



Available at
WWW.MATHEMATICSWEB.ORG
POWERED BY SCIENCE @ DIRECT®

Journal of
MATHEMATICAL
ANALYSIS AND
APPLICATIONS

J. Math. Anal. Appl. 285 (2003) 128–140

www.elsevier.com/locate/jmaa

Semicrossed products generated by two commuting automorphisms

Mohamed Ridha Alaimia^a and Justin R. Peters^{b,*}

^a *Department of Mathematical Sciences, King Faud University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia*

^b *Department of Mathematics, Iowa State University, Ames, IA 50011, USA*

Received 29 January 2003

Submitted by Z.-J. Ruan

Abstract

In this paper, we study the semicrossed product of a finite dimensional C^* -algebra for two types of \mathbb{Z}_+^2 -actions, and identify them with matrix algebras of analytic functions in two variables. We look at the connections with semicrossed by \mathbb{Z}_+ -actions.

© 2003 Elsevier Inc. All rights reserved.

1. Introduction

The study of semicrossed products with respect to \mathbb{Z}_+ -actions was begun in [3,7], and higher dimensional actions have also been considered (e.g., [5,8]). In this note we continue the program of [1,2,4], which studied \mathbb{Z}_+ -actions on finite dimensional C^* algebras, to the case of \mathbb{Z}_+^2 -actions.

For a transitive action σ on a finite set X , $\text{card}(X) = k$, the crossed product $\mathbb{Z} \times_{\sigma} C(X)$ is $M_k \otimes C(\mathbb{T})$. The semicrossed product, which is the closed subalgebra corresponding to nonnegative powers of the automorphism, is not $M_k \otimes A(\mathbb{D})$ (where $A(\mathbb{D})$ is the disk algebra), but rather a proper subalgebra (cf. Section 2). For a transitive \mathbb{Z}^2 -action on X , the crossed product is $M_k \otimes C(\mathbb{T}^2)$, and the semicrossed product is a matrix algebra of bianalytic functions, which is a proper subalgebra of $M_k \otimes A(\mathbb{D}^2)$ (where $A(\mathbb{D}^2)$ is the bidisk algebra), $\text{card}(X) = k$.

* Corresponding author.

E-mail addresses: alaimia@kfupm.edu.sa (M.R. Alaimia), peters@iastate.edu (J.R. Peters).

However, different transitive actions, though having the same C^* -crossed product, may give rise to nonisomorphic semicrossed products. In fact, we exhibit two transitive \mathbb{Z}^2 -actions σ, σ' on a finite set X for which the semicrossed products are not isomorphic.

In Proposition 4.3 we show that if $\sigma = (\sigma_1, \sigma_2)$ is a \mathbb{Z}^2 -action on X , $\text{card}(X) = k$, then the semicrossed product with respect to the action on $C(X)$ is the norm closure of the algebra generated by

$$zP_1, \quad wP_2, \quad D \in \mathcal{D},$$

where P_i is the $k \times k$ permutation matrix associated with σ_i , $i = 1, 2$, and \mathcal{D} is the algebra of diagonal matrices in M_k . The closure is in $M_k \otimes A(\mathbb{D}^2)$.

Ideally, a classification theorem for these \mathbb{Z}^2 -actions on X would establish necessary and sufficient conditions for two semicrossed products to be (completely) isometrically isomorphic, in terms of some invariants of the permutation matrices. Such a theorem has not been achieved here. Our results, which enable us to distinguish the semicrossed products coming from certain (nonconjugate) \mathbb{Z}^2 -actions, are obtained by computing the codimensions of maximal ideals of the algebra.

One of the main theorems for semicrossed products of the form $\mathbb{Z}_+ \times_\sigma C(X)$ (for X a compact metric space) is that two such semicrossed products are isomorphic if and only if the actions are conjugate. For \mathbb{Z}^2 -actions, the question is open. The work here lends support to the conjecture that the analogue of the theorem for \mathbb{Z} -actions may hold for \mathbb{Z}^2 -actions as well.

2. Some matrix function algebras

We will let \mathbb{D} be the open unit disk, and $\mathbb{D}^2 = \mathbb{D} \times \mathbb{D}$, the bidisk. $A(\mathbb{D})$ will denote the disk algebra, the subalgebra of $C(\bar{\mathbb{D}})$ of functions which are analytic on the interior of \mathbb{D} , and $A(\mathbb{D}^2)$ the bidisk algebra, i.e., the subalgebra of $C(\bar{\mathbb{D}}^2)$ of bianalytic functions. We recall the matrix algebras of analytic functions studied in [4],

$$\mathcal{B}_k = \left\{ (f_{ij})_{i,j=0}^{k-1} : f_{ij} \in A(\mathbb{D}), f_{ij} \sim \sum_{n=0}^{\infty} a_n^{(ij)} z^{l(i,j)+nk} \right\},$$

where $l(i, j) \in \{0, \dots, k - 1\}$ and $l(i, j) \equiv i - j \pmod{k}$. Note we are labeling the matrix entries from 0 to $k - 1$ rather than from 1 to k .

For $k \in \mathbb{Z}^+$, let A_k denote the subalgebra of $A(\mathbb{D})$ whose nonzero Fourier coefficients are multiples of k . Then the algebra \mathcal{B}_k takes the form

$$\begin{pmatrix} A_k & z^{k-1}A_k & \cdots & zA_k \\ zA_k & A_k & \cdots & z^2A_k \\ \vdots & \vdots & \ddots & \vdots \\ z^{k-1}A_k & z^{k-2}A_k & \cdots & A_k \end{pmatrix}. \tag{1}$$

The algebras \mathcal{B}_k were first studied in [4] and later in [1,2], arise as semicrossed products of actions of \mathbb{Z}^+ on \mathbb{C}^k .

Next we introduce a two variable analog of this algebra,

$$\mathcal{B}_{k,2} = \left\{ (f_{ij})_{i,j=0}^{k-1} : f_{ij} \in A(\mathbb{D}^2), f_{ij} \sim \sum_{\substack{m,n \geq 0 \\ m+n \equiv (i,j) \pmod{k}}}^{\infty} a_{mn}^{(i,j)} z^m w^n \right\}.$$

Another algebra which will play an auxiliary role is

$$A_k(\mathbb{D}^2) := \left\{ f \in A(\mathbb{D}^2) : f \sim \sum_{\substack{m,n \geq 0 \\ m+n \equiv 0 \pmod{k}}}^{\infty} a_{mn} z^m w^n \right\}.$$

Using Fejer's theorem in two variables [10, Vol. 2, p. 304] and the fact that Cesaro means of a function in $A_k(\mathbb{D}^2)$ are polynomials in the same algebra, we see that the polynomials in $A_k(\mathbb{D}^2)$ are dense in $A_k(\mathbb{D}^2)$. Similarly, the subalgebra of $\mathcal{B}_{k,2}$ whose matrix entries are polynomials is dense in $\mathcal{B}_{k,2}$.

3. Maximal ideals

One of the main tools in studying and distinguishing various matrix function algebras is the space of maximal ideals. First we recall [1,2] the maximal ideal structure of \mathcal{B}_k . Let $F = (f_{ij}) \in \mathcal{B}_k$ and $0 \leq i_0 \leq k-1$. Set $\psi_0^{i_0} : \mathcal{B}_k \rightarrow \mathbb{C}$, $\psi_0^{i_0}(F) = f_{i_0 i_0}(0)$, and for $\lambda \in \bar{\mathbb{D}} \setminus \{0\}$, set $\psi_\lambda(F) = F(\lambda)$, that is, $\psi_\lambda(F)$ is the $k \times k$ matrix $(f_{ij}(\lambda))$. Then $J_0^{i_0} := \ker(\psi_0^{i_0})$, and $J_\lambda := \ker(\psi_\lambda)$, $\lambda \in \bar{\mathbb{D}} \setminus \{0\}$, are all the maximal ideals of \mathcal{B}_k .

To study the maximal ideals of $\mathcal{B}_{k,2}$ we first look at $A_k(\mathbb{D}^2)$.

Lemma 3.1. $A_k(\mathbb{D}^2)$ is a Banach subalgebra of $A(\mathbb{D}^2)$ and its maximal ideals are of the form

$$N_{\lambda,\mu} = \{f \in A_k(\mathbb{D}^2) : f(\lambda, \mu) = 0\} \quad \text{for } (\lambda, \mu) \in \bar{\mathbb{D}}^2.$$

Proof. It is routine to check that $A_k(\mathbb{D}^2)$ is a Banach subalgebra. For the maximal ideals, let χ be a multiplicative linear functional on $A_k(\mathbb{D}^2)$. Let λ be a k th root of $\chi(z^k)$ and μ a k th root of $\chi(w^k)$. Note that $\chi(zw^{k-1})$ has the form $\omega\lambda\mu^{k-1}$, where ω is a k th root of unity. Indeed, $(\chi(zw^{k-1}))^k = \chi(z^k)(\chi(w^k))^{k-1} = (\lambda\mu^{k-1})^k$. Thus, replacing λ by $\omega\lambda$ we may assume that $\chi(zw^{k-1}) = \lambda\mu^{k-1}$. Then

$$\chi(z^2 w^{k-2}) = (\chi(zw^{k-1}))^2 \chi(w^k)^{-1} = \lambda^2 \mu^{k-2}.$$

Continuing in this way, we have that $\chi(z^r w^{k-r}) = \lambda^r \mu^{k-r}$, $0 \leq r \leq k$. Hence if p is any polynomial in $A_k(\mathbb{D}^2)$, $\chi(p) = p(\lambda, \mu)$. By continuity this holds for all functions in $A_k(\mathbb{D}^2)$. \square

Remark 3.2. For $k > 1$, the correspondence between maximal ideals and points in the bidisk is not bijective. Indeed, for $(\lambda, \mu) \in \bar{\mathbb{D}}^2$ and ω a k th root of unity, the maps $f \in A_k(\mathbb{D}^2) \mapsto f(\lambda, \mu)$ and $f \in A_k(\mathbb{D}^2) \mapsto f(\omega\lambda, \omega\mu)$ are identical. Only in the case $k = 1$, the bidisk algebra, is the correspondence bijective.

Next we identify the maximal ideals of $\mathcal{B}_{k,2}$. Let $F = (f_{ij})_{i,j=0}^{k-1} \in \mathcal{B}_{k,2}$. Observe that the map $F \mapsto F(0, 0)$ maps F to a $k \times k$ matrix which is zero except along the diagonal. Thus if we fix $\ell, 0 \leq \ell \leq k - 1$, the map

$$\psi_0^\ell : \mathcal{B}_{k,2} \rightarrow \mathbb{C}, \quad \psi_0^\ell(F) = f_{\ell\ell}(0, 0), \quad F = (f_{ij})_{i,j=0}^{k-1}$$

is a (nonzero) multiplicative linear functional, and hence $\ker(\psi_0^\ell)$ is a maximal ideal. We will call this ideal “of zero type” and denote it by $\mathcal{J}_{(0,0)}^\ell$.

The second type of mapping is parameterized by $(\lambda, \mu) \in \bar{\mathbb{D}}^2 \setminus \{(0, 0)\}$ and takes the form

$$\psi_{\lambda,\mu} : \mathcal{B}_{k,2} \rightarrow \mathcal{M}_k, \quad \psi_{\lambda,\mu}(F) = F(\lambda, \mu).$$

By this we mean of course that each of the component functions of F is evaluated at the point (λ, μ) . It is routine to check that this mapping is surjective and multiplicative. Thus $\ker(\psi_{\lambda,\mu})$ provides us with another type of maximal ideal, which we denote $\mathcal{J}_{(\lambda,\mu)}$.

Proposition 3.3. *Let \mathcal{J} be a maximal ideal of $\mathcal{B}_{k,2}$. Then either*

$$\mathcal{J} = \mathcal{J}_{(\lambda,\mu)} \quad \text{for some } (\lambda, \mu) \neq (0, 0) \text{ in the closed bidisk}$$

or

$$\mathcal{J} = \mathcal{J}_{(0,0)}^\ell \quad \text{for some } \ell \in \{0, \dots, k - 1\}.$$

Proof. Let $e_{ij}, 0 \leq i, j \leq k - 1$, be matrix units for \mathcal{M}_k . That is, e_{ij} is the matrix which is 1 in the (i, j) position and 0 elsewhere. Since \mathcal{J} is proper, it cannot contain all the e_{ii} , for then it would contain the identity. We will assume \mathcal{J} does not contain $e_{00} \otimes 1$; the other cases are similar.

For any $F = (f_{ij}) \in \mathcal{J}$ we have

$$(e_{00} \otimes 1)F(e_{j0} \otimes z^j) = e_{00} \otimes z^j f_{0j},$$

so $e_{00} \otimes z^j f_{0j} \in \mathcal{B}_{k,2} \mathcal{J} \mathcal{B}_{k,2} = \mathcal{J}$.

Similarly, $e_{00} \otimes w^j f_{0j} \in \mathcal{J}$. Also,

$$e_{00} \otimes z^{k+j-i} f_{ij} = (e_{0i} \otimes z^{k-i})F(e_{j0} \otimes z^j) \in \mathcal{J} \quad \text{for } i, j \in \{0, \dots, k - 1\}.$$

Similarly, $e_{00} \otimes w^{k+j-i} f_{ij} \in \mathcal{J}$.

Consider the closed ideal \mathcal{I} in $A_k(\mathbb{D}^2)$ generated by

$$z^j f_{0j}, \quad w^j f_{0j}, \quad z^{k+j-i} f_{ij}, \quad \text{and } w^{k+j-i} f_{ij} \quad \text{for } 0 \leq i, j \leq k - 1,$$

and where f_{ij} are the entries of $F = (f_{ij})$ as F ranges through \mathcal{J} . Note that $e_{00} \otimes \mathcal{I} \subset \mathcal{J}$. Since by assumption \mathcal{J} does not contain $e_{00} \otimes 1$, it follows that $1 \notin \mathcal{I}$, and \mathcal{I} is proper in $A_k(\mathbb{D}^2)$. By Lemma 3.1, $\mathcal{I} \subset \{f \in A_k(\mathbb{D}^2) : f(\lambda, \mu) = 0\}$ for some $(\lambda, \mu) \in \bar{\mathbb{D}}^2$.

If $(\lambda, \mu) = (0, 0)$, then $\mathcal{J} \subset \mathcal{J}_{(0,0)}^0$. Indeed, elements of $\mathcal{J}_{(0,0)}^0$ have the same components as elements of $\mathcal{B}_{k,2}$, except for the $(0, 0)$ component. As \mathcal{J} is maximal, this forces $\mathcal{J} = \mathcal{J}_{(0,0)}^0$.

If $\lambda \neq 0$ and $F = (f_{ij}) \in \mathcal{J}$, then $\lambda^j f_{0j}(\lambda, \mu) = 0$ and $\lambda^{k+j-i} f_{ij}(\lambda, \mu) = 0$, which imply $F(\lambda, \mu) = 0$. Similarly, if $\mu \neq 0$, we conclude $F(\lambda, \mu) = 0$. But then $\mathcal{J} \subset \mathcal{J}_{\lambda,\mu}$, and so by maximality $\mathcal{J} = \mathcal{J}_{\lambda,\mu}$. \square

4. Semicrossed products by \mathbb{Z}_+^2 -actions

Let \mathcal{A} be a C^* -algebra, and $\text{Aut}(\mathcal{A})$ the group of star automorphisms of \mathcal{A} . An action α of \mathbb{Z}^2 on \mathcal{A} is a (group) homomorphism from \mathbb{Z}^2 to $\text{Aut}(\mathcal{A})$. The triple $(\mathcal{A}, \mathbb{Z}^2, \alpha)$ is called a C^* -dynamical system. In the case of \mathbb{Z}^2 , α defines two commuting automorphisms of \mathcal{A} , $\alpha_1 = \alpha(1, 0)$, $\alpha_2 = \alpha(0, 1)$. Conversely, any two commuting automorphisms of \mathcal{A} define a \mathbb{Z}^2 -action α .

We briefly recall the crossed product and semicrossed product constructions (see, e.g., [6,7] for more details). Let

$$l^1(\mathbb{Z}^2, \mathcal{A}, \alpha) = \left\{ F: \mathbb{Z}^2 \rightarrow \mathcal{A}, \text{ such that } \sum_{i,j \in \mathbb{Z}} \|F(i, j)\| < \infty \right\},$$

equipped with the usual multiplication

$$(\delta_{ij} \otimes f)(\delta_{kl} \otimes g) = \delta_{i+k, j+l} \otimes f\alpha(i, j)(g) = \delta_{i+k, j+l} \otimes f\alpha_1^i \alpha_2^j(g)$$

and involution

$$(\delta_{ij} \otimes f)^* = \delta_{-i, -j} \otimes f^*,$$

where $f, g \in \mathcal{A}$ and $\delta_{ij} \otimes f$ denotes the function $F: \mathbb{Z}^2 \rightarrow \mathcal{A}$ which takes the value f at the point $(i, j) \in \mathbb{Z}^2$ and zero elsewhere.

The crossed product $\mathbb{Z}^2 \times_\alpha \mathcal{A}$ is the completion of $l^1(\mathbb{Z}^2, \mathcal{A}, \alpha)$ with respect to the norm

$$\|F\| = \sup\{\|\pi(F)\|: \pi \text{ a nondegenerate star representation of } l^1(\mathbb{Z}^2, \mathcal{A}, \alpha)\}.$$

The subalgebra $l^1(\mathbb{Z}_+^2, \mathcal{A}, \alpha)$ of $l^1(\mathbb{Z}^2, \mathcal{A}, \alpha)$ consists of functions $F: \mathbb{Z}^2 \rightarrow \mathcal{A}$ such that $F(i, j) = 0$ if either i or j is negative. The subalgebra is not star-closed. The semicrossed product, $\mathbb{Z}_+^2 \times_\alpha \mathcal{A}$, has been defined in other contexts (e.g., [7]) as the completion of the l^1 -algebra with respect to the norm

$$\|F\|_c := \sup\{\|\pi(F)\|: \pi \text{ a contractive representation of } l^1(\mathbb{Z}_+^2, \mathcal{A}, \alpha)\}.$$

Since nondegenerate star representations are contractive, we have $\|F\| \leq \|F\|_c$. But in this case the reverse inequality also holds.

Lemma 4.1. *For any $F \in l^1(\mathbb{Z}_+^2, \mathcal{A}, \alpha)$, we have $\|F\|_c = \|F\|$, and consequently $\mathbb{Z}_+^2 \times_\alpha \mathcal{A}$ is the completion of $l^1(\mathbb{Z}_+^2, \mathcal{A}, \alpha)$ in $\mathbb{Z}^2 \times_\alpha \mathcal{A}$.*

Proof. Let ρ be a contractive representation of $l^1(\mathbb{Z}_+^2, \mathcal{A}, \alpha)$. It is then also contractive with respect to $\|\cdot\|_c$, and hence it can be extended to a contractive representation of $\mathbb{Z}_+^2 \times_\alpha \mathcal{A}$. By Ling and Muhly [5], ρ is completely contractive, and by Arveson–Stinespring theorem [3], there exists a C^* dilation $\tilde{\rho}$. If ρ acts on the Hilbert space \mathcal{H} , and $\tilde{\rho}$ acts on \mathcal{K} , there is an isometry V from \mathcal{H} into \mathcal{K} such that $\rho(F) = V^* \tilde{\rho}(F) V$. Hence, $\|\rho(F)\| \leq \|\tilde{\rho}(F)\| \leq \|F\|$.

It follows that $\|F\|_c \leq \|F\|$. \square

4.1. Canonical representations of semicrossed products

By Pedersen [6, Theorem 7.7.7], and the fact that \mathbb{Z}^2 is amenable, it follows that the full crossed product coincides with the reduced crossed product. Hence if (π, \mathcal{H}) is a faithful representation of \mathcal{A} , then $(\tilde{\pi} \times U, \ell^2(\mathbb{Z}^2, \mathcal{H}))$ is a faithful representation of $\mathbb{Z} \times_{\alpha} \mathcal{A}$, where $U : \mathbb{Z}^2 \rightarrow \mathcal{L}(\ell^2(\mathbb{Z}^2, \mathcal{H}))$, $U(s, t) := U_1^s U_2^t$, $U_1(\delta_{m,n} \otimes h) = \delta_{m+1,n} \otimes h$, and $U_2(\delta_{m,n} \otimes h) = \delta_{m,n+1} \otimes h$. We think of U_1, U_2 as the horizontal and vertical shifts.

$\tilde{\pi}$ is a representation of \mathcal{A} on $\ell^2(\mathbb{Z}^2, \mathcal{H})$ given by

$$\tilde{\pi}(a)(\delta_{m,n} \otimes h) = \delta_{m,n} \otimes \pi(\alpha(-m, -n)a) = \delta_{m,n} \otimes \pi(\alpha_1^{-m} \alpha_2^{-n} a).$$

Finally, $\tilde{\pi} \times U$ is a representation of $\mathbb{Z}_+^2 \times_{\alpha} \mathcal{A}$ on $\ell^2(\mathbb{Z}^2, \mathcal{H})$ defined on generating elements $\delta_{i,j} \otimes a$ by

$$(\tilde{\pi} \times U)(\delta_{i,j} \otimes a) = \tilde{\pi}(a)U_1^i U_2^j.$$

In this paper we are concerned with actions of \mathbb{Z}^2 on finite dimensional C^* -algebras, and we will identify the resulting semicrossed products with matrix algebras of analytic functions. We begin with the very simplest case, $\mathcal{A} = \mathbb{C}$.

Example 4.2. If the C^* -algebra $\mathcal{A} = \mathbb{C}$, the actions α_1, α_2 are trivial. Take π to be the identity representation of \mathbb{C} on (the Hilbert space) $\mathcal{H} = \mathbb{C}$. Then $\tilde{\pi}(a)(\delta_{m,n} \otimes h) = \delta_{m,n} \otimes \pi(a)h$. Let $W : L^2(\mathbb{T}^2) \rightarrow \ell^2(\mathbb{Z}^2)$ be the two dimensional Fourier transform,

$$W\left(\sum_{m,n \in \mathbb{Z}} x_{mn} z^m w^n\right) = \sum_{m,n \in \mathbb{Z}} \delta_{m,n} \otimes x_{mn}.$$

The representation $\text{Ad } W \circ (\tilde{\pi} \times U)$ takes the element $\sum_{i,j=0}^N \delta_{ij} \otimes a_{ij}$ in the semicrossed product to the multiplication operator M_p on $L^2(\mathbb{T}^2)$, $M_p g = pg$, where $p(z, w) = \sum_{i,j=0}^N a_{ij} z^i w^j$, $N \in \mathbb{Z}^+$. It follows that the semicrossed product $\mathbb{Z}_+^2 \times_{\alpha} \mathbb{C}$, which is the closure of the set of elements of the above form, is identified with the closure of the polynomials in two variables in the supremum norm, that is, the bidisk algebra $A(\mathbb{D}^2)$. Of course the crossed product is the C^* -envelope, $C(\mathbb{T}^2)$.

Next we characterize the general case of semicrossed products $\mathbb{Z}_+^2 \times_{\alpha} C(X)$, where X is finite. Fix an integer $k > 1$ and let σ_1, σ_2 be two commuting permutations of $\{0, 1, \dots, k-1\}$. Let $X = \{x_0, \dots, x_{k-1}\}$ and $\alpha = (\alpha_1, \alpha_2)$ the \mathbb{Z}^2 -action on $\mathcal{A} = C(X)$ given by $\alpha_i(f)(x_j) = f(x_{\sigma_i(j)})$, $i = 1, 2$.

Proposition 4.3. The algebra $\mathbb{Z}_+^2 \times_{\alpha} C(X)$ is unitarily equivalent to the operator algebra on $\bigoplus_{j=0}^{k-1} L^2(\mathbb{T}^2)$ with the generators $V_1 = zP_1$, $V_2 = wP_2$, and $D(f)$, $f \in C(X)$, where P_i are the permutation matrices

$$P_i = \sum_{j=0}^{k-1} e_{j, \sigma_i(j)}, \quad i = 1, 2,$$

and

$$D(f) = \sum_{j=0}^{k-1} f(x_j) e_{j,j}, \quad f \in C(X).$$

Proof. Let π be the faithful representation of $C(X)$ on $\mathcal{H} = \mathbb{C}^k$, $\pi(f)h = (f(x_0)h_0, \dots, f(x_{k-1})h_{k-1})$ for $h = (h_0, \dots, h_{k-1}) \in \mathcal{H}$, $f \in C(X)$. Then $\tilde{\pi}$ is a representation of $C(X)$ on $\tilde{\mathcal{H}} = \ell^2(\mathbb{Z}^2, \mathcal{H})$, and

$$\tilde{\pi}(f)(\delta_{m,n} \otimes h) = \delta_{m,n} \otimes \pi(\alpha_1^m \alpha_2^n f)h.$$

Let $\mathbf{h}_j = (0, \dots, 0, 1, 0, \dots, 0)$, $j = 0, \dots, k-1$, be the standard basis vectors for \mathcal{H} , and let $W : \bigoplus_{j=0}^{k-1} L^2(\mathbb{T}^2) \rightarrow \ell^2(\mathbb{Z}^2, \mathcal{H})$ be the unitary operator given by

$$W(0, \dots, 0, z^m w^n, 0, \dots, 0) = \delta_{m,n} \otimes \mathbf{h}_{j'},$$

where if $z^m w^n$ occurs in the j th column on the left (with the first column corresponding to $j = 0$), then $j' = \sigma_1^{-m} \sigma_2^{-n}(j)$.

We now calculate

$$\begin{aligned} W^* U_1 W(0, \dots, 0, z^m w^n, 0, \dots, 0) &= W^* U_1 (\delta_{m,n} \otimes \mathbf{h}_{j'}) \\ &= W^* (\delta_{m+1,n} \otimes \mathbf{h}_{j'}) = (0, \dots, 0, z^{m+1} w^n, 0, \dots, 0), \end{aligned}$$

where the term $z^{m+1} w^n$ appears in the $\sigma_1(j)$ column. In other words, $W^* U_1 W = z(\sum_{j=0}^{k-1} e_{j, \sigma_1(j)}) = zP_1 = V_1$. Similarly, $W^* U_2 W = V_2$.

Moreover, for $f \in C(X)$ and $g_j \in L^2(\mathbb{T}^2)$, $j = 0, \dots, k-1$, it is straightforward that

$$W^* \tilde{\pi}(f) W(g_0, \dots, g_{k-1}) = (f(x_0)g_0, \dots, f(x_{k-1})g_{k-1}),$$

so that $W^* \tilde{\pi}(f) W = D(f)$. \square

Example 4.4. Let $\sigma_1 = \sigma_2 = \sigma = (k-1, k-2, \dots, 1, 0)$ be the forward shift on $\{0, 1, \dots, k-1\}$. Then

$$P_1 = P_2 = \begin{pmatrix} 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

It is straightforward to verify that the algebra generated by $V_1 = zP_1$, $V_2 = wP_2$, and the set of all diagonal matrices generate the subalgebra of $\mathcal{B}_{k,2}$ for which the functions f_{ij} are polynomials in z, w . Thus if α_1, α_2 are the automorphisms of $C(X)$ corresponding to σ_1, σ_2 , it follows from Proposition 4.3 that the semicrossed product $\mathbb{Z}_+^2 \times_\alpha C(X)$ is isometrically isomorphic to $\mathcal{B}_{k,2}$.

Corollary 4.5. Let $\mathcal{A} = M_n \otimes C(X)$ and $\alpha = (\alpha_1, \alpha_2)$ be as in the previous example, so $\alpha_i(B \otimes f) = B \otimes \alpha_i(f)$, $i = 1, 2$. Then $\mathbb{Z}_+^2 \times_\alpha \mathcal{A}$ is isometrically isomorphic to $M_n \otimes \mathcal{B}_{k,2}$.

The proof is clear and thus is omitted.

Remark 4.6. In Example 4.4 and in the corollary the condition ‘isometrically isomorphic’ can be replaced by ‘completely isometrically isomorphic.’ Indeed, this is immediate from the unitary equivalence in Proposition 4.3.

4.2. Tensor products of \mathcal{B}_k ’s

As the operator algebras \mathcal{B}_k were identified as semicrossed products by (transitive) \mathbb{Z} -actions on \mathbb{C}^k (cf. Section 2), it may seem intuitively clear that the tensor product of \mathcal{B}_k with \mathcal{B}_ℓ would correspond to semicrossed products by \mathbb{Z}^2 ‘product actions’ on $\mathbb{C}^k \otimes \mathbb{C}^\ell$. We will in fact establish such a result. First, however, we examine the maximal ideal structure of these tensor products.

Realizing the operator algebra \mathcal{B}_k (cf. Section 2) as a $k \times k$ matrix function algebra acting on the direct sum $\bigoplus_{j=0}^{k-1} L^2(\mathbb{T})$, the algebraic tensor product $\mathcal{B}_k \otimes \mathcal{B}_\ell$ acts naturally on $\bigoplus_{j=0}^{k\ell-1} L^2(\mathbb{T})$. The spatial tensor product $\mathcal{B}_k \check{\otimes} \mathcal{B}_\ell$ is the completion of the algebraic tensor product in this representation. Alternatively, $\mathcal{B}_k \check{\otimes} \mathcal{B}_\ell$ can be viewed as the completion of the algebraic tensor product in $M_{k\ell} \otimes A(\mathbb{D}^2)$.

As before, let e_{ij} , $0 \leq i, j \leq k - 1$, denote matrix units for M_k , and similarly let $\bar{e}_{i'j'}$, $0 \leq i', j' \leq \ell - 1$, denote matrix units for M_ℓ . Thus, the algebraic tensor product $\mathcal{B}_k \otimes \mathcal{B}_\ell$ is spanned by linear combinations of elements of the form

$$e_{ij} \otimes \bar{e}_{i'j'} \otimes z^{i-j[\text{mod } k]+mk} w^{i'-j'[\text{mod } \ell]+n\ell}, \quad m, n \in \mathbb{Z}, m, n \geq 0,$$

where as before $i - j[\text{mod } k]$ lies in $\{0, \dots, k - 1\}$ (respectively, $i' - j'[\text{mod } \ell] \in \{0, \dots, \ell - 1\}$). Thus, $\mathcal{B}_k \check{\otimes} \mathcal{B}_\ell$ is a $k\ell \times k\ell$ matrix of functions $f_{(ij)(i'j')} \in A(\mathbb{D}^2)$ whose Fourier series have the form

$$f_{(ij)(i'j')}(z, w) \sim \sum_{m,n \geq 0} a_{mn} z^{i-j[\text{mod } k]+mk} w^{i'-j'[\text{mod } \ell]+n\ell}.$$

Observe that functions in the diagonal subalgebra of $\mathcal{B}_k \check{\otimes} \mathcal{B}_\ell$ belong to $A_k(\mathbb{D}) \check{\otimes} A_\ell(\mathbb{D})$.

Lemma 4.7. *The maximal ideals of $A_k(\mathbb{D}) \check{\otimes} A_\ell(\mathbb{D})$ are kernels of the evaluation homomorphisms $f \mapsto f(\lambda, \mu)$, $(\lambda, \mu) \in \mathbb{D}^2$.*

Proof. This is in the spirit of Lemma 3.1. \square

As with the earlier lemma, the correspondence between maximal ideals and points in the closed bidisk is not bijective.

For $0 \leq i_0 \leq k - 1$ and $0 \leq i'_0 \leq \ell - 1$, define

$$\psi_{00}^{i_0 i'_0} : \mathcal{B}_k \check{\otimes} \mathcal{B}_\ell \rightarrow \mathbb{C} \quad \text{by } \psi_{00}^{i_0 i'_0}(F) = f_{(i_0 i_0)(i'_0 i'_0)}(0, 0), \text{ where } F = (f_{(ij)(i'j')}).$$

Define

$$\begin{aligned} \psi_{0\mu}^{i_0} : \mathcal{B}_k \check{\otimes} \mathcal{B}_\ell &\rightarrow M_\ell, & \psi_{0\mu}^{i_0}(F) &= (f_{(i_0 i_0)(i'j')}(0, \mu))_{i'j'=0}^{\ell-1}, \\ \psi_{\lambda 0}^{i'_0} : \mathcal{B}_k \check{\otimes} \mathcal{B}_\ell &\rightarrow M_k, & \psi_{\lambda 0}^{i'_0}(F) &= (f_{(ij)(i'_0 i'_0)}(\lambda, 0))_{ij=0}^{k-1}, \end{aligned}$$

and

$$\psi_{\lambda\mu}: \mathcal{B}_k \check{\otimes} \mathcal{B}_\ell \rightarrow M_{k\ell}, \quad \psi_{\lambda\mu}(F) = F(\lambda, \mu) = (f_{(ij)(i'j')}(\lambda, \mu))_{(ij)(i'j')=(0)(0)}^{(k-1)(\ell-1)}.$$

Denote the kernels of these homomorphisms, respectively, by $J_{00}^{i_0 i'_0}$, $J_{0\mu}^{i_0}$, $J_{\lambda 0}^{i'_0}$, and $J_{\lambda\mu}$.

Proposition 4.8. *Every maximal ideal of $\mathcal{B}_k \check{\otimes} \mathcal{B}_\ell$ is of the form $J_{00}^{i_0 i'_0}$, $J_{0\mu}^{i_0}$, $J_{\lambda 0}^{i'_0}$, or $J_{\lambda\mu}$, for $\lambda, \mu \in \bar{\mathbb{D}} \setminus \{0\}$.*

Proof. Note that the ψ -homomorphisms are all onto simple algebras, so their kernels are maximal ideals.

The proof is similar to that of Proposition 3.3. Let J be a maximal ideal of $\mathcal{B}_k \check{\otimes} \mathcal{B}_\ell$. Since J cannot contain the identity, there are indices i_0, i'_0 such that for all $F \in J$, $F = (f_{(ij)(i'j')})_{(i_0 i_0)(i'_0 i'_0)} \neq 1$. Indeed, if that were not the case, so we could find $F_{ii'}$ whose $((ii), (i'i'))$ component was 1, $0 \leq i \leq k-1$, $0 \leq i' \leq \ell-1$, then we would obtain

$$\sum_{i, i'=0}^{i=k-1, i'=\ell-1} (e_{ii} \otimes \bar{e}_{i'i'}) F_{ii'} (e_{ii} \otimes \bar{e}_{i'i'}) = I \in J,$$

a contradiction.

Without loss of generality, we may assume that for all $F \in J$, $F = (f_{(ij)(i'j')})_{(00)(00)} \neq 1$. Let \mathcal{I} be the ideal in $A_k(\bar{\mathbb{D}}) \check{\otimes} A_\ell(\bar{\mathbb{D}})$ generated by $\{f = f_{(00)(00)} \text{ for some } F = (f_{(ij)(i'j')}) \in J\}$. By assumption \mathcal{I} is proper, so it is contained in some maximal ideal given by evaluation at $(\lambda, \mu) \in \bar{\mathbb{D}}^2$.

If $(\lambda, \mu) = (0, 0)$ then $J \subset J_{00}^{00}$ and hence $J = J_{00}^{00}$ by maximality of J . If $(\lambda, \mu) = (\lambda, 0)$ for some $\lambda \in \bar{\mathbb{D}} \setminus \{0\}$, we claim $J = J_{\lambda, 0}^0$. Let $F \in J$, $F = (f_{(ij)(i'j')})$, and select $0 \leq i, j \leq k-1$. From the ideal property of J we obtain

$$(z^{k-i} \otimes e_{0i} \otimes \bar{e}_{00}) F (z^j \otimes e_{j0} \otimes \bar{e}_{00}) = z^{k+j-i} f_{(ij)(00)} \otimes e_{00} \otimes \bar{e}_{00} \in J,$$

whence $\lambda^{k+j-i} f_{(ij)(00)}(\lambda, 0) = 0$, and in particular $f_{(ij)(00)}(\lambda, 0) = 0$. This shows that $J \subset J_{\lambda, 0}^0$, and by maximality of J equality prevails.

The other cases are analogous. \square

Corollary 4.9. *For positive integers $k, \ell, p, k, \ell > 1$, the Banach algebras $\mathcal{B}_{p,2}, \mathcal{B}_k \check{\otimes} \mathcal{B}_\ell$ are not isomorphic.*

Proof. $\mathcal{B}_{p,2}$ has maximal ideals of at most two distinct codimensions, whereas $\mathcal{B}_k \check{\otimes} \mathcal{B}_\ell$ has maximal ideals of at least three distinct codimensions. \square

4.3. Perpendicular actions

In Example 4.4 the algebra $\mathcal{B}_{k,2}$ was identified with the semicrossed product $\mathbb{Z}_+^2 \times_\alpha C(X)$, where $\alpha = (\alpha_1, \alpha_2)$ in which $\alpha_1 = \alpha_2$ was implemented by a cyclic permutation acting transitively on X . In this section we consider the opposite extreme, namely, where

$\alpha = (\alpha_1, \alpha_2)$, α_i is implemented by a permutation σ_i , $i = 1, 2$, such that $\sigma = (\sigma_1, \sigma_2)$ is transitive as a \mathbb{Z}^2 action, but neither σ_i is transitive, and furthermore the orbits of σ_1, σ_2 overlap minimally, that is, in a single point. To state this formally, we make the following

Definition 4.10. Let X be a finite set, σ_i , $i = 1, 2$, commuting permutations on X , and $\sigma = (\sigma_1, \sigma_2)$ the induced \mathbb{Z}^2 -action on X . We say that σ is a *perpendicular* action if there exist finite sets X_1, X_2 , cyclic (transitive) permutations τ_i on X_i , $i = 1, 2$, and a bijection $h: X \rightarrow X_1 \times X_2$ such that for all $x_0 \in X$,

$$h(\sigma_1(x_0)) = (\tau_1(x_1), x_2) \quad \text{and} \quad h(\sigma_2(x_0)) = (x_1, \tau_2(x_2)),$$

where $(x_1, x_2) = h(x_0)$. In other words, $h \circ \sigma = \tau \circ h$, where $\tau(m, n) = \tau_1^m \times \tau_2^n$ on $X_1 \times X_2$.

In the language of dynamics, σ is a perpendicular action if it is conjugate to a product of transitive actions on finite sets.

While the following proposition is elementary, we are not aware of it in the dynamical systems literature.

Proposition 4.11. Let X be a finite set, σ_i , $i = 1, 2$, commuting permutations, and $\sigma = (\sigma_1, \sigma_2)$ the induced \mathbb{Z}^2 -action. Then σ is a perpendicular action if and only if

- (i) For any $x, y \in X$, $\mathcal{O}_{\sigma_1}^x \cap \mathcal{O}_{\sigma_2}^y$ consists of a single point, where $\mathcal{O}_{\sigma_1}^x$ (respectively, $\mathcal{O}_{\sigma_2}^y$) denotes the orbit of x under σ_1 (respectively, the orbit of y under σ_2);
- (ii) All $\mathcal{O}_{\sigma_1}^x$ ($x \in X$) have the same cardinality, and all $\mathcal{O}_{\sigma_2}^y$, $y \in Y$, have the same cardinality.

Proof. First, if σ is a perpendicular action it is transitive, and it is clear that (i) and (ii) are satisfied.

Suppose now that (i) and (ii) are satisfied. Observe that σ is a transitive action. Indeed, if $x, y \in X$ by (i) there are $m, n \in \mathbb{Z}$ such that $\sigma_1^m(x) = \sigma_2^n(y)$; that is $y = \sigma(m, -n)(x)$, which is transitivity.

Choose $x_0 \in X$ and let $X_1 = \mathcal{O}_{\sigma_1}^{x_0}$ and $X_2 = \mathcal{O}_{\sigma_2}^{x_0}$. Define a map $h: X \rightarrow X_1 \times X_2$ by $h(x) = (y_1, y_2)$, where y_1 is the unique element in $\mathcal{O}_{\sigma_1}^{x_0} \cap \mathcal{O}_{\sigma_2}^x$ and y_2 is the unique element in $\mathcal{O}_{\sigma_1}^x \cap \mathcal{O}_{\sigma_2}^{x_0}$. We show that the mapping h is a bijection.

Injectivity. Let $h(x) = h(x')$. Then $(y_1, y_2) = (y'_1, y'_2)$, where $\mathcal{O}_{\sigma_1}^{x_0} \cap \mathcal{O}_{\sigma_2}^x = \{y_1\} = \{y'_1\} = \mathcal{O}_{\sigma_1}^{x_0} \cap \mathcal{O}_{\sigma_2}^{x'}$. Since the orbits $\mathcal{O}_{\sigma_2}^x$ and $\mathcal{O}_{\sigma_2}^{x'}$ have a point in common, they are equal. Similarly, $\mathcal{O}_{\sigma_1}^x = \mathcal{O}_{\sigma_1}^{x'}$. So we have $\mathcal{O}_{\sigma_1}^{x'} \cap \mathcal{O}_{\sigma_2}^x = \mathcal{O}_{\sigma_1}^x \cap \mathcal{O}_{\sigma_2}^x = \{x\}$, and on the other hand $\mathcal{O}_{\sigma_1}^{x'} \cap \mathcal{O}_{\sigma_2}^x = \mathcal{O}_{\sigma_1}^{x'} \cap \mathcal{O}_{\sigma_2}^{x'} = \{x'\}$. Hence $x = x'$.

Surjectivity. Let $(y_1, y_2) \in X_1 \times X_2$. Let z denote the unique element in the intersection $\mathcal{O}_{\sigma_1}^{y_2} \cap \mathcal{O}_{\sigma_2}^{y_1}$. As y_1 is in $\mathcal{O}_{\sigma_2}^z$ and y_2 is in $\mathcal{O}_{\sigma_1}^z$, we have $\mathcal{O}_{\sigma_1}^{x_0} \cap \mathcal{O}_{\sigma_2}^z = \{y_1\}$ and $\mathcal{O}_{\sigma_1}^z \cap \mathcal{O}_{\sigma_2}^{x_0} = \{y_2\}$. Hence $h(z) = (y_1, y_2)$.

Let us now show that the action is perpendicular. Take $(y_1, y_2) \in X_1 \times X_2$ and set $\hat{\sigma}_i := h^{-1} \circ \sigma_i \circ h$, $i = 1, 2$. Let $h = (h_1, h_2)$; that is, $h_i = \pi_i \circ h$, where π_i is the i th coordinate projection, $i = 1, 2$. First we show that $\pi_2 \circ \hat{\sigma}_1(y_1, y_2) = y_2$. Call $z = h^{-1}(y_1, y_2)$.

As $\mathcal{O}_{\sigma_1}^z = \mathcal{O}_{\sigma_1}^{\sigma_1(z)}$, it follows $\{h_2(\sigma_1(z))\} = \mathcal{O}_{\sigma_1}^{\sigma_1(z)} \cap \mathcal{O}_{\sigma_2}^{x_0} = \mathcal{O}_{\sigma_1}^z \cap \mathcal{O}_{\sigma_2}^{x_0} = \{h_2(z)\}$. Hence, $h_2(\sigma_1(z)) = h_2(z)$. Therefore there is a map $\tau_1 : X_1 \times X_2 \rightarrow X_1$ such that $\hat{\sigma}_1(y_1, y_2) = (\tau_1(y_1, y_2), y_2)$.

It remains to show that τ_1 depends only on y_1 , and hence can be viewed as a map $X_1 \rightarrow X_1$. We show that $\tau_1(y_1, y_2) = \tau_1(y_1, y'_2)$. Let $h^{-1}(y_1, y_2) = z$, $h^{-1}(y_1, y'_2) = z'$. Since σ is a transitive action, $z' = \sigma(m, n)z = \sigma_1^m \sigma_2^n(z)$ for some $(m, n) \in \mathbb{Z}^2$. But as we have noted above, $h_2(z) = h_2(\sigma_1(z))$ and similarly $h_1(z) = h_1(\sigma_2(z))$, so it is an action of σ_2 that maps z to z' , say $z' = \sigma_2^n(z)$. Since σ_1, σ_2 commute, $\sigma_1(z') = \sigma_1(\sigma_2^n(z)) = \sigma_2^n(\sigma_1(z))$, so that $\sigma_1(z')$, $\sigma_1(z)$ are on the same σ_2 -orbit. Hence, $\tau_1(y_1, y_2) = h_1(\sigma_1(z')) = \tau_1(y_1, y'_2)$. \square

Remark 4.12. (1) If σ is a perpendicular \mathbb{Z}^2 action which is conjugate to a product action $\tau_1 \times \tau_2$ on $X_1 \times X_2$ and also conjugate to $\tau'_1 \times \tau'_2$ on $X'_1 \times X'_2$, then $\text{card}(X_i) = \text{card}(X'_i)$, $i = 1, 2$.

(2) Is there a version of Proposition 4.11 for topological dynamics?

Example 4.13. Take $X = \{0, 1, \dots, 5\}$, $\sigma_1 = (0, 1, 2)(3, 4, 5)$ and $\sigma_2 = (0, 3)(1, 4)(2, 5)$. Then the conditions of the proposition are satisfied, and with $X_1 = \{0, 1, 2\}$, $X_2 = \{0, 3\}$, and $x_0 = 0$ we have $h(i) = (i, 0)$, $i = 0, 1, 2$, and $h(i) = (i, 3)$, $i = 3, 4, 5$.

4.4. Semicrossed products by perpendicular actions

Earlier (in Section 4.1) we reviewed the construction of the (semi)crossed product as the completion of an ℓ_1 -algebra in a canonical representation. In the case of a perpendicular \mathbb{Z}^2 -action, the canonical representation can be taken to be a product representation, and this allows us to view the (semi)crossed product as the tensor product of (semi)crossed products with respect to \mathbb{Z} -actions. In what follows we consider (semi)crossed products of $C(X)$, where X is a compact metric space.

Let X_i be a compact metric space, σ_i a homeomorphism of X_i , and α_i the induced automorphism of $C(X_i)$, $\alpha_i(f) = f \circ \sigma_i$, $f \in C(X_i)$, $i = 1, 2$. Let π_i be a faithful representation of $C(X_i)$ on a Hilbert space \mathcal{H}_i . Denote by $K(\mathbb{Z}, C(X_i), \alpha_i)$ the dense subalgebra of $\ell_1(\mathbb{Z}, C(X_i), \alpha_i)$ consisting of finite sums of the generators $\sum_m \delta_m^i \otimes f_m$. Then, as in Section 4.1, the (semi)crossed product $\mathbb{Z} \times_{\alpha_i} C(X_i)$ (respectively, $\mathbb{Z}^+ \times_{\alpha_i} C(X_i)$) is the completion of $K(\mathbb{Z}, C(X_i), \alpha_i)$ (respectively, $K(\mathbb{Z}^+, C(X_i), \alpha_i)$) in the representation $\tilde{\pi}_i \times U_i$ on the Hilbert space $\mathcal{H}_i = \ell^2(\mathbb{Z}, \mathcal{H}_i)$.

$$\begin{aligned} & (\tilde{\pi}_1 \times U_1) \otimes (\tilde{\pi}_2 \times U_2) (K(\mathbb{Z}, C(X_1), \alpha_1)) \otimes (K(\mathbb{Z}, C(X_2), \alpha_2)) \\ &= ((\tilde{\pi}_1 \otimes \tilde{\pi}_2) \times U) (K(\mathbb{Z}^2, C(X_1) \otimes C(X_2), \alpha)), \end{aligned} \quad (2)$$

where α is the \mathbb{Z}^2 -action $\alpha = (\alpha_1, \alpha_2)$, $\tilde{\pi}_1 \otimes \tilde{\pi}_2$ acts on the Hilbert space $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_1 \hat{\otimes} \tilde{\mathcal{H}}_2$, and $U(m, n) = U_1^m \otimes U_2^n$. Since the algebraic tensor product $C(X_1) \otimes C(X_2)$ is dense in $C(X_1 \times X_2)$, and $\pi_1 \otimes \pi_2$ is a faithful representation of $C(X_1 \times X_2)$, it follows that the completion of the right-hand side of (2) is the crossed product $\mathbb{Z}^2 \times_{\alpha} C(X)$, $X = X_1 \times X_2$.

Proposition 4.14. *With notation as above,*

$$(\mathbb{Z} \times_{\alpha_1} C(X_1)) \check{\otimes} (\mathbb{Z} \times_{\alpha_2} C(X_2)) = \mathbb{Z}^2 \times_{\alpha} C(X)$$

and

$$(\mathbb{Z}_+ \times_{\alpha_1} C(X_1)) \check{\otimes} (\mathbb{Z}_+ \times_{\alpha_2} C(X_2)) = \mathbb{Z}_+^2 \times_{\alpha} C(X).$$

Proof. The completion of $(K(\mathbb{Z}, C(X_1), \alpha_1)) \otimes (K(\mathbb{Z}, C(X_2), \alpha_2))$ in the norm provided by the representation $(\tilde{\pi}_1 \times U_1) \otimes (\tilde{\pi}_2 \times U_2)$ on $\mathcal{L}(\tilde{\mathcal{H}})$ is a C^* -norm on the algebraic tensor product $(\mathbb{Z} \times_{\alpha_1} C(X_1)) \otimes (\mathbb{Z} \times_{\alpha_2} C(X_2))$. However, the crossed product $\mathbb{Z} \times_{\alpha_i} C(X_i)$ is nuclear (e.g., [9, Proposition 2.1.2]), so there is only one completion, and hence the completion is $\mathbb{Z}^2 \times_{\alpha} C(X)$. The statement about semicrossed products follows from the fact that the semicrossed product norm is, by Lemma 4.1, the restriction of the C^* -crossed product norm. \square

Corollary 4.15. *Let X be a finite set, σ a perpendicular \mathbb{Z}^2 -action on X , so that σ is conjugate to $\tau_1 \times \tau_2$ acting on $X_1 \times X_2$, where τ_i is a transitive \mathbb{Z} -action on X_i . If $\text{card}(X_1) = k$, $\text{card}(X_2) = \ell$, then the semicrossed product $\mathbb{Z}_+^2 \times_{\sigma} C(X)$ is identified with $\mathcal{B}_k \check{\otimes} \mathcal{B}_{\ell}$.*

Proof. This follows immediately from the above and the identification of $\mathbb{Z}_+ \times_{\sigma_i} C(X_i)$ with \mathcal{B}_k ($i = 1$) or \mathcal{B}_{ℓ} ($i = 2$). \square

Corollary 4.16. *With the same assumptions as in Corollary 4.15, $\mathbb{Z}_+^2 \times_{\sigma} C(X)$ is generated by $\{zP_k \otimes wP_{\ell}, D_k \otimes D_{\ell} : D_k \in \text{diag}(M_k), D_{\ell} \in \text{diag}(M_{\ell})\}$, where $\text{diag}(M_k)$ (respectively, $\text{diag}(M_{\ell})$) is the algebra of diagonal matrices in M_k (respectively, M_{ℓ}). $P_k \in M_k$ and $P_{\ell} \in M_{\ell}$ have the form of the matrix in Example 4.4.*

Proof. This follows from the form of \mathcal{B}_k (see (1)) and the fact that the subalgebra of \mathcal{B}_k consisting of matrix functions with polynomial entries are dense in \mathcal{B}_k . \square

Corollary 4.17. *There are transitive \mathbb{Z}^2 -actions σ, σ' on a finite set X such that the semicrossed products $\mathbb{Z}^2 \times_{\sigma} C(X)$ and $\mathbb{Z}^2 \times_{\sigma'} C(X)$ are not isomorphic.*

Proof. This follows from Corollary 4.9, Example 4.4, and Corollary 4.15. \square

Acknowledgments

M.R. Alaimia would like to thank Prof. Peters for his invitation to Iowa State University where part of this work has been done. He would like also to acknowledge the support of KFUPM.

References

- [1] M.R. Alaimia, Automorphisms of certain Banach algebras of analytic functions, *Linear Algebra Appl.* 298 (1999) 87–97.
- [2] M.R. Alaimia, Weak semicrossed product and a generalisation of a result of Sarason, *Linear Algebra Appl.* 318 (2000) 181–193.

- [3] W.B. Arveson, K.B. Josephson, Operator algebras and measure preserving automorphisms II, *J. Funct. Anal.* 4 (1969) 100–134.
- [4] L.M. DeAlba, J. Peters, Classification of semicrossed products of finite dimensional C^* algebras, *Proc. Amer. Math. Soc.* 95 (1985) 557–564.
- [5] K.-S. Ling, P.S. Muhly, An automorphic form of Ando’s theorem, *Integral Equations Operator Theory* 12 (1989) 424–434.
- [6] G.K. Pedersen, *C^* -Algebras and Their Automorphisms*, Academic Press, London, 1979.
- [7] J. Peters, Semi-crossed products of C^* -algebras, *J. Funct. Anal.* 59 (1984) 498–534.
- [8] S.C. Power, Classification of analytic crossed product algebras, *Bull. London Math. Soc.* 24 (1992) 368–372.
- [9] M. Rørdam, E. Stormer, Classification of nuclear C^* -algebras, in: *Operator Algebras and Non-Commutative Geometry*, in: *Encyclopedia of Mathematical Sciences*, Vol. VII, Springer-Verlag, Berlin, 2002.
- [10] A. Zygmund, *Trigonometric Series*, Cambridge Univ. Press, Cambridge, 1988.