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Some Qualitative Properties of Solutions of Quasilinear Elliptic Equations and Applications

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We study quasilinear elliptic equations of Leray–Lions type in $W^{1,p}(\Omega)$, maxi-

generating singularities of solutions, and lower bounds on constants appearing in Schauder, Agmon, Douglis, and Nirenberg estimates.

1. INTRODUCTION

In this article we consider the nonlinear equation

$$\begin{cases} -\operatorname{div} a(x, u, \nabla u) = f(x, u) + g(x, u)\,|\nabla u|^p & \text{in } \mathscr{D}'(\Omega), \\ f(x, u) \in L^1(\Omega), \qquad g(x, u)\,|\nabla u|^p \in L^1(\Omega), \\ u \in W^{1,p}(\Omega), \end{cases} \tag{1}$$

where Ω is an open, possibly unbounded set in $\mathbf{R}^N$, with sufficiently regular boundary $\partial\Omega$, $N \geqslant 1$, $1 < p < \infty$, and $a(x, \eta, \xi)$ a Carathéodory function satisfying the conditions of Leray–Lions type (see [13]),

$$\exists \alpha > 0, \qquad a(x, \eta, \xi) \cdot \xi \geqslant \alpha\,|\xi|^p, \qquad \text{a.e. in } \Omega, \quad \eta \in \mathbf{R}, \quad \xi \in \mathbf{R}^N, \tag{2}$$

$$\begin{cases} \exists a_1 \geqslant 0, \exists a_2 > 0, \exists h \in L^{p'}(\Omega), \forall \eta \in \mathbf{R}, \quad \forall \xi \in \mathbf{R}^N, \\ |a(x, \eta, \xi)| \leqslant h(x) + a_1\,|\eta|^{p-1} + a_2\,|\xi|^{p-1} \qquad \text{a.e. in } \Omega, \end{cases} \tag{3}$$

$$\begin{cases} \forall \eta \in \mathbf{R}, \forall \xi, \xi^* \in \mathbf{R}^N, \quad \xi \neq \xi^*, \\ (a(x, \eta, \xi) - a(x, \eta, \xi^*)) \cdot (\xi - \xi^*) > 0, \qquad \text{a.e. in } \Omega \end{cases} \tag{4}$$

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Here $f(x, \eta)$ and $g(x, \eta)$ are also Carathéodory functions and

$$|g(x, \eta)| \leqslant \hat{g}(\eta), \qquad \text{a.e. in } \Omega, \qquad \eta \in \mathbf{R}, \tag{5}$$

with $\hat{g}$ to be specified later.

First, in Section 2 we prove two types of maximum principles for solutions of (1) in $W^{1,p}(\Omega) \cap C(\bar{\Omega})$ and then apply these results to obtain nonexistence of solutions (including nonexistence of positive solutions and spherically symmetric solutions) and existence of solutions in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

In Section 3 we deal with the control of lower (upper) bound for ess sup (ess inf) of solutions and apply these results to the investigation of *sign-changing* solutions. Precisely, for arbitrary given four real numbers $m_0 < m_1 \leqslant M_1 < M_0$ we find some sufficient conditions on $f(x, \eta)$ and $g(x, \eta)$ such that for any solution u of (1) satisfying $\operatorname{ess\,inf}_{\partial\Omega} u \geqslant m_0$ and $\operatorname{ess\,sup}_{\partial\Omega} u \leqslant M_0$ we have: $m_0 \leqslant \operatorname{ess\,inf}_\Omega u < m_1$ and $M_1 < \operatorname{ess\,sup}_\Omega u \leqslant M_0$. In particular, for $m_1 = M_1 = 0$ we obtain that such solutions of (1) change sign in Ω.

In Section 4 we obtain lower bounds of oscillation of solutions, and derive lower bounds on constants appearing in Schauder a priori estimate and in Agmon, Douglis and Nirenberg a priori estimate. We find some natural conditions on the right hand side of (1) that insure existence of singularity of solutions in a given point. In Section 5 we present several extensions and variations of the main control result. We also provide several examples indicating that the best oscillation estimate is obtained using a family of deformation retracts of Ω.

In proving the main results, we will use a combination of two simple methods: the method of integration on the level set $\{u > t\}$, often used in the symmetrization and relative rearrangement technique (see Talenti [24] and Rakotoson and Temam [22]) and second, the method of localization on the arbitrary given ball $B_r(x)$ in Ω, often used in the qualitative and harmonic analysis (see Lions [14] and Kenig [12]). Both techniques are exploited simultaneously by a suitable choice of test functions φ from $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, and applying various inequalities.

Finally, we refer to a few interesting results concerning our subject here. First, it was proved in Granas and Guennoun [10] that in the case of $N = 1$ there exists at last one solution that has the controlled norm in the space $L^\infty(\Omega)$. Next, in Gilbarg and Trudinger [9] and Barles and Murat [3], we find various maximum principles for problems having the general structure as in (1). Regarding the results of Barles and Murat [3], note that we do not impose any sign condition on $g(x, \eta)$. Finally, in Ni and Serrin [18] one can find some nonexistence results of classical radial solutions.

2. MAXIMUM PRINCIPLES

In this section we are interested in proving two types of maximum principles and in deriving some nonexistence results for (1). Firstly, we present an easy and useful weak maximum principle of the first type, and its dual.

Throughout this paper, whenever we have a condition involving behaviour of u on the boundary of Ω, we automatically assume the analogous condition of u at infinity for Ω unbounded. In other words, we include infinity as belonging to the boundary of unbounded domains. For example, if $\operatorname{ess\,inf}_{\partial\Omega} u \geqslant m_0$ (as in Theorem 1) then we also assume $\lim_{R\to\infty} \operatorname{ess\,inf}_{\{|x|\geqslant R\}\cap\Omega} u \geqslant m_0$, similarly as in [7, Corollaire 4, p. 289].

THEOREM 1. *Assuming* (2), (3) *and* (5), *let the functions* $f(x,\eta)$ *and* $\hat{g}(\eta)$ *satisfy*

$$\begin{cases} f(x,\eta)\geqslant 0 & \text{a.e. in } \Omega, \qquad \eta\in(-\infty, m_0), \\ \hat{g}\in L^1(-\infty, m_0)\cap L^\infty(-\infty, m_0), \end{cases} \tag{6}$$

where m_0 *is a given real number* ($m_0\leqslant 0$ *in the case when* Ω *is unbounded*). *Then for each supersolution* u *of* (1) *such that* $\operatorname{ess\,inf}_{\partial\Omega} u\geqslant m_0$ *we have* $\operatorname{ess\,inf}_\Omega u\geqslant m_0$.

THEOREM 2 (Dual Result). *Assuming* (2), (3), *and* (5), *let the functions* $f(x,\eta)$ *and* $\hat{g}(\eta)$ *satisfy*

$$\begin{cases} f(x,\eta)\leqslant 0 & \text{a.e. in } \Omega, \qquad \eta\in(M_0,\infty), \\ \hat{g}\in L^1(M_0,\infty)\cap L^\infty(M_0,\infty), \end{cases} \tag{7}$$

where M_0 *is a given real number* ($M_0\geqslant 0$ *in the case when* Ω *is unbounded*). *Then for each subsolution* u *of* (1) *such that* $\operatorname{ess\,sup}_{\partial\Omega} u\leqslant M_0$ *we have* $\operatorname{ess\,sup}_\Omega u\leqslant M_0$.

Proof of Theorem 1. Let $t, h\in\mathbf{R}$, $h>0$, and define functions

$$S^-_{t,h}(\tau)=\begin{cases} -1, & \text{for } \tau<t-h, \\ \dfrac{1}{h}(\tau-t), & \text{for } t-h\leqslant\tau<t, \\ 0, & \text{for } \tau\geqslant t, \end{cases} \tag{8}$$

$$G^-(s)=\begin{cases} \dfrac{1}{\alpha}\displaystyle\int_{-\infty}^{s}\hat{g}(\eta)\,d\eta, & \text{for } s\leqslant m_0, \\ G^-(m_0), & \text{for } s>m_0, \end{cases} \tag{9}$$

$$\varphi=e^{-G^-(u)}S^-_{t,h}(u), \qquad u\in W^{1,p}(\Omega). \tag{10}$$

If u is our supersolution, then for all $t \leqslant m_0$, since $\operatorname{ess\,inf}_{\partial\Omega} u \geqslant m_0$, we have that $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $\varphi \leqslant 0$. Applying the test function φ to

$$-\operatorname{div} a(x, u, \nabla u) \geqslant f(x, u) + g(x, u)\, |\nabla u|^p \tag{11}$$

we obtain

$$\frac{1}{h} \int_{\{t-h \leqslant u < t\}} |\nabla u|^p \, dx \leqslant \frac{e^{G^-(t)}}{\alpha} \left[-\int_{\{u < t-h\}} f(x, u)\, e^{-G^-(u)} \, dx + \int_{\{t-h \leqslant u < t\}} e^{-G^-(u)}\, |f(x, u)| \, dx \right]. \tag{12}$$

Using (6) and passing to the limit as $h \to 0$ we derive

$$\frac{d}{dt} \left(\int_{\{u<t\}} |\nabla u|^p \, dx \right) \leqslant 0, \qquad \text{for a.e.} \quad t < m_0. \tag{13}$$

Since the function $t \mapsto \int_{\{u<t\}} |\nabla u|^p \, dx$ is nondecreasing, we conclude that

$$\frac{d}{dt} \left(\int_{\{u<t\}} |\nabla u|^p \, dx \right) = 0, \qquad \text{a.e.} \quad t < m_0, \tag{14}$$

that is,

$$\int_{\{u<t\}} |\nabla u|^p \, dx = const. \geqslant 0, \qquad \text{a.e.} \quad t < m_0. \tag{15}$$

The family of sets $A_t = \{u < t\}$ is decreasing as $t \to -\infty$, and $A_{-\infty} = \{u = -\infty\}$ is negligible due to $u \in L^p(\Omega)$, so we have that

$$const. = \lim_{t \to -\infty} \int_{\{u<t\}} |\nabla u|^p \, dx = \int_{A_{-\infty}} |\nabla u|^p \, dx = 0. \tag{16}$$

Therefore $\int_\Omega |\nabla (u-t)^-|^p \, dx = \int_{\{u<t\}} |\nabla u|^p \, dx = 0$ for a.e. $t < m_0$.

Since Ω is open, it can be represented as at most countable union $\bigcup_{i \in I} \Omega_i$ of its components of connectedness Ω_i. We conclude that $(u-t)^- = const_i(t) \geqslant 0$ a.e. in Ω_i for all $i \in I$, and a.e. $t < m_0$.

If for some $i \in I$ and $t < m_0$ we have $const_i(t) > 0$, then we obtain $u = t - const_i(t) < t$ in Ω_i, and this yields a contradiction:

$$m_0 > t > \operatorname*{ess\,inf}_{\partial\Omega_i} u \geqslant \operatorname*{ess\,inf}_{\partial\Omega} u \geqslant m_0.$$

Hence, we have $const_i(t)=0$ for all $i\in I$ and a.e. $t<m_0$. This is equivalent to $u\geqslant t$ a.e. in Ω for a.e. $t<m_0$. Taking essential supremum over a.e. $t<m_0$ in $\operatorname{ess\,inf}_\Omega u\geqslant t$ we obtain the desired inequality $\operatorname{ess\,inf}_\Omega u\geqslant m_0$. ▮

The proof of Theorem 2 is obtained analogously by defining $\varphi=e^{-G^+(u)}S^+_{t,h}(u)\geqslant 0$, where

$$S^+_{t,h}(\tau)=\begin{cases}0, & \text{for} \quad \tau\leqslant t,\\ \dfrac{1}{h}(\tau-t), & \text{for} \quad t<\tau\leqslant t+h,\\ 1, & \text{for} \quad \tau>t+h,\end{cases} \tag{17}$$

$$G^+(s)=\begin{cases}\dfrac{1}{\alpha}\displaystyle\int_s^\infty \hat{g}(\eta)\,d\eta, & \text{for} \quad s\geqslant M_0,\\ G^+(M_0), & \text{for} \quad s<M_0.\end{cases} \tag{18}$$

From the preceding two results we immediately derive the following consequences:

Corollary 1 (Nonexistence of Nontrivial Solutions). *Assuming* (2), (3), *and* (5), *let the functions* $f(x,\eta)$ *and* $\hat{g}(\eta)$ *satisfy*

$$-\operatorname{sgn}(\eta)\,f(x,\eta)\geqslant 0, \qquad \textit{a.e. in } \Omega,\ \eta\in\mathbf{R} \qquad \textit{and} \qquad \hat{g}\in L^1(\mathbf{R})\cap L^\infty(\mathbf{R}). \tag{19}$$

Then $u=0$ *is the unique solution of* (1) *in* $W_0^{1,p}(\Omega)$.

Corollary 2 (Nonpositive Solutions). *Assuming* (2), (3), *and* (5), *let the functions* $f(x,\eta)$ *and* $\hat{g}(\eta)$ *satisfy*

$$f(x,\eta)\leqslant 0 \ \textit{a.e. } x \textit{ in } \Omega, \quad \eta\in\mathbf{R}^+ \qquad \textit{and} \qquad \hat{g}\in L^1(\mathbf{R}^+)\cap L^\infty(\mathbf{R}^+). \tag{20}$$

Then for each solution $u\in W_0^{1,p}(\Omega)$ *of* (1) *we have* $u\leqslant 0$.

Corollary 3 (Nonnegative Solutions). *Assuming* (2), (3), *and* (5), *let the functions* $f(x,\eta)$ *and* $\hat{g}(\eta)$ *satisfy*

$$f(x,\eta)\geqslant 0 \quad \textit{a.e. in } \Omega,\ \eta\in(-\infty,0) \qquad \textit{and} \qquad \hat{g}\in L^1(-\infty,0)\cap L^\infty(-\infty,0). \tag{21}$$

Then for each solution $u\in W_0^{1,p}(\Omega)$ *of* (1) *we have* $u\geqslant 0$.

COROLLARY 4 (Existence Result in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$). *Assuming* (2)–(7) *with* $m_0 \leqslant 0$ *and* $M_0 \geqslant 0$, Ω *bounded, let*

$$|f(x,\eta)| \leqslant \hat{f}(x) \quad \text{a.e. } x\in\Omega,\ \forall \eta \in (m_0, M_0),$$
$$\hat{f}\in L^1(\Omega), \qquad \hat{g}\in L^1(\mathbf{R})\cap L^\infty(\mathbf{R}). \tag{22}$$

Then there exists a solution $u \in W_0^{1,p}(\Omega)$ *of* (1) *such that* $m_0 \leqslant u(x) \leqslant M_0$ *a.e. in* Ω, *and each solution of* (1) *has its values contained in* $[m_0, M_0]$ *a.e.*

Proof. It is easy to see that m_0 and M_0 are subsolution and supersolution of (1) respectively. To prove this, it suffices to show that the coercivity condition (2) of the Carathéodory vector function $a(x,\eta,\xi)$ implies that

$$a(x,\eta,0)=0 \qquad \text{a.e.} \quad x\in\Omega,\ \forall\eta\in\mathbf{R}. \tag{23}$$

Indeed, let us fix $x\in\Omega$ and $\eta\in\mathbf{R}$, and assume, contrary to the claim, that $a(x,\eta,0)\neq 0$. Then there exists $\varepsilon>0$ such that $|\xi|\leqslant\varepsilon$ implies $a(x,\eta,\xi)\neq 0$. On the other hand, for $|\xi|=\varepsilon$ we have $a(x,\eta,\xi)\cdot\xi\geqslant\alpha\varepsilon^p>0$, and the Krasnoselski principle (see, e.g., [16, p. 250]) implies that there exists ξ_0, $|\xi_0|<\varepsilon$ such that $a(x,\eta,\xi_0)=0$, a contradiciton.

The claim follows from existence result in [5]. ▮

Note that the conclusion of Corollary 1 does not hold in general in the case when $\hat{g}\notin L^1(\mathbf{R})$. In fact, if we take for example $a(x,\eta,\xi)=|\xi|^{p-2}\xi$, $f(x,\eta)=0$, $g(x,\eta)=\hat{g}(\eta)=1$, and $\Omega=B_R(0)$, $0<R<1$, then (1) reduces to

$$-\operatorname{div}(|\nabla v|^{p-2}\nabla v)=|\nabla v|^p \qquad \text{in } B_R(0). \tag{24}$$

However, if $1<p\leqslant N$ then this equation has an unbounded solution $v\in W_0^{1,p}(B_R(0))$, and it can be defined explicitely by

$$v(x)=\begin{cases} \displaystyle\int_R^{|x|} \frac{N-p}{r^{(N-1)/(p-1)}-r}\,dr & \text{if} \quad 1<p<N \\ (N-1)\left(\log\log\dfrac{1}{|x|}-\log\log\dfrac{1}{R}\right) & \text{if} \quad p=N. \end{cases} \tag{25}$$

For the proof of this interesting fact see [6].

Now we can state maximum principle of the second type and its dual. Note that we drop the assumption $\hat{g}\in L^1(\mathbf{R})$.

THEOREM 3. *Assuming* (2), (3) *and* $\hat{g} \in L^\infty(\mathbf{R})$ *in* (5) ($\hat{g} \in L^\infty(-\infty, 0)$), *let the function* $f(x, \eta)$ *satisfy*

$$\begin{cases} \text{there exist } \varepsilon > 0 \text{ and an open, bounded set } A \subset \Omega \text{ such that } A_\varepsilon \subset \Omega \text{ and} \\ f(x, \eta) \geqslant 0 \text{ a.e. in } A_\varepsilon,\ \eta \in \mathbf{R}\ (\eta \in (-\infty, 0)), \end{cases} \tag{26}$$

where A_ε *denotes the* ε*-neighbourhood of* A. *Then for each* (*nonpositive*) *supersolution* u *of* (1) *satisfying* $\operatorname{ess\,inf}_{A_\varepsilon} u > -\infty$ *we have* $\operatorname{ess\,inf}_A u \geqslant \operatorname{ess\,inf}_{A_\varepsilon \setminus \bar{A}} u$.

THEOREM 4 (Dual Result). *Assuming* (2), (3), *and* $\hat{g} \in L^\infty(\mathbf{R})$ *in* (5) ($\hat{g} \in L^\infty(\mathbf{R}^+)$), *let the function* $f(x, \eta)$ *satisfy*

$$\begin{cases} \text{there exist } \varepsilon > 0 \text{ and an open, bounded set } A \subset \Omega \text{ such that } A_\varepsilon \subset \Omega \text{ and} \\ f(x, \eta) \leqslant 0 \text{ a.e. in } A_\varepsilon,\ \eta \in \mathbf{R}\ (\eta \in \mathbf{R}^+). \end{cases} \tag{27}$$

Then for each (*nonnegative*) *subsolution* u *of* (1) *satisfying* $\operatorname{ess\,sup}_{A_\varepsilon} u < \infty$ *we have* $\operatorname{ess\,sup}_A u \leqslant \operatorname{ess\,sup}_{A_\varepsilon \setminus \bar{A}} u$.

Proof of Theorem 3. First of all, let Φ be a function $\Phi \in \mathcal{D}(\Omega)$, $0 \leqslant \Phi \leqslant 1$ satisfying

$$\begin{cases} \Phi(x) = 1, & x \in A; \qquad \Phi(x) = 0, \qquad x \in \Omega \setminus \bar{A}_\varepsilon, \\ \Phi(x) > 0, & x \in A_\varepsilon. \end{cases} \tag{28}$$

Applying the test function $\varphi = -(e^{m(u-t)^-} - 1)\, \Phi^p \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ to (11) we derive that for $m > 0$ large enough we have

$$H(t) \leqslant c \cdot F(t), \qquad \forall t \in \mathbf{R}, \tag{29}$$

where $c = 1/(m\alpha - \|\hat{g}\|_{L^\infty}) > 0$, and

$$\begin{cases} H(t) = \displaystyle\int_{\{u<t\}} |\nabla u|^p\, \Phi^p\, dx \qquad \text{and} \\ F(t) = \displaystyle\int_{\{u<t\}} |a(x, u, \nabla u)|\, \Phi^{p-1} p\, |\nabla \Phi|\, e^{m(u-t)^-}\, dx \\ \qquad\quad - \displaystyle\int_{\{u<t\}} f(x, u)\, (e^{m(u-t)^-} - 1)\, \Phi^p\, dx. \end{cases} \tag{30}$$

Since the second integral appearing in the definition of $F(t)$ is nonnegative, while the first one is zero for all $t < \operatorname{ess\,inf}_{A_\varepsilon \setminus A} u$ (note that $\{u < t\} \cap (A_\varepsilon \setminus \bar{A}) = \varnothing$), we conclude that

$$F(t) \leqslant 0, \qquad \forall t \in (-\infty, \operatorname*{ess\,inf}_{A_\varepsilon \setminus \bar{A}} u). \tag{31}$$

Now $H(t)=0$ for all $t\in(-\infty, \operatorname{ess\,inf}_{A_\varepsilon\setminus\bar{A}} u)$, that is:

$$|\nabla(u-t)^-|^p\,\Phi^p=0, \qquad \text{a.e. in } \Omega, \qquad \forall t\in(-\infty, \operatorname*{ess\,inf}_{A_\varepsilon\setminus\bar{A}} u). \tag{32}$$

Since $\Phi(x)>0$ for $x\in A_\varepsilon$ from the preceding equality we derive that $(u-t)^-=const_i(t)$ a.e. on each component of connectedness A_ε^i of A_ε, where $i\in I$ and the index set I is at most countable. Assume that $const_i(t)>0$ for some $i\in I$ and some $t<\operatorname{ess\,inf}_{A_\varepsilon\setminus\bar{A}} u$. Then $u=t-const_i(t)$ $<t$ a.e. in A_ε^i, and we obtain a contradiction:

$$\operatorname*{ess\,inf}_{A_\varepsilon\setminus\bar{A}} u\leqslant \operatorname*{ess\,inf}_{A_\varepsilon^i\setminus\bar{A}} u<t.$$

Therefore $const_i(t)=0$ for all $i\in I$ and all $t<\operatorname{ess\,inf}_{A_\varepsilon\setminus\bar{A}} u$. This is equivalent to

$$u\geqslant t \qquad \text{a.e. in } A_\varepsilon, \qquad \forall t\in(-\infty, \operatorname*{ess\,inf}_{A_\varepsilon\setminus\bar{A}} u). \tag{33}$$

Finally, taking ess inf in (33) over A_ε for fixed t, and then supremum over t, we obtain the claim. ∎

Theorem 4 is proved by using $\varphi=(e^{m(u-t)^+}-1)\,\Phi^p$. Now from the preceding two results we can derive the following consequences:

COROLLARY 5 (Strong Maximum Principle). *Assume* (2), (3), *and* (5) *hold with* $\hat{g}\in L^\infty(\mathbf{R})$.

(a) *If* $f(x,\eta)\geqslant 0$, ($f(x,\eta)\leqslant 0$) *a.e. in* Ω, $\eta\in\mathbf{R}$ *and* $u\in W^{1,p}(\Omega)\cap C(\bar{\Omega})$ *is a supersolution* (*subsolution*) *of* (1), *then it does not possess a local minimum* (*maximum*) *in* Ω;

(b) *If* $f(x,\eta)=0$ *a.e. in* Ω, $\eta\in\mathbf{R}$ *and* $u\in W^{1,p}(\Omega)\cap C(\bar{\Omega})$ *is a solution of* (1), *then it possesses neither local minimum nor local maximum in* Ω.

COROLLARY 6 (Nonexistence of a Solution That Is Positive, Decreasing, and Spherically Symmetric near the Origin). *Assuming* (2), (3), *and* (5) *with* $\hat{g}\in L^\infty(\mathbf{R}^+)$, *let the function* $f(x,\eta)$ *be such that*

$$\begin{cases} \exists R>0 \textit{ such that } B_R(0)\subset\Omega & \textit{and} \\ f(x,\eta)\leqslant 0, \qquad \textit{a.e. in } B_R(0), & \eta\in\mathbf{R}^+. \end{cases} \tag{34}$$

Then there is no solution $u\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ *of* (1) *that is positive, decreasing, and spherically symmetric in* $B_R(0)$.

COROLLARY 11. *Assuming* (2), (3), *and* $\hat{g} \in L^\infty(\mathbf{R})$ *in* (5), *let* A *be a regular, bounded, open subset of* Ω *such that* $A \subset\subset \Omega$.

If $f(x, \eta) \geqslant 0$, (*or* $f(x, \eta) \leqslant 0$) *for a.e.* $x \in A$, $\eta \in \mathbf{R}$, *then for each supersolution* (*subsolution*) u *of* (1) *in* $W_0^{1,p}(\Omega) \cap C(\Omega)$ *we have*

$$\min_{\bar{A}} u \geqslant \min_{\partial A} u \ (\max_{\bar{A}} u \leqslant \max_{\partial A} u). \tag{35}$$

3. CONTROL OF LOWER (UPPER) BOUND FOR ESS SUP (ESS INF) OF SOLUTIONS

In Theorem 1 and Theorem 2 we were able to obtain a global control over the upper (lower) bound of essential supremum (essential infimum) of solutions. Now we would like to find some sufficient conditions on $f(x, \eta)$ and $g(x, \eta)$ that permit local control over the lower (uper) bound of essential supremum (essential infimum) of solutions. We then describe a class of quasilinear problems whose solutions are sign–changing in Ω. Next, in the following section we derive a priori estimates of oscillation of solutions from below. Using this we find some sufficient conditions ensuring that all solutions be singular in a given point.

We start with the main result of this section and its dual.

THEOREM 5 (Control of Essential Supremum of Solutions). *Assume that conditions* (2) *and* (3) *are satisfied. Let* $x_1 \in \Omega$, $R_1 > 0$ *be such that* $B_{2R_1}(x_1) \subseteq \Omega$. *Let us choose* $m_0, M_1 \in \mathbf{R}$, $m_0 < M_1$, *so that the following conditions are fulfilled,*

$$\begin{aligned} &\exists f_1 \in L^1(B_{R_1}(x_1)), \qquad f(x,\eta) \geqslant f_1(x) \\ &\qquad a.e. \quad x \in B_{R_1}(x_1), \qquad \eta \in I_1 = (m_0, M_1), \end{aligned} \tag{36}$$

$$\begin{aligned} &f(x,\eta) \geqslant 0 \qquad a.e. \quad x \in B_{2R_1}(x_1) \backslash B_{R_1}(x_1), \qquad \eta \in I_1, \\ &g(x,\eta) \equiv 0 \qquad a.e. \quad x \in B_{2R_1}(x_1), \qquad \eta \in I_1, \end{aligned} \tag{37}$$

$$\int_{B_{R_1}(x_1)} f_1(x)\,dx > \inf_{s>0} \frac{1}{s}\,[\bar{\delta} H + D(|I_1| + s)^p], \tag{38}$$

where

$$H = \int_{B_{2R_1}(x_1)} [h(x)^{p'} + a_1^{p'} \bar{m}^p]\,dx, \qquad \bar{m} = \max\{|m_0|, |M_1|\},$$

$$D = \left(\frac{p}{\delta}\right)^{p-1} \frac{(2^N - 1)\,|B_{R_1}(x_1)|}{R_1^p}, \qquad \delta = \frac{p'}{3^{p'-1}}\,\bar{\delta}, \qquad \bar{\delta} = \frac{\alpha}{a_2^{p'}}. \tag{39}$$

Then each supersolution of (1) *has the property*

$$\textit{if } \operatorname*{ess\,inf}_{B_{2R_1}(x_1)} u \geqslant m_0 \qquad \textit{then} \quad \operatorname*{ess\,sup}_{B_{2R_1}(x_1)} u > M_1 \tag{40}$$

Proof. To simplify, we denote R_1 by R, and $B_{R_1}(x_1)$ by B_R. The proof rests on the use of localization function Φ of the ball B_R. It is easy to see that for any $c_0 > 1$ there exists a function $\Phi \in C_0^\infty(\Omega)$ with the following properties (a more general result is stated in Lemma 5 in Section 5 below),

$$\begin{aligned} &0 \leqslant \Phi \leqslant 1, \\ &\Phi(x) = 1 \quad \text{for} \quad x \in B_R \quad \text{and} \quad \Phi(x) = 0 \quad \text{for} \quad x \in \Omega \backslash B_{2R}, \\ &\Phi(x) > 0 \quad \text{on } B_{2R} \quad \text{and} \quad |\nabla \Phi| \leqslant \frac{c_0}{R} \quad \text{on } \Omega. \end{aligned} \tag{41}$$

Assume contrary to the claim in the theorem that $u \in W^{1,p}(\Omega)$ is a supersolution of (1) satisfying $u \geqslant m_0$ a.e. on B_{2R} and $u \leqslant M_1$ a.e. on B_{2R}. Let us choose any $t \in \mathbf{R}$ and define a function

$$\varphi = -(u-t)^- \, \Phi^p. \tag{42}$$

Since $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $\varphi \leqslant 0$, we can multiply the inequality corresponding to (1) by φ and integrate by parts. Using $u \leqslant M_1$ a.e. on B_{2R}, this yields

$$\begin{aligned} &\int_{\{u<t\} \cap B_{2R}} a(x, u, \nabla u) \cdot \nabla u \, \Phi^p \, dx \\ &\quad \leqslant \int_{\{u<t\} \cap B_{2R}} [\, |a(x, u, \nabla u)| \, (t-u) \, p\Phi^{p-1} \, |\nabla \Phi| \\ &\qquad - f(x, u)(t-u) \, \Phi^p] \, dx. \end{aligned} \tag{43}$$

Applying Young's inequality we see that the first term in square brackets is dominated by

$$\frac{\delta}{p'} \, |a(x, u, \nabla u)|^{p'} \, \Phi^p + \left(\frac{p}{\delta}\right)^{p-1} \, (t-u)^p \, |\nabla \Phi|^p. \tag{44}$$

Now using the elementary inequality

$$|a(x, u, \nabla u)|^{p'} \leqslant 3^{p'-1} [h(x)^{p'} + a_1^{p'} \, |u|^p + a_2^{p'} \, |\nabla u|^p]$$

we obtain that

$$
\begin{aligned}
L \int_{\{u<t\} \cap B_{2R}} & |\nabla u|^p \Phi^p \, dx \\
& \leqslant \bar{\delta} \int_{B_{2R}} [h(x)^{p'} + a_1^{p'} \bar{m}^p] \, dx \\
& \quad + \left(\frac{p}{\delta}\right)^{p-1} \int_{\{u<t\} \cap B_{2R}} (t-u)^p \, |\nabla \Phi|^p \, dx \\
& \quad - \int_{\{u<t\} \cap B_{2R}} f(x, u)(t-u) \, \Phi^p \, dx, \qquad (45)
\end{aligned}
$$

where $L = \alpha - \bar{\delta} a_2^{p'} = 0$. Since $u \leqslant M_1$ a.e. on B_{2R}, then from (36) and (37) we conclude that for any $t > M_1$

$$
0 \leqslant \bar{\delta} H + \left(\frac{p}{\delta}\right)^{p-1} (t-m_0)^p \left(\frac{c_0}{R}\right)^p |B_{2R} \backslash B_R| - (t-M_1) \, F_1, \qquad (46)
$$

where $F_1 = \int_{B_R} f_1(x) \, dx$.

Let $s > 0$ be arbitrary and substitute $t = M_1 + s$. Note that $|B_{2R} \backslash B_R| = (2^N - 1) \, |B_R|$. As c_0 can be chosen arbitrarily close to 1, we can let $c_0 \to 1$. Now we obtain a contradiction by estimating F_1 and taking the infimum over $s > 0$,

$$
F_1 \leqslant \inf_{s>0} \frac{1}{s} \left[\bar{\delta} H + \left(\frac{p}{\delta}\right)^{p-1} \frac{(2^N-1) \, |B_R|}{R^p} \, (s + |I_1|)^p \right]. \quad \blacksquare
$$

THEOREM 6 (Control of Essential Infimum of Solutions). *Assume that conditions* (2) *and* (3) *are satisfied. Let* $x_2 \in \Omega$, $R_2 > 0$ *be such that* $B_{2R_2}(x_2) \subseteq \Omega$. *Let us choose* $m_1, M_0 \in \mathbf{R}$, $m_1 < M_0$ *so that the following conditions are fulfilled,*

$$
\begin{aligned}
& \exists f_2 \in L^1(B_{R_2}(x_2)), \qquad f(x, \eta) \leqslant f_2(x) \\
& \qquad \text{a.e.} \quad x \in B_{R_2}(x_2), \qquad \eta \in I_2 = (m_1, M_0), \qquad (47) \\
& f(x, \eta) \leqslant 0 \qquad \text{a.e.} \quad x \in B_{2R_2}(x_2) \backslash B_{R_1}(x_1), \qquad \eta \in I_2, \\
& g(x, \eta) \equiv 0 \qquad \text{a.e.} \quad x \in B_{2R_2}(x_2), \qquad \eta \in I_2, \qquad (48)
\end{aligned}
$$

$$
\int_{B_{R_2}(x_2)} f_2(x) \, dx < - \inf_{s>0} \frac{1}{s} \, [\bar{\delta} H + D(|I_1| + s)^p], \qquad (49)
$$

where D, δ, *and* $\bar{\delta}$ *are as in Theorem* 5, *and*

$$H = \int_{B_{2R_2}(x_2)} [h(x)^{p'} + a_1^{p'} \bar{m}^p] \, dx, \qquad \bar{m} = \max\{|m_1|, |M_0|\}.$$

Then each subsolution of (1) *has the property*

$$\textit{if} \ \operatorname*{ess\,sup}_{B_{2R_2}(x_2)} u \leqslant M_0 \qquad \textit{then} \qquad \operatorname*{ess\,inf}_{B_{2R_2}(x_2)} u < m_1. \tag{50}$$

Corollary 8 (Simultaneous Control of ess sup and ess inf of Solutions of (1) in $W^{1,p}(\Omega)$). *Assume that four real numbers*

$$m_0 < m_1 \leqslant M_1 < M_0, \tag{51}$$

are given such that sign conditions (6) *and* (7) *hold. Let the conditions of the preceding two theorems be satisfied on the balls* $B_{2R_1}(x_1)$ *and* $B_{2R_2}(x_2)$, *that we assume to be disjoint. Then for each solution* $u \in W^{1,p}(\Omega)$ *of* (1) *such that* $\operatorname{ess\,inf}_{\partial\Omega} u \geqslant m_0$ *and* $\operatorname{ess\,sup}_{\partial\Omega} u \leqslant M_0$ *we have*

$$m_0 \leqslant \operatorname*{ess\,inf}_{\Omega} u < m_1 \qquad \textit{and} \qquad M_1 < \operatorname*{ess\,sup}_{\Omega} u \leqslant M_0. \tag{52}$$

Remark 1. Our sufficient conditions (6) and (7) for (52) to hold are almost necessary in the following sense. Assume that $m_1 \leqslant u \leqslant M_1$ a.e. on the boundary of Ω, and assume that (52) holds. Then neither $f(x, \eta) \leqslant 0$ a.e. $x \in \Omega$, $\eta \in (M_1, M_0)$, nor $f(x, \eta) \geqslant 0$ a.e. $x \in \Omega$, $\eta \in (m_0, m_1)$. This can easily be seen by contradiction, applying Theorem 1 or Theorem 2 to problem (1), where we modify $f(x, \eta)$ to $\bar{f}(x, \eta)$ by $\bar{f}(x, \eta) = f(x, M_0)$ for a.e $x \in \Omega$, $\eta \geqslant M_0$ or $\bar{f}(x, \eta) = f(x, m_0)$ for a.e $x \in \Omega$, $\eta \leqslant m_0$, respectively, and analogously for $g(x, \eta)$.

Corollary 9 (Sign-Changing Solutions). *Let the conditions of the preceding corollary be fulfilled with* $m_1 = M_1 = 0$. *Then for each solution* $u \in W^{1,p}(\Omega)$ *of* (1) *such that* $\operatorname{ess\,inf}_{\partial\Omega} u \geqslant m_0$ *and* $\operatorname{ess\,sup}_{\partial\Omega} u \leqslant M_0$ *we have* $|\Omega^+| \neq 0$ *and* $|\Omega^-| \neq 0$ *where* $\Omega^{\pm} = \{x \in \Omega : u(x) > 0 \ (u(x) < 0)\}$.

It is easy to see that in the last two theorems the expression $3^{p'-1}$ can be changed to $2^{p'-1}$ if either $h(x) \equiv 0$ on $B_{2R_1}(x_1)$ or $a_1 = 1$. If both $h(x) \equiv 0$ on $B_{2R_1}(x_1)$ and $a_1 = 0$, then we can use a better estimate for $|a(x, u, \nabla u)|$ in (44) involving 1 instead of $3^{p'-1}$. In this case the infimum in (38) is attained for

$$s = \frac{|I_1|}{p-1}, \tag{53}$$

and we obtain the following result.

COROLLARY 10. *Assume that the conditions of Theorem 5 are satisfied with* $h(x) \equiv 0$ *on* $B_{2R_1}(x_1)$, $a_1 = 0$, *and with* (38) *replaced by*

$$\int_{B_{R_1}(x_1)} f_1(x)\,dx > p\left(\frac{p}{\bar{\delta}}\right)^{p-1} \frac{(2^N-1)\,|B_{R_1}(x_1)|}{R_1^p}\,|I_1|^{p-1}, \tag{54}$$

and $\delta = p'\bar{\delta}$ *in* (39). *Then each supersolution of* (1) *has property* (40).

EXAMPLE 1. Corollary 10 can be exploited to the study of problem of reachability in control theory of elliptic equations. Let us consider the control system governed by

$$-\Delta u = f(x, u) + q(x), \qquad u \in H_0^1(\Omega),$$

and $q(x) \in \mathcal{U}$. Here

$$\mathcal{U} = \{q \in L^p(\Omega) : \exists c \in \mathbf{R}^+,\ q(x) = c \cdot \chi_{B_R(x_1)}(x)\}$$

is the set of admissible controls, where the ball $B_{2R}(x_1) \subseteq \Omega$ is given in advance, and χ_{B_R} is the characteristic function of $B_R(x_1)$. We study the problem of reachability of the prescribed set $I = [M_1, \infty)$ on the ball $B_{2R}(x_1)$, $M_1 \geqslant 0$, i.e., finding the admissible control $q(x) \in \mathcal{U}$ such that for the corresponding solution $u(x)$ (if it exists) we have $\operatorname{ess\,sup}_{B_{2R}} u \in I$. For the sake of simplicity we assume that all solutions of our control problem are nonnegative (for this it suffices to assume that $f(x, \eta) \geqslant 0$ for a.e. $x \in \Omega$, $\eta \leqslant 0$, see Corollary 3). We also assume that $f(x, \eta) \geqslant 0$ for a.e. $x \in B_{2R}(x_1) \backslash B_R(x_1)$, $\eta \in (0, M_1)$, $\operatorname{ess\,inf}_{\eta \in (0, M_1)} f(x, \eta) \in L^1(B_R(x_0))$. Using Corollary 10 it is easy to see that for any control $q(x) = c \cdot \chi_{B_R(x_1)}(x) \in \mathcal{U}$ such that

$$c \geqslant \frac{1}{|B_R(x_1)|}\left[DM_1 - \int_{B_R(x_1)} \operatorname*{ess\,inf}_{\eta \in (0, M_1)} f(x, \eta)\,dx\right]$$

we have that the corresponding solution (if it exists) reaches the set I, i.e. $\operatorname{ess\,sup}_\Omega u \geqslant M_1$. Here $D = 4(2^N - 1)(|B_R(x_1)|/R^2)$.

Proof. Indeed, take any $\varepsilon > 0$ and apply Corollary 10 to $m_0 = 0$ and $M_1 - \varepsilon$ with

$$f_1(x) = q(x) + \operatorname*{ess\,inf}_{\eta \in (0, M_1 - \varepsilon)} f(x, \eta).$$

Then we have $\operatorname{ess\,sup}_\Omega u > M_1 - \varepsilon$, and the result follows by letting $\varepsilon \to 0$. ∎

If we have a linear control problem with $f(x,\eta)=f(x)\in L^p(\Omega)$, $p>N$, where we assume Ω to be bounded, of class $C^{1,1}$, and $f(x)\geqslant 0$ on $B_{2R}(x_1)\backslash B_R(x_1)$, then $\max_{B_{2R}} u$ depends continuously and monotonically on $q\in\mathcal{U}$, i.e., on c. Let $M_0=\max_\Omega u(x)$ corresponding to $c=0$. Then for every $M_1\geqslant M_0$ there exists a unique $c\in\mathbf{R}$ such that for $q=q_c$ we have $\max_{B_{2R}(x_1)} u(x)=M_1$. Continuous and monotone dependence follows easily using maximum principle, Theorem 9.15, and Corollary 7.11 in [9].

4. OSCILLATION OF SOLUTIONS, SINGULARITIES

In this section we apply the main result of the preceding section. An important role is played by the notion of oscillation of a function $u\colon \Omega\to\mathbf{R}$ on Ω:

$$\operatorname*{osc}_\Omega u=\operatorname*{ess\,sup}_\Omega u-\operatorname*{ess\,inf}_\Omega u. \tag{55}$$

LEMMA 1. *Assume that conditions* (2) *and* (3) *are satisfied with* $a_1=0$. *Let* $x_1\in\Omega$, $R>0$ *and* $m_0\in\mathbf{R}$ *be such that* $B_{2R}(x_1)\subseteq\Omega$ *and*

$$h(x)\equiv 0 \quad \text{and} \quad g(x,\eta)\equiv 0 \quad \text{for a.e.} \quad x\in B_{2R}(x_1),\ \eta\in(m_0,\infty), \tag{56}$$

$$f(x,\eta)\geqslant 0 \quad \text{for a.e.} \quad x\in B_{2R}(x_1), \quad \eta\in(m_0,\infty). \tag{57}$$

If $u\in W^{1,p}(\Omega)$ *is a supersolution of* (1) *such that* $u\geqslant m_0$ *a.e. in* $B_{2R}(x_1)$, *then*

$$\operatorname*{osc}_{B_{2R}(x_1)} u\geqslant bR^{p'}\operatorname*{ess\,inf}_{B_R(x_1)\times(m_0,\,\infty)} f(x,\eta)^{p'/p}, \tag{58}$$

where

$$b=\frac{\bar\delta}{p^{p'}(2^N-1)^{p'/p}},\qquad \bar\delta=\frac{\alpha}{a_2^{p'}}. \tag{59}$$

Proof. Let us define

$$K=\operatorname*{ess\,inf}_{B_R(x_1)\times(m_0,\,\infty)} f(x,\eta).$$

If $K=0$ the claim is trivial. Let us assume therefore that $K>0$ and define $f_1(x)\equiv K$ on $B_R(x_1)$. We choose M_1, $M_1>m_0$, such that for $I_1=(m_0,M_1)$ the following inequality holds:

$$K>p\left(\frac{p}{\bar\delta}\right)^{p-1}\frac{2^N-1}{R^p}\,|I_1|^{p-1}.$$

Then by Corollary 10 we obtain that for any such M_1 our supersolution u has property (40). The above inequality is equivalent to

$$M_1 < bR^{p'}K^{p'/p} + m_0.$$

It is clear that for every $\varepsilon > 0$ there exists M_1 satisfying the above inequality and $M_1 > bR^{p'}K^{p'/p} + m_0 - \varepsilon$. Using property (40) of u we then obtain

$$\operatorname*{ess\,sup}_{B_{2R}(x_1)} u > M_1 > bR^{p'}K^{p'/p} + m_0 - \varepsilon,$$

and if we let $\varepsilon \to 0$, we arrive to

$$\operatorname*{ess\,sup}_{B_{2R}(x_1)} u \geqslant bR^{p'}K^{p'/p} + m_0. \tag{60}$$

This means that $\operatorname{osc}_{B_{2R}(x_1)} u \geqslant \operatorname{ess\,sup}_{B_{2R}(x_1)} u - m_0 \geqslant bR^{p'}K^{p'/p}$. ∎

LEMMA 2 (Dual Result). *Assume that conditions* (2) *and* (3) *are satisfied with $a_1 = 0$. Let $x_2 \in \Omega$, $R > 0$, and $M_0 \in \mathbf{R}$ be such that $B_{2R}(x_2) \subseteq \Omega$ and*

$$h(x) \equiv 0 \quad \textit{and} \quad g(x, \eta) \equiv 0 \quad \textit{for a.e.} \quad x \in B_{2R}(x_2), \quad \eta \in (-\infty, M_0), \tag{61}$$

$$f(x, \eta) \leqslant 0 \quad \textit{for a.e.} \quad x \in B_{2R}(x_1), \quad \eta \in (-\infty, M_0). \tag{62}$$

If $u \in W^{1,p}(\Omega)$ is a subsolution of (1) *such that $u \leqslant M_0$ a.e. in $B_{2R}(x_2)$, then retaining the same b and $\bar{\delta}$ as in the preceding lemma we have*

$$\operatorname*{osc}_{B_{2R}(x_2)} u \geqslant b\, R^{p'} \operatorname*{ess\,inf}_{B_R(x_2) \times (-\infty, M_0)} |f(x, \eta)|^{p'/p}. \tag{63}$$

Most of the remaining theorems and corollaries in this section also have their duals, but we do not formulate them here. Now we want to extend our oscillation estimate to problems whose right-hand side may depend on the gradient as well.

THEOREM 7 (Local Oscillation Estimate). *Let* (2), (3) *hold with $a_1 = 0$ and $h(x) \equiv 0$ on $B_{2R}(x_1) \subseteq \Omega$. Consider the quasilinear problem*

$$\begin{gathered} -\operatorname{div} a(x, v, \nabla v) = F(x, v, \nabla v) \quad \textit{in} \quad \mathscr{D}'(\Omega), \\ F(x, v, \nabla v) \in L^1(\Omega), \quad v \in W^{1,p}(\Omega), \end{gathered} \tag{64}$$

where $F(x,\eta,\xi)$ is a Carathéodory function (measurable with respect to x and continuous with respect to remaining variables). Assume that $m_0 \in \mathbf{R}$ is such that $F(x,\eta,\xi) \geqslant 0$ for a.e. $x \in B_{2R}(x_1)$, $\eta \geqslant m_0$ and $\xi \in \mathbf{R}^N$. If v is a supersolution in $W^{1,p}(\Omega)$ such that $v \geqslant m_0$ on $B_{2R}(x_1)$, then we have

$$\operatorname*{osc}_{B_{2R}(x_1)} v \geqslant b\, R^{p'} \operatorname*{ess\,inf}_{B_R(x_1)\times(m_0,\infty)\times\mathbf{R}^N} F(x,\eta,\xi)^{p'/p}, \tag{65}$$

where b is a constant defined by (59).

Proof. Let us define $f(x,\eta)=F(x,\eta,\nabla v)$, where v is a supersolution. Then $u=v$ is a supersolution of (1) with $g(x,\eta)\equiv 0$, and Lemma 1 applies. ∎

As a consequence we obtain variational a priori bound for oscillation on the whole of Ω.

Corollary 11 (Global Oscillation Estimate). *Let (2), (3) hold with $a_1=0$ and $h(x)\equiv 0$ on Ω. Assume that there exists $m_0 \in \mathbf{R}$ such that $F(x,\eta,\xi)\geqslant 0$ on $\Omega\times(m_0,\infty)\times\mathbf{R}^N$. Then we have global oscillation estimate for supersolutions of (64) satisfying $v \geqslant m_0$ on Ω:*

$$\operatorname*{osc}_{\Omega} v \geqslant b \sup_{\substack{x_1\in\Omega\\ R>0\\ B_{2R}(x_1)\subseteq\Omega}} R^{p'} \operatorname*{ess\,inf}_{\substack{x\in B_R(x_1)\\ \eta\in(m_0,\infty)\\ \xi\in\mathbf{R}^N}} F(x,\eta,\xi)^{p'/p}. \tag{66}$$

In particular, if the variable x in F is separated, that is, $F(x,\eta,\xi)=K(x)\,f(\eta,\xi)$, with $K(x)\geqslant 0$ on Ω, $f(\eta,\xi)\geqslant 0$ for $\eta\geqslant m_0$, $\xi\in\mathbf{R}^N$, then

$$\operatorname*{osc}_{\Omega} v \geqslant b\,\Big[\sup_{\substack{x_1\in\Omega\\ R>0\\ B_{2R}(x_1)\subseteq\Omega}} R^{p'} \operatorname*{ess\,inf}_{x\in B_R(x_1)} K(x)^{p'/p}\Big] \operatorname*{ess\,inf}_{\substack{\eta\in(m_0,\infty)\\ \xi\in\mathbf{R}^N}} f(\eta,\xi)^{p'/p}. \tag{67}$$

An immediate consequence of Theorem 7 is the following result, where the notion of inner radius of Ω is introduced. It shows that a priori estimate of oscillation of a solution depends heavily on the geometry of Ω. More general results will be stated in Section 5.

Corollary 12 (A Priori Estimate Involving Inner Radius of Domain). *Let (2), (3) hold with $h(x)\equiv 0$ on Ω and $a_1=0$, and let $F(x,\eta,\xi)$ be a Carathéodory function such that $F(x,\eta,\xi)\geqslant 0$ on $\Omega\times(m_0,\infty)\times\mathbf{R}^N$. Then for every supersolution v of (64) we have*

$$\operatorname*{osc}_{\Omega} v \geqslant b\left(\frac{r_0(\Omega)}{2}\right)^{p'} \operatorname*{ess\,inf}_{\Omega\times(m_0,\infty)\times\mathbf{R}^N} F(x,\eta,\xi)^{p'/p}. \tag{68}$$

Here $r_0(\Omega)$ is inner radius of Ω, i.e., the radius of largest ball that can be inscribed into Ω:

$$r_0(\Omega) = \sup\{r > 0 : \exists x_1 \in \Omega,\, B_r(x_1) \subseteq \Omega\}. \tag{69}$$

If the sup in (69) is achieved for $r_0 = r_0(\Omega)$ and x_1 (which is the case for any bounded Ω), then it suffices to assume that $F(x, \eta, \xi) \geqslant 0$ on $B_{r_0}(x_1) \times (m_0, \infty) \times \mathbf{R}^N$, and to have $B_{r_0/2}(x_1)$ instead of Ω under ess inf in (68). Also note that if there exists $K > 0$ such that $F(x, \eta, \xi) \geqslant K$ for x on an open subset Ω' of Ω such that $r_0(\Omega') = \infty$, $\eta \geqslant m_0$, $\xi \in \mathbf{R}^N$, then (65) implies that $\operatorname{osc}_{\Omega'} v = \infty$.

EXAMPLE 2. In the case when the left-hand side of (64) is $-\Delta v$ we have $p = 2$, $\alpha = a_2 = 1$, and $b = 1/4(2^N - 1)$, which yields the following a priori estimate for supersolutions such that $v \geqslant m_0$ on Ω:

$$\operatorname{osc}_\Omega v \geqslant \frac{r_0(\Omega)^2}{16(2^N - 1)} \operatorname*{ess\,inf}_{\Omega \times (m_0, \infty) \times \mathbf{R}^N} F(x, \eta, \xi). \tag{70}$$

It is interesting that our a priori estimates of oscillations of solutions imply lower bounds on constants appearing in Schauder estimates and Agmon, Douglis, and Nirenberg estimates (see, e.g., [9, 16]). We illustrate this on the boundary value problem

$$\begin{aligned} -\Delta u &= f(x) \qquad \text{in } \Omega, \\ u &= \phi \qquad \text{on } \partial\Omega. \end{aligned} \tag{71}$$

As is well known, if Ω is a bounded domain of class $C^{2,\theta}$, $\theta \in (0, 1)$, and $f \in C^{0,\theta}(\Omega)$, $\phi \in C^{2,\theta}(\partial\Omega)$, then for the corresponding solution we have Schauder's a priori estimate:

$$\|u\|_{C^{2,\theta}(\Omega)} \leqslant c(\|f\|_{C^{0,\theta}(\Omega)} + \|\phi\|_{C^{2,\theta}(\partial\Omega)}), \tag{72}$$

where c does not depend on f and ϕ.

COROLLARY 13. *Under the above conditions on Ω we have the following lower bound on the constant c appearing in Schauder's estimate*:

$$c \geqslant \frac{r_0(\Omega)^2}{16(2^N - 1)}. \tag{73}$$

Proof. We take $f \equiv 1$ and $\phi \equiv 0$. Since $f(x)$ is positive, then $u \geqslant 0$, which implies $\operatorname{osc}_\Omega u = \|u\|_{L^\infty}$, hence $\|u\|_{2,\theta} \geqslant \operatorname{osc}_\Omega u$. It suffices to substitute $f \equiv 1$ into (72) and combine with Corollary 12:

$$c = c\,\|f\|_{C^{0,\theta}(\Omega)} \geqslant \|u\|_{C^{2,\theta}(\Omega)} \geqslant \|u\|_{L^\infty(\Omega)} = \operatorname*{osc}_{\Omega} u \geqslant \frac{r_0(\Omega)^2}{16(2^N-1)}. \quad \blacksquare$$

Now we assume that Ω is a domain of class C^2, $\phi \equiv 0$, and $1 < q < \infty$. Then there exists $c_q > 0$ such that for all $f \in L^q(\Omega)$ we have the following Agmon, Douglis, and Nirenberg estimate for solutions of (71),

$$\|u\|_{W^{2,q}} \leqslant c_q\,\|f\|_{L^q}, \tag{74}$$

where c_q does not depend on f.

Corollary 14. *Assume that Ω is a bounded domain of class C^2 and $q > N/2$ and $\phi \equiv 0$. Then the constant c_q appearing in Agmon, Douglis, and Nirenberg estimate has the lower bound*

$$c_q \geqslant \frac{r_0(\Omega)^2}{16(2^N-1)\,D_q\,|\Omega|}, \tag{75}$$

where D_q is the imbedding constant of $W^{2,q}(\Omega) \subseteq C(\bar{\Omega})$.

Proof. Since $\|u\|_{L^\infty} \leqslant D_q\,\|u\|_{W^{2,q}}$, $\operatorname{osc}_\Omega u = \|u\|_{L^\infty}$, the result follows again by substituting $f \equiv 1$ and using Corollary 12. $\blacksquare$

Example 3. Let us consider the problem

$$-\Delta_p u = \lambda e^u \quad \text{in} \quad \mathscr{D}'(\Omega),$$
$$e^u \in L^1_{\mathrm{loc}}(\Omega), \qquad u \in W^{1,p}_{\mathrm{loc}}(\Omega),$$

where we assume that $\lambda > 0$, and Ω is an unbounded domain such that $r_0(\Omega) = \infty$. Then this problem has no essentially bounded solutions. This follows immediately from Corollary 12, which can be easily extended to spaces of locally integrable functions. For $p = 2$, $\Omega = \mathbf{R}^N$, $N > 2$, this problem has been studied by Mignot and Puel in [15]. They discovered that for any $x_0 \in \mathbf{R}^N$ the function $u(x) = -2 \ln |x - x_0| - \ln \lambda + \ln 2(N-2)$ is a solution. Note that this function is unbounded, which is in accordance with our result. Also note that for $\Omega = \mathbf{R}^N \backslash \overline{B_R(x_0)}$ this solution is also unbounded and has no singularity in Ω.

Now we describe a class of quasilinear equations of the form (64) such that any supersolution u possesses a *singularity* in a given point $x_0 \in \bar{\Omega}$, that is, $\operatorname{osc}_{x_0} u = \infty$, where oscillation in the point x_0 is defined by

$$\operatorname*{osc}_{x_0} u = \lim_{r \to 0} \operatorname*{osc}_{B_r(x_0) \cap \Omega} u.$$

We also introduce the following notion. We say that a point $x_0 \in \partial\Omega$ has the *weak cone property* if there exists $d \in (0, 1)$ and a sequence of balls $B_{r_k}(x_k) \subset \Omega$ such that $x_k \to x_0$, $r_k \to 0$ as $k \to \infty$, and $r_k > d\,|x_k - x_0|$ for all k. It is easy to see that if a boundary point x_0 has the cone property, then it has the weak cone property. The converse is not true. Cusps do not have weak cone property.

Corollary 15 (Generating Singularities of Solutions). *Let* (2) *and* (3) *hold with* $a_1 = 0$ *and* $h(x) \equiv 0$ *on* $B_R(x_0) \cap \Omega$. *Assume that* $F(x, \eta, \xi)$ *has a singularity of order* γ *at* $x_0 \in \bar{\Omega}$ *with* $\gamma > p$, *that is, there exists a constant* $C > 0$ *and* $s \geqslant 0$ *such that*

$$F(x, \eta, \xi) \geqslant \frac{C}{|x - x_0|^\gamma \cdot |\log |x - x_0||^s}$$

$$\text{a.e.} \quad x \in B_R(x_0) \cap \Omega,\ \forall \eta \in (m_0, \infty),\ \forall \xi \in \mathbf{R}^N. \tag{76}$$

If $x_0 \in \partial\Omega$, *we also assume that* x_0 *has the weak cone property. Then any supersolution* u *of* (64), *such that* $u \geqslant m_0$ *on* $B_R(x_0) \cap \Omega$, *is singular in* x_0. *If* $\gamma = p$ *and* $s = 0$, *then* $\operatorname{osc}_{x_0} u > 0$,

$$\operatorname*{osc}_{x_0} u \geqslant \begin{cases} bC^{p'/p}, & \textit{for} \quad x_0 \in \Omega, \\ \dfrac{bC^{p'/p}}{(d^{-1}+1)^{p'}}, & \textit{for} \quad x_0 \in \partial\Omega. \end{cases}$$

Proof. (a) Let $\gamma > p$ and assume that $x_0 \notin \partial\Omega$. Note that for any $s \in \mathbf{R}$ the function $r \mapsto C/r^\gamma\,|\log r|^s$ is decreasing on the interval $(0, r_0)$, $r_0 = e^{-s/\gamma}$. Applying Theorem 7, see (65), and taking $r < \frac{1}{2}\min(R, r_0, d(x_0, \partial\Omega))$, we obtain

$$\operatorname*{osc}_{B_{2r}(x_0)} u \geqslant b\, C^{p'/p}\, r^{p'(1-\gamma/p)}\, |\log r|^{-sp'/p} \to +\infty \qquad \text{as} \quad r \to 0. \tag{77}$$

(b) Let the point $x_0 \in \partial\Omega$ have the weak cone property, and $\gamma > p$. We can assume without loss of generality that $r_k < (d^{-1}+1)^{-1}\, r_0$ for all k, where $r_0 = e^{-s/\gamma}$. Then the infimum of the right-hand side of (76) restricted to the ball $\overline{B_{r_k}(x_k)}$ is achieved for $y_k = x_0 + (|x_k - x_0| + r_k)((x_k - x_0)/|x_k - x_0|) \in \partial B_{r_k}$, and we have

$$\operatorname*{osc}_{B_{2r_k}(x_k)} u \geqslant br_k^{p'} C^{p'/p}[(|x_k - x_0| + r_k)|^{\gamma} \cdot |\log(|x_k - x_0| + r_k)|^s]^{p'/p}$$

$$\geqslant \frac{bC^{p'/p}}{(d^{-1}+1)^{\gamma}} \cdot r_k^{p'(1-\gamma/p)} \cdot |\log(d^{-1}+1)\, r_k|^{-sp'/p} \to \infty$$

$$\text{as} \quad k \to \infty,$$

where we used the inequality $|x_k - x_0| < d^{-1} r_k$. This implies that $\operatorname{osc}_{x_0} u = \infty$. ▮

Remark 2. (a) Note that since integrability of $F(x, u, \nabla u)$ implies integrability of the right-hand side of (86), which is equivalent to $\gamma < N$, then we necessarily have $p < N$ and $\gamma \in (p, N)$ in the above corollary. This is in accordance with the imbedding theorem of Sobolev spaces for $p > N$, since in this case each supersolution $u \in W^{1,p}(\Omega)$ of (64) is in $L^{\infty}(\Omega)$, and therefore it cannot have a singularity.

(b) Analogous result can be stated for subsolutions such that $u \leqslant M_0$ a.e. in $B_R(x_0)$, with $C < 0$ and reverse inequality in (76), and $\eta \in (-\infty, M_0)$. Owing to (77) it is natural to conjecture that provided (76) then each solution of (64) has singularity of order $(\gamma - p)\frac{p'}{p} = \frac{\gamma - p}{p-1}$.

(c) We can still relax the growth condition on $F(x, \eta, \xi)$ near x_0 in Corollary 15. If $x_0 \in \bar{\Omega}$, it suffices to assume that there exists $d \in (0, 1)$, a sequence of balls $B_{2r_k}(x_k) \subset \Omega$ such that $x_k \to x_0$, $r_k \to 0$, $r_k > d\,|x_k - x_0|$ (if $x_k \equiv x_0$ for all k, this condition is superflouous), $F(x, \eta, \xi) \geqslant 0$ for a.e. $x \in \cup_k B_{2r_k}(x_k)$, $\eta \geqslant m_0$, $\xi \in \mathbf{R}^N$, and

$$\lim_{k \to \infty} r_k^p \operatorname*{ess\,sup}_{B_{r_k}(x_k) \times (m_0, \infty) \times \mathbf{R}^N} F(x, \eta, \xi) = \infty.$$

Even more general sufficient condition on $F(x, \eta, \xi)$, which ensures that a given point $x_0 \in \bar{\Omega}$ is singular for any weak solution in Corollary 15, can be seen in Example 4.

5. EXTENSIONS AND EXAMPLES RELATED TO THEOREM 5, GEOMETRY OF DOMAIN

A. Here we extend the control result that was stated in Theorem 5 for balls in Ω to bounded, open subsets A. It will enable us to obtain better estimates then with balls for some classes of functions $F(x, \eta, \xi)$, since we shall have opportunity to choose subsets A that are deformation retracts of Ω.

We shall need the following two elementary results.

LEMMA 3. *Let (X, d) be a metric space, A and B disjoint subsets such that $d(A, B) > 0$. Then the mapping*

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)} \tag{78}$$

is Lipschitz continuous and its smallest Lipschitz constant is equal to $1/d(A, B)$.

LEMMA 4. *Let $f\colon \mathbf{R}^N \to \mathbf{R}$ be a Lipschitz function with the Lipschitz constant L. If $\rho \in C_0^\infty(\Omega)$ is a regularizing function (i.e., $\rho \geqslant 0$, $\operatorname{supp} \rho = \overline{B_\varepsilon(0)}$, $\int_{\mathbf{R}^N} \rho(x)\, dx = 1$), then the convolution $\rho * f$ is also lipschitzian with the same Lipschitz constant L.*

This will permit us to construct a suitable localization function.

LEMMA 5 (Smooth Localization of Measurable Subsets). *Let Ω be an open subset of $\mathbf{R}^N$. Assume that A is a measurable subset of Ω and $r > 0$ such that $A_r \subseteq \Omega$, where A_r is r-neighbourhood of A. Then for any $c_0 > 1$ there exists a function $\Phi \in C^\infty(\Omega)$ such that*

$$0 \leqslant \Phi \leqslant 1, \tag{79}$$

$$\Phi = 0 \quad \text{on} \quad \Omega \backslash A_r, \qquad \Phi = 1 \quad \text{on} \quad A, \tag{80}$$

$$|\nabla \Phi| \leqslant \frac{c_0}{\mathrm{r}}. \tag{81}$$

Proof. It suffices to prove that for any $\varepsilon > 0$ small enough there exists Φ having properties (79), (80) and $|\nabla \Phi| \leqslant \frac{1}{r - 2\varepsilon}$. First we define a continuous localization function $f\colon \Omega \to \mathbf{R}$ by

$$f(x) = \frac{d(x, \Omega \backslash A_{r-\varepsilon})}{d(x, A_\varepsilon) + d(x, \Omega \backslash A_{r-\varepsilon})}, \tag{82}$$

where we choose $\varepsilon < r/2$. From Lemma 3 we see that it is lipschitzian and its Lipschitz constant is

$$L = \frac{1}{d(A_\varepsilon, \Omega \backslash A_{r-\varepsilon})} = \frac{1}{r - 2\varepsilon}.$$

Let us define

$$\Phi = \rho_\varepsilon * f,$$

where ρ_ε is a regularizing function having support equal to $\overline{B_\varepsilon(0)}$. By Lemma 4 for every $x \in \Omega$ we have

$$|\nabla\Phi(x)| \leqslant \sup_{y\in\Omega} |\nabla\Phi(y)| \leqslant L = \frac{1}{r-2\varepsilon},$$

and Φ has desired properties. ∎

Now we can formulate the result generalizing our Theorem 5.

THEOREM 8 (Control of Essential Supremum of Solutions). *Assume that conditions* (2) *and* (3) *are satisfied. Let A be a measurable subset of Ω and $r > 0$ such that $A_r \subseteq \Omega$ and $A_r \backslash A$ is bounded. Let us choose m_0, $M_1 \in \mathbf{R}$, $m_0 < M_1$, so that the following conditions are fulfilled*:

$$\exists f_1 \in L^1(A), \quad f(x,\eta) \geqslant f_1(x) \quad \text{a.e. } x \in A, \quad \eta \in I_1 = (m_0, M_1), \tag{83}$$

$$f(x,\eta) \geqslant 0 \quad \text{for a.e. } x \in A_r \backslash A, \eta \in I_1, \quad g(x,\eta) \equiv 0 \quad \text{a.e. } x \in A_r, \eta \in I_1, \tag{84}$$

$$\int_A f_1(x)\,dx > \inf_{s>0} \frac{1}{s}\,[\bar{\delta}H + D(|I_1| + s)^p], \tag{85}$$

where

$$\begin{gathered} H = \int_{A_r} [h(x)^{p'} + a_1^{p'}\bar{m}^p]\,dx, \qquad \bar{m} = \max\{|m_0|, |M_1|\}, \\ D = \left(\frac{p}{\delta}\right)^{p-1} \frac{|A_r \backslash A|}{r^p}, \qquad \delta = \frac{p'}{3^{p'-1}}\,\bar{\delta}, \qquad \bar{\delta} = \frac{\alpha}{a_2^{p'}}. \end{gathered} \tag{86}$$

Then each supersolution of (1) *has the property*:

$$\text{if } \operatorname*{ess\,inf}_{A_r} u \geqslant m_0 \qquad \text{then} \quad \operatorname*{ess\,sup}_{A_r} u > M_1. \tag{87}$$

Proof. The proof is the same as in Theorem 5. One only has to change B_R to A, B_{2R} to A_r, while the passage $c_0 \to 1$ is justified by Lemma 5. ∎

Using the same proofs as before, one can carry over all consequences of Theorem 5 to this more general setting. Since we need A to be bounded in the proof of the analogue of Lemma 1, from now on we will assume that A is bounded and open. We formulate only the following oscillation result, which includes its dual as well.

THEOREM 9. *Let* (2), (3) *hold with* $a_1 = 0$ *and* $h(x) \equiv 0$ *on* A_r, *where* A *is bounded, open, and* $A_r \subseteq \Omega$. *Assume that* $F(x, \eta, \xi)$ *is a Carathéodory function and* m_0, M_0 *are in* $\bar{\mathbf{R}}$, $m_0 < M_0$, *such that* $F(x, \eta, \xi)$ *does not change sign on* $A_r \times (m_0, M_0) \times \mathbf{R}^N$. *If* v *is a solution of* (64) *in the space* $W^{1,p}(\Omega)$, *such that* $m_0 \leqslant v \leqslant M_0$ *on* A_r, *then we have a local oscillation estimate,*

$$\operatorname*{osc}_{A_r, v} \geqslant \frac{\alpha}{(a_2\, p)^{p'}}\, Q_p(A, r)^{p'/p} \operatorname*{ess\,inf}_{A \times (m_0, M_0) \times \mathbf{R}^N} |F(x, \eta, \xi)|^{p'/p}, \tag{88}$$

where

$$Q_p(A, r) = \frac{r^p\, |A|}{|A_r \backslash A|}. \tag{89}$$

An immediate consequence is

COROLLARY 16 (Variational A Priori Bound). *If* $h(x) \equiv 0$ *on* Ω, m_0, $M_0 \in \bar{\mathbf{R}}$, $m_0 < M_0$, *then we have a global oscillation estimate of solutions of* (72) *satisfying* $m_0 \leqslant u \leqslant M_0$ *on* Ω,

$$\operatorname*{osc}_{\Omega} u \geqslant \frac{\alpha}{(a_2\, p)^{p'}} \sup_{(A, r) \in \mathscr{A}_F} \Big[Q_p(A, r) \operatorname*{ess\,inf}_{A \times (m_0, M_0) \times \mathbf{R}^N} |F(x, \eta, \xi)| \Big]^{p'/p}, \tag{90}$$

where $\mathscr{A}_F$ *is the family of all pairs* (A, r), $A \subseteq \Omega$ *is bounded and open,* $r > 0$ *such that* $A_r \subseteq \Omega$ *and* $F(x, \eta, \xi)$ *does not change sign on* $A_r \times (m_0, M_0) \times \mathbf{R}^N$.

EXAMPLE 4 (Generating Singularities). Let $x_0 \in \bar{\Omega}$ and assume that there exists a sequence of bounded, open sets $A^k \in \mathscr{A}_F$, and $r_k > 0$, $k = 1, 2, \ldots$, such that $\operatorname{diam}(\{x_0\} \cup A^k_{r_k}) \to 0$ as $k \to \infty$, where diam is the diameter of a set. If there exist m_0, $M_0 \in \bar{\mathbf{R}}$ such that

$$\lim_{k \to \infty} Q_p(A^k, r_k) \operatorname*{ess\,inf}_{A^k \times (m_0, M_0) \times \mathbf{R}^N} |F(x, \eta, \xi)| = \infty,$$

then by Theorem 9 any solution u of (64), such that $m_0 \leqslant u \leqslant M_0$ a.e. on $\bigcup_k A^k_{r_k}$, has a point of singularity in x_0, i.e., $\operatorname{osc}_{x_0} u = \infty$. This generalizes Corollary 15, see also Remark 2(c). In particular, if both m_0 and M_0 are finite, then (64) has no solutions u such that $u(x) \in [m_0, M_0]$ a.e. in $\bigcup_k A^k_{r_k}$.

Remark 3. (a) We can also allow m_0 and M_0 to depend on (A, r) in the above results. Also, if u is a solution for which we know an a priori bound of the form $|\nabla u| \leqslant c$ a.e. in Ω with Euclidean norm, then it is easy to see that oscillation estimates of solutions in Theorems 7 and 9 and in all their consequences still hold with essential infimum of $F(x, \eta, \xi)$ taken over the smaller set $B_R(x_1) \times (m_0, M_0) \times B_c(0)$ and $A \times (m_0, M_0) \times B_c(0)$, respectively.

(b) Theorem 9 can be extended to strip-like domains $A = A' \times (0, \infty)$, such that $A' \subset \mathbf{R}^{N-1}$ is bounded, open, $A_r \subseteq \Omega$, and with $Q_p(A, r)$ in (88) replaced by $Q_p(A', r) = r^p \, |A'|/|A'_r \setminus A'|$. To prove this, it suffices to fix $t > 0$ and apply Theorem 9 to bounded, open set $A^t = A' \times (0, t)$, and then let $t \to \infty$. It is easy to see that $Q_p(A^t, r) \to Q_p(A', r)$ as $t \to \infty$.

B. It is of obvious interest to study the quantity

$$q_p(\Omega) = \sup_{(A, r) \in \mathscr{A}_1} Q_p(A, r) = \sup_{(A, r) \in \mathscr{A}_1} \frac{r^p \, |A|}{|A_r \setminus A|} \tag{91}$$

which represents a p-numeric characteristic of the set Ω. We illustrate this in the case $p = 2$, corresponding to the Laplace operator. For the sake of simplicity we assume that $F(x, \eta, \xi) \geqslant K > 0$ for a.e. $x \in \Omega$, $\eta \geqslant m_0$, $\xi \in \mathbf{R}^N$, and let u be any supersolution of (1) such that $u \geqslant m_0$ on Ω. Note that in this case $\mathscr{A}_F = \mathscr{A}_K = \mathscr{A}_1$, and

$$\operatorname*{osc}_{\Omega} u \geqslant \tfrac{1}{4} q_2(\Omega) \cdot K.$$

Example 5. Let us consider estimates of oscillations of solutions on a two-dimensional annulus $\Omega \subset \mathbf{R}^2$ with radii $R_1 < R_2$. Since its inner radius is $\frac{1}{2}(R_2 - R_1)$, then using (70), i.e., approximation of Ω by balls, we obtain

$$\operatorname*{osc}_{\Omega} u \geqslant \tfrac{1}{192} (R_2 - R_1)^2 \, K. \tag{92}$$

However, a better estimate can be obtained if we use subannuli A of Ω such that $A_r = \Omega$. Using elementary differential calculus it is easy to see that the quotient $r^2 \, |A|/|A_r \setminus A|$ attains its maximum over this family of subannuli for $r = \frac{1}{4}(R_2 - R_1)$. Then from (90) we obtain an estimate of oscillation which is three times better than in (92):

$$\operatorname*{osc}_{\Omega} u \geqslant \tfrac{1}{64} (R_2 - R_1)^2 \, K. \tag{93}$$

This example indicates that the reason for improved oscillation estimate lies in the fact that we used the family of deformation retracts of a Ω, which is best adjusted to the topology of Ω. The examples that follow confirm our conjecture that this will hold in the case of any bounded Ω, provided $F(x, \eta, \xi) \geqslant K > 0$ for a.e. $x \in \Omega$, and all η, ξ. It is natural to ask whether the optimal value of $q_2(\Omega)$ is obtained if we take the supremum only over the family of subsets A of Ω that are deformation retracts of Ω and homeomorphic to Ω?

Finding the optimal value $q_2(\Omega)$ for arbitrary set Ω seems to be difficult. We do not know its value even in the case of rectangle. However, it is easy to obtain some lower bounds.

EXAMPLE 6. If Ω_{ab} is a rectangle with sides $a \geqslant b$, then the family of subrectangles having the sides $a-2r$ and $b-2r$, $r \leqslant b/2$ yields the following estimate for supersolutions of (64) such that $u \geqslant m_0$ on Ω_{ab},

$$\operatorname{osc}_{\Omega_{ab}} u \geqslant \tfrac{1}{4} \bar{Q}(r_0(a, b))\, K, \tag{94}$$

where

$$\bar{Q}(r) = \frac{r^2(a-2r)(b-2r)}{ab-(a-2r)(b-2r)},$$

and $r_0(a, b)$ is the unique solution of $\bar{Q}'(r) = 0$ in $(0, b/2)$, i.e., of the qubic equation $16r^3 - 16(a+b)\, r^2 + 4(a+b)^2\, r - ab(a+b) = 0$. If we try with the family of subrectangles that are homothetic to Ω_{ab} with respect to its centre, then we obtain a more explicit, but less precise estimate:

$$\operatorname*{osc}_{\Omega_{ab}} u \geqslant \frac{7-3\sqrt{5}}{16\,(\sqrt{5}+1)}\, b^2 K \tag{95}$$

which is still slightly better than the one that we obtain from Corollary 12, using inner radius $b/2$. Note that $r_0(a, b) \to b/4$ as $a \to \infty$, which yields an estimate corresponding to (94) on a strip-like domain $\Omega = \mathbf{R}^+ \times (0, b)$ or $\mathbf{R} \times (0, b)$.

More generally, if $\Omega = \mathbf{R}^+ \times \Omega'$ or $\mathbf{R} \times \Omega'$ is a strip-like domain, where $\Omega' \subseteq \mathbf{R}^{N-1}$ is bounded and open, then it is easy to see that $q_2(\Omega) \geqslant q_2(\Omega')$, and therefore

$$\operatorname*{osc}_{\Omega} u \geqslant \tfrac{1}{4} q_2(\Omega')\, K. \tag{96}$$

EXAMPLE 7. For a two-dimensional disk $\Omega = B_R$ we obtain similarly as above that

$$\operatorname*{osc}_{B_R} u \geqslant \frac{7-3\sqrt{5}}{4(\sqrt{5}+1)} R^2 K. \tag{97}$$

The analogous estimate holds if instead of a disk we have a torus Ω in $\mathbf{R}^3$ defined by two radii R_1, R, $R_1 > R$,

$$\operatorname*{osc}_{\Omega} u \geqslant \frac{7-3\sqrt{5}}{4(\sqrt{5}+1)} R^2 K. \tag{98}$$

If $\Omega = B_R$ is a three-dimensional ball then we have

$$\operatorname*{osc}_{B_R} u \geqslant \frac{1}{4} \sup_{r \in (0, R)} \frac{r^2 (R-r)^3}{R^3 - (R-r)^3} K \geqslant 0.00892857 \cdot R^2 K. \tag{99}$$

The optimal value of r is obtaind from a real solution $t = r/R$ of $2t^3 - 8t^2 + 12t - 3 = 0$. If B_R is an N-dimensional ball, then analogously

$$\operatorname*{osc}_{B_R} u \geqslant \frac{1}{4} \sup_{r \in (0, R)} \frac{r^2 (R-r)^N}{R^N - (R-r)^N} K = \frac{1}{4} \sup_{t \in (0, 1)} \frac{t^2 (1-t)^N}{1 - (1-t)^N} R^2 K. \tag{100}$$

All oscillation estimates are better than those involving inner radius in Corollary 12.

Let us describe some simple properties of $Q_q(A, r)$ and $q_p(\Omega)$. If $\Omega_1 \subseteq \Omega_2$ then $q_p(\Omega_1) \leqslant q_p(\Omega_2)$. If $A_r \cap B_r = \varnothing$ then $Q_q(A \cup B, r) \leqslant Q_p(A, r) + Q_p(B, r)$. Therefore, if $\Omega_1 \cap \Omega_2 = \varnothing$ then $q_p(\Omega_1 \cup \Omega_2) \leqslant q_p(\Omega_1) + q_p(\Omega_2)$. For any two open sets Ω_1 and Ω_2 we have $q_p(\Omega_1 \cup \Omega_2) \geqslant \max\{q_p(\Omega_1), q_p(\Omega_2)\}$. If there is an isometry between Ω_1 and Ω_2, then $q_p(\Omega_1) = q_p(\Omega_2)$.

COROLLARY 17. (i) *Let* Ω *be a bounded, open set in* $\mathbf{R}^N$. *Then we have*

$$q_2(\Omega) \leqslant 4 \min\{c, c_s D_s |\Omega_1|\}, \tag{101}$$

where c *the Schauder constant corresponding to an arbitrary bounded, open set* Ω_1 *of class* $C^{2,\theta}$ *containing* Ω, *see* (72), $s > N/2$, c_s *is the Agmon, Douglis, and Nirenberg constant corresponding to* Ω_1, *see* (74), *and* D_s *is the constant of imbedding* $W^{2,s}(\Omega_1) \subset C(\bar{\Omega}_1)$.

(ii) *If* $\operatorname{Cap}_p(A)$ *is p-capacity of a bounded and open set* A *in* Ω, *then for any* $r \in (0, d(A, \partial\Omega))$ *we have*

$$Q_p(A, r) \cdot \operatorname{Cap}_p(A) \leqslant |A|.$$

(iii) *Let* $1 \leqslant s < N$. *Then for any* $1 < p < \infty$,

$$Q_p(A, r) \leqslant C_{s,N} \cdot r^{p-s} |A|^{s/N}, \qquad q_p(\Omega) \leqslant C_{s,N} \cdot r_0(\Omega)^{p-s} |\Omega|^{s/N},$$

where $r_0(\Omega)$ *is inner radius of* Ω. *In particular*, $q_p(\Omega) < \infty$ *for any bounded* Ω, *and even for unbounded domains such that* $|\Omega| < \infty$.

(iv) *For domains* Ω *whose inner radius is infinite we have* $q_p(\Omega) = \infty$.

Proof. (i) Using Schauder's a priori estimate with $f \equiv 1$, $\phi = 0$, combined with Theorem 9 in the same way as in the proof of Corollary 13, we obtain $q_2(\Omega) = 4c$. Arguing as in the proof of Corollary 14 we obtain $q_2(\Omega) \leqslant 4c_s D_s |\Omega_1|$.

(ii) Lemma 5 implies that for any bounded, open A such that $A \subset\subset \Omega$ we have

$$\mathrm{Cap}_p(A) \leqslant \inf_{r \in (0, d(A, \partial\Omega))} \frac{|A_r \backslash A|}{r^p},$$

and the claim follows. The definition of capacity can be seen for example in [8].

(iii) Using the inequality $\mathrm{Cap}_s(A) \geqslant C |A|^{1-s/N}$, $C = C_{s,N} > 0$, for $1 \leqslant s < N$, see Theorem 4.57(vi) in [8], we conclude that for any such s,

$$Q_s(A, r) \leqslant C_{s,N} |A|^{s/N},$$

where $C_{s,N} > 0$ does not depend on A, r and Ω. In particular, $q_s(\Omega) \leqslant C |A|^{s/N}$. The claim follows from $Q_p(A, r) = r^{p-s} Q_s(A, r)$.

(iv) Take $r = 1$ and a sequence of balls $A^k = B_k(x_k) \subset \Omega$, $k \in \mathbf{N}$. Then $Q_p(A^k, r) \to \infty$ as $k \to \infty$. ∎

Remark 4. Note that property (iii) implies the following interesting estimate: $|A_r \backslash A| \geqslant C_{s,N} \cdot r^{p-s} |A|^{s/N}$, where $1 \leqslant p < \infty$, $1 \leqslant s < N$ and $C_{s,N} > 0$ does not depend on Ω. We conjecture that for all undbounded domains Ω with finite inner radius necessarily $q_p(\Omega) < \infty$.

C. Using Theorem 9 we can prove the following result.

Corollary 18. (Nonexistence of Solutions). *Let* $\Omega = \mathbf{R}^N$, *or* $\mathbf{R}^N \backslash D$, *where* D *is a compact subset of* $\mathbf{R}^N$, $h(x) \equiv 0$ *in* Ω *and* $a_1 = 0$. *Let there exist* γ, $\gamma < p$, $C > 0$, $k \in \mathbf{N}_0$, $m_0 \in \mathbf{R}$ *and* $R_0 > 0$ *such that*

$$F(x, \eta, \xi) \geqslant \frac{\mathrm{C}}{|x|^\gamma \cdot |\log |x||^k}, \qquad |x| \geqslant R_0, \eta \geqslant m_0, \xi \in \mathbf{R}^N. \tag{102}$$

Then (64) *has no solutions in* $W^{1,p}(\Omega)$ *such that* $u \geqslant m_0$ *on* Ω.

Proof. (a) Let us prove that $\operatorname{ess\,sup}\{u(x) : |x| \geqslant r\} \to \infty$ as $r \to \infty$. We use Theorem 9 with $M_0 = \infty$, $A = B_{3r}(0) \setminus \overline{B_{2r}(0)}$, where we choose $r > R_0$ large enough, so that also $D \subseteq B_r(0)$. Then

$$\frac{r^p\,|A|}{|A_r \setminus A|} \operatorname*{ess\,inf}_{A_r \times (m_0,\,\infty) \times \mathbf{R}^N} F(x, \eta, \xi)$$

$$\geqslant \frac{(3^N - 2^N)\,C}{4^\gamma (4^N - 3^N + 2^N - 1)} \cdot \frac{r^{p-\gamma}}{|\log(4r)|^k} \to \infty \qquad \text{as} \quad r \to \infty.$$

Therefore $\operatorname{osc}_{A_r} u \to \infty$ as $r \to \infty$.

(b) Assume that u is a solution. Since $p > N$ (note that since a solution u of (72) is such that $F(x, u, \nabla u) \in L^1(\Omega)$, we have that (102) implies that $N < \gamma$), then $u(x) \to 0$ as $|x| \to \infty$ (see [13, p. 189]). This contradicts (a). ∎

Example 8. Let us consider the nonlinear problem

$$\begin{aligned} -\Delta_p u &= K(x)\, e^{cu} \qquad \text{in} \quad \mathscr{D}'(\Omega), \\ K(x)\, e^{cu} &\in L^1(\Omega), \qquad u \in W^{1,p}(\Omega), \end{aligned} \tag{103}$$

where Ω is as in the preceding corollary, $c > 0$ and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$. It is related to problem of constructing a metric with prescribed curvature function $K(x)$; see [17, 19, 23]. It follows from the above corollary that if

$$K(x) \geqslant \frac{C}{|x|^\gamma \cdot |\log|x||^k}, \qquad x \in \mathbf{R}^N, \qquad |x| \geqslant R_0, \tag{104}$$

where $N < \gamma < p$, and $C > 0$, $k \in \mathbf{N}_0$, $R_0 > 0$, then problem (103) has no solutions in $W^{1,p}(\Omega)$ such that $u \geqslant m_0$ on Ω for any $m_0 \in \mathbf{R}$. This means there are no solutions that are uniformly bounded from below. In particular, the problem has no positive solutions in $W^{1,p}(\Omega)$. This complements the corresponding nonexistence results by Oleinik [19] and Sattinger [23], as well as Ni [17].

D. It is possible to obtain another variant of Theorem 5, where we relax the condition $g(x, \eta) \equiv 0$ on B_{2R}. Of course, we can also deal with arbitrary bounded, open subsets A instead of balls, as in Theorem 8.

Theorem 10 (Control of Essential Supremum of Solutions). *Assume that conditions* (2) *and* (3) *are satisfied. Let* $x_1 \in \Omega$, $R_1 > 0$ *be such that*

$B_{2R_1}(x_1) \subseteq \Omega$. *Let us choose* m_0, $M_1 \in \mathbf{R}$, $m_0 < M_1$, *so that the following conditions are fulfilled*:

$$\exists f_1 \in L^1(B_{R_1}(x_1)), \qquad f(x, \eta) \geqslant f_1(x)$$
$$\text{a.e.} \quad x \in B_{R_1}(x_1), \eta \in I_1 = (m_0, M_1), \tag{105}$$

$$f(x, \eta) \geqslant 0 \qquad \text{a.e. on} \quad B_{2R_1}(x_1) \backslash B_{R_1}(x_1), \eta \in I_1, \tag{106}$$

$$\exists g_0 > 0, \qquad |g(x, \eta)| \leqslant g_0 < \frac{\alpha - \bar{\delta} a_2^{p'}}{|I_1|} \qquad \text{a.e.} \quad x \in B_{2R_1}(x_1), \eta \in I_1, \tag{107}$$

$$\int_{B_{R_1}(x_1)} f_1(x)\, dx > \inf_{0 < s < (\alpha - \bar{\delta} a_2^{p'})/g_0 - |I_1|} \frac{1}{s} \left[\bar{\delta} H + D(|I_1| + s)^p \right], \tag{108}$$

where

$$H = \int_{B_{2R_1}(x_1)} [h(x)^{p'} + a_1^{p'} \bar{m}^p]\, dx, \qquad \bar{m} = \max\{|m_0|, |M_1|\},$$

$$D = \left(\frac{p}{\delta}\right)^{p-1} \frac{(2^N - 1)\, |B_{R_1}(x_1)|}{R_1^p}, \qquad \delta = \frac{p'}{3^{p'-1}}\, \bar{\delta}, \qquad \bar{\delta} < \frac{\alpha}{a_2^{p'}}. \tag{109}$$

Then each supersolution of (1) *has property* (40).

Proof. Repeating the proof of Theorem 5 in this situation, we obtain (45) with $L = \alpha - g_0(t - m_0) - \bar{\delta} a_2^{p'}$. We choose t such that $L \geqslant 0$ and $t > M_1$, that is,

$$M_1 < t \leqslant \frac{\alpha - \bar{\delta} a_2^{p'}}{g_0} + m_0.$$

This interval for t is nonempty if and only if $M_1 < (1/g_0)(\alpha - \bar{\delta} a_2^{p'}) + m_0$, which is satisfied by (108). If we set $s = t - M_1$, then $0 < s \leqslant (\alpha - \bar{\delta} a_2^{p'})/g_0 + m_0 - M_1$. ▮

It is easy to derive the following analogue of Corollary 10.

COROLLARY 19. *Assume that the conditions of the preceding theorem are satisfied with* $h(x) \equiv 0$ *a.e. on* $B_{2R_1}(x_1)$, $a_1 = 0$, *with the condition* (107) *on* $g(x, \eta)$ *replaced by*

$$|g(x, \eta)| \leqslant \frac{\alpha - \bar{\delta} a_2^{p'}}{p'\, |I_1|}, \tag{110}$$

with (108) *replaced by* (54), *and* $\delta = p' \bar{\delta}$ *in* (109). *Then each supersolution of* (1) *has property* (40).

Proof. The infimum in (109) taken over all $s > 0$ is achieved for $s_0 = |I_1|/(p-1)$. The interval under infimum in (109) contains s_0 if and only if g_0 does not exceed the value on the right-hand side of (111), so that we can take g_0 equal to the right-hand side. ∎

This permits to carry over all results of Section 4 to allow nonzero $g(x, \eta)$ in B_{2R}. For example, we have the following analogue of Theorem 7:

THEOREM 11. *Let (2), (3) hold with $a_1 = 0$ and $h(x) \equiv 0$ on $B_{2R}(x_1) \subseteq \Omega$. Consider the quasilinear problem*

$$-\operatorname{div} a(x, v, \nabla v) = F(x, v, \nabla v) + g(x, v)\,|\nabla v|^p \qquad \text{in } \mathscr{D}'(\Omega), \tag{111}$$
$$F(x, v, \nabla v) + g(x, v)\,|\nabla v|^p \in L^1(\Omega), \qquad v \in W^{1,p}(\Omega),$$

where $F(x, \eta, \xi)$ and $g(x, \eta)$ are Carathéodory functions. Assume that $m_0 \in \mathbf{R}$, $K > 0$ are such that

$$F(x, \eta, \xi) \geqslant K \quad \text{for a.e. } x \in B_{2R}(x_1), \quad \eta \geqslant m_0 \quad \text{and} \quad \xi \in \mathbf{R}^N, \tag{112}$$

$$|g(x, \eta)| \leqslant \frac{\alpha - \bar{\delta} a_2^{p'}}{p' b R^{p'} K^{p'/p}} \qquad \text{for a.e.. } x \in B_{2R}(x_1), \qquad \eta \geqslant m_0, \tag{113}$$

where

$$0 < \bar{\delta} < \alpha / a_2^{p'}, \qquad b = \frac{\bar{\delta}}{p^{p'}(2^N - 1)}.$$

If v is a supersolution in $W^{1,p}(\Omega)$ such that $v \geqslant m_0$ on $B_{2R}(x_1)$, then we have a local oscillation estimate,

$$\operatorname*{osc}_{B_{2R}(x_1)} v \geqslant b\, R^{p'} K^{p'/p}. \tag{114}$$

Remark 5. (Generating Singularities of Solutions). From this result we can easily derive the following analogue of Corollary 15 about generating singularities. Retaining all conditions of Theorem 11 except (113) and (114), which we replace by

$$F(x, \eta, \xi) \geqslant \frac{C}{|x - x_1|^{\gamma} \cdot |\log |x - x_1||^{s}}$$

and

$$|g(x, \eta)| \leqslant \frac{\alpha - \bar{\delta} a_2^{p'}}{p' b C^{p'/p}}\, |x - x_1|^{p'/p(\gamma - p)} \cdot |\log |x - x_1||^{sp'/p},$$

respectively, with $\gamma > p$ and $s \geqslant 0$, then any supersolution v of (111), such that $v \geqslant m_0$ on $B_{2R}(x_1)$, is singular in x_1. The proof of this fact follows easily by applying Theorem 11 to $K = K(r) = C/r^\gamma \, |\log r|^s$, and using $r \in (0, R)$ instead of R. It is possible to consider the case when $x_0 \in \partial\Omega$ is a weakly conic point, as in Corollary 15.

E. Estimates of local oscillations enable us to obtain a lower bound of total variation Var u of solutions of ordinary differential equations (here $N = 1$ and Ω is an interval). To this end we introduce two families of sets:

$\mathscr{A}_F^d =$ any disjoint subfamily of $\mathscr{A}_F$, such that for every $(A, r) \in \mathscr{A}_F^d$ the set A is an interval, disjoint being understood in the sense that if (A, r) and $(B, \rho) \in \mathscr{A}_F^d$ and $(A, r) \neq (B, \rho)$, then $A_r \cap B_\rho = \varnothing$.

$\mathscr{A}_F^D =$ the family of all disjoint subfamilies $\mathscr{A}_F^d$ of $\mathscr{A}_F$.

Corollary 20 (Lower Bound for Total Variation of a Solution). *Let $N = 1$ and assume that* (2) *and* (3) *hold with $h(x) \equiv 0$ on Ω and $a_1 = 0$. Then for any solution $u \in W^{1,p}(\Omega)$ of* (64) *such that $m_0 \leqslant u \leqslant M_0$ on Ω, $m_0, M_0 \in \overline{\mathbf{R}}$, we have*

$$\operatorname*{Var}_{\Omega} u \geqslant \frac{\alpha}{2^{p'/p}(a_2 p)^{p'}} \sup_{\mathscr{A}_F^d \in \mathscr{A}_F^D} \sum_{(A, r) \in \mathscr{A}_F^d} r\Big[\, |A| \operatorname*{ess\,inf}_{A \times (m_0, M_0) \times \mathbf{R}} |F(x, \eta, \xi)| \Big]^{p'/p}. \tag{115}$$

Proof. The claim follows immediately from Theorem 9, see (88), and the following obvious inequality:

$$\operatorname*{Var}_{\Omega} u \geqslant \sum_{(A, r) \in \mathscr{A}_F^d} \operatorname*{osc}_{A_r} u,$$

and from the fact that if A is an interval, then $r^p \, |A|/|A_r \backslash A| = \frac{1}{2} r^{p-1} \, |A|$. ∎

If for some (A, r) we know that $m_0(A, r) \leqslant u \leqslant M_0(A, r)$ on A_r, then we can use the interval $(m_0(A, r), M_0(A, r))$ instead of (m_0, M_0) in the above estimate. In the case when we have the problem

$$\begin{aligned} -u'' &= F(x, u, u') \qquad \text{in } \mathscr{D}'(a, b), \\ F(x, u, u') &\in L^1(a, b), \end{aligned} \tag{116}$$

$a, b \in \bar{\mathbf{R}}$, then the total variation of any solution u such that $m_0 \leqslant u \leqslant M_0$ in $\Omega = (a, b)$, $m_0, M_0 \in \bar{\mathbf{R}}$, has the following lower bound:

$$\operatorname*{Var}_{\Omega} u \geqslant \frac{1}{8} \sup_{\mathscr{A}_F^d \in \mathscr{A}_F^D} \sum_{(A, r) \in \mathscr{A}_F^d} r\, |A| \operatorname*{ess\,inf}_{A_r \times (m_0, M_0) \times \mathbf{R}} |F(x, \eta, \xi)|. \tag{117}$$

Of course, it is possible to formulate variants of the above estimates corresponding to more general problem (111) with $N = 1$.

F. (1) All results in this article that are of local nature, i.e. formulated on balls or on bounded, open sets A in $\mathbf{R}^N$, hold if we have $L^1_{loc}(\Omega)$ instead of $L^1(\Omega)$ in (1) and (64). We can also treat solutions u in $W^{1,p}_{loc}(\Omega)$ instead of $W^{1,p}(\Omega)$.

(2) Our main results can be formulated for more general equations than (1). First, it is clear from the proofs that we can allow functions f and g to depend also on ξ in Theorems 1–5, i.e., we can have $f(x, u, \nabla u)$ and $g(x, u, \nabla u)$ instead of $f(x, u)$ and $g(x, u)$ in (1). Moreover, it is possible to treat equations of the form

$$\begin{gathered} -\operatorname{div} a(x, u, \nabla u) + a_0(x, u, \nabla u) = H(x, u, \nabla u) + T, \\ H(x, u, \nabla u) \in L^1_{loc}(\Omega), \qquad u \in W^{1,p}_{loc}(\Omega), \end{gathered} \tag{118}$$

with $T \in W^{-1,p'}(\Omega)$, and the same conditions on $a(x, \eta, \xi)$ and $a_0(x, \eta, \xi)$ as in [5], except that we allow $\alpha_0 \geqslant 0$, and not only $\alpha_0 > 0$ in $a_0(x, \eta, \xi) \cdot \eta \geqslant \alpha_0 |\eta|^p$. If we deal with supersolutions u such that $u \geqslant m_0$ on A_r, we assume the condition of the form

$$H(x, \eta, \xi) \geqslant \underline{f}(x, \eta, \xi) + \underline{g}(x, \eta, \xi)\, |\xi|^p, \tag{119}$$

for a.e. $x \in A_r$, $\eta \geqslant m_0$, $\xi \in \mathbf{R}^N$, and $T \geqslant 0$ on A_r. If we deal with subsolutions u such that $u \leqslant M_0$ on A_r, we assume

$$H(x, \eta, \xi) \leqslant \bar{f}(x, \eta, \xi) + \bar{g}(x, \eta, \xi)\, |\xi|^p, \tag{120}$$

for a.e. $x \in A_r$, $\eta \leqslant M_0$, $\xi \in \mathbf{R}^N$, and $T \leqslant 0$ on A_r.

(3) It is easy to formulate Theorem 5 for degenerate quasilinear elliptic equations too. First, we replace conditions (2) and (3) by

$$a(x, \eta, \xi) \cdot \xi \geqslant \alpha(x, \eta, \xi)\, |\xi|^p, \tag{121}$$

$$|a(x, \eta, \xi)| \leqslant h(x) + a_1 |\eta|^{p-1} + \alpha(x, \eta, \xi)\, |\xi|^{p-1}, \tag{122}$$

where $h \in L^{p'}(\Omega)$, $a_1 \geqslant 0$, and $\alpha(x, \eta, \xi)$ is a Carathéodory function for which there exists $a_0 > 0$ such that $0 \leqslant \alpha(x, \eta, \xi) \leqslant a_0$ for a.e. $x \in \Omega$, $\eta \in \mathbf{R}$, $\xi \in \mathbf{R}^N$. Theorems 5 and 8 (and all their consequences) still hold if we replace the last equality in (39) and (86) by $\bar{\delta} = 1/a_0^{p'/p}$. To see this, it suffices to note that the constant L in (45) now becomes $L(x) = \alpha(x, \eta, \xi) - \bar{\delta} \cdot \alpha(x, \eta, \xi)^{p'}$, which should be under the integral sign, and we need the condition $L(x) \geqslant 0$.

(4) Our main results can also be generalized to elliptic systems of quasilinear equations and variational inequalities, which will be a subject of a forthcoming paper.

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