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On Gopalsamy and Liu's conjecture for global stability in a population model

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Abstract

For a logistic equation with a piecewise constant argument which models the dynamics of a population of a single species undergoing a density-dependent harvesting, Gopalsamy and Liu (J. Math. Anal. Appl. 224 (1998) 59–80), have offered a condition of the growth rate of species and conjectured that this is a necessary and sufficient condition of not only the asymptotic stability but also the global asymptotic stability for the positive equilibrium of the equation. But until now, there were no mathematical answers except computer simulations.

In this paper, we establish a mathematically rigorous proof of a partial affirmative answer of this conjecture.

MSC: 34K20; 34K60; 39A11

Keywords: Asymptotic stability; Global asymptotic stability; Logistic equation with a piecewise constant delay

1. Introduction

Consider the following differential equation with a piecewise constant argument:

$$\begin{aligned} &\frac{\mathrm{d}N(t)}{\mathrm{d}t} = rN(t)\{1 - aN(t) - bN([t])\}, \quad t > 0, \\ &N(0) = N_0 > 0, \end{aligned} \tag{1.1}$$

where $r, a, b > 0$ and $[t]$ denote the maximal integer less than or equal to t.

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This equation models the dynamics of a population undergoing a density-dependent harvesting and $N(t)$ denotes the biomass (or population density) of a single species at time t. The differential equation (1.1) is reduced to a difference one (see Eq. (2.3)) which is then studied.

For the case $a=0$ in Eq. (1.1), several authors have investigated the stability and oscillatory characteristics of Eq. (1.1) (see [1,2,5,7,8,11–13] and the references cited therein). Gopalsamy [3] proved that for $a=0$ in Eq. (1.1), the positive equilibrium $N^*=1/b$ of Eq. (1.1), is globally asymptotically stable, if and only if, $r\leqslant 2$ (see also [9]).

Recently, using elementary methods of differential calculus, Gopalsamy and Liu [4] gave a sufficient condition for all positive solutions of Eq. (1.1) to converge to the positive equilibrium, and generalized a result known for a simpler special model with no harvesting.

For the solution $N(t)$ and the positive equilibrium $N^*=1/(a+b)$ of Eq. (1.1), put

$$x(n)=bN(n),\ x^*=bN^* \quad \text{and} \quad \alpha=a/b, \quad \text{for } b>0.$$

Gopalsamy and Liu [4] have considered a Lyapunov function $V(n)=(x(n)-x^*)^2$, calculated the change $V(n+1)-V(n)$ along the solution $x(n)$, and showed that for $\alpha>0$, N^* is globally asymptotically stable, if

$$\begin{cases} r<+\infty & \text{for } \alpha\geqslant 1,\\ r\leqslant r^*(\alpha)\equiv \dfrac{1}{\alpha}\ln(1+2\alpha)+\ln\dfrac{1+\alpha}{1-\alpha} & \text{for } 0<\alpha<1.\end{cases}$$

Moreover, using computer aided method, they showed that for $0<\alpha<1$, N^* is asymptotically stable, if

$$r\leqslant \hat{\hat{r}}(\alpha)\equiv \frac{1+\alpha}{\alpha}\ln\frac{1+\alpha}{1-\alpha}. \tag{1.2}$$

They also conjectured that for $0<\alpha<1$, Eq. (1.2) is sufficient for N^* to be globally asymptotically stable. Liu and Gopalsamy [6] investigated and justified this conjecture by computer plots of solutions for various values of r and α.

Note that

$$\begin{aligned} r^*(\alpha)&\leqslant 2+\ln\frac{1+\alpha}{1-\alpha}\leqslant\frac{1}{2}\left\{2(1+\alpha)+\frac{1+\alpha}{\alpha}\ln\frac{1+\alpha}{1-\alpha}\right\}\\ &<\hat{\hat{r}}(\alpha)=2(1+\alpha)\left(1+\frac{\alpha^2}{3}+\frac{\alpha^4}{5}+\cdots\right), \quad \text{for } 0<\alpha<1. \end{aligned} \tag{1.3}$$

Muroya [10] established necessary and sufficient conditions for the persistence and contractivity of solutions, respectively (see [10, Theorem 3.1]). The contractivity condition is a sufficient condition of the global asymptotic stability for the positive equilibrium.

In this paper, we establish the following two theorems:

Theorem 1.1. *For $0<a<b$, the positive equilibrium N^* of Eq.* (1.1) *is asymptotically stable, if and only if, Eq.* (1.2) *holds for $0<\alpha=a/b<1$.*

Theorem 1.2. *For $0<a\leqslant\bar{\alpha}_1 b$, the positive equilibrium N^* of Eq.* (1.1) *is globally asymptotically stable, if and only if Eq.* (1.2) *holds, where $0<\alpha=a/b\leqslant\bar{\alpha}_1$ and $\bar{\alpha}_1=0.634817\cdots$ is a unique positive solution* *of the equation*

$$\hat{\hat{r}}(\alpha) = \frac{2(2+\alpha)}{2-\alpha}. \tag{1.4}$$

Theorem 1.2 is a partial affirmative answer to the Gopalsamy and Liu's conjecture in [4].

For each $\bar{\alpha}_1 < \alpha < 1$, we give a condition in Lemma 2.6 which is satisfied numerically. Using this condition by computer simulations, we have tested several cases of α. As a result, for $\bar{\alpha}_1 < \alpha < 1$, we believe as in Liu and Gopalsamy [6], that the Gopalsamy and Liu's conjecture: $r \leqslant \hat{\hat{r}}(\alpha)$, is also a necessary and sufficient condition of the global asymptotic stability for the positive equilibrium N^* of Eq. (1.1). But for $\bar{\alpha}_1 < \alpha < 1$, a mathematically rigorous proof of this still remains as an open question.

The organization of this paper is as follows. In Section 2, using a basic relation (see Eqs. (2.3) and (2.4) in Lemma 2.1) and related several lemmas (see Lemmas 2.2–2.10), we establish necessary and sufficient conditions for the positive equilibrium of Eq. (1.1) to be asymptotically stable for $0 < \alpha < 1$ and to be globally asymptotically stable for $0 < \alpha \leqslant \bar{\alpha}_1 < 1$.

2. Lemmas and proofs of theorems

Let us consider a density-dependent harvesting model denoted by Eq. (1.1). Then, by Gopalsamy and Liu [4], we see that

$$N(t) = N(n)\exp\left\{r\int_n^t (1 - aN(s) - bN(n))\,\mathrm{d}s\right\},\quad n \leqslant t < n+1,\qquad n = 0, 1, 2, \ldots,$$

$N(t) > 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{N(t)}\exp\{r(1-bN(n))(t-n)\}\right] = ar\ \exp\{r(1-bN(n))(t-n)\},\ \ t\in[n, n+1). \tag{2.1}$$

We have the first basic lemma (see [10, Lemma 3.1]).

Lemma 2.1. *Assume that*

$$\begin{cases} 1 + aN(n)\dfrac{\exp\{r(1-bN(n))\}-1}{1-bN(n)} > 0 & \text{if } b = 0,\ \text{or } b \neq 0 \text{ and } N(n) \neq 1/b, \\ 1 + aN(n)r > 0 & \text{if } b \neq 0 \text{ and } N(n) = 1/b. \end{cases} \tag{2.2}$$

Then,

$$N(n+1) = \begin{cases} \dfrac{N(n)\exp\{r(1-bN(n))\}}{1 + aN(n)\{(\exp\{r(1-bN(n))\}-1)/(1-bN(n))\}} & \text{if } b = 0,\ \text{or} \\ & b \neq 0 \text{ and } N(n) \neq 1/b, \\ \dfrac{N(n)}{1 + aN(n)r} & \text{if } b \neq 0 \text{ and } N(n) = 1/b \end{cases} \tag{2.3}$$

and

$$N(n+1)-N^* = \begin{cases} \dfrac{1-bN(n)\dfrac{\exp\{r(1-bN(n))\}-1}{1-bN(n)}}{1+aN(n)\dfrac{\exp\{r(1-bN(n))\}-1}{1-bN(n)}}(N(n)-N^*), & N(n)\neq 1/b, \\ \dfrac{1-bN(n)r}{1+aN(n)r}(N(n)-N^*), & N(n)=1/b. \end{cases} \tag{2.4}$$

Proof. If $r=0$, then we see $N(n+1)=N(n)$.

Now, assume $r>0$.

From Eqs. (2.1) and (2.2), we have Eq. (2.3). Then,

$$\begin{cases} N(n+1) = \frac{N(n)+\{(a+b)N^*-bN(n)\}N(n)\{(\exp\{r(1-bN(n))\}-1)/(1-bN(n))\}}{1+aN(n)\{(\exp\{r(1-bN(n))\}-1)/(1-bN(n))\}} & \text{if } b=0, \text{ or } b\neq 0 \text{ and } N(n)\neq 1/b, \\ N(n+1) = \frac{N(n)+\{(a+b)N^*-bN(n)\}N(n)r}{1+aN(n)r} & \text{if } b\neq 0 \text{ and } N(n)=1/b, \end{cases}$$

from which we have Eq. (2.4). □

Hereafter in this section, we consider the case $0<a<b$. Note that if $r=0$, then we see $N(n+1)=N(n)$. For simplicity, we assume $r>0$ and put

$$0<\alpha=a/b<1, \ \ x(n)=bN(n)>0, \ \ x^*=1/(1+\alpha)>0$$

and for $r>0$,

$$f(t;r)=\begin{cases}(1-t)\frac{e^{rt}-1}{t}, & t\neq 0,\\ r, & t=0.\end{cases} \tag{2.5}$$

Note that $f(t;r_1)\leqslant f(t;r_2)$ for any $0<r_1\leqslant r_2$ and $t<1$.

Since for $0<\alpha<1$, $f(t;r)>0$ for $t<1$,

$$1+\alpha f(1-x(n);r)>0,$$

then Eq. (2.3) is equivalent to

$$x(n+1)=\frac{x(n)\exp\{r(1-x(n))\}}{1+\alpha f(1-x(n);r)} \tag{2.6}$$

and Eq. (2.4) is equivalent to

$$x(n+1)-x^*=F(1-x(n);r)(x(n)-x^*), \tag{2.7}$$

where

$$F(t;r)=\frac{1-f(t;r)}{1+\alpha f(t;r)}. \tag{2.8}$$

Now, we offer the following lemmas.

Lemma 2.2. *For $-\infty < t < +\infty$, it holds that*

$$\begin{cases} f(t;r) = r + \sum_{k=1}^{\infty} \dfrac{(r-k-1)r^k t^k}{(k+1)!}, & f(0;r) = r, \\ f'(t;r) = \sum_{k=0}^{\infty} \dfrac{(r-k-2)r^{k+1} t^k}{(k+2)k!}, & f'(0;r) = \dfrac{(r-2)r}{2}, \\ f''(t;r) = \sum_{k=0}^{\infty} \dfrac{(r-k-3)r^{k+2} t^k}{(k+3)k!}, & f''(0;r) = \dfrac{(r-3)r^2}{3}, \\ f'''(t;r) = \sum_{k=0}^{\infty} \dfrac{(r-k-4)r^{k+3} t^k}{(k+4)k!}, & f'''(0;r) = \dfrac{(r-4)r^3}{4}. \end{cases} \tag{2.9}$$

Proof. Since for $t \neq 0$,

$$f(t;r) = \frac{1-t}{t}(\mathrm{e}^{rt} - 1),$$

we have for $t \neq 0$,

$$\begin{aligned} f(t;r) &= \frac{1-t}{t}\left(\sum_{k=0}^{\infty} \frac{(rt)^k}{k!} - 1\right) \\ &= \frac{1-t}{t} \sum_{k=1}^{\infty} \frac{r^k t^k}{k!} = \sum_{k=1}^{\infty}\left(\frac{r^k t^{k-1}}{k!} - \frac{r^k t^k}{k!}\right) \\ &= r + \sum_{k=1}^{\infty}\left(\frac{r^{k+1} t^k}{(k+1)!} - \frac{r^k t^k}{k!}\right) = r + \sum_{k=1}^{\infty} \frac{(r-k-1)r^k t^k}{(k+1)!}. \end{aligned}$$

Since $f(0;r) = r$, we obtain the first equation of Eq. (2.9) for any $-\infty < t < +\infty$. Moreover, for $a_k = (r-k-1)r^k/(k+1)!,\ k \geqslant 1$ we have

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{(r-k-2)r}{(r-k-1)(k+2)} = 0,$$

which implies that $\lim_{k\to\infty} 1/\sqrt[k]{|a_k|} = +\infty$ and an infinite series $r + \sum_{k=1}^{\infty} a_k t^k$ is convergent for any $-\infty < t < +\infty$. Thus, the function $f(t)$ is infinitely differentiable on $(-\infty, +\infty)$ and

$$f'(t;r) = \sum_{k=0}^{\infty} \frac{(r-k-2)r^{k+1} t^k}{(k+2)k!},$$

$$f''(t;r) = \sum_{k=0}^{\infty} \frac{(r-k-3)r^{k+2} t^k}{(k+3)k!},$$

$$f'''(t;r) = \sum_{k=0}^{\infty} \frac{(r-k-4)r^{k+3} t^k}{(k+4)k!}$$

and we obtain Eq. (2.9). □

The following lemma is a part of [10, Lemma 2.2], which gives useful properties of the function $f(t;r)$.

Lemma 2.3 (*See Muroya [10, Lemma 2.2]*). *For* $0<\alpha<1$, *there exist a strictly monotone increasing function* $\hat{r}(\alpha)$ *of* α *on the interval* $(-1,1)$, *and* $\hat{t}(\alpha)<1$ *such that*

$$f(\hat{t}(\alpha);\hat{r}(\alpha))=2/(1-\alpha) \quad \textit{and} \quad f'(\hat{t}(\alpha);\hat{r}(\alpha))=0$$

and

$$\begin{cases} f(t;\hat{r}(\alpha))<2/(1-\alpha) & \textit{for } t<1 \textit{ and } t\neq\hat{t}(\alpha),\\ f'(t;\hat{r}(\alpha))>0 \quad \textit{for } -\infty<t<\hat{t}(\alpha) & \textit{and} \quad f'(t;\hat{r}(\alpha))<0 \textit{ for } \hat{t}(\alpha)<t<1. \end{cases}$$

For $0<\alpha<1$, put

$$\begin{aligned} &\hat{\hat{r}}(\alpha)=\frac{1+\alpha}{\alpha}\ln\frac{1+\alpha}{1-\alpha},\\ &t^*=1-x^*=\alpha/(1+\alpha) \quad \text{and} \quad t^{**}=2t^*=2\alpha/(1+\alpha). \end{aligned} \tag{2.10}$$

Then,

$$0<\hat{\hat{r}}(\alpha)<+\infty \quad \text{and} \quad 0<t^*<t^{**}<1. \tag{2.11}$$

Now, we have the second basic lemma.

Lemma 2.4. *For* $0<\alpha<1$, *it holds that*

$$\begin{cases} f(t^*;\hat{\hat{r}}(\alpha))=f(t^{**};\hat{\hat{r}}(\alpha))=2/(1-\alpha),\\ f(t;\hat{\hat{r}}(\alpha))>2/(1-\alpha) \quad \textit{for } t^*<t<t^{**},\\ f(t;\hat{\hat{r}}(\alpha))<2/(1-\alpha) \quad \textit{otherwise}, \end{cases} \tag{2.12}$$

and

$$\begin{cases} f'(t^*;\hat{\hat{r}}(\alpha))=\dfrac{(1+\alpha)^2\left(\dfrac{1}{\alpha}\ln\dfrac{1+\alpha}{1-\alpha}-2\right)}{(1-\alpha)\alpha}>0,\\ f''(t^*;\hat{\hat{r}}(\alpha))=\dfrac{(1+\alpha)^3\left(\dfrac{1}{\alpha}\ln\dfrac{1+\alpha}{1-\alpha}-2\right)\left(\ln\dfrac{1+\alpha}{1-\alpha}-2\right)}{(1-\alpha)\alpha^2},\\ f'(t^{**};\hat{\hat{r}}(\alpha))=\dfrac{(1+\alpha)^2}{2\alpha(1-\alpha)}\left(\dfrac{1+\alpha}{\alpha}\ln\dfrac{1+\alpha}{1-\alpha}-\dfrac{2}{1-\alpha}\right)<0. \end{cases} \tag{2.13}$$

Further, for any $r<\hat{\hat{r}}(\alpha)$ *and* $0<\alpha<1$,

$$1+\alpha f(t;r)>0 \quad \textit{for any } t<1. \tag{2.14}$$

Proof. We have

$$e^{\hat{\hat{r}}(\alpha)t^*}=\frac{1+\alpha}{1-\alpha} \quad \text{and} \quad e^{\hat{\hat{r}}(\alpha)t^{**}}=\left(\frac{1+\alpha}{1-\alpha}\right)^2$$

and hence, for $0<\alpha<1$,

$$f(t^*;\hat{\hat{r}}(\alpha))=(1-t^*)\frac{e^{\hat{\hat{r}}(\alpha)t^*}-1}{t^*}=\frac{1}{\alpha}\left(\frac{1+\alpha}{1-\alpha}-1\right)=\frac{2}{1-\alpha},$$

$$f(t^{**};\hat{\hat{r}}(\alpha))=(1-2t^*)\frac{e^{\hat{\hat{r}}(\alpha)2t^*}-1}{2t^*}=\frac{1-\alpha}{2\alpha}\left\{\left(\frac{1+\alpha}{1-\alpha}\right)^2-1\right\}=\frac{2}{1-\alpha}.$$

On the other hand, by Eq. (2.5), we have $tf(t;r)=(1-t)(e^{rt}-1)$, and

$$\begin{aligned}&tf'(t;r)+f(t;r)=(1-t)re^{rt}-(e^{rt}-1)=\{(1-t)r-1\}e^{rt}+1,\\&tf''(t;r)+2f'(t;r)=(1-t)r^2e^{rt}-2re^{rt}=\{(1-t)r-2\}re^{rt}.\end{aligned}\tag{2.15}$$

Thus, by Eq. (2.10) and direct computations, we have for $0<\alpha<1$,

$$f'(t^*;\hat{\hat{r}}(\alpha))=\frac{(1+\alpha)^2\left(\frac{1}{\alpha}\ln\frac{1+\alpha}{1-\alpha}-2\right)}{(1-\alpha)\alpha}>0,$$

$$f''(t^*;\hat{\hat{r}}(\alpha))=\frac{(1+\alpha)^3\left(\frac{1}{\alpha}\ln\frac{1+\alpha}{1-\alpha}-2\right)\left(\ln\frac{1+\alpha}{1-\alpha}-2\right)}{(1-\alpha)\alpha^2}.$$

Moreover, from

$$t^{**}f'(t^{**};r)+f(t^{**};r)=(1-t^{**})re^{rt^{**}}-(e^{rt^{**}}-1),$$

we have

$$\frac{2\alpha}{1+\alpha}f'(t^{**};r)+\frac{2}{1-\alpha}=\frac{1-\alpha}{1+\alpha}r\left(\frac{1+\alpha}{1-\alpha}\right)^2-\left\{\left(\frac{1+\alpha}{1-\alpha}\right)^2-1\right\},$$

from which we obtain the second equation of Eq. (2.13). Hence, we get Eqs. (2.12) and (2.13).

Since $f(t;r)>0$ for $t<1$, Eq. (2.14) holds for any $r\leqslant\hat{\hat{r}}(\alpha)$ and $0<\alpha<1$. □

Note that $2<\hat{r}(\alpha)<\hat{\hat{r}}(\alpha)$ for any $0<\alpha<1$ and $\lim_{\alpha\to+0}\hat{r}(\alpha)=\lim_{\alpha\to+0}\hat{\hat{r}}(\alpha)=2$.

For simplicity, for $0<\alpha<1$, put

$$\begin{aligned}&f(t)=f(t;\hat{\hat{r}}(\alpha)),\\&G(t)=F(t)(t-t^*)+t^* \quad\text{and}\quad F(t)=\frac{1-f(t)}{1+\alpha f(t)}.\end{aligned}\tag{2.16}$$

Then, Eq. (2.12) implies

$$\begin{cases}f(t^*)=f(t^{**})=2/(1-\alpha),\\ f(t)>2/(1-\alpha) \quad \text{for } t^*<t<t^{**},\\ f(t)<2/(1-\alpha) \quad \text{otherwise.}\end{cases}\tag{2.17}$$

Thus,

$$\begin{cases}F(t^*)=F(t^{**})=-1 \quad\text{and}\quad F(t)<-1 \quad \text{for } t^*<t<t^{**},\\ G(t^*)=t^*,\ G(t^{**})=0 \quad\text{and}\quad G(t)<t^* \quad\text{for } t^*<t<t^{**}\end{cases}\tag{2.18}$$

and $t^*=\alpha/(1+\alpha)$ is a unique solution of $t=G(t)$ for $t<1$.

Now, by Eqs. (2.6)–(2.8) and (2.16), we see that for $t_n = 1 - x(n) < 1$, Eq. (2.7) is equivalent to

$$t_{n+1} = G(t_n), \quad n = 0, 1, 2, \ldots. \tag{2.19}$$

By definitions, we have that

$$\begin{cases} t_{n+1} - t^* = F(t_n)(t_n - t^*) \quad \text{and} \\ t_{n+2} - t^* = F(t_{n+1})(t_{n+1} - t^*) = F(G(t_n))F(t_n)(t_n - t^*). \end{cases} \tag{2.20}$$

Then, from Eq. (2.20), we easily get the following lemma on the "contractivity" of $|t_n - t^*| = |x(n) - x^*|$.

Lemma 2.5. (a) *Assume that for* $t_n < 1$,

$$|F(t_n)| < 1 \quad \textit{for } t_n \neq t^*. \tag{2.21}$$

Then,

$$|t_{n+1} - t^*| < |t_n - t^*|.$$

(b) *Suppose that for* $t_n < 1$,

$$|F(G(t_n))F(t_n)| < 1 \quad \textit{for } t_n \neq t^*.$$

Then,

$$|t_{n+2} - t^*| < |t_n - t^*|.$$

Note that for $0 < \alpha < 1$ and $t^* < t < t^{**}$, $f(t) > f(t^*) = 2/(1 - \alpha)$.
For $f(t) > 1/(1 - \alpha)$, put

$$S(t) = f(G(t)) - \frac{f(t)}{(1 - \alpha)f(t) - 1}. \tag{2.22}$$

We have the following lemma.

Lemma 2.6. $S(t) < 0$, *for* $t^* < t < t^{**}$, *if and only if*, $|F(G(t))F(t)| < 1$, *for* $t^* < t < t^{**}$.
In this case, the positive equilibrium N^* *of Eq.* (1.1) *is globally asymptotically stable*.

Proof. By Eq. (2.17), for $t^* < t < t^{**}$, $f(t) > 2/(1 - \alpha) > 1$ and $G(t) < t^*$. Since $f(t) > 1$ for $t < t^*$, we have that for $0 < \alpha < 1$ and $t^* < t < t^{**}$, $f(t^*) = f(G(t^*)) > f(G(t)) > 1$.

Then, by Eqs. (2.16) and (2.17), it holds that for $t^* < t < t^{**}$, $F(G(t)) < 0$ and $F(t) < -1$, and hence, $F(G(t))F(t) \geqslant 0$. Thus, we can easily see that for $t^* < t < t^{**}$, $|F(G(t))F(t)| < 1$ is equivalent to

$$f(G(t)) + f(t) > (1 - \alpha)f(G(t))f(t) \tag{2.23}$$

which is equivalent to $S(t) < 0$, because of $f(t) > 2/(1 - \alpha)$ for $t^* < t < t^{**}$.

In this case, by Lemma 2.5(b), Eqs. (2.7), (2.8), (2.10) and (2.16), N^* is globally asymptotically stable. □

We have the following lemma.

Lemma 2.7. *For* $0<\alpha<1$,

$$f(G(t^{**}))=f(0)=\hat{\hat{r}}(\alpha)<\frac{2}{1-\alpha}=f(t^*) \tag{2.24}$$

and

$$\begin{cases} S(t^*)=S'(t^*)=0 \quad and \quad S(t^{**})=f(0)-f(t^*)<0, \\ S''(t^*)=2\left\{f''(t^*)-2\dfrac{1-\alpha}{1+\alpha}(f'(t^*))^2\right\}<0. \end{cases}$$

Proof. By Eqs. (2.9), (2.16) and (2.18), $f(0)=\hat{\hat{r}}(\alpha)$ and $G(t^{**})=0$. Moreover, by Eq. (1.3),

$$(1-\alpha)\hat{\hat{r}}(\alpha)=2(1-\alpha^2)\left(1+\frac{\alpha^2}{3}+\frac{\alpha^4}{5}+\cdots\right)=2\left(1-\frac{2}{1\cdot 3}\alpha^2-\frac{2}{3\cdot 5}\alpha^4-\cdots\right)<2,$$

from which we have $\hat{\hat{r}}(\alpha)<2/(1-\alpha)=f(t^*)$ and hence Eq. (2.24) holds. Since

$$f(t^*)=\tfrac{2}{1-\alpha},\ F(t^*)=-1,\ G(t^*)=t^*,\ G'(t^*)=-1,$$
$$G''(t^*)=2F'(t^*)=-\frac{2(1+\alpha)f'(t^*)}{\{1+\alpha f(t^*)\}^2}=-\frac{2(1-\alpha)^2f'(t^*)}{1+\alpha},$$

we have that for $f(t)>1/(1-\alpha)$,

$$S'(t)=f'(G(t))G'(t)+\frac{f'(t)}{\{(1-\alpha)f(t)-1\}^2},$$
$$S''(t)=f''(G(t))(G'(t))^2+f'(G(t))G''(t)+\frac{f''(t)\{(1-\alpha)f(t)-1\}-2(1-\alpha)(f'(t))^2}{\{(1-\alpha)f(t)-1\}^3}$$

and hence,

$$S(t^*)=S'(t^*)=0 \quad \text{and} \quad S''(t^*)=2\left\{f''(t^*)-2\frac{1-\alpha}{1+\alpha}(f'(t^*))^2\right\}.$$

Since

$$\frac{2(1+\alpha)}{2-\alpha}<2(1+\alpha)<\frac{1+\alpha}{\alpha}\ln\frac{1+\alpha}{1-\alpha} \quad \text{for } 0<\alpha<1,$$

we have that for $0<\alpha<1$,

$$S''(t^*)=-2\frac{(1+\alpha)^3}{(1-\alpha)\alpha^2}\left(\frac{2-\alpha}{\alpha}\ln\frac{1+\alpha}{1-\alpha}-2\right)\left(\frac{1}{\alpha}\ln\frac{1+\alpha}{1-\alpha}-2\right)<0. \qquad \square$$

Proof of Theorem 1.1. Since by Lemma 2.7, $S(t^*)=S'(t^*)=0$ and $S''(t^*)<0$ and the continuity of $S''(t)$ at $t=t^*$, there is a positive constant δ such that $t^*+\delta<1$ and

$$S(t)<0 \quad \text{for any } t^*-\delta<t<t^* \quad \text{and} \quad t^*<t<t^*+\delta$$

which is equivalent to $|F(G(t))F(t)|<1$ for any $t^*-\delta<t<t^*$ and $t^*<t<t^*+\delta$. Hence by Lemma 2.5 and as similar to the proof of Lemma 2.6, we have the conclusion of this theorem. $\square$

By Theorem 1.1, we establish a mathematical proof of the numerical results in Gopalsamy and Liu [4] and Liu and Gopalsamy [6] for the asymptotic stability of the positive equilibrium N^* of Eq. (1.1).

Now, consider sufficient conditions of $S(t) < 0$ for $t^* < t < t^{**}$.

Lemma 2.8. (i) *For* $a,\ b > 0$ *and* $a \neq b$, *it holds*

$$\frac{a+b}{2} > \sqrt{ab} > \frac{2ab}{a+b} = \frac{2}{1/a + 1/b}. \tag{2.25}$$

(ii) *For* $t^* < t < t^{**}$,

$$f(G(t)) + f(t) < 2f(t^*) \quad \textit{implies } S(t) < 0. \tag{2.26}$$

Proof. (i) Eq. (2.25) is easily proved.

(ii) Suppose that for $t^* < t < t^{**}$, $f(G(t)) + f(t) < 2f(t^*)$. Then, for $a = f(G(t)),\ \ b = f(t)$, we have $(a+b)/2 < 2/(1-\alpha)$. By Eq. (2.25), $(a+b)/2 < 2/(1-\alpha)$ implies $2ab/(a+b) < 2/(1-\alpha)$, that is, $a + b > (1-\alpha)ab$ which is equivalent to Eq. (2.23). Thus, we obtain Eq. (2.26). □

Lemma 2.9. *If*

$$\begin{cases} G'(t) < 0 \quad \textit{for } t^* < t < t^{**} \quad \textit{and} \\ f(t) \leqslant f(t^*) + f'(t^*)(t - t^*) \quad \textit{for } 0 = G(t^{**}) < t < t^{**}, \end{cases} \tag{2.27}$$

then $f(G(t)) + f(t) < 2f(t^*)$, *for* $t^* < t < t^{**}$, *and the positive equilibrium* N^* *of Eq.* (1.1) *is globally asymptotically stable.*

Proof. $G'(t) < 0$ for $t^* < t < t^{**}$ implies $0 = G(t^{**}) < G(t) < G(t^*) = t^*$. By Eq. (2.27), we have that for $t^* < t < t^{**}$,

$$\begin{aligned} &f(t) \leqslant f(t^*) + f'(t^*)(t - t^*), \\ &f(G(t)) \leqslant f(t^*) + f'(t^*)(G(t) - t^*). \end{aligned}$$

Since $F(t) < -1$, for $t^* < t < t^{**}$ and $G(t) = F(t)(t - t^*) + t^*$, we have

$$f(t) + f(G(t)) \leqslant 2f(t^*) + f'(t^*)(1 + F(t))(t - t^*) < 2f(t^*),$$

and by Lemmas 2.6 and 2.8, we see that N^* is globally asymptotically stable. □

For a sufficient condition of Eq. (2.27), further we offer the following lemma.

Lemma 2.10. *Assume*

$$f''(t^*) \leqslant 0 \quad \textit{and} \quad f'''(t) \leqslant 0 \quad \textit{for } 0 = G(t^{**}) \leqslant t \leqslant t^{**}, \tag{2.28}$$

then

$$G'(t) < 0 \quad \textit{for } t^* < t < t^{**}. \tag{2.29}$$

Moreover, the second part of Eq. (2.27) *holds, if and only if,*

$$f(G(t^{**})) \leqslant f(t^*) + f'(t^*)(G(t^{**}) - t^*). \tag{2.30}$$

Proof. By assumption $f'''(t)\leqslant 0$, for $0\leqslant t\leqslant t^{**}$, $f''(t)$ is a strictly decreasing function on $[0,t^{**}]$, and by $f''(t^*)\leqslant 0$, we have that $f''(t)\leqslant 0$, for $t^*\leqslant t\leqslant t^{**}$.

Moreover, by Lemma 2.2 and Eq. (2.13), there exists a unique $\bar{t}_1$ such that

$$0<t^*<\bar{t}_1<t^{**} \quad \text{and} \quad f'(\bar{t}_1)=0. \tag{2.31}$$

Hence, we easily get

$$f(t)\geqslant f(t^{**}) \quad \text{and} \quad 0\leqslant -f'(t)\leqslant -f'(t^{**}) \quad \text{for } \bar{t}_1\leqslant t\leqslant t^{**}.$$

Therefore, by Eq. (2.17), for $\bar{t}_1\leqslant t\leqslant t^{**}$,

$$\begin{cases} 0\leqslant F'(t)=-\dfrac{(1+\alpha)f'(t)}{(1+\alpha f(t))^2}\leqslant -\dfrac{(1+\alpha)f'(t^{**})}{(1+\alpha f(t^{**}))^2}=F'(t^{**})=-\dfrac{(1-\alpha)^2}{1+\alpha}f'(t^{**}),\\ \text{and} \quad F(t)\leqslant F(t^{**})=-1. \end{cases}$$

On the other hand, for $t^*<t<\bar{t}_1$, $f'(t)>0$, and hence,

$$F'(t)=-\frac{(1+\alpha)f'(t)}{(1+\alpha f(t))^2}<0 \quad \text{and} \quad F(t)<F(t^*)=F(t^{**})=-1.$$

Therefore, by Eqs. (2.13) and (2.16), we have that for $t^*<t<t^{**}$,

$$\begin{aligned} G'(t)&=F'(t)(t-t^*)+F(t)\leqslant F'(t^{**})(t^{**}-t^*)+F(t^{**})=G'(t^{**})\\ &=\alpha\left(\frac{1-\alpha}{1+\alpha}\right)^2(-f'(t^{**}))-1=-\frac{(1-\alpha)(1+\alpha)}{2\alpha}\ln\frac{1+\alpha}{1-\alpha}<0. \end{aligned}$$

Thus, we obtain Eq. (2.29).

Now, assume the second part of Eq. (2.27). We see that the second part of Eq. (2.27) and the continuity of $f(t)$ at $t=G(t^{**})=0$ imply Eq. (2.30).

Inversely, suppose that Eq. (2.30) holds.

Suppose first that $f''(G(t^{**}))=(\hat{\hat{r}}(\alpha)-3)\hat{\hat{r}}(\alpha)/3>0$, that is, $\hat{\hat{r}}(\alpha)>3$. Then by $f''(t^*)\leqslant 0$, there exists a unique solution $t=\bar{t}_2$ of the equation $f''(t)=0$, and $G(t^{**})<\bar{t}_2\leqslant t^*$. Since for $\bar{t}_2\leqslant t\leqslant t^{**}$, $f''(t)\leqslant 0$, we have that for $\bar{t}_2\leqslant t\leqslant t^{**}$,

$$\begin{aligned} f(t)&=f(t^*)+\frac{f'(t^*)}{1!}(t-t^*)+\frac{f''(\xi)}{2!}(t-t^*)^2\\ &\leqslant f(t^*)+\frac{f'(t^*)}{1!}(t-t^*),\ \ \bar{t}_2<\xi<t^{**}. \end{aligned}$$

Since $f''(t)>0$, for $G(t^{**})\leqslant t<\bar{t}_2$, and

$$f(\bar{t}_2)\leqslant f(t^*)+f'(t^*)(\bar{t}_2-t^*) \quad \text{and} \quad f(G(t^{**}))\leqslant f(t^*)+f'(t^*)(G(t^{**})-t^*),$$

by the lower convexity of $f(t)$ for $G(t^{**})\leqslant t\leqslant \bar{t}_2$, it follows that for $G(t^{**})\leqslant t\leqslant \bar{t}_2$,

$$\begin{aligned} f(t) &\leqslant \frac{f(\bar{t}_2)-f(G(t^{**}))}{\bar{t}_2-G(t^{**})}(t-G(t^{**}))+f(G(t^{**})) \\ &= \frac{f(\bar{t}_2)(t-G(t^{**}))+f(G(t^{**}))(\bar{t}_2-t)}{\bar{t}_2-G(t^{**})} \\ &\leqslant \frac{\{f(t^*)+f'(t^*)(\bar{t}_2-t^*)\}(t-G(t^{**}))+\{f(t^*)+f'(t^*)(G(t^{**})-t^*)\}(\bar{t}_2-t)}{\bar{t}_2-G(t^{**})} \\ &= f(t^*)+f'(t^*)(t-t^*). \end{aligned}$$

Thus, in this case, the second part of Eq. (2.27) holds. If $f''(G(t^{**}))\leqslant 0$, that is, $\hat{\hat{r}}(\alpha)\leqslant 3$, then put $\bar{t}_2=G(t^{**})$ in the above discussion and similarly we obtain the second part of Eq. (2.27). □

From Lemmas 2.5 and 2.6 and 2.8–2.10, we can prove Theorem 1.2 which is a partial answer to the Gopalsamy and Liu's conjecture in [4] that for $0<\alpha<1$, Eq. (1.2) is sufficient for the positive equilibrium N^* of Eq. (1.1) to be globally asymptotically stable.

Proof of Theorem 1.2. Since by (1.4), $\bar{\alpha}_1=0.634817\cdots<\frac{2}{3}$, and for $0<\alpha\leqslant\bar{\alpha}_1$, $\hat{\hat{r}}(\alpha)\leqslant 2(2+\alpha)/(2-\alpha)<4$. Thus, by Eqs. (1.2) and (2.9), we have $f'''(t)\leqslant 0$, for $t\geqslant 0$. Since for $0<\alpha\leqslant\bar{\alpha}_1$,

$$2(1+\alpha)<\hat{\hat{r}}(\alpha)\leqslant\frac{2(2+\alpha)}{2-\alpha}<4<\frac{2(1+\alpha)}{\alpha},$$

we have that by Eq. (2.13),

$$f''(t^*)=\frac{(1+\alpha)(\hat{\hat{r}}(\alpha)-2(1+\alpha))(\hat{\hat{r}}(\alpha)-\frac{2(1+\alpha)}{\alpha})}{(1-\alpha)\alpha}<0.$$

By Eqs. (2.12) and (2.13), $f(t^*)=2/(1-\alpha)$ and $f'(t^*)t^*=\{\hat{\hat{r}}(\alpha)-2(1+\alpha)\}/(1-\alpha)$. Therefore,

$$f(t^*)-f'(t^*)t^*=\{2(2+\alpha)-\hat{\hat{r}}(\alpha)\}/(1-\alpha)\geqslant\hat{\hat{r}}(\alpha)=f(0).$$

Thus, all the conditions in Lemma 2.10 are satisfied. Hence by Lemma 2.9, we get the conclusion. □

Note that Eq. (2.30) is equivalent to $\hat{\hat{r}}(\alpha)\leqslant 2(2+\alpha)/(2-\alpha)$, from which we have that $0<\alpha\leqslant\bar{\alpha}_1<\frac{2}{3}$ and $\hat{\hat{r}}(\alpha)<4$, and hence, $f''(t^*)<0$ and $f'''(t)\leqslant 0$, for $t\geqslant 0$.

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