be xy. Since $G - xy$ is a cover graph and G is not, it follows that $k_T = 1$.

In each case, we can find a spanning tree T by depth-first search satisfying $k_T = d_{\min}(G)$. We are done by Theorem 2. ■

In Fisher et al. [3], a connected graph G was shown to be fully orientable if $\chi(G) < g(G)$ and a special construct, called the *color-first tree*, was used in its proof. Since it is well known that $\chi(G) < g(G)$ implies $d_{\min}(G) = 0$, Theorem 3 is a generalization of their result and a color-first tree is actually a particular way of defining an acyclic orientation without dependent arcs.

Lemma 4. *Let $G_1, G_2, \ldots, G_k$ be all the components of G. Then G is fully orientable if all $G_1, G_2, \ldots, G_k$ are fully orientable.*

Proof. It is obvious that $d_{\min}(G) \leqslant \sum_{i=1}^{k} d_{\min}(G_i)$. If $d_{\min}(G) < \sum_{i=1}^{k} d_{\min}(G_i)$, then, for some j, there is an acyclic orientation D_j of G_j such that $d(D_j) < d_{\min}(G_j)$, which is impossible. Therefore, $d_{\min}(G) = \sum_{i=1}^{k} d_{\min}(G_i)$. For any d satisfying $d_{\min}(G) \leqslant d \leqslant d_{\max}(G) = \|G\| - |G| + k = \sum_{i=1}^{k} d_{\max}(G_i)$, we can rewrite d as $\sum_{i=1}^{k} d_i$, where $d_{\min}(G_i) \leqslant d_i \leqslant d_{\max}(G_i)$ for each i. Since every G_i is fully orientable, there is an acyclic orientation D_i of G_i with $d(D_i) = d_i$. The union of all these acyclic orientations is an acyclic orientation D of G with $d(D) = d$. ■

Corollary 5. *Let G be a graph with components $G_1, G_2, \ldots, G_k$. If $d_{\min}(G_i) \leqslant 1$ for each $1 \leqslant i \leqslant k$, then G is fully orientable.*

Proof. Since $d_{\min}(G_i) \leqslant 1$ for each $1 \leqslant i \leqslant k$, every component of G is fully orientable by Theorem 3. By Lemma 4, G is fully orientable. ■

Corollary 6. *Let G be a connected graph such that, for some vertex v, $G - v$ is a cover graph. Suppose that $G - v$ has m components $G_1, G_2, \ldots, G_m$. Then every number d satisfying $d_G(v) - m \leqslant d \leqslant \|G\| - |G| + 1$ is achievable as $d(D)$ for some acyclic orientation D of G.*

Proof. Let T be a depth-first search tree of G such that v is the root of T. All the edges in $E(G) \setminus E(T)$ incident with v constitute the first $d_G(v) - m$ edges in the lexicographical order determined by T. Consider the subgraph G' obtained from G by removing these edges. Then the vertex v in G' is joined to each component G_i of $G - v$ with a single edge. The graph $G - v$ is assumed to be a cover graph; hence each G_i is a cover graph. For each $1 \leqslant i \leqslant m$, let D_i be an acyclic orientation of G_i satisfying $d(D_i) = d_{\min}(G_i) = 0$. We get an acyclic orientation D' of G' by orienting each component G_i with D_i and v into a source. Since each edge incident with v is a cut edge in G', every cycle of G' lies entirely inside some G_i. It follows that $d(D') = \sum_{i=1}^{k} d_{\min}(G_i) = d_{\min}(G') = 0$, and hence G' is a cover graph. Therefore, $k_T \leqslant d_G(v) - m$. We are done by Theorem 2. ■

Let $K_{r(n)}$ denote the complete r-partite graph each of whose partite sets has n vertices. Chang et al. [1] proved that $K_{r(n)}$ is non-fully-orientable when $r \geq 3$ and $n \geq 2$. These are the only non-fully-orientable graphs known so far and they all have girth 3. An immediate consequence is that, when m is a composite number, there exist m-degenerate graphs which are non-fully-orientable. Moreover, it can be checked that $K_{3(2)}$ is the smallest non-fully-orientable graph. Any acyclic orientation of $K_{3(2)}$ has 4, 6, or 7 dependent arcs. Based upon the above data, we conclude this paper by posing the following open questions.

Question 1. For a given odd prime p, does there exist a non-fully-orientable p-degenerate graph that is not $(p - 1)$-degenerate?

Question 2. For any given integer $g \geq 4$, does there exist a non-fully-orientable graph G whose girth is g?

Question 3. Does there exist a non-fully-orientable graph G whose $d_{\min}(G)$ is 2 or 3?

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