JOURNAL OF DIFFERENTIAL EQUATIONS 57, 248–257 (1985)

Motion under Gravity on a Saddle

ALISTAIR GRAY

Department of Mathematics, La Trobe University, Bundoora, Victoria, 3083, Australia

Received July 12, 1983

1. INTRODUCTION

This paper is a sequel to that of Gray, Jones, and Rimmer [2] in which it was shown that the problem of a particle moving under gravity on an elliptical paraboloid with axis vertical is soluble. In this paper it is shown that the corresponding problem for an hyperbolic paraboloid is also soluble. The methods employed in [2] are used here.

2. CHARTS FOR THE HYPERBOLIC PARABOLOID

Let $l, m > 0$ and choose $0 < a < b$, with

$$b^2 - a^2 = m^2 + l^2, \qquad l^2 < b^2 - a^2,$$

and let M be the hyperbolic paraboloid

$$M = \left\{ P \in \mathbb{R}^3 : \frac{x(P)^2}{l^2} - \frac{y(P)^2}{m^2} = 2z(P) - a^2 - m^2 \right\}, \tag{2.1}$$

where x, y, z are the cartesian coordinate functions on $\mathbb{R}^3$. Note that M is a submanifold of $\mathbb{R}^3$ and the restriction of the map $(x, y)\colon \mathbb{R}^3 \to \mathbb{R}^2$ to M as domain forms a chart for the whole of M.

The *paraboloidal coordinates* of a point $P \in \mathbb{R}^3$ may be defined as in [3] with

$$-\infty < \nu(P) \leqslant a^2 < \lambda(P) < b^2 \leqslant \mu(P) < \infty,$$

where

$$\begin{aligned} x^2 &= (\mu - b^2)(b^2 - \nu)(b^2 - \lambda)/(b^2 - a^2) \\ y^2 &= (\mu - a^2)(a^2 - \nu)(\lambda - a^2)/(b^2 - a^2) \\ z &= \mu + \nu + \lambda - a^2 - b^2. \end{aligned} \tag{2.2}$$

0022-0396/85 $3.00

The hyperbolic paraboloid M is given by $\lambda = a^2 + m^2 = b^2 - l^2$. We now restrict all coordinate systems to this domain and hence get

$$\begin{aligned} x^2 &= l^2(\mu - b^2)(b^2 - v)/(b^2 - a^2) \\ y^2 &= m^2(\mu - a^2)(a^2 - v)/(b^2 - a^2) \\ 2z &= \mu + v + l^2 - b^2, \end{aligned} \tag{2.3}$$

where

$$-\infty < v \leqslant a^2 < b^2 \leqslant \mu < \infty. \tag{2.4}$$

From (2.3) it follows that for $\mu > b^2$,

$$\frac{x^2}{l^2(\mu - b^2)} - \frac{y^2}{m^2(v - a^2)} = 1.$$

Hence under the chart (x, y): $M \to \mathbb{R}^2$, the image of the set of points of M, for which μ is a constant, is an hyperbola (Fig. 1). From (2.3) it similarly follows that for $-\infty < v < a^2$ (Fig. 2),

$$\frac{-x^2}{l^2(b^2 - v)} + \frac{y^2}{m^2(a^2 - v)} = 1.$$

Putting $\mu = b^2$ gives the y axis by (2.3) and $v = a^2$ gives the x axis.

Unfortunately the map (μ, v): $M \to \mathbb{R}^2$ is not one-to-one, hence cannot be a chart. To remedy this we define the following open subsets of M:

$$\begin{aligned} U_1 &= \{P \in M: x(P) > 0 \text{ and } y(P) > 0\} \\ U_2 &= \{P \in M: x(P) < 0 \text{ and } y(P) > 0\} \\ U_3 &= \{P \in M: x(P) < 0 \text{ and } y(P) < 0\} \\ U_4 &= \{P \in M: x(P) > 0 \text{ and } y(P) < 0\}, \end{aligned}$$

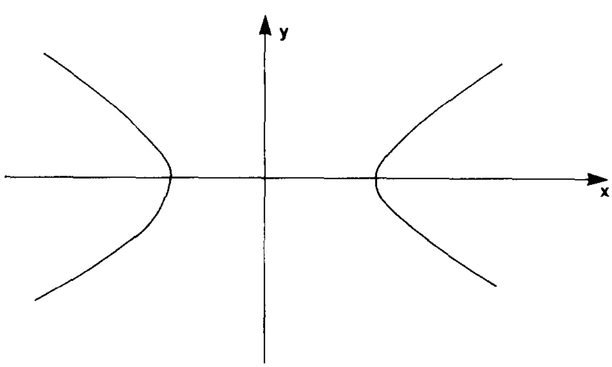

FIG. 1. An hyperbola, $\mu = $ constant.

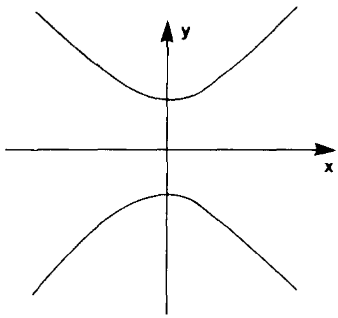

FIG. 2. An hyperbola, $\nu = \text{constant}$.

each of which consists of the points of M lying above one of the four quadrants of the (x, y)-plane. The restriction of (μ, ν) to any one of these four sets forms a chart for M (Fig. 3). Note that the points of M with $x = 0$ or $y = 0$ are excluded from each of the four open sets U_1, U_2, U_3, U_4. We will study the motion on each of the four sets and then patch the results together at the boundaries of the sets by using the methods as outlined in [2].

3. EQUATIONS OF MOTION

Our notation is as in [2, Sect. 3]. Thus TM denotes the tangent bundle of M and τ_M: $TM \to M$ is the natural projection. If (U, ϕ) is a chart for M with $\phi = (q_1, q_2)$: $U \to \mathbb{R}^2$ then $(TU, T\phi)$ is a chart for TM and we write $T\phi = (q_1, q_2, \dot{q}_1, \dot{q}_2)$.

Consider now the mechanical system consisting of a particle of unit mass moving on the hyperbolic paraboloid M under a uniform gravitational field

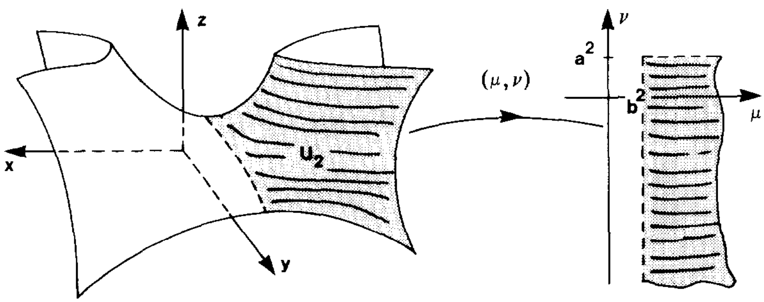

FIG. 3. A chart for the hyperbolic paraboloid M.

of intensity g directed vertically downwards. The Lagrangian L is given by

$$L = T - V, \qquad \text{where} \quad T = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2), \quad V = g\left(z - \frac{a^2}{2} - \frac{m^2}{2}\right), \tag{3.1}$$

where the functions are to be restricted to the domain TM.

The position of the particle on M at time t is $(\tau_M \circ c)(t)$, $c: I \to TM$, for some interval I, satisfies a certain second-order differential equation on the manifold M. As in [2, Sect. 3] this means that for each chart $(TU, (q_1, q_2, \dot{q}_1, \dot{q}_2))$ for TM,

$$\left(\frac{\partial L}{\partial \dot{q}_k} \circ c\right)' - \frac{\partial L}{\partial q_k} \circ c = 0 \qquad (k = 1, 2), \tag{3.2}$$

where the dash denotes the usual differentiation of functions from $I \subseteq \mathbb{R}$ to $\mathbb{R}$. Since the differential equation is of second order, $(\tau_M \circ c)' = c$, from which it can be shown that

$$(q_k \circ c)' = \dot{q}_k \circ c \qquad (k = 1, 2). \tag{3.3}$$

Cartesian coordinates. From (2.1) and (3.1) it follows that

$$\begin{aligned} T &= \frac{1}{2}\left[\dot{x}^2 + \dot{y}^2 + \left(\frac{\dot{x}x}{l^2} - \frac{y\dot{y}}{m^2}\right)^2\right] \\ V &= \frac{g}{2}\left(\frac{x^2}{l^2} - \frac{y^2}{m^2}\right). \end{aligned} \tag{3.4}$$

We now write out the differential equations (3.2) using the chart $(x, y, \dot{x}, \dot{y})$ for TM. To simplify notation we denote $x \circ c$ and $y \circ c$ by $\bar{x}$ and $\bar{y}$, respectively. One use of (3.4) and (3.3) we find after some manipulation that (3.2) becomes

$$\begin{aligned} \bar{x}'' &= -\frac{\bar{x}}{l^2}\left[g + \frac{\bar{x}'^2}{l^2} - \frac{\bar{y}'^2}{m^2}\right] \bigg/ \left[1 + \frac{\bar{x}^2}{l^4} + \frac{\bar{y}^2}{m^4}\right] \\ \bar{y}'' &= \frac{\bar{y}}{m^2}\left[g + \frac{\bar{x}'^2}{l^2} - \frac{\bar{y}'^2}{b^2}\right] \bigg/ \left[1 + \frac{\bar{x}^2}{l^4} + \frac{\bar{y}^2}{m^4}\right]. \end{aligned} \tag{3.5}$$

From these equations it is clear that the differential equation for c satisfies the criteria for existence and uniqueness of solutions through each point of TM. From the energy integral, $(T + V) \circ c = \text{constant}$, it now follows that we can take the domain I of each solution c to be the whole of $\mathbb{R}$, that is, the flow on TM is complete. Furthermore, this flow has symmetries corresponding to those of M and is reversible with respect to time. Other obvious properties of the flow are:

(i) *Along the* x *axis there is a family of (unstable) oscillating solutions.*

(ii) *The origin is the only equilibrium point.*

Paraboloidal coordinates. To obtain more detailed information about the solutions we now express T and V in terms of paraboloidal coordinates. We regard the map (μ, ν) as being restricted to one of the sets U_i of Section 2. This induces the charts $(TU_i, (\mu_i, \nu_i, \dot{\mu}_i, \dot{\nu}_i))$ for TM, $i=1, 2, 3, 4$. From (3.4) and (2.2) (with $\lambda=0$) we find that on each $TU_i \subseteq TM$.

$$\begin{aligned} T&=\frac{1}{2}(\mu_i-\nu_i)\left[\frac{\mu_i-m^2-a^2}{(\mu_i-a^2)(\mu_i-b^2)}\left(\frac{\dot{\mu}_i}{2}\right)^2+\frac{a^2+m^2-\nu_i}{(\nu_i-a^2)(\nu_i-b^2)}\left(\frac{\dot{\nu}_i}{2}\right)^2\right] \\ V&=\frac{g}{2}(\mu_i-\nu_i)^{-1}[(a^2+b^2)\,\nu_i-\nu_i^2-(a^2+b^2)\,\mu_i+\mu_i^2]. \end{aligned} \tag{3.6}$$

The Eqs. (3.2) are now

$$\left(\frac{\partial L}{\partial \dot{\mu}_i}\circ c\right)'-\frac{\partial L}{\partial \mu_i}\circ c=0, \qquad \left(\frac{\partial L}{\partial \dot{\nu}_i}\circ c\right)'-\frac{\partial L}{\partial \nu_i}\circ c=0. \tag{3.7}$$

From (3.6) it follows that these equations are of the separable Liouville type described in Whittaker [4, pp. 67–68] and hence are soluble. Following the procedure outlined by Whittaker [4, p. 68] we define a coordinate transformation (ϕ, ψ) as follows:

$$\begin{aligned} \phi\circ\mu_i(P)&=\frac{1}{2}\int_{b^2}^{\mu_i(P)}\sqrt{\frac{\xi-m^2-a^2}{(\xi-b^2)(\xi-a^2)}}\,d\xi \\ \psi\circ\nu_i(P)&=\frac{1}{2}\int_{a^2}^{\nu_i(P)}\sqrt{\frac{a^2+m^2-\xi}{(b^2-\xi)(a^2-\xi)}}\,d\xi. \end{aligned} \tag{3.8}$$

Now set

$$\begin{aligned} u_i&=\phi\circ\mu_i \\ v_i&=\psi\circ\nu_i. \end{aligned} \tag{3.9}$$

The Eqs. (3.6) now become

$$\begin{aligned} T&=\tfrac{1}{2}(\phi^{-1}\circ u_i-\psi^{-1}\circ v_i)(\dot{u}_i^2+\dot{v}_i^2) \\ V&=\frac{g}{2}(\phi^{-1}\circ u_i-\psi^{-1}\circ v_i)^{-1}[(a^2+b^2)\,\psi^{-1}\circ v_i \\ &\quad-(\psi^{-1}\circ v_i)^2-(a^2+b^2)(\phi^{-1}\circ u_i)+(\phi^{-1}\circ u_i)^2]. \end{aligned} \tag{3.10}$$

The Lagrangian equations can then be integrated [4, p. 68] to give

$$\begin{aligned}
&\tfrac{1}{2}(\phi^{-1}\circ\bar{u}_i-\psi^{-1}\circ\bar{v}_i)^2(\bar{u}_i')^2\\
&\qquad=h(\phi^{-1}\circ\bar{u}_i)-\tfrac{1}{2}g[(\phi^{-1}\circ\bar{u}_i)^2-\phi^{-1}\circ\bar{u}_i(a^2+b^2)]+k\\
&\tfrac{1}{2}(\phi^{-1}\circ\bar{u}_i-\psi^{-1}\circ\bar{v}_i)^2(\bar{v}_i')^2\\
&\qquad=-h(\psi^{-1}\circ\bar{v}_i)-\tfrac{1}{2}g[-\psi^{-1}\circ\bar{v}_i)^2-k+\psi^{-1}\circ\bar{v}_i(a^2+b^2)]-k,
\end{aligned}\tag{3.11}$$

where $\bar{u}_i$ and $\bar{v}_i$ denote $u_i\circ c$ and $v_i\circ c$, respectively, while $h=(T+V)\circ c$, the energy and k are constants for a given solution c.

Now reverting back to our charts $(TU_i,(\mu_i,v_i,\dot{\mu}_i,\dot{v}_i))$, we obtain from (3.11) above

$$\begin{aligned}
\frac{1}{8}(\bar{\mu}_i-\bar{v}_i)^2(\bar{\mu}_i')^2&=\frac{(\bar{\mu}_i-b^2)(\bar{\mu}_i-a^2)}{(\bar{\mu}_i-m^2-a^2)}[h\bar{\mu}_i-\tfrac{1}{2}g(\bar{\mu}_i^2-\bar{\mu}_i(a^2+b^2))+k]\\
\frac{1}{8}(\bar{\mu}_i-\bar{v}_i)^2(\bar{v}_i')^2&=\frac{(b^2-\bar{v}_i)(a^2-\bar{v}_i)}{(a^2+m^2-\bar{v}_i)}[-h\bar{v}_i+\tfrac{1}{2}g(\bar{v}_i^2-\bar{v}_i(a^2+b^2))-k],
\end{aligned}\tag{3.12}$$

where $\bar{\mu}_i$ and $\bar{v}_i$ denote $\mu_i\circ c$ and $v_i\circ c$, respectively.

From (3.4) it follows that

$$h=\frac{1}{2}\left[(\bar{x}')^2+(\bar{y}')^2+\left(\frac{\bar{x}\bar{x}'}{l^2}-\frac{\bar{y}\bar{y}'}{m^2}\right)+g\left(\frac{\bar{x}^2}{l^2}-\frac{\bar{y}^2}{m^2}\right)\right],\tag{3.13a}$$

while from (3.12) and (2.3) it follows after some calculation that

$$k=\frac{1}{2}l^2m^2\left[1+\left(\frac{\bar{x}}{l^2}\right)^2-\left(\frac{\bar{y}}{m^2}\right)^2\right]\left[g+\left\{\left(\frac{\bar{x}'}{l}\right)^2-\left(\frac{\bar{y}'}{m}\right)^2\right\}\right].\tag{3.13b}$$

Hence (h,k) maps onto the set $\mathbb{R}\times\mathbb{R}$.

The process of obtaining (3.12) from (3.7) essentially involves multiplying by $\bar{\mu}_i'$ and $\bar{v}_i'$ which may assume the value zero. Hence these equations are not necessarily equivalent to each other but clearly

$$(\bar{\mu}_i,\bar{v}_i)\ \textit{satisfies}\ (3.7)\Rightarrow(\bar{\mu}_i,\bar{v}_i)\ \textit{satisfies}\ (3.12).\tag{3.14}$$

Note that

$$\bar{v}_i-\bar{\mu}_i>0$$

in view of (2.3), (2.4), and (2.5).

4. Behaviour at Boundaries

Since the paraboloidal coordinates are not valid on the x and y axes it is necessary to establish the behaviour of the solutions at points on these axes by using the global cartesian chart. An account of how this may be done is in [2, Sect. 4].

5. Classification of Solutions

We first define a map $f\colon \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} f(\xi) &= 8(\xi - b^2)(\xi - a^2)(k + (h + \tfrac{1}{2}g(a^2 + b^2))\,\xi - \tfrac{1}{2}g\xi^2) \\ &= 4g(\xi - b^2)(\xi - a^2)\left(\frac{2k}{g} + \left(\frac{2h}{g} + (a^2 + b^2)\right)\xi - \xi^2\right) \end{aligned} \tag{5.1}$$

so that Eqs. (3.12) may be written as

$$\begin{aligned} (\bar{\mu}_i - \bar{v}_i)^2(\bar{\mu}_i')^2 &= f \circ \bar{\mu}_i/(\bar{\mu}_i - m^2 - a^2) \\ (\bar{\mu}_i - \bar{v}_i)^2(\bar{v}_i')^2 &= -15f \circ \bar{v}_i/(a^2 + m^2 - \bar{v}_i). \end{aligned} \tag{5.2}$$

Recall from Section 3 that h, k can take any real values, by suitable choice of initial conditions and from Section 2 that

$$\begin{aligned} \bar{\mu}_i - m^2 - a^2 &> 0 \\ a^2 + m^2 - \bar{v}_i &> 0. \end{aligned}$$

By using the same methods as those in [2, Sects. 5 and 6], we obtain a classification of the solution types. It is easy to show that every solution of (3.2) belongs to one of these classes. The resulting classification is illustrated in Figs. 4 and 5.

Case 1. $\alpha < a^2 < b^2 < \beta$. Let $\bar{\mu}(0) \in (b^2, \beta)$, $\bar{v}(0) \in (-\infty, \alpha)$, and then let $c\colon \mathbb{R} \to TM$ be one of the solutions determined by the procedure as outlined in [2, Sect. 6]. By Lemma 5.2 of [2], $\bar{\mu}$ oscillates between b^2 and β while $\bar{v}$ is bounded above by α and is unbounded below. Hence as t varies $(\bar{x}, \bar{y})(t)$ moves around in the region enclosed by the hyperbola $\mu = \beta$ touching each branch alternately. It touches the hyperbola $v = \alpha$ once.

Case 2. $\alpha < a^2 < b^2 = \beta$. Here the motion is along the y axis and the particle touches the hyperbola $v = \alpha$ once.

Case 3. $\alpha = a^2 < b^2 = \beta$. Two solutions are possible. The first is the

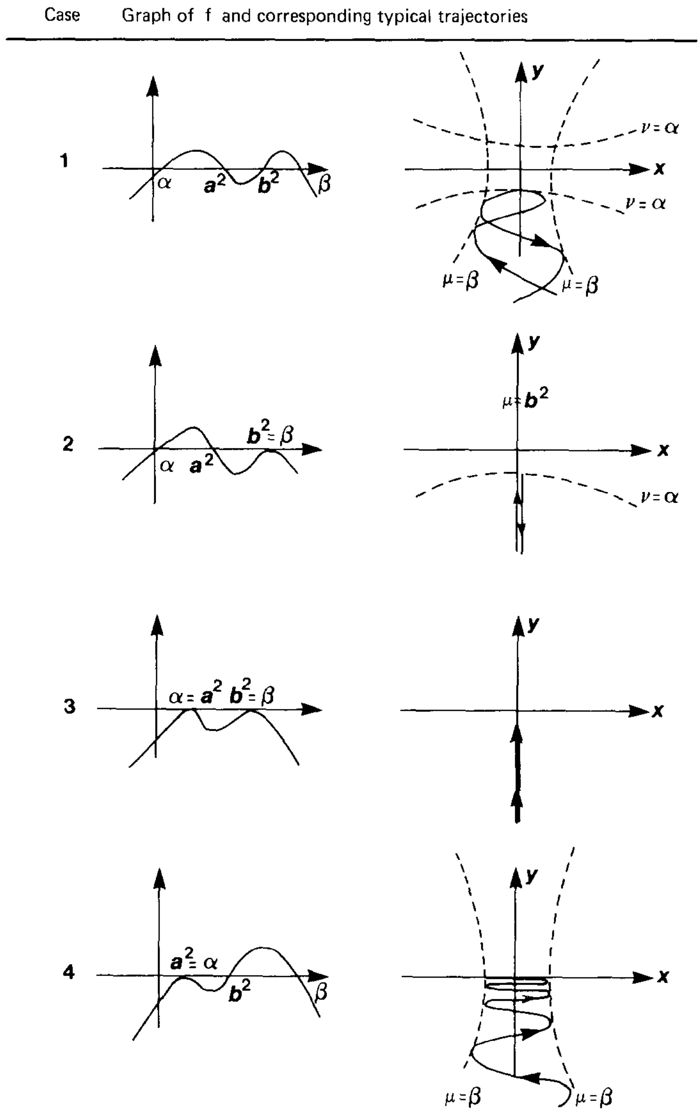

FIG. 4. Qualitative behaviour.

unstable equilibrium at $(\bar{x}, \bar{y})(t) = (0, 0)$ and the other is motion along the y axis with the particle asymptotically approaching the origin.

Cases 4 *and* 5. $a^2 = \alpha < b^2 < \beta$. A continuity argument based on Cases 1 and 3 suggests that motion is either oscillatory along the x axis (Case 5)

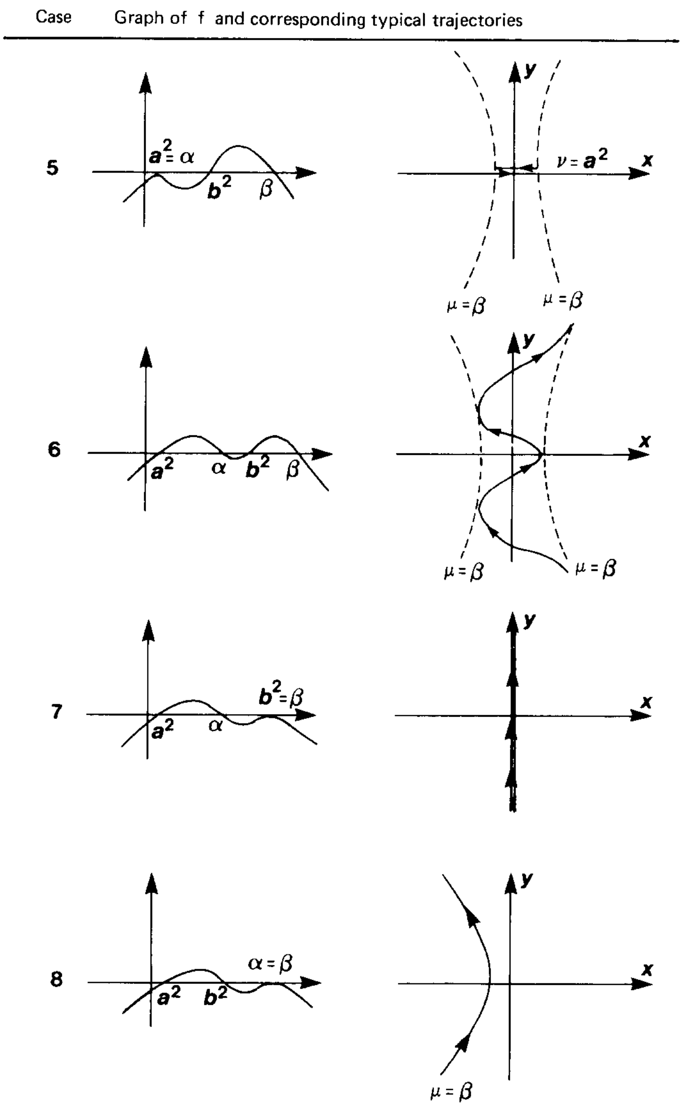

FIG. 5. Qualitative behaviour.

or approaches the x axis asymptotically as shown (Case 4).

Case 6. $a^2 < \alpha < b^2 < \beta$. Let $\bar{\mu}(0) \in (b^2, \beta)$, $\bar{\nu}(0) \in (-\infty, a^2)$. Let $c: \mathbb{R} \to TM$ be one of the solutions determined by the procedure as outlined in [2, Sect. 6]. Application of Lemma 5.2 of [2] shows that $\bar{\mu}$ oscillates between b^2 and β while $\bar{v}$ is bounded above by a^2 and is unbounded below. Hence as t varies $(\bar{x}, \bar{y})(t)$ moves around in the region enclosed by the hyperbola $\mu = \beta$ touching each branch alternately. It crosses the x axis, which corresponds to $v = a^2$.

Case 7. $a^2 < \alpha < b^2 = \beta$. A similar argument to that in Case 2 shows that motion is along the y axis but crosses the x axis.

Case 8. $a^2 < b^2 < \alpha = \beta$. Let $\bar{v}(0) \in (-\infty, a^2)$ and $\bar{\mu}(0) = \alpha$, and let $c: \mathbb{R} \to TM$ be one of the solutions so determined. Since $\bar{\mu}'$ is real, (4.2) and Fig. 5 imply that $\bar{\mu}$ can only assume the value β, and hence must be constant. Thus as t varies $(\bar{x}, \bar{y})(t)$ describes the hyperbola $\mu = \beta$.

References

1. R. Abraham and J. E. Marsden, "Foundations of Mechanics," Benjamin, Reading, Mass. 1967.
2. A. Gray, A. Jones, and R. Rimmer, Motion under gravity on a paraboloid, *J. Differential Equations* **45** (1982), 168–181.
3. P. Moon and D. E. Spencer, "Field Theory Handbook," (2nd ed.), Springer-Verlag, Berlin/Heidelberg/New York, 1971.
4. E. T. Whittaker, "A Treatise on the Analytical Dynamics of Particles and Rigid Bodies," Cambridge Univ. Press, Cambridge, 1927.