The Maximal Subgroups of the Finite 8-Dimensional Orthogonal Groups $PO_8^+(q)$ and of Their Automorphism Groups

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PART 1. PRELIMINARIES

1.1. Introduction

During this century, and even before, a substantial amount of work has been devoted to finding the maximal subgroups of finite simple groups and their automorphism groups. Some of the earliest published results of this nature appear in Wiman [42] and Moore [33], where the maximal subgroups of the groups $L_2(q) = PSL_2(q)$ are determined for all $q$. Then Mitchell [31, 32] and Hartley [15] found the maximal subgroups of $L_3(q)$, $U_3(q) = PSU_3(q)$, and also $PSp_4(q)$ for odd $q$. Several decades later, analogous results appeared for various other low-dimensional classical groups, including $PGL_2(q)$ with $q$ even, $L_4(q)$, $U_4(q)$, and $L_5(q)$. (The bibliographies of [10] and [43] serve as good sources of reference.)

Recently Aschbacher [2] made a significant contribution to the solution of the problem of finding the maximal subgroups of any group whose socle is a classical simple group. The main theorem of [2] says the following. Let $G_0$ be a finite classical simple group with natural projective module $V$, and let $G$ be a group with socle $G_0$ (i.e., $G_0 \leq G \leq \text{Aut}(G_0)$). Assume that if $G_0 \cong PO_8^+(q)$ then $G$ does not contain a triality automorphism of $G_0$. If $M$ is a maximal subgroup of $G$ not containing $G_0$, then one of the following holds:

(a) $M$ is a known group with a well-described projective action on $V$;

(β) the socle $S = \text{soc}(M)$ of $M$ is a non-abelian simple group whose projective representation in $PGL(V)$ corresponds to an absolutely irreducible representation of the covering group of $S$ in $GL(V)$.

So roughly speaking, the main theorem of [2] "reduces" the problem of finding the maximal subgroups of $G$ to that of finding its absolutely
irreducible simple subgroups. Thus one is left to answer: Which quasisimple groups have an absolutely irreducible representation in $GL(V)$? If the dimension $\dim(V)$ of $V$ is small enough, then the answer can be obtained by invoking the classification of finite simple groups. In this way, we have determined the maximal subgroups of $G$ when $\dim(V) \leq 12$. In [24] we treat the case in which $G_0$ is isomorphic to one of these low-dimensional classical groups other than $P\Omega_8^+(q)$, and in this paper we handle the case $G_0 \cong P\Omega_8^+(q)$. There are essentially two reasons for giving $P\Omega_8^+(q)$ this special attention. First, the geometry associated with $P\Omega_8^+(q)$ is perhaps the richest low-dimensional classical geometry, and thus many groups occur under $(a)$. Second, unlike the other classical groups, our analysis must go beyond the scope of the main theorem of [2], because that theorem does not cover the case in which $G$ contains a triality automorphism of $P\Omega_8^+(q)$. So in some sense, this paper serves to fill in the gap occurring in the main theorem of [2]. Our proof uses the classification of finite simple groups and the statement of our results appears in the results matrix, Table I, described in Section 1.5.

Although we have mentioned only the classical finite simple groups so far, there are in fact numerous results concerning the maximal subgroups of other simple groups. For instance, the maximal subgroups of the following exceptional groups of Lie type have been found: $Sz(q) = 2B_2(q)$ [37], $G_2(q)$ [3, 10, 25, 30], $2G_2(q)$ [25], $3D_4(q)$ [23] and $2F_4(q)$ [35]. Moreover, a classification (but not an explicit enumeration) of the maximal subgroups of the alternating and symmetric groups appears in [29], and presently the maximal subgroups of 21 of the sporadic simple groups are known (see [9]).

1.2. Notation and Prerequisites

Our conventions for expressing the structure of groups run as follows. (Note that all groups in this paper are finite.) If $H$ and $K$ are arbitrary groups, then $H.K$ denotes any extension of $H$ by $K$. The expressions $H:K$ and $H.K$ denote split and nonsplit extensions, respectively, while $H_0 K$ denotes a central product of $H$ and $K$. Also, $(1/m)H$ refers to a subgroup of index $m$ in $H$. The symbol $[m]$ denotes an arbitrary group of order $m$, while $Z_m$ or simply "$m"$ denotes a cyclic group of that order. The dihedral group of order $m$ is written $D_m$. If $r$ is prime, then $(Z_r)^e$ or simply "$r^e"$ denotes an elementary abelian group of order $r^e$, and $2_{+}^{1+e}$ denotes an extraspecial group of order $2^7$ isomorphic to $D_8 \circ D_8 \circ D_8$. We write $L_m(r^e)$ for the group $L_m(r^e)$ or $U_m(r^e)$, according as $e$ is $+$ or $-$. Let $V_i$ be an $m$-dimensional vector space over $F_1 = GF(q_1)$ and $Q_1: V_1 \rightarrow F_1$ a quadratic form with associated bilinear form $(\cdot, \cdot)_1$. Assume that $Q_1$ is nondegenerate, which is to say $(\cdot, \cdot)_1$ is nondegenerate. If $v \in V_1$ and $Q(v) = 0$, then $v$ and the 1-space $<v>$ are called singular. Otherwise $v$
and $\langle v \rangle$ are nonsingular. The norm of $v$ is $(v, v)_1$. A subspace $W$ is non-degenerate (n.d.) if the restriction $(Q_1)_W$ of $Q_1$ to $W$ is a nondegenerate quadratic form on $W$. However, if $Q_1$ vanishes on $W$, then $W$ is totally singular (t.s.).

Denote by $\Gamma L(V_1, F_1)$ the group of all nonsingular semilinear transformations of $V_1$ and define

$$
\Gamma(V_1, Q_1, F_1) = \{ g \in \Gamma L(V_1, F_1); Q_1(v^g) = \lambda_g Q_1(v)^{\sigma_g} \text{ for all } v \in V_1, \text{ where } \lambda_g \in F_1^* \text{ and } \sigma_g \in \text{Aut}(F_1) \text{ depend only on } g \},
$$

$$
\Delta(V_1, Q_1, F_1) = \{ g \in \Gamma(V_1, Q_1, F_1); \sigma_g = 1 \} \leq GL(V_1),
$$

$$
O(V_1, Q_1, F_1) = \{ g \in \Delta(V_1, Q_1, F_1); \lambda_g = 1 \},
$$

$$
SO(V_1, Q_1, F_1) = \{ g \in O(V_1, Q_1, F_1); \det(g) = 1 \}.
$$

$$
\Omega(V_1, Q_1, F_1) = [O(V_1, Q_1, F_1), O(V_1, Q_1, F_1)].
$$

If $\dim(V_1) = m$ is odd, then $\Omega_m(q_1)$ denotes the abstract group isomorphic to $\Omega(V_1, Q_1, F_1)$. If $m = 2h$ is even, then the corresponding abstract group is written $\Omega^{\varepsilon}_m(q_1)$, where $\varepsilon$ is + or − according as $Q_1$ has (Witt) defect 0 or 1 (i.e., according as the maximal t.s. subspaces of $V_1$ have dimension $h$ or $h - 1$). We also write $PY = Y/Z(Y)$, where $Y$ is either $\Omega_m(q_1)$ or $\Omega^{\varepsilon}_m(q_1)$. Similar remarks hold for the groups $SO_m(q_1)$, $SO^\varepsilon_m(q_1)$, $O_m(q_1)$, etc. Recall the isomorphisms $O^{\varepsilon}_2(q) \cong D_2(q_{-\varepsilon})$, $\Omega_2(q) \cong L_2(q)$, $\Omega^\varepsilon_2(q) \cong SL_2(q)$, $SL_2(q)$, $\Omega_{4^\varepsilon}(q) \cong L_2(q^2)$, $\Omega_5(q) \cong PSp_4(q)$, and $PQ_8^+(q) \cong L_4^\varepsilon(q)$.

Define the discriminant $\text{disc}(Q_1)$ of $Q_1$ to be the determinant $(\text{mod}(F^*)^2)$ of the matrix of $(\ , \ )_1$ with respect to some basis of $V_1$ (see [1, p. 107]). When $q_1$ is odd and $m$ is even, then the defect of $Q_1$ is determined by $\text{disc}(Q_1)$.

**Lemma 1.2.1.** When $q_1$ is odd and $m$ is even, $Q_1$ has defect 0 if and only if one of the following holds:

(i) $m \equiv 0 \text{ mod } 4$ and $\text{disc}(Q_1)$ is a square;

(ii) $m \equiv 2 \text{ mod } 4$, $q_1 \equiv 1 \text{ mod } 4$, and $\text{disc}(Q_1)$ is a square;

(iii) $m \equiv 2 \text{ mod } 4$, $q_1 \equiv 3 \text{ mod } 4$, and $\text{disc}(Q_1)$ is a nonsquare.

For a thorough description of the basic properties of the orthogonal groups, see [1, Chap. V]. There is also a good compendium of information in the introduction of [9] and in [20].

Now fix an 8-dimensional vector space $V$ over $F = GF(q)$, where $q = p^n$ and $p$ is prime. Assume that $Q : V \to F$ is a nondegenerate quadratic form of defect 0 (thus the maximal t.s. subspaces have dimension 4). Write $(\ , \ )$ for the associated bilinear form and put $X = X(V, Q, F)$, where $X$ ranges over the symbols $\Gamma, \Delta, O, SO$, and $\Omega$. The corresponding projective groups will
be denoted $PX$. When $q$ is odd, $-1 \in \Omega$ by Theorem 5.19 of [1]. When $q$ is even we regard $-1 = 1$, so that we may write $\langle -1 \rangle = Z(\Omega)$ for all $q$. Define

$$G_0 = P\Omega \cong P\Omega^+_q(q),$$

a simple group. Also define $Z$ to be the group of scalars in $GL(V)$ and put $d = (2, q-1)$. Then

$$\Gamma/A \cong P\Gamma/P\Lambda \cong \text{Aut}(F) \cong Z_n,$$

$$|A:OZ| = |P\Lambda:PO| = d,$$

$$|O:SO| = |PO:PSO| = d,$$

$$|SO:\Omega| = |PSO:G_0| = 2,$$

$$|G_0| = \frac{1}{d^2} q^{12}(q^2 - 1)(q^4 - 1)^2(q^6 - 1).$$

Also define $A = \text{Aut}(G_0)$. Thus $G_0 \leq P\Gamma \leq A$, and in fact $|A:P\Gamma| = 3$; the group $A$ is generated by $P\Gamma$ and a triality automorphism of $G_0$ (see [7, Theorem 12.5.1]). Throughout this paper, $G$ denotes a group satisfying

$$G_0 \leq G \leq A.$$

Further $M$ is a maximal subgroup of $G$ not containing $G_0$ and $M_0 = M \cap G_0$. The term triality automorphism refers to any element of $A$ inducing a symmetry of order 3 on the Dynkin diagram of $G_0$, and we let $\mathcal{T}$ be the set of triality automorphisms in $A$.

Let $W$ be a n.d. subspace of $V$ of dimension $m$. If $m$ is even, then $W$ inherits an $O_m^+$-geometry from $V$, where $\varepsilon$ is $+$ or $-$. In this case we call $W$ an $em$-space. If $m$ is odd, then $q$ must be odd and we call $W$ a $+m$-space (resp. $-m$-space) if $\text{disc}(Q_w)$ is a square (resp. nonsquare). We write $\Omega(W) = \Omega(W, Q_w, F)$ and regard $\Omega(W)$ as a subgroup of $\Omega$: elements of $\Omega(W)$ act naturally on $W$ and centralize $W^\perp$. Similar remarks hold for $SO(W)$, $O(W)$, and so on. We write $-1_w$ for the element in $GL(V)$ which acts as $-1$ on $W$ and $+1$ on $W^\perp$. (When $q$ is even, $-1_w = 1$.) Clearly $-1_w \in O(W) \leq O$.

We extend the definition of “$+1$-space” to subspaces of $V$ in even characteristic. Namely, when $p = 2$ we call $W$ a $+1$-space provided $W$ is a nonsingular $1$-space.

If $H \leq O$ and $U \leq V$ is $H$-invariant, define

$$H(U) = H/C_H(U).$$

Thus $H(U)$ acts faithfully on $U$ and if $U$ is n.d. then $H(U) \leq O(U)$. 
If \( V_1, \ldots, V_k \) are subspaces of \( V \), then for any \( H \) contained in \( \Gamma \) or \( \Pi \), \( N_H(V_1, \ldots, V_k) \) is the set of elements of \( H \) which permute the spaces \( V_i \) amongst themselves, while \( N_p(V_1, \ldots, V_k) \) is the set of elements of \( H \) which fix each \( V_i \). If \( V = V_1 \oplus \cdots \oplus V_k \), then the spaces \( V_i \) form a decomposition of \( V \), and we usually let \( \partial \) designate such a decomposition. We define the centralizer of \( \partial \) by

\[
C_H(\partial) = N_H(V_1, \ldots, V_k),
\]

and the normalizer or stabilizer of \( \partial \) by

\[
N_H(\partial) = N_H(V_1, \ldots, V_k),
\]

and if \( H \leq N_p(\partial) \) (or \( H \leq N_p(\partial) \)) then we write

\[
H^\partial = H/C_H(\partial).
\]

Thus \( H^\partial \) acts faithfully on the set \( \{V_1, \ldots, V_k\} \). If each \( V_i \) has dimension \( m \) for some fixed \( m \), then \( \partial \) is called an \( m \)-decomposition. If each \( V_i \) in an \( m \)-decomposition is t.s., then \( \partial \) is called an \( sm \)-decomposition (s \( \equiv \) singular). If, on the other hand, each \( V_i \) is an \( sm \)-space and the sum is also an orthogonal sum (i.e., \( V_i \) is orthogonal to \( V_j \) for all \( i \neq j \)), then \( \partial \) is called an \( \epsilon m \)-decomposition. If \( U \) and \( W \) are subspaces of \( V \), then the expression \( U \perp W \) indicates that \( U \cap W = 0 \) and \( U \) is orthogonal to \( W \). Thus an \( \epsilon m \)-decomposition may be written

\[
V = V_1 \perp \cdots \perp V_k,
\]

where \( km = 8 \). Note that if \( \partial \) is an \( \epsilon 1 \)-decomposition, then \( q \) is odd.

**Remark.** (i) If \( X \) is a group, then \( O(X) \) usually refers to the largest normal odd-order subgroup of \( X \). However, in this paper the symbol \( O(X) \) appears only when \( X \) is a n.d. subspace of \( V \). Thus \( O(X) \) is always a subgroup of \( O = O(V, Q, F) \) which acts faithfully on \( X \).

(ii) The terms "\( m \)-space" and "\( +m \)-space" have different meanings. When a subspace \( W \) is called an \( m \)-space, then no assumptions about non-degeneracy or nonsingularity are to be made about \( W \). However, when \( W \) is called a \( +m \)-space, then \( W \) is a n.d. subspace (or a nonsingular 1-space in even characteristic) according to the definitions above.

(iii) As in (ii), there is an analogous distinction between "\( m \)-decomposition" and "\( +m \)-decomposition."

In the event that \( W \leq V \) is a \( +m \)-space, where \( m = 2h \) is even, \( W \) contains a standard basis

\[
\beta = (e_1, \ldots, e_h, f_1, \ldots, f_h).
\]
where each $e_i$ and $f_j$ is a t.s. vector and $(e_i, f_j) = \delta_{ij}$. The proof of the following lemma is left to the reader.

**Lemma 1.2.2.** Suppose that $g \in GL(W)$ fixes both $\langle e_1, ..., e_n \rangle$ and $\langle f_1, ..., f_n \rangle$. Then $g \in O(W)$ if and only if

$$g = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \quad (a \in GL_n(q))$$

with respect to $\beta$, where $t$ denotes transpose. Further, $g \in \Omega(W)$ if and only if $\det(a)$ is a square.

Now suppose that $(e_1, ..., e_4, f_1, ..., f_4)$ is a standard basis of $V$. Observe that the map $d_2$ defined by

$$e_i \mapsto \lambda e_i, \quad f_i \mapsto f_i, \quad (1 \leq i \leq 4)$$  \hfill (1a)

multiplies $Q$ by $\lambda$. Hence if $\lambda$ is a nonsquare, then $d_2 \in A \setminus OZ$. Some of the elements in $O \setminus Q$ are the reflections: if $v \in V$ is nonsingular, then the reflection in $v$ is the element $r_v \in O \setminus Q$ given by

$$r_v(x) = x - ((v, v) / Q(v)) v,$$

for all $x \in V$. When $q$ is odd, elements of $SO \setminus Q$ are those which have spinor norm a nonsquare (see [1, p. 193ff]). When $q$ is even, elements of $SO \setminus O$ are those which interchange two families of maximal t.s. subspaces of $V$ (see [20] and Section 1.6).

For any $X \subseteq I'$, let $X$ be the image of $X$ in $P'$. If $Y \subseteq PO$ let $Y$ be the full preimage of $Y$ in $O$, and if $Y \subseteq P \Delta$ but $Y \subseteq PO$, let $Y$ be the full preimage of $Y$ in $A$. If $y \in P \Delta$, let $y \in \Delta$ be a preimage of $y$, and if $y \in PO$, choose $y$ to lie in $O$. Moreover, if $y \in O$ and $|y|$ is odd, then choose $y$ so that $|y| = |y|$. The letter $\beta$ usually denotes an ordered basis of $V$, and $\text{diag}_\rho(a_1, ..., a_8)$ denotes the corresponding diagonal matrix with respect to $\beta$.

If $H$ is any group and $\rho$ a set of primes, then $O_\rho(H)$ is the largest normal $\rho$-subgroup of $H$, and $O_\rho(H)$ is the subgroup of $H$ generated by all $\rho'$-elements of $H$. Also $\text{soc}(H)$ is the socle of $H$, the group generated by all minimal normal subgroups of $H$. If $H$ is an $r$-group for some prime $r$, then $\Omega_1(H) = \langle h \in H : h^r = 1 \rangle$. If $H$ is a subgroup of $K$ and $k \in K$, then $C_H^K(k) = C_H^K(\langle k \rangle) = \langle h \in H : k^h = k \text{ or } k^h = k^{-1} \rangle$. If $H \leq A$, then $\text{Hom}_A(V) = \{ g \in \text{End}_A(V) : \text{gh} = gh \text{ for all } h \in H \}$. Thus if $H$ is irreducible on $V$, then $\text{Hom}_A(V)$ is a field extension of $F$ by Schur's Lemma. We conclude this section with an easy yet useful result.
LEMMA 1.2.3. Suppose that $H \leq G_0$ and $O^2(H) = H$. If $c \in C_{Pa}(H)$ then $\hat{c} \in C_{\hat{a}}(\hat{H})$.

Proof. Clearly we may assume that $q$ is odd. Take $h \in H$ with $|h|$ odd, so that $\hat{h}^c = \pm \hat{h}$. By our convention, $|\hat{h}| = |h|$ and so $|\hat{-h}|$ is even. Therefore $\hat{h}^c = \hat{h}$ and the result now follows because $H$ is generated by elements of odd order. \qed

1.3. Some Terminology and Lemmas

(Some of the material here is based on [41].) In this section, $K$ denotes an arbitrary (finite) group, $H$ a normal subgroup of $K$, $T$ a maximal subgroup of $K$ not containing $H$ and $T_0 = T \cap H$.

**Definition.** If $L, J \subseteq K$ then $L, J = \bigcap_{\sigma \in J} L^\sigma$, the largest subgroup of $L$ which is normalized by $J$.

**Lemma 1.3.1.** Assume that $H$ is non-abelian and simple and let $L$ satisfy $T < L \leq H$. Then

(i) $1 \neq T_0 = L_T$;

(ii) $T/T_0 \cong K/H$;

(iii) if $1 < J \leq T_0$ and $J \leq T$, then $T_0 = N_H(J)$;

(iv) if $O_r(T_0)$ is a nontrivial Sylow $r$-subgroup of $T_0$ for some prime $r$, then $T_0$ is a Sylow $r$-normalizer in $H$.

(v) $O_r(T_0) \neq 1$ for some prime $r$, then $O_r(L) \leq O_r(T_0)$.

Proof. Lemma 2.1 of [41] shows that $T_0 \neq 1$, hence $L_T \neq 1$ as $T_0 \leq L_T$. Clearly $T \leq TL_T \leq K$, and so $TL_T$ equals $T$ or $K$. However, $1 \neq L_T \cong TL_T$, hence $TL_T \neq K$ by the simplicity of $H$. Hence $T \cong TL_T$ and $T_0 = H \cap TL_T = (H \cap T) L_T = L_T$, proving (i). Assertion (ii) is obvious and (iii) is an immediate consequence of the simplicity of $H$ and the maximality of $T$. As for (iv), we see that $O_r(T_0) \in \text{Syl}_r(H)$ by (iii) and the fact that every proper subgroup of an $r$-group is properly contained in its normalizer. Thus by (iii), $T_0 = N_H(O_r(T_0))$ is a Sylow $r$-normalizer in $H$. To prove (v) put $R = O_r(T_0)$ and $S = O_r(L)$, so that $S \cap T_0 \leq R$. By (iii), $T_0 = N_H(R)$, hence

$N_{RS}(R) = RS \cap T_0 = R(S \cap T_0) \neq R$.

Thus $RS = R$, as desired. \qed

**Terminology.** The group $T$ is called an $H$-novelty if $T_0$ is nonmaximal in $H$. If $L$ is any subgroup of $H$, then we say $L$ extends from $H$ to $K$ if $HN_K(L) = K$. If $L$ is self-normalizing and nonmaximal in $H$, yet $N_K(L)$ is
maximal in $K$, then we say $L$ extends to an $H$-novelty in $K$. Also define $[L]_H$ to be the $H$-class of groups containing $L$, that is,

$$[L]_H = \{L^h : h \in H\}.$$

When using this terminology we often omit reference to the group $H$, in which case it is understood that $H = G_0$. Thus "novelty," "extends to $K"," and "[L]_H" are short for "$G_0$-novelty," "extends from $G_0$ to $K"," and "[L]_{G_0}." We make use of this next result implicitly throughout the paper.

**Lemma 1.3.2.** Assume that $L \leq H$.

(i) If $H \leq J \leq K$ and $L$ extends from $H$ to $K$, then $L$ extends from $H$ to $J$.

(ii) Assume that $H \leq J_i \leq K$ for $i = 1, 2$, and that $L$ extends from $H$ to $J_i$, for $i = 1, 2$. Then $L$ extends from $H$ to $\langle J_1, J_2 \rangle$.

(iii) If $L$ is maximal and self-normalizing in $H$, and $L$ extends from $H$ to $K$, then $N_K(L)$ is maximal in $K$.

(iv) The $K$-class $[L]_K$ splits into $|K:N_K(L)|$ classes in $H$.

(v) If $L$ does not extend to $K$, then $N_K(L)$ is nonmaximal in $K$.

The following lemma helps to show that certain subgroups of $G_0$ cannot extend to a novelty in any $G$ (see, e.g., 1.6.1).

**Lemma 1.3.3.** Assume that $1 < L \leq J < H$, and $H$ is non-abelian and simple. Also suppose that $J$ extends from $H$ to $K$ and $L$ extends from $J$ to $N_K(J)$.

(i) If $N_H(L) = L$, then $N_K(L) \leq N_K(J)$.

(ii) If $T_0 = L$, then $L = J$.

**Proof:** (i) Define $P = N_K(J)$ and $N = N_P(L)$. By assumption, $PH = K$ and $NJ = P$. Therefore $NH = K$, which means $N_K(L) H = K$. Because $I$ is self-normalizing in $H$, we also have $N \cap H = N_K(L) \cap H = L$. Therefore $|N| = |K| |L| / |H| = |N_K(L)|$, whence $N_K(L) = N \leq P$, as required.

(ii) By 1.3.1(iii) $L$ is self-normalizing in $H$. Thus by (i) and the maximality of $T$ we conclude $T = N_K(J)$. Thus $J \geq L = T_0 = N_H(J) \geq J$, and (ii) follows. 

1.4. **The Groups $A = \text{Aut}(G_0)$, $\text{Out}(G_0)$ and $\Sigma$**

Let $e_1, \ldots, e_4, f_1, \ldots, f_4$ be a standard basis of $V$ (see Sect. 1.2) and consider the map $\phi \in \Gamma L(V)$ given by

$$\left( \sum_{i=1}^{4} \alpha_i e_i + \beta_i f_i \right) ^{\phi} = \sum_{i=1}^{4} \alpha_i^{\phi} e_i + \beta_i^{\phi} f_i,$$
where \( x, y \in F \) and \( \langle \sigma \rangle = \text{Aut}(F) \). Clearly \( Q(v^\sigma) = Q(v)^\sigma \) for all \( v \in V \), hence \( \Gamma = A : \langle \phi \rangle \). Thus \( P\Gamma = PA : \Phi \), where \( \Phi = \langle \phi \rangle \). We claim that

\[
[PA, \Phi] \leq G_0.
\]  

(1b)

When \( q \) is even, \( PA = PO = PSO \cong G_0,2 \); hence \( PA/G_0 \) is a normal subgroup of order 2 in \( P\Gamma /G_0 \) so (1b) holds. Now assume that \( q \) is odd and observe \( (r_v)^\phi = r_v^\phi \) for each nonsingular \( v \in V \). Thus the spinor norm of \( r_v(r_v)^\phi \) is \( (v, v)(v^\phi, v^\phi) = (v, v)(v, v)^\phi \in (F^*)^2 \), and so \( [r_v, \phi] \in \Omega \). Since the reflections generate \( O \), it follows that \( [O, \phi] \leq \Omega \) and so \( [PO, \Phi] \leq G_0 \).

Also \( (d_j)^\phi = d_j^\lambda \), where \( d_j \) is as in (1a), hence \( [d_j, \phi] = d_j^\lambda \). Now \( \lambda^\phi = \mu^2 \) for some \( \mu \in F^* \), and so \( [d_j, \phi] = \mu g \) where \( g \) acts as the scalar \( \mu \) on \( \langle e_1, \ldots, e_4 \rangle \) and the scalar \( \mu^{-1} \) on \( \langle f_1, \ldots, f_4 \rangle \). By 1.2.2, \( g \in \Omega \) and hence \( [A, \phi] \leq \Omega Z \). This proves (1b).

The group \( \Phi \) consists of field automorphisms of \( G_0 \) in the sense of [7, p. 200], and so by [7, Theorem 12.2.3], \( \Phi \) centralizes a triality automorphism. It follows from (1b) that

\[
[A, \Phi] \leq G_0.
\]

(1c)

In particular \( G_0 \Phi \leq A \).

Define \( D \) to be the group of inner and diagonal automorphisms of \( G_0 \). Thus \( D \leq PA \) and \( D \leq A \) and

\[
D/G_0 \cong \begin{cases} 1 & \text{if } q \text{ is even} \\ 2^2 & \text{if } q \text{ is odd}. \end{cases}
\]

It is well known that \( A/D \cong S_3 \times \mathbb{Z}_n \), and when \( q \) is odd the \( S_3 \) acts faithfully on \( D/G_0 \cong 2^2 \). Consequently \( D \leq A' \) and

\[
A'/D \cong \mathbb{Z}_3 \quad \quad \quad \quad \text{and} \quad \quad \quad \quad A'/G_0 \cong \begin{cases} \mathbb{Z}_3 & \text{if } q \text{ is even} \\ A_4 & \text{if } q \text{ is odd}. \end{cases}
\]

(1d)

We also define

\[
\theta = A'PA.
\]

(1e)

Thus

\[
\theta/G_0 \cong \begin{cases} S_3 & \text{if } q \text{ is even} \\ S_4 & \text{if } q \text{ is odd}. \end{cases}
\]

(1f)

Evidently \( A = \theta : \Phi \) and

\[
\text{Out}(G_0) = A/G_0 = \theta/G_0 \times \Phi G_0/G_0 \cong \begin{cases} S_3 \times \mathbb{Z}_n & \text{if } q \text{ is even} \\ S_4 \times \mathbb{Z}_n & \text{if } q \text{ is odd}. \end{cases}
\]
It is convenient to define $B$ as the kernel of the action of $A$ on the Dynkin diagram of $G_0$. Thus

$$A/B \cong S_3 \quad \text{and} \quad B = D\Phi.$$  \hfill (1g)

By (1f) there exists a homomorphism $\pi$ from $\Theta$ to $\Sigma$ with kernel $G_0$, where $\Sigma$ is $S_3$ or $S_4$ according as $q$ is even or odd. When $q$ is odd, $\pi(D)$ is the normal 4-group $V_4$ in $\Sigma \cong S_4$, whence

$$\pi(D) = \begin{cases} 1 & \text{if } q \text{ is even} \\ V_4 = \langle(12)(34), (13)(24)\rangle & \text{if } q \text{ is odd}. \end{cases} \hfill (1h)$$

Now let $r \in O$ be a reflection in a vector whose norm is a square in $F^*$. Then $r \in PO/D$ and so $\pi(r)$ is a 2-cycle in $\Sigma$. Without loss we put

$$\pi(r) = (12),$$

and $PA = D \langle r \rangle$ we obtain

$$\pi(PA) = \begin{cases} \langle(12)\rangle & \text{if } q \text{ is even} \\ D_8 = \langle(12), (13)(24)\rangle & \text{if } q \text{ is odd}. \end{cases} \hfill (1i)$$

Assume that $q$ is odd and let $s \in O$ be a reflection in a vector with norm $\lambda$, a nonsquare in $F$. Then $\pi(s)$ (like $\pi(r)$) is a 2-cycle and as $r \hat{s} \in PSO\setminus G_0$, we have $\pi(r) \neq \pi(s)$. However, $[r, s] \in \Omega$, hence $[\pi(r), \pi(s)] = 1$ and so

$$\pi(s) = (34).$$

Thus

$$\pi(PSO) = \begin{cases} \pi(O) = \pi(PA) = \langle(12)\rangle & \text{if } q \text{ is even} \\ \langle(12)(34)\rangle & \text{if } q \text{ is odd}. \end{cases} \hfill (1k)$$

and

$$\pi(PO) = \begin{cases} \langle(12)\rangle & \text{if } q \text{ is even} \\ \pi(G_0\langle r, \hat{s} \rangle) = \langle(12), (34)\rangle & \text{if } q \text{ is odd}. \end{cases} \hfill (1l)$$

We now enlarge the domain of $\pi$ from $\theta$ to $A$ by enlarging the kernel from $G_0$ to $G_0\Phi$. Thus for $g \in \Theta$ and $\psi \in \Phi$, we have $\pi(g\psi) = \pi(g)$. For any subgroup $\Pi \leq \Sigma$, define $G_\Pi = \pi^{-1}(\Pi) \leq A$. If $\Pi = \langle a \rangle$, then write $G_a = G_\Pi$. Thus

$$G_1 = \ker(\pi), \quad A/G_1 \cong \theta/G_0 \cong \Sigma,$$  \hfill (1m)
and the following holds:

<table>
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<td>$G_1 \langle \tilde{r} \rangle$</td>
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<td>$G_{&lt; (12), (34)&gt;}$</td>
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<td>$G_{\Sigma}$</td>
<td>$A$</td>
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</table>

Also observe that the set of triality automorphisms $\mathcal{T}$ in $A$ satisfies

$$\mathcal{T} = \{ a \in A : \pi(a) \text{ is a 3-cycle} \}.$$  \hspace{1cm} (1n)

It is often useful to exploit the structure of $\text{Out}(G_0)$ to obtain information about the subgroups of $G_0$. We do so with the help of these next few results. The first appears in [14, Theorem 9.1].

**Proposition 1.4.1.** Suppose that $\tau \in \mathcal{T}$ has order 3 and put $C = C_{G_0}(\tau)$. Then one of the following occurs:

(i) $C \cong 3D_4(q_1)$ where $q = q_1^3$;

(ii) $C \cong G_2(q)$;

(iii) $p = 3$ and $C \cong [q^5].\text{SL}_2(q)$;

(iv) $q \equiv \pm 1 \mod 3$ and $C \cong P\text{GL}_3(q)$, where $\varepsilon = \pm$.

Conversely, each of these groups do in fact occur as centralizers in $G_0$ of triality automorphisms of order 3.

**Lemma 1.4.2.** Let $H$ be a subgroup of $G$ not containing $G_0$, and assume that $H_0 = H \cap G_0$ has an $H$-invariant subgroup $N$ such that $C_D(N) = 1$.

(i) If $c \in C_A(N)$ then $N_{G_0}(N) \leq C_{G_0}(c)$, and if $c \in \mathcal{T}$, then $|N_{P_A}(N) : N_{G_0}(N)| \leq 2$.

(ii) If $H$ is maximal in $G$, then at least one of the following holds:

(a) $H_0$ appears in 1.4.1 (but not 1.4.1(iii));

(b) $H_0 = C_{G_0}(x)$ for all $x \in C_A(N) \setminus 1$.

**Proof.** (i) Clearly $[N_{G_0}(N), c] \leq C_{G_0}(N) = 1$, hence $N_{G_0}(N) \leq C_{G_0}(c)$. Now suppose that $c \in \mathcal{T}$ and take $g \in N_{P_A}(N)$. If $\pi(g)$ does not normalize $\langle \pi(c) \rangle$, then $q$ is odd, $\Sigma \cong S_4$ and $[\pi(c), \pi(g), \pi(c)] \in V_4 \setminus 1$. Consequently $1 \neq [c, g, c] \in G_{\nu_4} \cap A' = D$, hence $[c, g, c] \in C_D(N) \setminus 1$, a contradiction.
Therefore \( \pi(g) \) normalizes \( \langle \pi(c) \rangle \). If \( [\pi(g), \pi(c)] = 1 \), then \( g \in \ker(\pi) \cap \mathcal{P} = G_0 \), and it now follows that \( |N_{\mathcal{P}_A}(N) : N_{G_0}(N)| \leq 2 \).

(ii) Since \( |\mathcal{A}' : D| = 3 \), the condition \( C_\mathcal{A}(N) = 1 \) ensures that \( |C_\mathcal{A}(N)| \mid 3 \). If \( |C_\mathcal{A}(N)| = 1 \), then \( [H, C_\mathcal{A}(N)] = 1 \). Thus for \( x \in C_\mathcal{A}(N) \setminus \{1\} \) the maximality of \( H \) yields \( H = C_\mathcal{A}(x) \). Consequently \( H_0 = C_{G_0}(x) \) and (b) occurs. Assume therefore that \( C_\mathcal{A}(N) = \langle \tau \rangle \) has order 3. Clearly \( H = N_G(\langle \tau \rangle) \), hence \( H_0 = C_{G_0}(\tau) \). Thus \( H_0 \) appears in 1.4.1, but not 1.4.1(iii) by [4] and 1.3.1(v). Therefore (a) holds. \( \square \)

**Lemma 1.4.3.** Assume that \( H \leq G_0 \) and that the \( \theta \)-class \( [H]_{\theta} \) splits into 4 classes in \( G_0 \). Suppose further that \( [H]_{\theta} = [H]_{\mathcal{A}} \). Then \( A / G_{1} \cong S_4 \) acts naturally on these 4 classes.

**Proof.** Let \( K \) be the kernel of the action of \( A \) on the 4 \( G_0 \)-classes. Clearly \( \theta/G_0 \) acts naturally as \( S_4 \) on the 4 classes, hence so does \( A/K \). By (1c), \( \Phi K / K \leq Z(A/K) \cong Z(S_4) = 1 \), hence \( G_1 = G_0 \Phi \leq K \). Since \( A / G_1 \cong \Sigma \cong A/K \), we have \( G_1 = K \), as desired. \( \square \)

1.5. The Results Matrix and Our Theorem

We now present the main result of this paper. We exhibit a collection \( \mathcal{G} \) of subgroups of \( G_0 \) such that \( M_0 = M \cap G_0 \) is \( G_0 \)-conjugate to some \( H \in \mathcal{G} \). In this case \( M = N_G(M_0) \) is \( G \)-conjugate to \( N_G(H) \) and \( M \cong H(G/G_0) \). Conversely, for a given \( H \in \mathcal{G} \) we determine precisely those groups \( G \) for which \( N_G(H) \) is maximal in \( G \). Thus for any \( G \), one can identify all classes of maximal subgroups of \( G \) not containing \( G_0 \).

It turns out that for a given \( H \in \mathcal{G} \), the maximality of \( H \) in \( G \) depends only on \( q \) and \( \pi(G) \). Thus we can express our results in a matrix—called the results matrix—whose rows are indexed by the groups \( H \in \mathcal{G} \) and whose columns are indexed by representatives of the conjugacy classes of subgroups of \( \Sigma \). If \( \Pi \leq \Sigma \) heads a column, then the \( (H, \Pi) \)-entry of the results matrix contains the values of \( q \) for which \( N_G(H) \) is a maximal subgroup of any group \( G \) with \( \pi(G) = \Pi \). The goal of this paper is to prove

**Theorem.** The results matrix holds.

The results matrix appears in Table I, and we now explain the notation used therein. Column I contains the name of the group \( H \in \mathcal{G} \) and column XV indicates where in the paper a discussion of the relevant group occurs. Column II usually gives the structure of \( H \in \mathcal{G} \). Sometimes, however, it is convenient to write the structure of the preimage \( \hat{H} \leq \Omega \); in these cases, the symbol “\(^\wedge\)” appears just before the structure is given. (Since \( A \) does not act on \( \Omega \), the structure of \( \hat{H} \) may be different from \( \hat{H}^a \) for
some \( a \in A \).) Two groups in \( \mathfrak{C} \) are separated by a horizontal line in the results matrix if and only if they are not \( A \)-conjugate.

Column III contains certain restrictions on \( q \), and the symbol * in the results matrix is an abbreviation for those values of \( q \) which appear in column III. The symbol \( \circ \) is an abbreviation for \( "q \text{ odd}" \); for example, \( \circ, q \geq 5 \) in row 19 and column X stands for \( "q \text{ odd and } q \geq 5." \)

We now describe the symbol \( \dagger \) which frequently appears in the results matrix. As we determine the groups in \( \mathfrak{C} \) in the course of this paper, we will also show that the action of \( A \) on \([\mathfrak{C}]=\{[H], H \in \mathfrak{C}\}\) contains \( G_1 = G_0 \Phi \) in its kernel. Thus \( \Sigma \) acts on \([\mathfrak{C}]\) via the homomorphism \( \pi \) defined in Section 1.4, and two groups \( H, K \in \mathfrak{C} \) are joined by the symbol \( \dagger \) in the \( n \)-th column if there exists an element of \( \Pi \) which takes \([H]\) to \([K]\). If this occurs, and if \( \pi(G) = \Pi \), then neither \( H \) nor \( K \) extends to \( G \), and hence \( N_G(K) \) and \( N_G(H) \) are nonmaximal in \( G \) (see 1.3.2(v)).

We remark that the subgroup \( S_3 \leq \Sigma \) which heads column VII is the subgroup \( \langle (123), (12) \rangle \). Thus \( S_3 = \Sigma \) when \( q \) is even, while \( |\Sigma: S_3| = 4 \) when \( q \) is odd.

Observe that we have compressed 12 classes in \( \mathfrak{C} \) to the single row 75. The symbol \( @ \) in column V of that row indicates that the 12 classes of \( A_{10} \) in \( G_0 \) are permuted transitively by \( A \) with stabilizer \( G_{(12)} \). Similarly, an \( (\dagger) \) appears in column VII of row 70 because the \( d^2 \) classes of \( PGL_3(q) \) in \( G_0 \) are permuted transitively by \( A \) with stabilizer \( G_{S_3} \). The same notation is used in rows 63, 71–74.

If \( G \cap \mathcal{F} = \phi \), then \( \pi(G) \) is a 2-group hence \( \pi(G) \) is \( \Sigma \)-conjugate to a subgroup of \( \pi(PFG) \). Thus \( G \) is \( A \)-conjugate to a subgroup of \( PFG \), hence we may divide our analysis into two cases: \( G \leq PFG \) and \( G \cap \mathcal{F} \neq \phi \). Accordingly, we obtain two collections, \( \mathfrak{C}_1 \) (2.4.1) and \( \mathfrak{C}_2 \) (4.2.1), whose union is \( \mathfrak{C} \), such that \( M_0 \) is \( G_0 \)-conjugate to some member of \( \mathfrak{C}_1 \) (resp. \( \mathfrak{C}_2 \)) if \( G \leq PFG \) (resp. \( G \cap \mathcal{F} \neq \phi \)). The case \( G \leq PFG \) is handled using the main theorem of [2] and the classification of finite simple groups, as described in Section 1.1. The case \( G \cap \mathcal{F} \neq \phi \) depends on results in Sections 2.2, 2.3 and Part 3, along with the fundamental ideas of 1.3.1.

1.6. Some Parabolic Subgroups of \( G_0 \) and Their Incidence

We fix a Borel subgroup \( B_0 \) of \( G_0 \) and consider the parabolic subgroups of \( G_0 \) containing \( B_0 \). The Dynkin diagram of \( G_0 \) is

```
    r_3
     |
  r_1 --- r_2
     |
    r_4
```

and \( P_{i,j,...} \) denotes the parabolic subgroup of \( G_0 \) corresponding to the set of
### Table I: The Results Matrix

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PETER B. KLEIDMAN
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<td></td>
</tr>
<tr>
<td>74 $K_3$, $1 \leq i \leq 8$</td>
<td>$g\langle q, \langle g \rangle \rangle$</td>
<td>$\Omega_4^- (q)$</td>
<td>$\Omega_4^+ (q)$</td>
<td>$\Omega_8^- (q)$</td>
<td>$\Omega_8^+ (q)$</td>
<td>$\Omega_{16}^- (q)$</td>
<td>$\Omega_{16}^+ (q)$</td>
<td>$\Omega_{32}^- (q)$</td>
<td>$\Omega_{32}^+ (q)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>75 $K_3$, $1 \leq i \leq 8$</td>
<td>$g\langle q, \langle g \rangle \rangle$</td>
<td>$\Omega_4^- (q)$</td>
<td>$\Omega_4^+ (q)$</td>
<td>$\Omega_8^- (q)$</td>
<td>$\Omega_8^+ (q)$</td>
<td>$\Omega_{16}^- (q)$</td>
<td>$\Omega_{16}^+ (q)$</td>
<td>$\Omega_{32}^- (q)$</td>
<td>$\Omega_{32}^+ (q)$</td>
<td></td>
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</tr>
</tbody>
</table>
nodes \{r_1, r_j, ...\}. The maximal parabolic subgroups of \(G_0\) have the following geometric interpretations:

\[
\begin{align*}
P_{1,2,3} &= N_{G_0}(S), \\
P_{1,2,4} &= N_{G_0}(T), \\
P_{2,3,4} &= N_{G_0}(U), \\
P_{1,3,4} &= N_{G_0}(W),
\end{align*}
\]

where \(S\) and \(T\) are t.s. 4-spaces (or t.s. solids), \(U\) is a t.s. point, \(W\) is a t.s. line, and \(U < W < S \cap T\). The group \(G_0\) is transitive on the sets \(\mathcal{P}\) of t.s. points and \(\mathcal{L}\) of t.s. lines and has just two orbits \(\mathcal{S}_1, \mathcal{S}_2\) of t.s. solids, with representatives \(S\) and \(T\). Two t.s. solids lie in the same orbit if and only if their intersection has even dimension. This fact ensures that each t.s. 3-space (or t.s. plane) lies in exactly two t.s. solids, one in each \(\mathcal{S}\). Thus the normalizer in \(G_0\) of a t.s. plane is the intersection of the normalizers of the two t.s. solids which contain it. Therefore

\[
P_{1,2} = P_{1,2,3} \cap P_{1,2,4} = N_{G_0}(S, T) = N_{G_0}(S \cap T).
\]

Furthermore \(G_0\) is transitive on t.s. planes.

Two parabolic subgroups are said to be incident if their intersection is again a parabolic subgroup. Also, two t.s. subspaces of \(V\) are incident if one contains the other, or if they are a pair of solids which intersect in a plane. It is easy to verify that two t.s. subspaces are incident if and only if their normalizers in \(G_0\) are incident.

If \(X \in \mathcal{P} \cup \mathcal{L} \cup \mathcal{S}_1 \cup \mathcal{S}_2\) and \(a \in A\), then \(X^a\) is defined by \(N_{G_0}(X^a) = N_{G_0}(X)^a\). In this way \(A\) acts on the set of t.s. points, lines and solids. (Of course this action agrees with the usual action of \(P\Gamma\).) Since \(A\) preserves incidence amongst the parabolics, \(A\) also preserves incidence in \(\mathcal{P} \cup \mathcal{L} \cup \mathcal{S}_1 \cup \mathcal{S}_2\). For example, let \(\tau \in \mathcal{S}\) induce the symmetry \(r_1 \mapsto r_4 \mapsto r_3\) on the Dynkin diagram. Then \(\mathcal{P}' = \mathcal{S}_1\) and \(\mathcal{S}' = \mathcal{S}_2\), and if \(X \in \mathcal{P}\), then \(X \leq X'\) if and only if \(\dim(X^\tau \cap X'^\tau) = 3\). We will make use of these remarks in the proof of 4.1.4.

The only proper parabolic subgroups which extend to \(A\) lie in \([P_{1,3,4}] \cup [P_2] \cup [B_0]\). The Borel subgroups (i.e., groups in \([B_0]\)) are the stabilizers of flags. A flag is a sequence of four subspaces \((V_1, \ldots, V_4)\) of \(V\) such that \(V_i < V_{i+1}\) for \(i \leq 3\), and \(V_i\) is a t.s. \(i\)-space. For example, \((U, W, S \cap T, S)\) is a flag. Observe that \(P_2 = P_{1,2,3} \cap P_{1,2,4} \cap P_{2,3,4} = N_{G_0}(U, S, T)\). Conversely, if \((U', S', T')\) is any triple of subspaces of \(V\) such that \(U' \in \mathcal{P}, S' \in \mathcal{S}_1, T' \in \mathcal{S}_2,\) and \(S' \cap T'\) is a plane containing \(U'\), then \(N_{G_0}(U', S', T') \in [P_2]\). Although \(P_2\) is nonmaximal in \(G_0\), \(P_2\) extends to a novelty in groups \(G \leq A\) which contain a triality automorphism (see 4.2.2). However, 1.3.3 shows that \(B_0\) never extends to a novelty.
1.6.1. Lemma. $M_0$ is not a Borel subgroup of $G_0$.

Proof. Since $B_0$ is a Sylow $p$-normalizer of $P_2$, the Frattini argument shows that $B_0$ extends from $P_2$ to $N_{G}(P_2)$ (recall $G_0 \leq G \leq A$). But $P_2$ extends to $A$, and hence to $G$ (1.3.2(i)). Thus $N_{G}(B_0) < N_{G}(P_2)$ by 1.3.3(i). \(\blacksquare\)

1.7. Counting Classes

In some of the discussions below, we will need to determine the number of conjugacy classes of absolutely irreducible subgroups of $G_0$ with a given structure. Here are some remarks about counting such classes.

1.7.1. Lemma. Let $H \leq \Omega$ be absolutely irreducible.

(i) $Q$ and its scalar multiples are the only quadratic forms fixed by $H$. Hence $N_{\Gamma(V)}(H) \leq 1$.

(ii) If $H$ is perfect, then the number of conjugacy classes of absolutely irreducible copies of $H$ in $\Delta$ is at most the number of such classes in $GL(V)$.

Proof. (i) Let $P$ be a nonzero quadratic form preserved by $H$. Let $F$ be the matrix of the bilinear form $(,)$ with respect to some basis of $V$, and let $E$ be the corresponding matrix for $P$ (with respect to the same basis). Thus

$$h(FE^{-1}) = F \quad \text{and} \quad hEh^{-1} = E$$

for all $h \in H$. Therefore,

$$h(Fe^{-1})h^{-1} = hFh^{-1}(h^{-1})'E^{-1}h^{-1} = FE^{-1},$$

and as $H$ is absolutely irreducible, $E = \lambda F$ for some $\lambda \in \mathbb{F}^\times$. Thus $P$ and $\lambda Q$ have the same associated bilinear form. The equality $P = \lambda Q$ is immediate for odd $q$ and follows from 4.9 of [2] for even $q$. The second assertion in (i) is clear, since $\Gamma = N_{\Gamma(V)}(\Omega)$.

(ii) Assume that $H^x \leq \Delta$ for some $x \in GL(V)$. Then $H \leq \Delta^{x^{-1}}$, and since $H$ is perfect, $H \leq \Omega^{x^{-1}}$. Therefore $H$ stabilizes the quadratic form $P(v) = Q(v^x)$, and by (i), $P = \lambda Q$ for some $\lambda \in \mathbb{F}^\times$. Therefore $x \in \Delta$ and (ii) holds. \(\blacksquare\)

Let $\rho, \sigma : H \to GL(V)$ be representations of a group $H$. We say that $\rho$ and $\sigma$ are quasiequivalent if there exists $\alpha \in \text{Aut}(H)$ such that $\sigma$ is equivalent to $\alpha \rho$. It is trivial to prove

1.7.2. Lemma. If $\rho$ and $\sigma$ are quasiequivalent, then $H^\rho$ and $H^\sigma$ are conjugate in $GL(V)$. In particular, if all the irreducible faithful representations of $H$ in $GL(V)$ are quasiequivalent, then there is a unique class of irreducible copies of $H$ in $GL(V)$. 

MAXIMAL SUBGROUPS OF $PO_8^+(q)$
PART 2. THE CASE $G \leq P\Gamma$

Throughout Part 2 (except for Section 2.3) $G$ satisfies

$$G_0 \leq G \leq P\Gamma.$$  

2.1. The Classical Subgroups

Aschbacher [2, pp. 472ff] describes eight collections $C_1, \ldots, C_8$ of subgroups of $G$, such that $M$ is either a member of one of these collections or $\text{soc}(M)$ is a non-abelian simple group. A subgroup of $G$ is called a classical subgroup if it is a member of one of these eight collections. We use these letters $R, I, F, T, S,$ and $E$ to denote members of $C_1$ (the reducible groups), $C_2$ (the imprimitive groups), $C_3$ (the normalizers of field extensions of $F$), $C_4$ or $C_7$ (the stabilizers of tensor product decompositions of $V$), $C_5$ (the stabilizers of subfields of $F$), and $C_6$ (whose preimages in $\Gamma$ normalize extraspecial groups), respectively. The collection $C_8$ is void for the orthogonal groups. (We warn the reader that the letters $R, I, F,$ etc., may be used in contexts other than the ones mentioned here.) The groups with simple socle satisfying the description on p. 469 of [2] will be called $C_9$-groups, since they comprise, in effect, Aschbacher's ninth collection.

Result 15.1 of [2] indicates how a triality automorphism of $G_0$ acts on various subgroups of $G_0$. (Regarding 15.1.13 of [2], however, we remark that a certain $P\Delta$-orbit of $S$-groups is not an $A$-orbit—see 2.2.10.) In particular, the proofs of 15.1.12 and 15.1.14 of [2] show that $T$-groups, other than normalizers of $Sp_2(q) \otimes Sp_4(q)$ in odd characteristic, are contained in members of $C_2, C_3$, or a suitable $C_9$-group. Thus "$T$-group" refers only to a normalizer of $Sp_2(q) \otimes Sp_4(q)$ with $q$ odd. In the table below, we explain our notation for the classical subgroups.

<table>
<thead>
<tr>
<th>Name</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{sm}$</td>
<td>stabilizer of a t.s. $m$-space</td>
</tr>
<tr>
<td>$R_{em}$</td>
<td>stabilizer of an $em$-space</td>
</tr>
<tr>
<td>$I_{sm}$</td>
<td>stabilizer of an $sm$-decomposition</td>
</tr>
<tr>
<td>$I_{sa}$</td>
<td>stabilizer of an $sa$-decomposition</td>
</tr>
<tr>
<td>$F_1$</td>
<td>preimage in $\Omega$ is normalizer of an irreducible $\Omega^+_4(q^2)$</td>
</tr>
<tr>
<td>$F_2$</td>
<td>preimage in $\Omega$ is normalizer of an irreducible $SU_4(q)$</td>
</tr>
<tr>
<td>$T$</td>
<td>preimage in $\Omega$ is normalizer of $Sp_2(q) \otimes Sp_4(q), q$ odd</td>
</tr>
<tr>
<td>$S_a$</td>
<td>normalizer of subfield group $P\Omega^+_6(q_0), q = q_0^p, q$ prime</td>
</tr>
<tr>
<td>$S_s$</td>
<td>normalizer of subfield group $\Omega^{+6}((q^2)), q$ a square</td>
</tr>
<tr>
<td>$E$</td>
<td>preimage in $\Omega$ is normalizer of $2^{1+6}, q = p \geq 3$</td>
</tr>
</tbody>
</table>
Remarks. (i) The letter $s$ occurs in the names $R_{s4}$, $I_{s4}$, as a mnemonic for totally singular.

(ii) The groups in the table are subgroups of $G_0$. The classical subgroups of $G$ are the normalizers in $G$ of the classical subgroups of $G_0$. For example, if $\hat{\sigma}$ is a $+2$-decomposition of $V$, then $N_{\sigma}(\hat{\sigma}) = N_{\sigma}(N_{G_0}(\hat{\sigma}))$.

2.2. The $A$-Conjugates of the Classical Subgroups

Here we determine the number of classes in $G_0$ of a given type of classical subgroup (e.g., the number of classes of $T$-groups), and determine where these classes are sent under the action of $A$. Recall (Sect. 1.3) that if $H \subseteq G_0$, then $[H]$ denotes the $G_0$-class containing $H$, also define 

$$[H]^A = \{[H^a] : a \in A\}.$$ 

If there is more than one $G_0$-class of a given type of classical subgroup or $C_s$-group, then we add superscripts to distinguish the classes. For example, there are two classes of $R_{s4}$-groups, corresponding to the two families $\mathcal{Y}_i$, $\mathcal{S}_2$ of t.s. solids (see Sect. 1.6). Thus we write $R_{s4}^1$ and $R_{s4}^2$ to denote representatives of the two classes.

Let $C$ be a classical subgroup of $G_0$ and suppose that $C$ extends to $G_1 = G_0 \Phi$. Thus $[C]^g = [C]$ for all $g \in G_1$, and if $a \in A$ then $C^a$ extends to $G_1^a = G_1$ (see (1c)). Therefore $A/G_1$ acts transitively on $[C]^A$ and so $\Sigma$ acts transitively on $[C]^A$ via the homomorphism $\pi$ given in Section 1.4. We thus describe the action of $A$ on $[C]^A$ in terms of the action of $\Sigma$. For example, let $C$ be an $R_{s4}$-group, (i.e., the stabilizer in $G_0$ of a t.s. point). As $G_0$ is transitive on $\mathcal{P}$ (the set of t.s. points), there is a unique class of $R_{s4}$-groups in $G_0$. And because $G_1$ acts on $\mathcal{P}$, it follows that $C$ extends to $G_1$. Thus $\Sigma$ acts transitively on $[C]^A$. Now $P \Gamma$ acts on $\mathcal{P}$, so $C$ extends to $P \Gamma$. Therefore $\pi(P \Gamma) \subseteq N_\Sigma([C])$, which means $N_\Sigma([C])$ equals $\pi(P \Gamma)$ or $\Sigma$. However, if $\tau \in \mathcal{T}$, then $C^\tau$ is an $R_{s4}$-group and so $[C] \neq [C^\tau]$. Therefore $N_\Sigma([C]) - \pi(P \Gamma)$, which equals $\langle (12) \rangle$ or $D_8$ according as $q$ is even or odd (see (1j)). Thus there are $|\Sigma : N_\Sigma([C])| = 3$ classes in $[C]^A$ and it is clear from Section 1.6 that $[C]^A = \{[R_{s1}], [R_{s4}^1], [R_{s4}^2]\}$. This shows

**Proposition 2.2.1.** There is a unique class of $R_{s1}$-groups in $G_0$ and just 2 classes of $R_{s4}$-groups. Furthermore, we have

\[
\begin{array}{ccc}
(12) & (123) & (13)(24) \\
R_{s1} & \times & \\
R_{s4}^1 & \times & \\
R_{s4}^2 & \times & \\
\end{array}
\]
If \( \alpha \) is one of the elements of \( \Sigma \) in the top row of the table above, and \( R \) is one of the groups on the left, then an "x" appears in the \((R, \alpha)\)-entry if and only if \( \alpha \) fixes \([R]\). This occurs if and only if \( R \) extends to \( G_\alpha = \pi^{-1}(\langle \alpha \rangle) \); that is, if and only if \( G_\alpha \leq N_A(R)G_0 \). As in the results matrix, two groups \( R, R^* \) on the left are joined by "\( \cdot \)" in the \( \alpha \)-th column if and only if \( \langle \alpha \rangle \) takes \([R]\) to \([R^*]\). The last column is to be ignored when \( q \) is even. We use diagrams of this sort in most of the propositions in this section. These next two results follow directly from Section 1.6.

**Proposition 2.2.2.** There is a unique class of \( R_s \)-groups in \( G_0 \) and \( R_{s_2} \) extends to \( A \).

**Proposition 2.2.3.** There is a unique class of \( R_3 \)-groups in \( G_0 \) and using the notation of Section 1.6 we have

\[
\begin{array}{ccc}
(12) & (123) & (13)(24) \\
R_{s_3} = P_{1,2} & \times & \\
P_{\gamma,1} & \bullet & \bullet \\
P_{2,4} & \bullet & \times
\end{array}
\]

**Proposition 2.2.4.** There are just \( 2d \) classes of \( C_q \)-groups in \( G_0 \) with socle \( \Omega_7(q) \). Let \( K_i, 1 \leq i \leq 2d \) be representatives of these classes. There are just \( d \) classes of \( R_{s_1} \)-groups. We have \( K_i \cong R_{s_1} \cong \Omega_7(q) \) and

\[
\begin{array}{ccc}
(12) & (123) & (13)(24) \\
R_{s_1} & \times & \\
K_1^1 & \bullet & \bullet & \bullet \\
K_1^2 & \bullet & \bullet & \times \\
R_{s_1} (q \text{ odd}) & \times & \\
K_1^3 (q \text{ odd}) & \bullet & \bullet & \\
K_1^4 (q \text{ odd}) & \bullet & \bullet & \times
\end{array}
\]
Remark. When $q$ is even, $O_7(q) = \Omega_7(q) = Sp_6(q)$. Also, recall $d = (2, q - 1)$.

Proof. It is known that $G_0$ is transitive on any full set of isometric 1-spaces, thus $G_0$ has just $d$ classes of $R_{+1}$-groups. Clearly $G_1 = G_0 \Phi$ acts on the set of $+1$-spaces in $V$, hence $R_{+1}$ extends to $G_1$. Therefore, as described above, $\Sigma$ acts transitively on $[R_{+1}]^4$. Since $PO$ acts on the $+1$-spaces, $R_{+1}$ extends to $PO$. Therefore $(12) \in N_\Sigma([R_{+1}])$ by (11), and if $q$ is odd then also $(34) \in N_\Sigma([R_{+1}])$. When $q$ is odd, $P\Delta$ fuses the two classes $[R_{+1}]$, $[R_{-1}]$, and so $(13)(24) \notin N_\Sigma([R_{+1}])$. Further, it is well known that if $\sigma \in \mathcal{F}$ then $R_{+1}^\sigma$ realizes an irreducible spin representation of $\Omega_7(q)$ in $G_0$, hence $N_\Sigma([R_{+1}])$ does not contain a 3-cycle (see (1n)). We conclude that $N_\Sigma([R_{+1}]) = \pi(PO)$. Thus there are just $|\Sigma: \pi(PO)| = 3d$ classes in $[R_{+1}]^4$. Clearly $2d$ of these classes are comprised of $C_9$-groups, so it remains to show that there are no other $C_9$-groups with socle $\Omega_7(q)$.

By Theorem 1.1 of [27], all (absolutely) irreducible representations of $B_3(q)$ in $GL(V)$ are quasiequivalent, where $B_3(q)$ is the full covering group of $\Omega_7(q)$. Hence by 1.7.2 and 1.7.1(ii), $P\Delta$ has a unique class of absolutely irreducible $\Omega_7(q)$. Thus any such copy of $\Omega_7(q)$ in $G_0$ must be $A$-conjugate to $R_{+1}$, and the proof is complete. 

Remark. The term $K_i$-group or the symbol $K_i$ refers to an arbitrary member of $[K_i]$ for some $i$. Similar terminology holds for $K_2, ..., K_8$, below.

**Proposition 2.2.5.** For odd $q$, there are just 2 classes of $R_{+3}$-groups and just 4 classes of $T$-groups. We have

\[
\begin{array}{ccc}
(12) & (123) & (13)(24) \\
R_{+3} & \times & \\
T^1 & & \\
T^2 & & \times \\
R_{-3} & \times & \\
T^3 & & \\
T^4 & & \times \\
\end{array}
\]

Proof. The group $G_0$ is transitive on $+3$-spaces and on $-3$-spaces,
hence there are just 2 classes of $R_0$-groups in $G_0$, namely $[R_{+3}]$ and $[R_{-3}]$. Since $PO.\Phi$ acts on the $+3$-spaces, $R_{+3}$ extends to $PO.\Phi$. In particular, $R_{+3}$ extends to $G_1$, hence $\Sigma$ acts transitively on $[R_{+3}]^A$ and $\langle(12), (34)\rangle \leq N_2([R_{+3}])$. Now $P\Delta$ fuses the two classes $[R_{+3}]$ and $[R_{-3}]$, hence $(13)(24) \notin N_2([R_{+3}])$. Therefore $N_2([R_{+3}]) = \langle(12), (34)\rangle$, so there are $|\Sigma : N_2([R_{+3}])| = 6$ classes in $[R_{+3}]^A$. By 15.1.6 of [2], the 4 classes in $[R_{+3}]^A$ other than $[R_{+3}]$ and $[R_{-3}]$ are classes of $T$-groups; we write $T^1, \ldots, T^4$ for representatives of these classes. Clearly $[T^1] \cup \cdots \cup [T^4]$ is a $P\Delta$-orbit of $T$-groups, and by Theorem B4.4 of [2], any $T$-group is $G_0$-conjugate to some $T^i$. Thus there are just 4 classes of $T$-groups, and it is clear that the action of $\Sigma$ on the 6 classes $[R_{+3}]^A$ is as stated in the Proposition.

**Proposition 2.2.6.** (i) If $q = p \equiv +1 \mod 8$, then $I_{e_1} \cong 2^6.S_8$ and there are precisely 4 classes of $I_{e_1}$-groups and 8 classes of $E$-groups. We have

<table>
<thead>
<tr>
<th>$I^1_{-1}$</th>
<th>$E^1$</th>
<th>$E^2$</th>
<th>$P^2_{+1}$</th>
<th>$E^3$</th>
<th>$E^4$</th>
<th>$I^1_{+1}$</th>
<th>$E^5$</th>
<th>$E^6$</th>
<th>$P^2_{-1}$</th>
<th>$E^7$</th>
<th>$E^8$</th>
</tr>
</thead>
</table>
(ii) If \( q = p \equiv \pm 3 \mod 8 \), then \( I_{e_1} \cong 2^6 \cdot A_8 \), and there are just 2 classes of \( I_{e_1} \)-groups and 4 classes of \( E \)-groups. We have

\[
\begin{array}{ccc}
I_{+1} & \times & \bullet \\
E^1 & \times & \bullet \\
E^2 & \times & \bullet \\
I_{-1} & \times & \bullet \\
E^3 & \times & \bullet \\
E^4 & \times & \bullet \\
\end{array}
\]

**Proof.** Let \( \bar{\sigma} \) be a \(+1\)-decomposition \( V = \langle v_1 \rangle \perp \cdots \perp \langle v_8 \rangle \), where \( (v_i, v_j) = 1 \) for all \( i \). Put \( J = N_\Omega(\bar{\sigma}) \cong 2 \cdot S_8 \) and \( I = J \cap \Omega = N_\Omega(\bar{\sigma}) \), so that \( \bar{I} \) is an \( I_{+1} \)-group in \( G_0 \). Clearly \( I \) contains a perfect subgroup \( 2^7 \cdot A_8 \). Further \( J = N_\Omega(I) \), hence by 1.3.2(iv) the \( PO \)-class \( \lfloor \bar{I} \rfloor_{PO} \) splits into \( |O: \Omega| \) classes in \( G_0 \). So as \( PO \) is transitive on \(+1\)-decompositions, \( G_0 \) contains precisely \( |O: \Omega| \) classes of \( I_{+1} \)-groups.

Observe \( v_1 \) interchanges \( v_1 \) and \( v_2 \) and \( J = I \langle r_{v_1-v_2}, r_{v_3} \rangle \). Now \( \langle r_{v_1-v_2}, r_{v_3} \rangle \) is a \( 4 \)-group and \( I \cap \langle r_{v_1-v_2}, r_{v_3} \rangle \leq \langle r_{v_1-v_2}, r_{v_3} \rangle \). The spinor norm of \( r_{v_1-v_2} r_{v_3} \) is 2 and so \( |I \cap \langle r_{v_1-v_2}, r_{v_3} \rangle| = 2 \) or 1 according as 2 is a square or nonsquare in \( F \).

Thus when \( p = q = \pm 3 \mod 8 \), we have \( |J\Omega: \Omega| = |J:I| = 4 \), which means \( J\Omega = O \). Hence \( G_0 \) has a unique class of \( I_{+1} \)-groups with representative \( \bar{I} = I_{+1} \), and similarly \( G_0 \) has a unique class of \( I_{-1} \)-groups. Now \( \Sigma \) acts on \( \lfloor I_{+1} \rfloor^4 \) because \( G_0 = G_1 \) (since \( q = p \)), and \( \langle (12), (34) \rangle = \pi(\bar{J}) \leq N_2(\lfloor \bar{I} \rfloor) = N_2(\lfloor I_{+1} \rfloor) \). Notice, however, that \( PD \) fuses \( \lfloor I_{+1} \rfloor \) and \( \lfloor I_{-1} \rfloor \), so as in the proofs of 2.2.4 and 2.2.5, \( (13)(24) \notin N_2(\lfloor I_{+1} \rfloor) \). Consequently \( N_2(\lfloor I_{+1} \rfloor) = \langle (12), (34) \rangle \) which means there are \( |\Sigma : \langle (12), (34) \rangle| = 6 \) classes in \( \lfloor I_{+1} \rfloor^4 \). By 15.1.11 of [2] the remaining 4 classes are \( E \)-groups, and as in 2.2.4 and 2.2.5, there are precisely 4 classes of \( E \)-groups. Thus the diagram in (ii) holds. Moreover \( I \cong 2^7 \cdot A_8 \) and \( \bar{I} \cong 2^6 \cdot A_8 \), hence the proof of (ii) is finished.

When \( p = q = \pm 1 \mod 8 \) then \( |J\Omega: \Omega| = |J:I| = 2 \), hence \( G_0 \) has \( |O:J\Omega| = 2 \) classes of \( I_{+1} \)-groups, with representatives \( \bar{I} = I_{+1}^1 \) and \( I_{+1}^2 \). Similarly \( G_0 \) has just 2 classes of \( I_{-1} \)-groups \( \lfloor I_{-1}^2 \rfloor \), \( \lfloor I_{-1}^2 \rfloor \). Thus there are at least 4 classes in \( \lfloor \bar{I} \rfloor^4 \). And by 15.1.11 of [2], \( \bar{I} \) is \( A \)-conjugate to an \( E \)-group, which means there are at least 5 classes in \( \lfloor \bar{I} \rfloor^4 \), whence \( |N_2(\lfloor \bar{I} \rfloor)| \leq 4 \). However \( N_{PO}(\bar{I}) = \bar{J} = I \langle r_{v_1} \rangle \), hence \( N_2(\lfloor \bar{I} \rfloor) \cap \lfloor \bar{I} \rfloor^4 \) has at least 5 classes.
\[ \langle (12), (34) \rangle = \pi(\mathcal{J}) = \langle \pi(\mathcal{R}_m) \rangle = \langle (12) \rangle. \]

It now follows that \( N_{x}(\mathcal{I}) = \langle (12) \rangle \), so there are 12 classes in \([\mathcal{I}]\). As before, the remaining 8 classes in \([\mathcal{I}]^d\) exhaust the classes of \(E\)-groups in \(G_0\). Since \((13)(24)\) interchanges the classes of \(I_{+1}\)-groups with the classes of \(I_{-1}\)-groups, it is easily seen that the diagram in (i) holds. Finally, \(I \cong 2^7.S_8\) and \(I \cong 2^6.S_8\), so the proof is complete.

**Proposition.** 2.2.7. (i) There is a unique class of \(I_{\pm2}\)-groups in \(G_0\) for each \(\varepsilon\).

(ii) \(\varepsilon \neq \{(2, +), (3, +)\}\) then \(I_{\pm2}\) extends to \(A\).

(iii) There is a unique class of \(I_{+4}\)-groups in \(G_0\) and \(I_{+4}\) extends to \(A\).

(iv) \(\varepsilon \neq \{(2, +, 2), (3, +, 2), (2, +, 4)\}\), then \(M_0\) is not contained in an \(I_{\pm\varepsilon}\)-group. This includes the case \(G \cap \mathcal{F} \neq \varnothing\).

**Proof.** Evidently \(G_0\) is transitive on \(\varepsilon2\)-decompositions for each \(\varepsilon\), hence (i) holds. Moreover (ii) and (iii) follow from 15.1.9–15.1.10 and 15.1.7 of [2], respectively.

(iv) Assume for a contradiction that \(M_0 \leq I_{\varepsilon\varepsilon}\). Note that \(M_0 \neq 1\) by 1.3.1(i). If \((q, \varepsilon, m) = (2, +, 4)\), then \(M_0 \leq I_{+4} \cong 3^2:2^5\). Thus \(M_0\) is either a 2-group or has a nontrivial normal Sylow 3-subgroup, contrary to 1.3.1(iv). If \((q, \varepsilon, m) = (2, +, 2)\), then \(M_0 \leq I_{+2} \leq R_{\varepsilon1}\). Now \(I_{+2} \cong 2^3.S_4\), and as \(M_0\) is not a 2-group nor does it have a nontrivial normal Sylow 3-subgroup (by 1.3.1(iv)), we have \(1 < |O_2(M_0)| \leq 2^5\). But \(R_{\varepsilon1} \cong 2^6.A_8\), and this contradicts 1.3.1(v). If \((q, \varepsilon, m) = (3, +, 2)\), then \(M_0 \leq I_{+2} \leq I_{+4}\) (see the proof of 15.1.9 of [2]). We have \(I_{+2} \cong [2^7].S_3\), so as before \(1 < |O_2(M_0)| \leq 2^7\). However, \(I_{+4} \cong [2^9]:[3^4]:[2^3]\) and we appeal to 1.3.1(v) again.

In this next proposition we compress three tables into one; the notation is self-explanatory.

**Proposition 2.2.8.** There is a unique class of \(R_{+2}\), \(R_{-2}\) and \(I_{-4}\)-groups in \(G_0\) and just two classes of \(I_{\pm4}\), \(F_{\pm2}\), and \(F_{-1}\)-groups. We have

\[
\begin{array}{ccc}
(12) & (13)(24) & (123) \\
R_{+2} & R_{-2} & I_{-4} & \times & \times \\
I_{+4}^1 & F_{+2}^1 & F_{-2}^1 & \times \\
& F_{+2}^1 & \bullet & \bullet \\
F_{-4}^1 & F_{+2}^1 & F_{-2}^1 & \times
\end{array}
\]
Proof. Evidently $G_0$ is transitive on $-2$-spaces and $P\Gamma$ acts on the $-2$-spaces. Therefore $R_{-2}$ extends to $P\Gamma$ and $|\text{Tr}(R_{-2})| = 3$. By the proof of 15.1.5 of [2], $R_{-2}$ is an $F_2$-group for each $\tau \in \mathcal{S}$, and thus $\text{Tr}(R_{-2}) = \{R_{-2}, [F_1], [F_2]\}$. Thus $P\Delta$ acts on $[F_1] \cup [F_2]$, and as $P\Delta$ is transitive on $F_2$-groups [2, Theorem B.4.3], the groups in $[F_1] \cup [F_2]$ exhaust the $F_2$-groups in $G_0$. Similar arguments apply to the $R_{+2}$-, $I_{s4}$-, $I_{-4}$- and $F_1$-groups (see [2, 15.1.4, 15.1.8]).

**Proposition 2.2.9.** Assume that $q = q_0^2$, with $\alpha$ prime. If $(\alpha, d) = 1$, then $G_0$ has a unique class of $S_2$-groups and $S_2 \cong P\Omega_8^+(q_0)$ is self-normalizing in $G_0$. If $\alpha = d = 2$, then there are 4 classes of $S_2$-groups and $S_2 = S_2 \cong P\Omega_8^+(q_0).2^2$; the group $A/G_1 \cong S_4$ acts naturally on these 4 classes.

**Proof.** Since $F_0 = GF(q_0)$ is a splitting field for $D_4(q_0)$ (the full covering group of $P\Omega_8^+(q_0)$), it follows from [2, Sect. 8] that every $P\Omega_8^+(q_0)$ in $G_0$ is the socle of an $S_2$-group. Thus by Theorem B.4.5(a) of [2], $P\Delta$ has a unique class of $P\Omega_8^+(q_0)$. Let $H$ be a natural copy of $\Omega_8^+(q_0)$ in $\Omega$ acting on $V_0$, the $F_0$-span of a standard basis $(e_1, \ldots, f_4)$ (Sect. 2). Then $H = \Omega(V_0, Q_0, F_0)$, where $Q_0$ is the restriction of $Q$ to $V_0$, and by 8.2 of [2]

$$N_\Delta(H) = A(V_0, Q_0, F_0) \mathbb{Z}.$$ 

Let $\lambda$ generate $F_0^*$, and put $v = e_1 + f_1$, $w = e_2 + \lambda f_2$. Then

$$A(V_0, Q_0, F_0) \subset \langle H, r_v, r_w, d_\lambda \rangle,$$

where $d_\lambda$ is as in (1a).

If $\alpha = d = 2$ (so that $q$ is odd), then $\lambda$ is a square in $F$, hence $\langle d_\lambda, r_v, r_w \rangle \leq \Omega Z$. Obviously $r_v \in O/\Omega$, hence $N_{G_0}(\bar{H}) = \langle \bar{H}, \bar{d}_\lambda, \bar{r}_v, \bar{r}_w \rangle \cong \bar{H}.2^2$. Further $N_{P\Delta}(\bar{H}) G_0 = G_0 \langle \bar{r}_v \rangle \cong G_0.2$, thus $G_0$ has $|P\Delta: G_0 \langle \bar{r}_v \rangle| = 4$ classes of $P\Omega_8^+(q_0)$ permuted transitively by $\theta$. Hence by 1.4.3, $A/G_1$ acts naturally on these classes.

If $q$ and $\alpha$ are odd, then $\lambda$ is a nonsquare in $F$, whence the spinor norm of $r_v, r_w$ is a nonsquare in $F$. Therefore $r_v, r_w \in SO/\Omega$ and $d_\lambda \in A/\Omega Z$, showing that $N_{G_0}(\bar{H}) = \bar{H}$ and that $\bar{H}$ extends to $P\Delta$. Thus $G_0$ has a unique class of $P\Omega_8^+(q_0)$ and such groups are self-normalizing in $G_0$.

Finally, if $q$ is even, then $r_v \in O/G_0$, and again $H$ extends to $O = P\Delta$. Similar remarks apply.

**Proposition 2.2.10.** If $q = q_0^2$ then $G_0$ has just $2d$ classes of $C_\sigma$-groups with socle $\Omega_8^-(q_0)$. Let $K_i$, $1 \leq i \leq 2d$, be representatives of these classes. There are just $d$ classes of $S_\gamma$-groups in $G_0$. We have $K_i \cong S_\gamma \cong \Omega_8^-(q_0)$, and
Proof. Let $H$ be a fixed copy of $\Omega_8^-(q_0)$. It is convenient to adopt the notation of [27, pp. 428ff]. Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be the fundamental dominant weights of $H$ which are obtained from the fundamental roots $r_1, r_2, r_3, r_4$. By Theorem 1.1 of [27], each 8-dimensional absolutely irreducible $FH$-module is quasiequivalent to one of $M(\lambda_1), M(\lambda_2), M(\lambda_4)$, as described on pp. 428 of [27]. And by the remarks preceding Theorem 2.2 of [27], $M(\lambda_3)^{(a)} \cong M(\lambda_4)$, where $q_0 = p^a$. Thus $M(\lambda_3)$ and $M(\lambda_4)$ are quasiequivalent $FH$-modules. Hence there are at most 2 quasiequivalence classes of $FH$-modules. Therefore by 1.7.1(ii), 1.7.2 and the fact that any $\Omega_8^-(q_0)$ in $G_0$ is absolutely irreducible, we conclude

$$PA \text{ has at most two classes of } \Omega_8^-(q_0).$$

(2a)

By [7, Theorem 14.5.2], there exists a graph-field involution $\gamma \in A$ which satisfies $[\gamma, \Phi] = 1$ and $C_{G_0}(\gamma) \cong \Omega_8^-(q_0)$. Assume now that $H = C_{G_0}(\gamma)$. Since $H$ is absolutely irreducible, $C_A(H) = Z = Z(GL(V))$, and thus by 1.2.3, $C_{A(H)} = 1$. Hence by 1.4.2(i),

$$N_{G_0}(H) = H,$$

(2b)

and as $[\gamma, \Phi] = 1$ we deduce that $H$ extends to $G_1 = G_0 \Phi$. Therefore $\Sigma$ acts transitively on $[H]^A$. Replacing $H$ by an $A$-conjugate if necessary, we can write $\pi(\gamma) = (12) \in \Sigma$, so that

$$(12) \in N_{\Sigma}(H).$$

(2c)

Observe that $[N_\Sigma(H), \gamma] \leq C_\Sigma(H) = 1$, whence $N_\Sigma(H) = C_\Sigma(\gamma)$. By 9.1.2(e) of [14], $C_\Sigma(\gamma) = \text{Indiag}(H) = H.d$. Thus when $q$ is odd, (2b) implies that $|N_\Sigma(H) \cap V_4| = 2$. Thus by (2c),

$$|N_\Sigma(H) \cap D_8| = 4 \quad (q \text{ odd}).$$

(2d)
We now argue that if \( \tau \in \mathcal{T} \), then

\[
[H]_{PA} \neq [H']_{PA}.
\]

(2c)

Otherwise, there exists \( x \in PA \) such that \( H^x = H' \), and thus \( H \leq C_{G_0}(\langle \gamma, \delta \rangle) \), where \( \delta = \gamma^{\tau x^{-1}}. \) Since \( \pi(\tau) \) is a 3-cycle, \( \pi(\delta) \) is a 2-cycle distinct from (12) or (34). Hence \( w = [\gamma, \delta] \in C_A(H) \) and \( \pi(w) \) is a 3-cycle. Since \( A' \cong G_0 \cdot 3 \) or \( G_0 \cdot A_4 \) (see (1d)) it follows that \( w^3 \in G_0 \). Therefore \( w^3 \in C_{G_0}(H) = 1 \), which means \( H \) is a subgroup of one of the groups occurring in 1.4.1. But none of these groups contains an \( \Omega_8^- \) (as seen) and so (2e) holds. Thus by (2a), \( PA \) has precisely 2 classes of \( \Omega_8^- \) with representatives \( H \) and \( H' \). Consequently \( A \) is transitive on subgroups \( \Omega_8^- \) of \( G_0 \) and \( N_\Sigma([H]) \) does not contain a 3-cycle. Thus (2c) and (2d) yield \( |N_\Sigma([H])| = 2d \). Therefore \( A \) has exactly \( |\Sigma|/2d = 3d \) classes of \( \Omega_8^- \) (as seen).

Finally suppose that \( H \) is an \( S^- \) group in \( G_0 \). Since \( PA \) is transitive on \( S^- \)-groups by [2, Theorem B.4.5.b], (2c) implies that \( H' \) is not an \( S^- \)-group, and it follows easily that \( H' \) is a \( C_\sigma \)-group. Similarly \( H'^2 \) is a \( C_\sigma \)-group, and so just \( d \) of the classes in \( [H]^4 \) are \( S^- \)-groups, while the remaining \( 2d \) classes are \( C_\sigma \)-groups. The diagram in the Proposition now follows.

We make use of this next result in Section 4, below.

**Lemma 2.2.11.** The \( R_{e_1}^- \), \( K_1^- \), \( R_{e_3}^- \), \( T^- \), \( K_2^- \) and \( S \)-groups are the centralizers in \( G_0 \) of elements in \( A \setminus G_0 \).

**Proof.** The \( R_{e_1}^- \) -groups are centralizers of reflections in \( PO \setminus G_0 \), hence the \( K_1^- \) -groups are also involution centralizers by 2.2.4. Similarly, an \( R_{e_3}^- \) -group is the centralizer of involution \( x \in PO \) where \( x = -1_{W'} \) for some n.d. 3-space or 5-space \( W \). Thus \( T^- \) -groups are involution centralizers by 2.2.5. The \( S^- \) and \( K_2^- \) -groups are involution centralizers in view of the proof of 2.2.10. Finally, the \( S_x^- \) -groups are centralizers in \( G_0 \) of field automorphisms of \( G_0 \) (see 9.1.1 of [14]).

### 2.3. The \( C_\sigma \)-Groups

In this section we drop the assumption \( G \leq \Pi \); thus \( G \) is any group satisfying \( G_0 \leq G \leq A \). We suppose that \( M_0 = M \cap G_0 \) is a \( C_\sigma \)-group, so that the socle \( S = \text{soc}(M_0) \) is a non-abelian simple group which satisfies

\[
\hat{S} \text{ is absolutely irreducible on } V; \quad (2f)
\]

the representation of \( \hat{S} \) on \( V \) is defined over no proper subfield of \( F \). \( (2g) \)

**Remark.** If \( \rho \) is an absolutely irreducible \( p \)-modular representation of a
group \( H \) and \( \chi \) is the character corresponding to \( \rho \), then \( \rho \) is defined over \( F_p(\chi) = F_p[\chi(h): h \in H] \), where \( F_p = GF(p) \) (see Theorem 2.7B of [11]).

This fact has two consequences, which we record in

**Lemma 2.3.1.**

(i) \( F = F_p[\text{trace}(s): s \in \hat{S}] \).

(ii) If \( F_1 \) is a splitting field for \( \hat{S} \), then \( F \leq F_1 \).

Also by Schur's Lemma, \( C_{GL(n,q)}(\hat{S}) = Z \). Hence by 1.2.3,

\[
C_{P_{d}}(S) = 1. \quad (2h)
\]

As a convenience, we call a \( p \)-modular representation of a group \( p \)-relevant if it is an absolutely irreducible representation of degree 8.

Using the classification of finite simple groups, we consider the various possibilities for \( S \).

**S of Lie Type in Characteristic \( p \)**

Let \( q_1 \) be an arbitrary power of \( p \), and set \( F_1 = GF(q_1) \).

**Proposition 2.3.3.** If \( S \cong L_3(q_1) \), then

(i) \( 2 < q = q_1 \equiv 1 \mod 3 \);

(ii) \( M_0 = S.3 \cong PGL_3(q) \);

(iii) there are just \( d \) classes of \( PGL_3(q) \) in \( G_0 \) permuted transitively by \( A/S \).

**Proof:** First assume that \( \varepsilon = + \). By Theorem 2.1 of [27], \( F_1 \leq F \). But \( F_1 \) is a splitting field for \( SL_3(q_1) \), so by 2.3.1(ii), \( F = F_1 \). Further, by Theorem 2.2 of [28], \( S \) acts on \( V \) via the adjoint representation. Namely,

\[
V = \{ P \in M_3(q): \text{Tr}(P) = 0 \}, \quad (2i)
\]

where \( M_3(q) \) is the set of \( 3 \times 3 \) matrices over \( F \), and \( S \) acts on \( V \) by conjugation. This representation is reducible when 3 divides \( q \), so \( (3, q) = 1 \). Note that \( S \) preserves the quadratic form

\[
Q \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a^2 + i^2 + ai + bd + cg + fh. \quad (2j)
\]

We leave it to the reader to verify that \( Q \) has defect 0 only if \( q \equiv 1 \mod 3 \). Thus (i) holds. The absolutely irreducible representations of \( SL_3(q) \) of degree \( \leq 8 \) are given in Theorem 2.2 of [28], and only the adjoint representations have \( Z(SL_3(q)) \cong Z_3 \) in their kernels. Therefore all groups
$L_3(q)$ in $GL(V)$ are absolutely irreducible. And since the adjoint representations of $L_3(q)$ are quasiequivalent, $GL(V)$ has a unique class of $L_3(q)$ by 1.7.2. Thus by 1.7.1(ii), $A$ has a unique class of $L_3(q)$, hence so does $P A$. It now follows from 1.4.1 that $S < C_{G_0}(T)$ for some triality automorphism $T$ of order 3. Thus by 1.4.2(i), $M_0 = C_{G_0}(T) = S.3 \cong PGL_3(q)$, proving (ii). Also 1.4.2(ii) yields

$$|N_{PA}(S):M_0| \leq 2. \quad (2k)$$

However, the transpose map $x: P \mapsto P'$ defined on the module $V$ given in (2i) induces a graph automorphism on $S$ and $\bar{x} \in PO \setminus G_0$. Thus equality holds in (2k) and so $G_0$ has just $|P A:G_0(\bar{x})| = d^2$ classes of $L_3(q)$, permuted transitively by $P A$. Thus by 1.4.3, $A/G_1$ acts on these classes and (iii) holds.

Now take $\varepsilon = -$, so that $S \cong U_3(q_1)$. By Theorem 2.2 of [27], $q_1 \leq q$, and since $GF(q_1^2)$ is a splitting field for $SU_3(q_1)$, we have $q \leq q_1^2$. Let $W$ be an 8-dimensional irreducible module for $SU_3(q_1)$ over $GF(q_1^2)$. By Theorems 13.1 and 13.3 of [36], the corresponding representation extends to one of $SL_3(q_1^2)$, so as before,

$$W = \{P \in M_3(q_1^2): \text{Tr}(P) = 0\},$$

and $(3, q_1) = 1$. However, $SU_3(q_1)$ acts on the $F_1$-space

$$U = \{P \in W: P = P'\},$$

where "-' is the involutory field automorphism of $GF(q_1^2)$ (see [28. Sect. 2b, case 1]). Thus the representation in $GL(W)$ is writable over $F_1$ with module $U$, hence $q - q_1$. (Also note that $q > 2$, for $U_3(2)$ is solvable.) Clearly $S$ stabilizes $Q_U$, where $Q$ is the quadratic form on $W$ given in (2j). It is straightforward to verify that $Q_U$ has defect 0 if and only if $q \equiv -1 \mod 3$. We now reason as for $L_3^+(q)$.

**Proposition 2.3.4.** (i) If $S \cong \Omega_7(q_1)$, then $M_0 = S$ is a $K_1$-group.

(ii) If $S \cong \Omega^-_8(q_1)$, then $M_0 = S$ is a $K_2$-group.

(iii) If $S \cong ^3D_4(q_1)$, then

(a) $q = q_1^3$;

(b) $M_0 = S$;

(c) there are just $2d^3$ classes of $^3D_4(q_1)$ in $G_0$ permuted transitively by $A/G_1$.

**Proof.** (i) By Theorem 2.1 in [27], $F_1 \leq F$. And as $F_1$ is a splitting field for $\Omega_7(q_1)$, we have $F = F_1$ by 2.3.1(ii). Thus $M_0 = S$ is a $K_1$-group by 2.2.4.
(ii) By Theorem 2.2(i) of [27], \( F_1 \leq F \). Since \( GF(q_1^3) \) is a splitting field for \( \Omega_8(q_1) \), \( F \leq GF(q_1^3) \) (again by 2.3.1(ii)). Order considerations show that \( q = q_1^3 \), hence \( M_0 = S \) is a \( K_z \)-group by 2.2.10.

(iii) By [27, Theorem 2.2(i)-(ii)] and the fact that \( GF(q_1^3) \) is a splitting field for \( 3D_4(q_1) \), we deduce that \( q = q_1^3 \). As in the proof of 2.2.10, each 8-dimensional irreducible FS-module is quasiequivalent to one of the modules \( M(\lambda_1), M(\lambda_3), M(\lambda_4) \). And if \( q_1 = p^a \), then as FS-modules, \( M(\lambda_1)^{(2a)} \cong M(\lambda_3)^{(a)} \cong M(\lambda_4) \). Thus all irreducible representations of \( S \) in \( GL(V) \) are quasiequivalent. So by 1.7.1(ii), 1.7.2 and the fact that every \( 3D_4(q_1) \) in \( G_0 \) is absolutely irreducible, \( PA \) has a unique class of \( 3D_4(q_1) \). So as in the proof of 2.2.10, we conclude that every \( 3D_4(q_1) \) is the centralizer in \( G_0 \) of a graph-field automorphism and hence extends to \( G_1 \). Therefore \( A/G_1 \) acts on the classes of \( 3D_4(q_1) \) in \( G_0 \). Moreover, 1.4.1 and 1.4.2(i) imply that \( N_{G_0}(S) = S \) and \( |N_{P_A}(S):S| \leq 2 \). Suppose for the moment that \( |N_{P_A}(S):S| = 2 \). By (2h), \( N_{P_A}(S) \) embeds in \( Aut(S) \), and it is well known that \( Aut(S) \cong S:Z_n \) (a split extension). Thus there is an involution \( x \in N_{P_A}(S) \setminus S \), and by 9.1.1 of [14], \( C_S(x) \cong 3D_4(\sqrt{q_1}) \). But this is impossible, for any \( 3D_4(\sqrt{q_1}) \) in \( G_0 \) must be absolutely irreducible, hence has trivial centralizer in \( PA \). Thus \( N_{P_A}(S) = S \) and \( G_0 \) has precisely \( |PA:G_0| = 2d^2 \) classes of groups \( 3D_4(q_1) \) upon which \( PA/G_0 \) acts regularly. The assertions in the Proposition now follow.  

**Definition.** A \( K_z \)-group (resp. \( K_4 \)-group) is a subgroup \( PGL_3^z(q) \) (resp. \( 3D_4(q_1) \)) as given in 2.3.3 (resp. 2.3.4(iii)).

The following result is useful for eliminating some of the smaller simple groups, such as \( L_2(q_1) \).

**Lemma 2.3.5.**

(i) If \( S \leq K_i \) for some \( i \leq 4 \), then \( M_0 = K_i \).

(ii) \( S \) is not contained in a \( T \)-group.

(iii) Suppose that the following hold:

- (a) \( PA \) has a unique class of absolutely irreducible copies of \( S \);
- (b) \( S \) embeds in \( \Omega_7(q) \);
- (c) \( S \) does not embed in \( R_m \) \((1 \leq m \leq 4, R_{\pm 2}, R_{\pm 3}, I_{\pm 4}, F_1 \) or \( F_2 \));
- (d) \( S \) does not embed in \( G_2(q) \).

Then \( M_0 = S \) is a \( K_1 \)-group.

**Proof.** Suppose that \( S \leq H \leq G_0 \), where \( H \) is either a \( T \)-group or a \( K_i \)-group for some \( i \leq 4 \). By (2h), we can appeal to 1.4.2(ii). If 1.4.2(ii)(a) holds, then \( M_0 \) is a \( K_3 \)- or \( K_4 \)-group because the subgroups \( G_2(q) \) of \( G_0 \) are reducible (see 3.1.1). Assume first that \( M_0 \) is a \( K_3 \)-group, so that \( S \cong L^z_5(q) \).
MAXIMAL SUBGROUPS OF $\Omega_8^+(q)$

We remarked in the proof of 2.3.3 that $L_2^+(q)$ has no faithful representation of degree $\leq 7$, hence $H \cong \Omega_7(q)$. Also, any faithful representation of $L_2^+(q)$ of degree 8 cannot be written over a proper subfield of $\mathbb{F}$ (see [27, Theorem 2.1]). Therefore $H \cong \Omega_8^+(q)$, whence $H$ is neither a $K_1$- nor a $K_2$-group. Lagrange's Theorem ensures that $M_0$ is not contained in a $T$- or $K_3$-group, hence $H$ is a $K_1$-group. That is, $M_0 = H$, as desired. A similar argument handles the case in which $M_0$ is a $K_4$-group, so we can assume that 1.4.2(ii)(b) holds. Now 1.4.1, 2.3.3(iii), 2.3.4(iii)(c) and 2.2.11 ensure that $H = C_9(x)$ for some $x \in A$, hence $M_0 = H$. Consequently (i) holds, and as $\text{soc}(T)$ is not simple, it follows that $H$ is not a $T$-group and so (ii) also holds.

(iii) Let $K$ be a $K_1$-group and let $L$ satisfy $S \cong L \leq K$. We claim that $L$ is irreducible. Otherwise, $(\gamma)$ ensures that $L$ is contained in an $R$-group $R$ say, and by 2.2.4, $R = K^\tau$ for some $\tau \in \mathcal{T}$. But then by 3.1.1(vi), $L \leq K \cap K^\tau \cong G_3^+$, contrary to $(\delta)$. Therefore $L$ is irreducible, and as $L$ is not contained in an $F_1$- or $F_2$-group (by $(\gamma)$), $L$ is absolutely irreducible. Therefore by $(x)$, $S$ is $P\Delta$-conjugate to $L$ and hence $S$ is contained in a $K_1$-group. Thus $M_0$ is a $K_1$-group by (i).

**Proposition 2.3.6.** We have $S \cong L_2(q_1)$.

**Proof.** Suppose otherwise. Since $F_1$ is a splitting field for $SL_2(q_1)$, we have $F \leq F_1$. Thus the proof of Theorem 2.1 of [27] yields $q_1 \in \{q, q^3\}$. The absolutely irreducible $p$-modular representations of $SL_2(q_1)$ are well known (see [5], for example), and one of the following holds (we use the notation of [27] as in the proofs of 2.2.10 and 2.3.4(iii)):

(a) $q_1 = q$ is even and $V \cong M(\lambda_1)^{(i)} \otimes M(\lambda_1)^{(j)} \otimes M(\lambda_1)^{(k)}$ as $FS$-modules for some $i, j, k$;

(b) $q_1 = q, \ p \geq 5, \ \tilde{S} \cong L_2(q)$ and $V \cong M(3\lambda_1)^{(i)} \otimes M(\lambda_1)^{(j)}$ as $FS$-modules for some $i, j$;

(c) $q_1 = q^3$ is even and $V \otimes F_1 \cong W_i$ as $F_1S$-modules, where $W_i = M(\lambda_1)^{(i)} \otimes M(\lambda_1)^{(i+n)} \otimes M(\lambda_1)^{(i+2n)}$ (recall $n = \log_p(q)$).

Evidently (b) cannot hold because of 2.3.5(ii). If (a) occurs, then by the proof of 15.1.14 of [2], $S$ is contained in a $K_1$-group, contrary to 2.3.5(i). Thus (c) holds. As the modules $W_i$ are quasiequivalent, $GL(V)$ has a unique class of absolutely irreducible $L_2(q^3)$. Thus there is just one class of absolutely irreducible $L_2(q^3)$ in $P\Delta$, hence 2.3.5(iii)$(x)$ holds. It is easy to verify that $(\beta), (\gamma)$ and $(\delta)$ also hold, and thus we have contradicted 2.3.5(iii).

**Proposition 2.3.7.** If $S$ is of Lie type in characteristic $p$, then $M_0$ is a $K_3$-group for some $i \leq 4$. 
The possibility that $S$ is $L_m(q_1)$ or $U_m(q_1)$ for $m \geq 4$ can be eliminated by Theorem 2.2 of [28]. When $q$ is even, the absolutely irreducible modules for $B_2(q_1)$ and $Sz(q_1)$ have dimension $4^m$ for some $m$ (see Theorem 3.2 of [8]), and hence these groups are also eliminated. The remaining groups can be discarded using Theorem 1.1 in [27] and Theorems 2.2, 2.6, and 2.10 in [28].

S Alternating or of Lie Type in Characteristic Prime to $p$.

By results in [26, 38, 39, 40], $S$ is one of the following groups:

- $A_n$, $5 \leq n \leq 11$,
- $L_2(r)$, $r \in \{4, 5, 7, 8, 9, 11, 13, 17\}$,
- $I_3(2)$, $I_3(3)$, $I_3(4)$, $U_3(3)$,
- $L_4(2)$, $U_4(2)$, $U_4(3)$,
- $Sp_6(2)$, $\Omega^+_8(2)$, $Sz(8)$.

To study these groups, we rely on results appearing in [9], which supply their ordinary character tables, along with the unpublished work of Parker [34], which provides most of the appropriate modular character tables.

**Proposition 2.3.8.** If $S \cong \Omega^+_8(2)$, then

(i) $q = p \geq 3$;

(ii) $M_0 = S$ and $N_A(S) \cong \text{Aut}(S) \cong S.S_3$;

(iii) $G_0$ has just 4 classes of $\Omega^+_8(2)$, permuted naturally by $A/G_0 \cong S_4$.

**Proof.** It follows from [9, 21, 22] that the double cover $2^*\Omega^+_8(2)$ has a unique $p$-relevant representation (namely, the reduction mod $p$ of the ordinary 8-dimensional representation). The character of this representation takes values in $F_p$, hence (i) holds by 2.3.1(i). The inclusion $\Omega^+_8(2) \leq P\Omega^+_8(p)$ is well known for all primes $p$, thus by 1.7.1(ii), 1.7.2 and the fact that every $\Omega^+_8(2)$ in $G_0$ is absolutely irreducible, $P\Delta$ has a unique class of $\Omega^+_8(2)$. According to [9], the $p$-relevant representation of $2^*\Omega^+_8(2)$ extends to representations of a double cover $2^*\Omega^+_8(2)$ of $O^+_8(2) = \Omega^+_8(2)$.2. (Note that the pair $(O^+_8(2), GO^+_8(2))$ in the notation of [9] corresponds to our pair $(\Omega^+_8(2), O^+_8(2))$.) The character value on an involution in $2^*\Omega^+_8(2) \backslash 2^*\Omega^+_8(2)$ is ±2, hence $2^*\Omega^+_8(2)$ does not embed in $SL_8(p)$. Therefore $P\Delta \not\supset O^+_8(2) \subsetneq G_0$. A triality automorphism of the full cover $2^*\Omega^+_8(2)$ of $\Omega^+_8(2)$ acts nontrivially on the normal 4-group, hence this representation of $2^2\Omega^+_8(2)$ does not extend to $2^2\Omega^+_8(2)$.3. Thus $\Omega^+_8(2) \not\leq G_0$, $M_0 = S$, $N_{pr}(S) = N_{pr}(S) = S.2 \cong O^+_8(2)$, and there are $|P\Delta : N_{pr}(S) G_0| = |P\Delta : G_0.2| = 4$ classes of $\Omega^+_8(2)$ in $G_0$, permuted...
naturally by $A/G_0 \cong S_4$. Therefore $N_A(S) \cong S.S_3$, and as $S$ is not centralized by a triality automorphism (see 1.4.1), $N_A(S) \cong \text{Aut}(S)$. 

**Proposition 2.3.9.** If $S \cong Sz(8)$, then

(i) $q = 5$;

(ii) $M_0 = S$ and $N_A(S) \cong \text{Aut}(S) = S.3$;

(iii) $G_0$ has just 8 classes of $Sz(8)$, permuted transitively by $A/G_0 \cong S_4$.

**Proof.** By [9, 34], the double cover $2' Sz(8)$ has a $p$-relevant representation only when $p = 5$. There is only one such representation, and it is writable over $F_5$. By [9, p. 281], this representation is orthogonal and is a faithful representation of $2' Sz(8)$. Obviously the central involution is sent to the scalar $-1 \in GL(V)$, hence $2' Sz(8) \not\leq \Omega^-_8(5)$ as $Z(\Omega^-_8(5)) = 1$. Therefore $Sz(8)$ embeds in $G_0$. Thus by 1.7.1(ii), 1.7.2 and the fact that any $Sz(8)$ in $G_0$ is absolutely irreducible, $PA$ has a unique class of $Sz(8)$. As in the proof of 2.3.8, $\text{Aut}(Sz(8)) = Sz(8).3 \leq PA$, hence $N_A(S) = S.3$, and as in 2.3.8, we have $C_A(S) = 1$ and so $N_A(S) \cong \text{Aut}(S)$.

**Proposition 2.3.10.** If $S \cong A_9$, then

(i) $q = 2$;

(ii) $M_0 = S$ and $N_A(S) \cong S_9$;

(iii) $G_0$ has just 3 classes of $A_9$, permuted naturally by $A/G_0 \cong S_3$.

**Proof.** Case $p = 2$. All 2-relevant representations of $A_9$ are writable over $F_2$, so $q = 2$ by 2.3.1. Assertions (ii) and (iii) now follow from [12].

Case $p = 3$. By [34], $2' A_9$ has a unique 3-relevant representation, and it is writable over $F_3$. Therefore $q = 3$ and $PA$ has a unique class of absolutely irreducible $A_9$ by 1.7.1(ii). Thus 2.3.5(iii)(a) holds, and as $A_9$ embeds in $\Omega^-_7(3)$, 2.3.5(iii)(b) also holds. It is easy to see that $A_9$ satisfies conditions (β) and (δ) of 2.3.5(iii), hence 2.3.5(iii) eliminates the case $p = 3$.

Case $p > 5$. By [9, 34], $2' A_9$ has just two quasiequivalence classes of $p$-relevant representations, and these are writable over $F_p$. Thus $q = p$, and because every $A_9$ in $PGL(V)$ is absolutely irreducible, there are at most two classes of $A_9$ in $PGL(V)$. Thus by 1.7.1(ii), there are at most two classes of $A_9$ in $PA$. Suppose for the moment that $A$ fixes a $PA$-class. This $PA$-class splits into 1, 2, 4, or 8 classes in $G_0$, one of which must be fixed by a triality automorphism. It follows that an $A_9$ in $G_0$ is normalized by a triality automorphism and as $3 \nmid |\text{Out}(A_9)|$, this $A_9$ is centralized by a triality automorphism, contrary to 1.4.1. It follows that $PA$ has two classes of $A_9$ which are fused into a single class in $A$. We now argue that $C_A(S) = 1$. For
take \( g \in C_A(S) \) and note that \( \pi(g) \) is not a 3-cycle, again by 1.4.1. Thus for a suitable \( a \in A \), we have \( \pi(g^a) \in D_8 \), and so \( g^a \in G_{D_8} = PT = PA \). However \( S^a \) is absolutely irreducible, whence \( C_{PA}(S^a) = 1 \), as required. Thus \( N_A(S) \leq \text{Aut}(S) \cong S_9 \). However, \( S_9 < O_2^+(2) < A \) and it follows that \( M \) normalizes a \( K_5 \)-group, hence is nonmaximal.

**Proposition 2.3.11.** If \( S \cong A_{10} \), then

(i) \( q = p = 5 \);

(ii) \( M_0 = S \) and \( N_A(S) \cong S_{10} \);

(iii) \( G_0 \) has just 12 classes of \( A_{10} \) permuted transitively by \( A/G_0 \cong S_4 \);

(iv) \( N_A(S) G_0 \) is \( A \)-conjugate to \( G_{(12)} = G_{0.2} \).

**Proof.** According to [9, 34], \( 2A_{10} \) has no \( p \)-relevant representation, except when \( p = 5 \). Further, it has just three \( 5 \)-relevant representations, all writable over \( F_5 \). Thus (i) holds by 2.3.1.

Let \( A_{10} \) act naturally on 10 basis vectors \( w_1, \ldots, w_{10} \) of a 10-dimensional space \( W \) over \( F_5 \). Then \( A_{10} \) preserves the nondegenerate quadratic form \( Q_2(\sum_{i=1}^{10} x_i w_i) = \sum_{i=1}^{10} x_i^2 \) on \( W \), which has defect 0. Now \( A_{10} \) fixes \( w = w_1 + \cdots + w_{10} \) and acts on the 8-space \( w^\perp/\langle w \rangle \). Evidently \( w^\perp/\langle w \rangle \) inherits a nondegenerate quadratic form \( Q_0 \) from \( Q_1 \), and \( Q_0 \) also has defect 0. Therefore \( A_{10} \leq G_0 \).

The argument in the proof of 2.3.10 (Case \( p \leq 5 \)) also applies here to show that there is a unique \( A \)-class of \( A_{10} \) and \( C_A(A_{10}) = 1 \). Clearly the representation of \( A_{10} \) in \( GL(w^\perp/\langle w \rangle) \) described above extends to \( S_{10} \), hence by 1.7.1(i), \( N_A(S) \cong S_{10} \). Let \( x \) be an involution in \( N_A(S) \). Note that an involution in \( S_{10}\backslash A_{10} \) has determinant \(-1\) on \( w^\perp/\langle w \rangle \), hence \( S_{10} \leq SL_4(5) \). Therefore \( x \in PO\backslash PSO \). Hence without loss, \( \pi(x) = (12) \in \Sigma \) and \( G_0 \) has \( |A:N_A(S)G_0| = |A:G_{(12)}| = 12 \) classes of \( A_{10} \). Assertions (iii) and (iv) have thus been proved.

**Definition.** The \( K_5 \)-, \( K_6 \)-, \( K_7 \)-, and \( K_8 \)-groups are the subgroups described in 2.3.8, 2.3.9, 2.3.10, and 2.3.11, respectively.

**Proposition 2.3.12.** If \( S \) is alternating or of Lie type in characteristic prime to \( p \), then \( S \) is a \( K_i \)-group where \( 5 \leq i \leq 8 \).

**Proof.** Results in [9, 34] and Brauer’s Theorem on blocks of defect one [11, Theorem 4.6B] ensure that the only groups in (21) (apart from \( O_8^+(2) \), \( S_2(8) \), \( A_9 \), \( A_{10} \)) whose covers have a \( p \)-relevant representation are \( A_6 \cong L_3(9) \) (\( p \neq 3 \)), \( A_7 \), \( A_8 \cong L_4(2) \) (\( p \neq 2 \)), \( L_5(7) \cong L_3(2) \) (\( p \neq 2, 7 \)), \( L_5(8) \) (\( p \neq 2 \)), \( L_2(17) \) (\( p \neq 17 \)), \( L_3(4) \) (\( p \neq 2 \)), and \( Sp_6(2) \) (\( p \neq 2 \)). (The restrictions on \( p \) occur because it is assumed that \( S \) is not of Lie type in characteristic \( p \).)
Observe that the projective representation of $S$ in $G_0$ corresponds to a representation of a double cover of $S$ in $\Omega$. Although the full cover of $L_3(4)$ does have a $p$-relevant representation, the double cover $2L_3(4)$ does not, and so $S \cong L_3(4)$. Further $S \cong L_3(17)$ because the indicator of each $p$-relevant representation of $SL_3(17)$ is $-1$, and this means that any absolutely irreducible $SL_3(17)$ in $GV(V)$ stabilizes a symplectic form but not a quadratic form.

Now assume that $S \cong L_3(2)$ or $Sp_6(2)$ ($p \neq 2$). Then $S$ has an irreducible representation in $\Omega_2(q)$, so $S$ satisfies 2.3.5(iii)(β). The group $G_2(q)$ has 2-rank 3 while $S$ has 2-rank at least 4, hence 2.3.5(iii)(δ) holds, and it is easy to check that (γ) holds. Further [9, 34] imply that $S$ has a unique $p$-relevant representation, whence (α) holds, and we have contradicted 2.3.5(iii).

Now take $S \cong L_2(7) \cong L_3(2)$ ($p \neq 2, 7$). By [34], $SL_2(7)$ has no 3-relevant representation, so $p \nmid |S|$. By [9, p. 3], $SL_2(7)$ has a unique $p$-relevant representation with indicator $+1$, and the character is $\chi_6$. Thus there is at most one class of absolutely irreducible $L_2(7)$ in $P\Delta$. Also $F = F_p(\chi_6) = F_p$ which means $q = p$. Now $L_2(7)$ has characters $\chi_2, \chi_3$ of degree 3 and $F_p[\chi_2] = F_p[\chi_3] = F_p[\chi_2 \chi_3]$. Thus there is a copy $C$ of $L_2(7)$ in $L = GL_2^+(p)$, where $e^1 = (p/7)$. If $\rho$ is a faithful 3-dimensional representation of $L$ over $F_p[\chi_2 \chi_3]$, then $\sigma = \rho \otimes \rho^* - 1$ gives a representation of $L/Z(L)$ in $GL(V)$, where $\rho^*$ is the dual of $\rho$. Moreover the restriction of $\sigma$ to $C$ affords the character $\chi_2 \otimes \chi_3 - 1 = \chi_6$. Thus we may write $C^\sigma \leq L^\sigma \leq GL(V)$, with $C^\sigma$ absolutely irreducible. By the remarks in the proof of 2.3.3, $\sigma$ realizes an adjoint representation of $L$, and thus we can assume that $L^\sigma \leq A$. Consequently $\overline{L^\sigma}$ is a $K_3$-group, and as there is just one class of absolutely irreducible $L_2(7)$ in $P\Delta$, we conclude that $S$ is $P\Delta$-conjugate to $\overline{C^\sigma}$. But then $S$ is contained in a $K_3$-group, against 2.3.5(i).

The case $S \cong A_5$ is treated in the same way as $L_2(7)$, using the characters $\chi_{14}$ and its dual $\chi_{14}^*$ given in [9, p. 5]. We omit further details. The double cover of $A_7$ has a $p$-relevant representation only when $p = 5$, and the representation is writable over $GF(5)$. As for $A_6$ and $L_2(7)$, it can be shown that any $C_9$-group in $G_0$ with socle $A_7$ is contained in a $K_3$-group $PGU_3(5)$.

Finally, assume that $S \cong L_2(8)$ ($p \neq 2$). By [34], $L_2(8)$ has no 3-relevant representation, hence $p \geq 5$. Also [9, 34] ensure that $P\Delta$ has a unique $p$-relevant representation and it is writable over $F_p$. Therefore $q = p$ and $P\Delta$ contains a unique class of absolutely irreducible $L_2(8)$, (see 1.7.1(ii) and 1.7.2). Now the $p$-relevant representation of $L_2(8)$ extends to $Aut(L_2(8)) \cong L_2(8).3$, hence $M_0 = S.3$ by 1.7.1(i). Further, this $p$-relevant representation of $L_2(8).3$ extends to a $p$-relevant representation of $A_9$. Thus if $C_A(S) = 1$, then $M = M_0 = S.3 < A_9 < G_0$. Hence $C_A(S) \neq 1$. If $S$ centralizes a triality automorphism, then so does $M_0$ by 1.4.2(i). But $L_2(8).3$ does not embed in any of the groups in 1.4.1, so there is an involution $j \in C_A(S)$. For suitable
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\[ a \in A, j^n \in PA \] and thus \( C_{P, A}(S^a) \neq 1. \) Since the smallest degree of a non-trivial representation of \( L_2(8) \) is 7, \( S^a \) is contained in an \( R_{a1} \)-group. Therefore \( S \) is contained in a \( K_1 \)-group, against 2.3.5(i). □

\( S \) Sporadic

Results in [16, 17, 18, 19, 28 (Sect. 5), 34] show that none of the double covers of the sporadic simple groups have \( p \)-relevant representations, except possibly \( 2J_2 \). However, it is not difficult to show that \( 2J_2 \) has no \( p \)-relevant representation by restricting a character of degree 8 of \( 2J_2 \) to the subgroup \( U_3(3) \). Thus \( S \) cannot be a sporadic simple group. We summarize the results of Section 2.3 in

**Proposition 2.3.13.** Let \( G \) be any group satisfying \( G_0 \leq G \leq A \). Assume that \( M \) is a maximal subgroup of \( G \) not containing \( G_0 \) and that \( M_0 = M \cap G_0 \) is a \( C_0 \)-group. Then \( M_0 \) is a \( K_i \)-group for some \( i \leq 8 \).

2.4. Maximality Amongst the Groups in \( \mathcal{C}_1 \).

Recall \( G_0 \leq G \leq P \Gamma \), \( M \) is maximal in \( G \) and \( G_0 \leq M \). If \( M \) is a classical subgroup of \( G \), then apart from a few exceptions, \( M_0 = M \cap G_0 \) appears in Section 2.2. The classical subgroups of \( G \) which do not appear in Section 2.2 are (i) members of \( C_4 \) with \( q \) even; (ii) members of \( C_4 \) which stabilize a tensor product decomposition \( V = V_1 \otimes V_2 \) with \( V_1 \) an orthogonal space; (iii) members of \( C_7 \); (iv) members of \( C_2 \) which stabilize an \( e_1 \)-decomposition with \( q > p \). However by 15.1.11, 15.1.12, and 15.1.14 of [2], these exceptions do not give rise to maximal subgroups of \( G \). Thus Section 2.2 describes all the classical subgroups of \( G_0 \) whose normalizer in \( G \) can be maximal in \( G \). Also Section 2.2 and 2.3 describe all the \( C_0 \)-groups whose normalizer in \( G \) can be maximal, hence when we have proved

**Proposition 2.4.1.** If \( G_0 \leq G \leq P \Gamma \) and \( M \) is a maximal subgroup of \( G \) not containing \( G_0 \), then \( M_0 \) is \( G_0 \)-conjugate to some member of \( \mathcal{C}_1 \), where \( \mathcal{C}_1 \) consists of the groups in Table II.

(Two groups in Table II are separated by a horizontal line if and only if they are not \( A \)-conjugate.)

For each group \( H \in \mathcal{C}_1 \), we will find those groups \( G \leq P \Gamma \) for which \( N_G(H) \) is maximal in \( G \). In doing so, we prove that certain rows in the results matrix (see Table I after Section 1.5) are correct. Recall (Section 1.4) that \( P \Gamma = G_{D_8} \) when \( q \) is odd and \( P \Gamma = G_{(12)} \) when \( q \) is even. Thus it suffices to consider the case

\[
\pi(G) \in \{ 1, \langle (12) \rangle, \langle (13)(24) \rangle, \\
\langle (1423) \rangle, \langle (12), (34) \rangle, V_4, D_8 \}.
\]

(2m)
<table>
<thead>
<tr>
<th>Name</th>
<th>Order</th>
<th>Non-abelian composition factors</th>
<th>Restrictions on $q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{s_1}$</td>
<td>$\frac{1}{d^2} q^{12}(q^4 - 1)(q^3 - 1)(q^2 - 1)(q - 1)$</td>
<td>$L_4(q)$</td>
<td></td>
</tr>
<tr>
<td>$R_{s_1}'$, $i \leq 2$</td>
<td>$\frac{1}{d^2} q^{12}(q^3 - 1)^3(q - 1)$</td>
<td>$L_3(q)^*$</td>
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<tr>
<td>$R_{s_2}$</td>
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<td>$L_3(q)^*$</td>
<td></td>
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<tr>
<td>$R_{s_3}$</td>
<td>$\frac{1}{d^2} q^{12}(q - 1)(q^4 - 1)(q^3 - 1)^2$</td>
<td>$L_3(q)$</td>
<td></td>
</tr>
<tr>
<td>$R_{s_4}$</td>
<td>$\frac{1}{d} q^9(q^6 - 1)(q^4 - 1)(q^2 - 1)$</td>
<td>$\Omega_3(q)$</td>
<td></td>
</tr>
<tr>
<td>$K_{s_1}$, $i \leq 2d$</td>
<td>$\frac{2}{d^2} q^6(q^4 - 1)(q^3 - 1)(q^2 - 1)(q - 1)$</td>
<td>$L_4(q)$</td>
<td></td>
</tr>
<tr>
<td>$R_1$, $i \leq 2$</td>
<td>$\frac{2}{d^2} q^6(q^4 - 1)(q^3 - 1)(q^2 - 1)(q - 1)$</td>
<td>$L_4(q)$</td>
<td></td>
</tr>
<tr>
<td>$P_3$, $i \leq 2$</td>
<td>$\frac{2}{d^2} q^6(q^4 - 1)(q^3 + 1)(q^2 - 1)(q + 1)$</td>
<td>$U_4(q)$</td>
<td></td>
</tr>
<tr>
<td>$R_{s_3}$, $R_{s_4}$</td>
<td>$\frac{1}{2} q^3(q^4 - 1)(q^3 - 1)^2$</td>
<td>$L_2(q)^*$</td>
<td>$q$ odd</td>
</tr>
<tr>
<td>$T_1$, $i \leq 4$</td>
<td>$j2^{11,12,13,5,7,2j = 3 + \left(\frac{3}{p}\right)}$</td>
<td>$A_8$</td>
<td>$q = p &gt; 2$</td>
</tr>
<tr>
<td>$E_1$, $i \leq 4$</td>
<td>$\frac{192}{d^2} (q - 1)^4$</td>
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<td></td>
</tr>
<tr>
<td>$E_3$</td>
<td>$\frac{192}{d^2} (q + 1)^4$</td>
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<tr>
<td>$I_{s_2}$</td>
<td>$\frac{4}{d^2} q^6(q^2 - 1)^4$</td>
<td>$L_3(q)^*$</td>
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</tr>
<tr>
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<tr>
<td>$F_1$, $i \leq 2$</td>
<td>$\frac{1}{d^2} q^6(q^6 - 1)(q^5 - 1)^2(q^3 - 1)^2$</td>
<td>$P\Omega_5^+(q_0)$</td>
<td>$q = q_0^+$ $\text{prime}$ $(a, d) = 1$</td>
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<tr>
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<td>$q^6(q^3 - 1)^2(q^2 - 1)^2(q - 1)$</td>
<td>$P\Omega_7^+(q_0)$</td>
<td>$q = q_0^+$ $q$ odd</td>
</tr>
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<td>$S_1^*$, $i \leq d$</td>
<td>$\frac{1}{d} q^6(q^4 - 1)(q^3 - 1)(q - 1)$</td>
<td>$\Omega_8^-(q_0)$</td>
<td>$q = q_0^+$</td>
</tr>
<tr>
<td>$K_{s_1}$, $i \leq d^2$</td>
<td>$q^6(q^2 - 1)(q^2 - 1)$</td>
<td>$L_3(q)$</td>
<td>$2 &lt; q = \equiv 1(3)$</td>
</tr>
<tr>
<td>$K_{s_2}$, $i \leq 2d^2$</td>
<td>$q^{12}(q^3 + q^2 + 1)(q^3 - 1)(q^2 - 1)$</td>
<td>$3D_4(q_4)$</td>
<td>$q = q_2^3$</td>
</tr>
<tr>
<td>$K_{s_3}$, $i \leq 4$</td>
<td>$2^{12}3^25^27$</td>
<td>$\Omega_5^+(2)$</td>
<td>$q = p &gt; 2$</td>
</tr>
<tr>
<td>$K_{s_4}$, $i \leq 8$</td>
<td>$2^65^2.713$</td>
<td>$Sz(8)$</td>
<td>$q = 5$</td>
</tr>
<tr>
<td>$K_{s_5}$, $i \leq 3$</td>
<td>$2^83^45^27$</td>
<td>$A_4$</td>
<td>$q = 2$</td>
</tr>
<tr>
<td>$K_{s_6}$, $i \leq 12$</td>
<td>$2^{10}3^45^27$</td>
<td>$A_10$</td>
<td>$q = 5$</td>
</tr>
</tbody>
</table>

* $L_2(q)$ is not simple when $q \leq 3$. 
Since the maximal parabolic subgroups of \( G_0 \) are maximal subgroups of \( G_0 \), 2.2.1 and 2.2.2 imply

**Proposition 2.4.2.** Rows 1, 2, 3 and 5 of the results matrix hold.

**Proposition 2.4.3.** Rows 6–8 in the results matrix hold.

**Proof.** By Section 1.6, the group \( R_3 \) is contained in precisely two overgroups in \( G_0 \), namely, an \( R_{s^1} \)-group and an \( R_{s^2} \)-group (see (1c)). So as \( R_3 \)-extends to \( PT \), \( N_G(R_3) \) is maximal in \( G \) if and only if \( G \) interchanges the classes \([R_{s^1}]\) and \([R_{s^2}]\). This occurs if and only if \( \pi(G) \notin V_q \), and so by (2m), \( N_G(R_3) \) is maximal if and only if \( \pi(G) \in \{\langle(12)\rangle, \langle(1423)\rangle, \langle(12), (34)\rangle, D_8\} \).

**Proposition 2.4.4.** The \( R_1^+ \) and \( K_1^+ \)-groups are maximal in \( G_0 \). Hence rows 9–14 in the results matrix hold.

**Proof.** By Lagrange’s Theorem, \( R_1^+ \) and \( K_1^+ \)-groups are not contained in any of the other groups in \( \mathcal{C}_1 \). Thus they are maximal in \( G_0 \). The result now follows from 2.2.4.

**Proposition 2.4.5.** Rows 19–21 of the results matrix hold.

**Proof.** Write \( R = R_{s^2} = N_G(W) \), where \( W \) is a +2-space. Thus \( N_G(R) = N_G(W) \). First, Lagrange’s Theorem ensures that \( R \) is not contained in any other group in \( \mathcal{C}_1 \) except \( R_{s^1}, R_{s^4}, R_{s^1} \) or \( K_1 \). If \( R \leq K_1 \), then choosing \( \tau \in \mathcal{F} \) suitably yields \( I_{s^1} = R^\tau \leq K_{s^1} = R_{s^1} \), which is impossible as \( I_{s^1} \) is irreducible. Therefore \( R \notin K_1 \) and similarly \( R \) is not contained in \( R_{s^1} \) or \( R_{s^4} \). When \( q \geq 4 \), \( R \notin R_{s^1} \) by Lagrange’s Theorem, and when \( q = 2 \), \( N_P(R) = N_G(R) = N_G(W) < N_G(Y) \), where \( Y \) is the unique +1-space in \( W \).

Thus it remains to consider the case \( q = 3 \). In this case, our remarks so far show that \( R \) has precisely two overgroups in \( G_0 \), namely, the stabilizers of the unique +1-space and the unique –1-space contained in \( W \). Thus as in the proof of 2.4.3, \( N_G(W) \) is maximal in \( G \) if and only if \( G \) interchanges the classes \([R_{s^1}]\) and \([R_{s^2}]\). This occurs if and only if \( G \leq PO \), and hence if and only if

\[
\pi(G) \notin \pi(PO) = \langle(12), (34)\rangle \quad \text{(see (11))}.
\]

Therefore (2m) ensures that \( N_G(R) \) is maximal in \( G \) if and only if \( \pi(G) \in \{\langle(13)(24)\rangle, \langle(1423)\rangle, V_q, D_8\} \), hence row 19 of the results matrix holds. Now take \( \tau \in \mathcal{F} \) with \( \pi(\tau) = (123) \). By 2.2.8 we have \( R^\tau = I_{s^2} \) for some \( i \in \{1, 2\} \), and without loss, \( i = 1 \). Thus (2n) ensures that \( N_G(I_{s^2}) \) is maximal in \( G \) if and only if \( \pi(G) \notin \pi(PO)^\tau = \langle(23), (14)\rangle \). Hence by (2m), \( N_G(I_{s^2}) \) is maximal in \( G \) if and only if \( \pi(G) \in \{\langle(13)(24)\rangle, V_q\} \), which
means row 20 holds. Similarly $I_{24}^2$ is maximal in $G$ if and only if 
\(\pi(G) \leq \pi(PO)^{132} = \langle (13), (24) \rangle\). Thus by (2m), $N_G(I_{24}^2)$ is maximal in $G$ if and only if $\pi(G) = V_4$. 

**Proposition 2.4.6.** The $R_{-2}$- and $F_2$-groups are maximal in $G_0$, hence rows 23–25 in the results matrix hold.

**Proof.** Use Lagrange’s Theorem and 2.2.8.

In some of the arguments below, we need to show that certain subgroups of $G_0$ are not contained in $K_5 \cong \Omega_8^+(2)$. So it is useful to have a list of the maximal subgroups of $\Omega_8^+(2)$. Applying 2.4.1 and 2.2.7 to $\Omega_8^+(2)$ yields

**Lemma 2.4.7.** Any maximal subgroup of $\Omega_8^+(2)$ has one of the following orders:

- $|R_{s1}| = 2^{12}.3^2.5.7$,
- $|R_{s2}| = 2^{12}.3^3$,
- $|R_{s3}| = 2^9.3^4.5.7$,
- $|R_{s4}| = 2^6.3^2.5^2$,
- $|R_{s5}| = 2^6.3^5$,
- $|A_9| = 2^6.3^4.5.7$.

**Proposition 2.4.8.** The $R_{+3}$- and $T$-groups are maximal in $G_0$. Hence rows 27–32 of the results matrix hold.

**Proof.** Write $R = R_{+3} = N_{G_0}(W)$, where $W$ is a $+3$-space. By Lagrange’s Theorem, $R$ is not contained in any other member of $\mathfrak{G}_1$ except possibly $R_{s1}$, $R_{-2}$, $F_2$, $K_1$, or $K_5$. Since $R$ is irreducible on $W$ and $W^\perp$, we eliminate $R_{s1}$ and $R_{-2}$ as possible overgroups of $R$ in $G_0$. Suppose that $R \leq F_2$. Then choosing $\tau \in \mathcal{T}$ suitably gives $T = R^\tau \leq F_2 = R_{-2}$. But $T$ (a tensor product group) is irreducible, hence this inclusion is impossible. An identical argument shows that $R \leq K_5$. Finally, $|R|$ divides $|K_5|$ only when $q = 3$, and in this case $|R| = 3^5.2^9.5$ divides none of the orders of the groups in 2.4.7. The result now follows from 2.2.5.

**Lemma 2.4.9.** Every $L$-group is irreducible, except for $I_{+2}$ when $q \leq 3$.

**Proof.** Let $I$ be the stabilizer in $G_0$ of the $+2$-decomposition $V = V_1 \perp \cdots \perp V_4$. The exceptional cases are pointed out in 2.2.7(iv), so take $q > 3$. Let $U \neq 0$ be an $I$-invariant subspace of $V$, and fix $u = v_1 + v_2 + v_3 + v_4 \in U \setminus \{0\}$, with $v_i \in V_i$. With no loss $v_1 \neq 0$, and it is clear that $\Omega(V_1) \times \cdots \times \Omega(V_4) \leq I$. Since $q > 3$ there exists $g \in \Omega(V_1)$ with $v_1^g \neq v_1$. Thus $0 \neq u^g - u \in U \cap V_1$. But $N_I(V_1)$ acts irreducibly as $O(V_1)$ on $V_1$, hence $V_1 \leq U$. Since $I$ is transitive on the spaces $V_i$, $U = V$ as required. The arguments for the remaining $I$-groups $I_{s1}$, $I_{-2}$, $I_{s4}$, $I_{s4}$, are similar and are left to the reader.
PROPOSITION 2.4.10. Rows 33–50 of the results matrix hold.

Proof. Let \( I = I_{+1} \) be the normalizer in \( G_0 \) of a \(+1\)-decomposition of \( V \). By 2.4.9, \( I \) is not contained in an \( R \)-group, and since \( I_{+2} \) is solvable for \( \varepsilon = \pm \), these groups cannot contain \( I \). The fact that \( A_8 \) has no nontrivial projective \( p \)-modular representation of degree \(<7\) in odd characteristic implies that \( I \) is not contained in any \( F_{17}, K_5^-, I_{54}^-, \) or \( I_{64}^- \) group. Since \( E \) is irreducible, it follows (conjugating by a suitable \( \tau \in \mathcal{F} \) as in the proofs of 2.4.5 and 2.4.8) that \( I \) is not contained in any \( K_6^-, F_{2^-}, \) or \( T \)-group. Order considerations show that \( I \) is not contained in \( K_6, K_7, \) or \( K_8 \), so it remains to consider \( K_5 \) as a possible overgroup of \( I \). When \( q = p \equiv \pm 1(8) \), \( |I| \mid |K_5| \), hence rows 39–50 hold by 2.2.6(i).

Thus assume that \( q = p \equiv 3(8) \), so \( I \cong 2^6: A_8 \). By 2.3.8, there is a \( K_5 \)-group \( K \cong \Omega_8^+ (2) \) which extends to \( G_{(12)} \cong G_{0,2} \). Thus \( N_{G_{(12)}}(K) = K_2 \cong \Omega_8^+(2) \). Let \( L \) be the stabilizer in \( N_{G_{(12)}}(K) \) of a singular vector in the \( O_8^+(2) \)-geometry associated with \( K \). Thus \( L \cong 2^6:S_8 \) and we put \( C = O_5(L) \), so that \( 2^6 \cong C \leq G_0 \). Since \( L \) acts irreducibly on \( C \), \( \hat{C} \) is either elementary abelian or extraspecial. The group \( Q_8 \circ D_8 : D_8 \) has no faithful orthogonal representation of degree \( 8 \) (see the proof of \([2, 11.8]\)), and so if \( \hat{C} \) is extraspecial then \( \hat{C} \cong 2_1^{+6} \). But then \( L \cap G_0 \) is an \( E \)-group, which is impossible because \( E \)-groups do not extend to \( G_{(12)} \) (see 2.2.6). Therefore \( \hat{C} \cong 2^7 \), hence \( \hat{C} \) can be diagonalized with respect to a basis \( \beta = (w_1, ..., w_8) \).

If \( F \) is the matrix of the bilinear form with respect to \( \beta \), then \( F \) centralizes \( \hat{C} \) whence \( F \) is a diagonal matrix. Further, \( r_i, r_j \in \hat{C} \leq \Omega \) for all \( i, j \), hence the \( 1 \)-spaces \( \langle w_i \rangle \) are all isometric. Thus \( \langle w_1 \rangle \perp \cdots \perp \langle w_8 \rangle \) is an \( \varepsilon \)-decomposition of \( V \) and \( N_{G_0}(C) \) is an \( E_{17} \)-group. Therefore \( L \cap G_0 \cong 2^6: A_8 \) is \( PM \)-conjugate to \( I \) and so column IV of rows 45–50 holds. Suppose that \( I \) is the stabilizer of the \(+1\)-decomposition \( \langle v_1 \rangle \perp \cdots \perp \langle v_8 \rangle \). Here \((v_1, v_1)\) is a square in \( F^* \), hence \( \tilde{r}_{v_1} \in G_{(12)} \) (see (1i)). Thus \( N_{G_{(12)}}(I) = I \langle \tilde{r}_{v_1} \rangle \cong 2^1: A_8 \) and observe that \( O_8^+(2) \) has no subgroup with this structure. Therefore \( I \) extends to a novelty in \( G_{(12)} \) as shown in row 33, column V of the results matrix. It follows that \( L \cap G_0 \) is an \( I_{-1} \)-group, and so \( N_{G_{(12)}}(I_{-1}) \) is nonmaximal in \( G_{(12)} \). Thus column V of rows 33–38 of the results matrix are correct. Since \( K_5 \)-groups do not extend to \( G_{(12)(34)} \), the groups \( I_{+1} \) and \( I_{-1} \) extend to novelties in \( G_{(12)(34)} \) and in \( G_{(12)(34)} = PO \). Conjugating by a suitable \( \tau \in \mathcal{F} \) with \( \pi (\tau) = (132) \), we conclude that \( E^2 = I_{+1} \) and \( E^4 = I_{-1} \) extend to novelties in \( (G_{(12)(34)}, \tau) = G_{(13)(24)} \), hence column VIII of rows 33–38 hold. The remaining columns are now accounted for and the proof is complete.

PROPOSITION 2.4.11. Row 55 of the results matrix holds.

Proof. Let \( I \) be the normalizer in \( G_0 \) of the \(+2\)-decomposition \( V = V_1 \perp \cdots \perp V_4 \). By 2.2.7(iv) we can take \( q \geq 4 \).
Assume in this paragraph that $q = 4$. Let $S \leq G_0$ be an $S_2$-group, so that $S \cong \Omega_8^+(2)$, and let $I^*$ be an $I_{-2}$-subgroup of $S$ (thus $I^*$ normalizes a $-2$-decomposition over $GF(2)$). Then $I^* \cong \frac{1}{2}(O_2^+(2) \wr S_4) \cong \frac{1}{2}(O_2^+(4) \wr S_4) \cong I$, and it follows that $I^*$ is actually the stabilizer of a $+2$-decomposition over $F$. Thus $I$ is $G_0$-conjugate to $I^*$ and we may assume that $I = I^*$. By 2.2.9, $S$ extends to $A$ and by 2.2.7(ii), $I$ extends from $S$ to $N_A(S) \cong \operatorname{Aut}(S) \cong S \cdot S_3$. Thus by 1.3.3(i), $N_A(I) \leq N_A(S)$, hence the restriction $q \geq 5$ in row 55.

Now assume that $q \geq 5$. By 2.4.9, $I$ is not contained in an $R$-group, thus by a previous argument, (applying triality) $I$ is not contained in a $K_1$, $F_2^*$, $I_4^*$, or $T$-group. Lagrange's Theorem eliminates all other members of $\mathcal{Q}$ as possible overgroups of $I$, as possible overgroups of $I$, save $I_{+4}$, $I$, $E$, $K_5$, and $S_2$. With respect to a standard basis, $O(V_i) = \langle (\lambda, 0), (0, \lambda) \rangle$, where $\langle \lambda \rangle = F^*$. As $q \geq 5$, $\text{tr}(\frac{1}{2}(\lambda, 0)) = \lambda + \lambda^{-1}$ generates $F$ over the ground field $F$, and it follows that $I$ is not contained in an $S$-group. We also claim that

$$I \leq I_{+4}. \tag{20}$$

For it is not hard to see that $\hat{I}$ contains $O(V_1) \times \cdots \times O(V_4)$, that $\hat{I}$ is transitive on the spaces $V_i$, and that $N_{\hat{I}}(V_i)$ acts as $O(V_i)$ on each $V_i$. I thus the proof of 2.4.9 shows that $\hat{I}$ is irreducible on $V$, and as $I_{+4}$ fixes a $+4$-space, we deduce that (20) holds. When $q \geq 7$, $|I| \leq |E|$ and $I \leq K_5$ by 2.4.7.

Thus it remains to consider the case $q = 5$. Note that the only possible overgroups of $I$ are $I_{+4}$, $E$, and $K_5$, and because none of these extend to $G_{(1423)}$ or $G_{(5)}$, $G_{D_8}$, $G_{A_5}$, or $G_{S_4}$, the corresponding columns IX, XI, XII, XIII, XIV of row 55 hold. The group $O(V_i) \cong O_2^+(5) \cong D_8$ is generated by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2p}$

with respect to an orthonormal basis, and so $N_G(\hat{I})$ is monomial. Therefore $N_PG(I) < N_PG(I_{+1})$ and as $PO = G_{(12), (34)}$, $N_G(I)$ is nonmaximal in $G$ when $\pi(G) \leq \langle (12), (34) \rangle$. Thus columns IV, V, and X hold. Choosing $a \in N_A(I)$ with $\pi(a) = (23)$ yields $N_G_{(12), (34)}(I) < N_G_{(12), (34)}(I_{+1})$, hence column VIII holds. Now by the proof of 2.4.10, $I < I_{+1} < K_5 \cong \Omega_8^+(2)$. Further $I \cong [2^{1+}]S_3$, whence $I$ is a parabolic subgroup of $K_5$ corresponding to the central node of the Dynkin diagram. Therefore $I$ extends from $K_5$ to $N_A(K_5)$. Thus by 1.3.1(i), $N_G(I) < N_G(K_5)$ when $G \leq N_A(K_5) G_0$. Conjugating $K_5$ by a suitable element in $N_A(I)$, we can assume that $N_A(K_5) G_{S_3}$, hence columns VI and VII hold.

**Proposition 2.4.12.** Row 56 of the results matrix holds.
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Proof. An argument similar to yet easier than the one in 2.4.11 shows that $I_{-2} = I$ is contained in no other group in $G$, save possibly $I_{-1}$, $E$, or $K_5$ when $q = 3$. So take $q = 3$. Observe that $O_2(3)$ is generated by the matrices given in (2p) with respect to an orthonormal basis, and so as in 2.4.11, $N_{I_0}(I) \leq N_{I_0}(I+1)$. Thus as above columns IV, V, VIII-XIV hold. Further $I_{-1} < I_1 < K_5$ and as before $I$ is a parabolic subgroup of $K_4$ corresponding to the central node of the Dynkin diagram. Thus the remarks in the proof of 2.4.11 may be applied here to show that columns VI and VII hold.

Proposition 2.4.13. Row 57 of the results matrix holds.

Proof. By 2.2.7(iv) we can take $q > 2$. The irreducibility of $I_{+4}$ and Lagrange's Theorem eliminate all other members of $G$ as possible overgroups of $I_{+4}$, save $K_5$ when $q = 3$. However, $I_{+4} \not\leq K_5$ by 2.4.7.

Proposition 2.4.14. The $I_{-}$ and $F_{-}$-groups are maximal in $G_0$, hence rows 58-60 of the results matrix hold.

Proof. Use Lagrange's Theorem, 2.4.7, and 2.2.8.

Proposition 2.4.15. The $S$-groups $S_{a}$, $S_-$, and the $C_i$-groups $K_i$, $3 \leq i \leq 8$, are maximal in $G_0$. Hence rows 62-75 of the results matrix hold.

Proof. It is straightforward to show that these groups are maximal, simply by running through the list given in 2.4.1. Perhaps the only subtlety arises when showing that $K_i$ is not contained in $K_1$ or $K_2$. However, this was established in the proof of 2.3.5.

Part 3. Some Subgroups of $G_0$ Which Are Normalized by a Triality Automorphism

In Section 2.2 we saw that most of the maximal subgroups of $G_0$ are not normalized by a triality automorphism. For this reason, if $G \cap T \neq \phi$ then most of the maximal subgroups of $G$ are novelties. In this part we describe what these novelties are. That is, we introduce some subgroups which are nonmaximal in $G_0$, yet whose normalizers in $G$ are maximal in $G$ for certain groups $G \leq A$ with $G \cap T \neq \phi$.

3.1. The $G_2$-groups

Definition. A $G_2$-group is a subgroup of $G_0$ isomorphic to $G_2(q)$.

Proposition 3.1.1. Let $N$ be a $G_2$-group.
(i) $N$ fixes a unique 1-space $X$, and $X$ is nonsingular.

(ii) $PA$ is transitive on $G_2$-groups.

(iii) $N_{G_0}(N) = N$.

(iv) $\pi(C_A(N)) = \pi(N_A(N)) \cong S_3$.

(v) There are just $d^2$ classes of $G_2(q)$ in $G_0$, permuted transitively by $A/G_1$.

(vi) If $K$ is a $K_1$-group and $\gamma \in \mathcal{F}$, then $K \cap K' \cong G_2(q)$ and $K \cap K' = N_K(X)$ for some nonsingular 1-space $X$.

Proof. (i) The smallest degree of a nontrivial $p$-modular representation of $N$ is $5 + d$, hence $N$ does not fix a nonzero t.s. space or a n.d. 2-, 3-, or 4-space. Thus (i) holds because $N$ has no irreducible $p$-modular 8-dimensional representation.

(ii) This follows from the facts that $PA$ is transitive on nonsingular 1-spaces and that $O_2(q)$ contains a unique class of $G_2(q)$. (The latter fact is proved in [24] for example).

(iii) By 1.4.1 and (ii), $N = C_{G_0}(\tau)$ for some $\tau \in \mathcal{F}$. If $X$ is the 1-space provided by (i), then $N_D(N) \leq N_D(X) \cong SO_7(q) \cong \Omega_7(q)$, $d$. However any $G_2(q)$ in $SO_7(q)$ has no centralizer in $SO_7(q)$, hence

$$C_\ell(N) = 1.$$  \hfill (3a)

Thus (iii) follows from 1.4.2(i).

(iv) Since $N \leq N_{G_0}(X)$ where $X$ is as in (i), there is a reflection $r \in O$ such that $\bar{r} \in C_{PO}(N)$. Therefore $\pi(C_A(N))$ contains the 2-cycle $\pi(\bar{r})$ and the 3-cycle $\pi(\tau)$, whence $\pi(C_A(N)) \cong S_3$ or $S_4$. If $\pi(N_A(N)) \cong S_4$, then there exists $g \in N_A(N)$ such that $\pi(g)$ does not normalize $\pi(\tau)$. But then as in the proof of 1.4.2(i), $[\tau, g, \tau] \in C_D(N) \setminus 1$, contrary to (3a). Thus $\pi(N_A(N)) \cong S_3$ and (iv) now follows.

(v) By 1.4.2(i), $|N_{PA}(N) : N| \leq 2$, and since $\bar{r} \in C_{PO}(N) \setminus G_0$, we have $N_{PA}(N) = N \times \langle \bar{r} \rangle$. Thus by 1.3.2(iv) and (ii), $G_0$ has precisely $|PA : G_0 \langle \bar{r} \rangle | = d^2$ classes of groups $G_2(q)$, permuted transitively by $PA$. Thus $A/G_1$ acts on these classes by 1.4.3.

(vi) Assume here that $G_2(q) \cong N < K$, and let $X \leq V$ be as in (i). The space $X$ is an $\ell$-1-space where $\ell$ is + or −, and we define $N_1 = N_K(X)$, an overgroup of $N$. By 2.2.4, there exists $v \in \mathcal{F}$ such that $N_{G_0}(X)^v = K$, hence $K = C_{G_0}(\bar{r}^v)$, where $r$ is as in (iv). Now $\bar{r} \in PO \leq A$, hence $\tilde{\bar{r}} \in \theta$ (recall from Sect. 1.4 that $\theta = A'PA \cong G_0 \cdot S_3$ or $G_0 \cdot S_4$). Thus $N_1$ centralizes $\tau_1 = \bar{r}^v \in \theta$, and since $\pi(\bar{r})$ is a 2-cycle and $\pi(v)$ is a 3-cycle, $\pi(\tau_1)$ is a 3-cycle. Therefore $\tau_1^3 \in \ker(\pi) \cap \theta = G_\circ$, and so $\tau_1^3 \in C_{G_\circ}(N_{1}) = 1$, by (3a). Thus $N_1$ is a subgroup of one of the groups appearing in 1.4.1. Since $N_1$
contains \( N \cong G_2(q) \), it follows that \( N_1 \cong G_2(q) \), hence \( N = N_1 \). Thus it suffices to show that \( K \cap K' \cong G_2(q) \). Now \( |K: N| = (1/d) q^2 (q^4 - 1) = |G_0: N_{G_0}(X)| \), whence \( K \) is transitive on \( \varepsilon \)-spaces in \( V \). Replacing \( y \) by \( y^{-1} \) if necessary, we can assume that \( K' \) is an \( R_{\varepsilon_1} \)-group, where \( \varepsilon = + \) or \( - \) (see 2.2.4). If \( \delta = \varepsilon \), then by the transitivity of \( K \) there exists \( k \in K \) such that \( K' = N_{G_0}(X)^k \). Hence \( K \cap K' = N^k \), as required. Now suppose that \( \delta = -\varepsilon \).

By 2.2.4, \( K \) extends to either \( G_{(13)(24)} \) or \( G_{(14)(23)} \). Since \( G_{(13)(24)} \) and \( G_{(14)(23)} \) interchange the classes \([R_{+1}]\) and \([R_{-1}]\), there exists \( j \in N_{G_0}(K) \) such that \( X' \) is a \( \varepsilon_1 \)-space. Thus the same argument as before shows \( K \cap K' \) is \( K \)-conjugate to \( N' = N_K(X') \), as desired.

3.2. The \( N_{+1} \) and \( N_{-2} \) groups

Write \( R = R_{+2} = N_{G_0}(W) \), where \( W \) is an \( \varepsilon_2 \)-space. Then \( R \) has a normal cyclic subgroup \( \bar{\Omega}(W) \cong \Omega_2(q) \cong (1/d) \mathbb{Z}_{q - \varepsilon_1} \). The group \( \bar{\Omega}(W) \) and its subgroups are in fact the only normal cyclic subgroups of \( R \), and when \((q, \varepsilon) \notin \{(2, +), (3, +)\} \) we define \( \eta(R) \) as the group of order \( r \) in \( \bar{\Omega}(W) \), where \( r \) is the largest prime divisor of \((1/d)(q - \varepsilon_1)\). When \((q, \varepsilon) = (3, +) \) then \( \bar{\Omega}(W) = 1 \), however, \( \bar{SO}(W) = \langle j \rangle \) for some involution \( j \in D \setminus G_0 \). In this case we define \( \eta(R) = \langle j \rangle \), and it is easily seen that \( \eta(R) = C_{\delta}(R) \). We do not define \( \eta(R) \) when \((q, \varepsilon) = (2, +) \). If \( g \in P \Gamma \) then \( R^\varepsilon \) is another \( R_{+2} \)-group and it is clear from the definition that \( \eta(R^\varepsilon) = \eta(R)^\varepsilon \). Thus we may extend the function \( \eta \) to all \( I_{+a} \) and \( F_2 \)-groups by putting \( \eta(R^a) = \eta(R)^a \) for all \( a \in A \).

Evidently \( H = C_{G_0}^\varepsilon(\eta(H)) \) where \( H \) is any \( R_{+2} \), \( I_{+a} \), or \( F_2 \)-group. The following lemma serves to collect some useful facts about these groups \( \eta(H) \).

**Lemma 3.2.1.** Let \( r \) and \( s \) be the largest prime divisors of \( q - 1 \) and \( q + 1 \), respectively. Also let \( \omega_m \) be a primitive \( m \)th root of unity in an algebraic closure of \( \mathbb{F} \). Then there exists generators \( w, x, y, z \) of \( \eta(R_{+2}), \eta(R_{-2}), \eta(F_2) \), and \( \eta(I_{+4}) \), respectively, which satisfy the following.

(i) \(|w| = |z| = r \) and \(|x| = |y| = s|.

(ii) \( \hat{w} \) acts on a \( +2 \)-space \( U \) with eigenvalues \( \omega_r, \omega_r^{-1} \) and \( [\hat{w}, U^\perp] = 1 \).

(iii) \( \hat{x} \in \Omega \) acts on a \( -2 \)-space \( W \) with eigenvalues \( \omega_s, \omega_s^{-1} \) and \( [\hat{x}, W^\perp] = 1 \).

(iv) \( K = C_{\Omega}(\hat{y}) \) acts irreducibly but not absolutely irreducibly as \( GU_4(q) \) on \( V \); moreover \( E = \text{Hom}_K(V) = \mathbb{F} \langle \hat{y} \rangle \) is a quadratic field extension of \( \mathbb{F} \) and there is a nondegenerate Hermitian \( E \)-form \( f \) on \( V \) such that \( f(v, v) = Q(v) \) for all \( v \in V \).

(v) There exists \( \lambda_1, \lambda_2 \in \mathbb{F} \) and t.s. 4-spaces \( V_1, V_2 \subseteq V \) such that
V = V₁ ⊕ V₂ and \( \hat{z} \) acts as the scalar \( \lambda \), on \( V_i \) (i = 1, 2). If \( r \) is odd then \( \hat{z} \in \Omega \) and \( (\lambda_1, \lambda_2) = (\omega_r, \omega_r^{-1}) \). If \( r = 2 \) and \( q \equiv 1(4) \), then \( \hat{z} \in \Omega \) and \( (\lambda_1, \lambda_2) = (i, -i) \) (i = \( \sqrt{-1} \)). If \( r = 2 \) and \( q \equiv 3(4) \), then \( q = 3, \hat{z} \in \Delta \setminus O \) and \( (\lambda_1, \lambda_2) = (-1, 1) \).

**Proof.** These assertions are clear, except possibly those in (iv). For information about the embedding of \( GU_4(q) \) in \( O \) we refer to Section 1.F of [20] or 7.6.2 of [2].

**DEFINITION.** The group \( N \leq G_0 \) is an \( N_1 \)-group if \( N = R \cap F \), with \( R \) an \( R_{-2} \)-group, \( F \) an \( F_2 \)-group, and \([\eta(R), \eta(F)] = 1\).

**PROPOSITION 3.2.2.** Let \( N \leq G_0 \) be an \( N_1 \)-group. Then

(i) \( \hat{N} \cong ((1/d)(Z_{q+1}) \times (1/d) GU_3(q)).2^d \),

(ii) \( G_0 \) is transitive on \( N_1 \)-groups;

(iii) \( N \) extends to \( A \).

**Proof.** (i) Write \( N = R \cap F \) as in the definition, and put \( \eta(R) = \langle x \rangle \) and \( \eta(F) = \langle y \rangle \) as in 3.2.1. Also let \( W, f, E, K \) be as in 3.2.1. Since \( y \in C_{G_0}(x) \), \( W \) is \( \hat{y} \)-invariant and hence \( W \) is a 1-space over \( E \). And because \( Q(w) \neq 0 \) for all \( w \in W \setminus \{0\} \), \( W \) is nondegenerate in the unitary geometry \((V, E, f)\). Therefore \( C_\alpha(\hat{x}) \cap C_\alpha(\hat{y}) = \bigcap_{Q(w)}(W) \cong GU_3(q) \times GU_3(q) \cong Z_{q+1} \times GU_3(q) \). The details concerning the various factors of 2 are left to the reader.

(ii) Let \( N^* \) be another \( N_1 \)-group. The proof of (i) shows that \( \hat{N}^* = N_0(E^*, f^*, W^*) \), where \((V, E^*, f^*)\) is a unitary geometry over the quadratic field extension \( E^* \) of \( F \) and the \(-2\)-space \( W^* \) is \( E^* \)-invariant and n.d. with respect to \( f^* \). Since \( G_0 \) is transitive on \(-2\)-spaces, we can suppose that \( W = W^* \). Now \( N_0(W) \) acts as \( O(W^\perp) \) on the \(-6\)-space \( W^\perp \), and by Theorems B4.3 and B8 of [2], \( O(W^\perp) \cong O_{6^-}(q) \) has a unique class of groups \( GU_3(q) \) acting irreducibly but not absolutely irreducibly on \( W^\perp \). Assertion (ii) now follows.

(iii) Fix \( a \in A \). Replacing \( a \) by \( a^{-1} \) if necessary, we can assume that \( F^a \) is an \( R_{-2} \)-group (see 2.2.8). Since \( R^a \) is an \( F_2 \)-group, it follows from the definition that \( N^a = R^a \cap F^a \) is an \( N_1 \)-group. Thus by (ii), \( a \) fixes \([N]\) whence (iii) holds.

**DEFINITION.** For \( q \geq 3 \), \( N \leq G_0 \) is an \( N_2 \)-group if \( N = R \cap I \), with \( R \) an \( R_{-2} \)-group, \( I \) an \( I_{4-} \)-group, and \([\eta(R), \eta(I)] = 1\).

**PROPOSITION 3.2.3.** Let \( N \leq G_0 \) be an \( N_2 \)-group. Then

(i) \( \hat{N} \cong ((1/d)(Z_{q-1}) \times (1/d) GL_3(q)).2^d \).
(ii) $G_0$ is transitive on $N_2$-groups;

(iii) $N$ extends to $A$.

Proof. Write $N = R \cap I$ as in the definition and put $\eta(R) = \langle w \rangle$ and $\eta(I) = \langle z \rangle$ as in 3.2.1.

(i) Let $r$, $U$, $V_1$, $V_2$ be as in 3.2.1.

Case $r = 2$ and $q \equiv 1(4)$. Then $\hat{w} = -1_U$, and as $[w, z] = 1$, we have $\hat{w}^2 = \pm \hat{w}$. But $\hat{w}$ and $-\hat{w}$ have different eigenvalues, hence $[\hat{w}, \hat{z}] = 1$. Now $\hat{z} \in \Omega$ and $\hat{z}$ acts as $i = \sqrt{-1} e_F$ on $V_1$ and $-i$ on $V_2$. Thus $\langle \hat{w}, \hat{z} \rangle$ may be diagonalized with respect to a basis $\beta = (v_1, \ldots, v_8)$. Since $z \in C_G(w) = NG_0(U)$, it follows that $\hat{w}$ and $\hat{z}$ act on $U$ and $U^\perp$. Thus we can take $U = \langle v_1, v_2 \rangle$ and $U^\perp = \langle v_3, \ldots, v_8 \rangle$. Relabeling $v_3, \ldots, v_8$ if necessary, we have

\[ \hat{w} = \text{diag}_g(-1, -1, 1, 1, 1, 1, 1, 1, 1), \]

\[ \hat{z} = \text{diag}_g(i, 1, 1, 1, -1, -1, i, i). \]

Evidently $N = NG_0 \{ U, X, Y \} = NG_0 \{ X, Y \}$, where $X = \langle v_3, v_4, v_5 \rangle$ and $Y = \langle v_6, v_7, v_8 \rangle$. If $\{a, b\} \subseteq \{3, 4, 5\}$, then $(v_a, v_b) = (v_b, v_a) = 0$. Thus $X$, and similarly $Y$, is t.s. Therefore $N = (\Omega(U) \times N_{\Omega(U^\perp)}(X, Y))^2$, and $N_{\Omega(U^\perp)}(X, Y) \cong (1/2) GL_5(q)$ by 1.2.2. Thus (i) holds.

Case $r = 2$ and $q \equiv 3(4)$. As in the previous case, $\hat{w} = -1_U$ and $[\hat{w}, \hat{z}] = 1$. Further $\hat{z} \in A \setminus Q$, $\hat{z}$ acts as $-1$ on $V_1$ and $+1$ on $V_2$, and $\hat{z}$ multiplies $Q$ by $-1$. Reasoning as before we obtain

\[ \hat{w} = \text{diag}_g(-1, -1, 1, 1, 1, 1, 1, 1, 1) \]

\[ \hat{z} = \text{diag}_g(1, -1, 1, 1, -1, -1, 1, 1, 1) \]

Further $N = NG_0 \{ U, X, Y \} = NG_0 \{ X, Y \}$, with $X$, $Y$ as above. If $\{a, b\} \subseteq \{3, 4, 5\}$, then $(v_a, v_b) = -(v_b, v_a) = 0$. So again, $X$ and $Y$ are t.s. and the rest is the same.

Case $r$ odd. Here $|\hat{w}| = |\hat{z}| = r$ and $[\hat{w}, \hat{z}] = 1$ by 1.2.3. Since $r | q - 1$, we can diagonalize $\langle \hat{w}, \hat{z} \rangle$ as before, and by 3.2.1 we arrive at

\[ \hat{w} = \text{diag}_g(\omega_r, \omega_r^{-1}, 1, 1, 1, 1, 1, 1, 1), \]

\[ \hat{z} = \text{diag}_g(\omega_r, \omega_r^{-1}, \omega_r, \omega_r, \omega_r, \omega_r^{-1}, \omega_r^{-1}, \omega_r^{-1}). \]

The previous argument now applies.

(ii) Let $N^* \leq G_0$ be another $N_2$-group. The proof of (i) yields $N^* = NG_0 \{ X^*, Y^* \}$, where $X^* \oplus Y^*$ is a $+6$-space, and where $X^*$ and $Y^*$ are t.s. planes. Since $G_0$ is transitive on $+6$-spaces, we can assume that
$X \oplus Y = X^* \oplus Y^* = W$. Now $N_G(W)$ acts as $O(W) \cong O_6^+(q)$ on $W$, and it is clear that $O(W)$ is transitive on decompositions of $W$ into a direct sum of two t.s. planes. It now follows that $N_{G_0}(X^*,Y^*)$ is $G_0$-conjugate to $N_{G_0}(X,Y)$, as required.

(iii) Argue as in the proof of 3.2.2(iii).

3.3. The $N_3$-Groups

**Definition.** An $N_3$-group is a Sylow $r$-normalizer in $G_0$, where $r$ is an odd prime divisor of $q^2 + 1$.

**Proposition 3.3.1.** We have

$$N_3 \cong (D_{(2/d)}(q^2 + 1) \times D_{(2/d)}(q^2 + 1)).2^2.$$

**Proof.** Let $W$ be a $-4$-space and observe $N_{G_0}(W,W^\perp)$ contains a Sylow $r$-subgroup $R$, where $r$ is an odd prime divisor of $q^2 + 1$. Since $r \mid |GL_2(q)|$, $\hat{R}$ is irreducible on $W$ and $W^\perp$. Further, since $\hat{R}$ does not act faithfully on $W$, it follows that $W$ and $W^\perp$ are nonisomorphic as $\hat{R}$-modules. Hence $N_{G_0}(R) \leq N_{G_0}(W,W^\perp) \cong (\Omega_4^+(q) \times \Omega_4^-(q)).2^2 \cong (L_2(q^2) \times L_2(q^2)).2^2$. The result now follows because the Sylow $r$-normalizer in $L_2(q^2)$ is $D_{(2/d)}(q^3 + 1)$.

3.4. The $N_4$-Groups

**Proposition 3.4.1.** Assume that $q$ is odd.

(i) $G_0$ contains exactly 4 classes of involutions, called $2A$, $2B$, $2C$, $2D$ (following the conventions of [9]).

(ii) $C_{G_0}(2A) = I_{+4}$ (by "$C_{G_0}(2A)$" we mean the centralizer of an element in the class $2A$).

(iii) $(C_{G_0}(2B), C_{G_0}(2C), C_{G_0}(2D)) = (R_{+2}, I_{+4}, I_{+4})$ or $(R_{-2}, F_1^1, F_1^2)$ according as $q \equiv 1$ or $3 \mod 4$.

(iv) If $g \in 2A$ then $\hat{g} \in \Omega$ has order 2 and trace 0.

(v) The class $2A$ is characteristic in $G_0$.

(vi) The group $A$ acts on $\{2B, 2C, 2D\}$ and a triality automorphism cyclically permutes these three classes.

(vii) The group $D$ has just 6 classes $2E, ..., 2J$ of involutions not contained in $G_0$ and $(C_{G_0}(2E), C_{G_0}(2F), C_{G_0}(2G)) = (I_{-4}, F_1^1, F_1^2)$, while $(C_{G_0}(2H), C_{G_0}(2I), C_{G_0}(2J))$ is as in (iii) with 1 and 3 interchanged.

**Proof.** Ten visible classes of involutions in $D$ are defined in assertions (ii), (iii), and (vii), and by [13, Theorem 8] $D$ has precisely 10 involution
classes. (Recall $D$ is the group of inner and diagonal automorphisms of $G_0$.) Thus (i), (ii), (iii), and (vii) hold. Further (v) and (vi) follow directly from 2.2.7(iii) and 2.2.8. As for (iv), it is clear that $g$ acts as $-1$ on a $+4$-space and as $+1$ on its orthogonal complement.

**Definition.** A subgroup of $G_0$ is 2A-pure if every non-identity element is an involution in the class 2A. A 4-group in $G_0$ is 2BCD-mixed if it has involutions in each of 2B, 2C, and 2D.

**Proposition 3.4.2.** Assume that $q$ is odd and let $P \leq G_0$ be 2A-pure. Then $|P| \leq 8$ and if $|P| = 8$ then the following hold.

(i) $P$ centralizes a unique 1-decomposition $\bar{\sigma}$;

(ii) if $\bar{\sigma}$ is an $\epsilon 1$-decomposition, then

(a) $N_{G_0}(P) \cong [2^9] : L_3(2)$;

(b) $P = Z(O_2(N_{G_0}(P)))$;

(c) $N_{G_0}(P)$ is irreducible and $N_{G_0}(P) \not\leq 1 + 4$.

**Proof.** By 3.4.2(iv), $|\tilde{P}| = 2$ for all $j \in P \setminus 1$, hence $\tilde{P}$ is elementary abelian and $\tilde{P}$ can be diagonalized with respect to a basis $\beta = (v_1, \ldots, v_8)$. Moreover $\text{tr}(\tilde{P}) = 0$ for all $j \in P \setminus 1$, and thus by considering the character table of $\tilde{P}$ we find that $|\tilde{P}| \leq 16$. Thus $|P| \leq 8$ and if $|P| = 8$ then $\tilde{P} = \langle -1, x, y, z \rangle$, where (relabelling the indices if necessary)

\[
\begin{align*}
x &= \text{diag}_\beta(1, 1, 1, 1, -1, -1, -1, -1), \\
y &= \text{diag}_\beta(1, 1, -1, -1, 1, 1, -1, -1), \\
z &= \text{diag}_\beta(1, -1, -1, 1, -1, 1, 1, -1).
\end{align*}
\]

(3b)

It is now straightforward to show that the decomposition $\bar{\sigma}$ given by $V = \bigoplus v_i \oplus \cdots \oplus v_8$ is the only 1-decomposition centralized by $P$, so (i) holds. Thus putting $N = N_{G_0}(P)$ we have $\tilde{N} \leq N_\sigma(\bar{\sigma})$, which means $\tilde{N}$ acts monomially. To prove (ii), we assume that $\bar{\sigma}$ is an $\epsilon 1$-decomposition (recall, this means $V = \bigoplus v_i \perp \cdots \perp v_8$ and $\langle v_i \rangle$ is isometric to $\langle v_j \rangle$ for all $i, j$). Define $C = C_{G_0}(\tilde{P})$ and observe $C = C_{G_0}(\sigma) \cong 2^2$. Thus $C_{G_0}(P)^\sigma = C_{G_0}(P)/C$, and it is clear that $C_{G_0}(P)^\sigma$ induces the group of permutations \((15)(26)(37)(48), (13)(24)(57)(68), (12)(34)(56)(78)\) $\cong 2^3$ on the 1-spaces $\langle v_1 \rangle, \ldots, \langle v_8 \rangle$. Thus $|C_{G_0}(P)| = |C| 2^3 = 2^9$. It is not difficult to show that $\text{Aut}_N(P) \cong \text{Aut}(P) \cong L_3(2)$, hence $N \cong [2^9] : L_3(2)$, proving (ii)(a). It is also not hard to see that each element of $\tilde{C} \setminus P$ is moved by a permutation in $C_{G_0}(P)^\sigma$, and conversely, each permutation in $C_{G_0}(P)^\sigma$ moves an element of $\tilde{C} \setminus P$. Thus $P = Z(C_{G_0}(P)) = Z(O_2(N_{G_0}(P)))$, proving (ii)(b).

Since $N$ is transitive on the vectors in $\beta$, an argument similar to the one
in 2.4.9 shows that \( N \) is irreducible. Also \( \hat{N} \) contains a subgroup \( L_3(2) \) which consists of permutation matrices acting intransitively on the vectors \( v_i \). This \( L_3(2) \) has an irreducible constituent of degree \( \geq 5 \), hence \( N \leq I_{+4} \). 

**Definition.** A 2A-pure group \( P \leq G_0 \) of order 8 is called *nice* if the 1-decomposition \( \hat{\delta} \) given in 3.4.2(i) is indeed an \( \varepsilon_1 \)-decomposition. An \( N_4 \)-group is the normalizer in \( G_0 \) of a nice group.

**Proposition 3.4.3.** Assume that \( q = p \) is odd. Then there are just 4 classes of \( N_4 \)-groups in \( G_0 \) permuted naturally by \( A/G_0 \cong S_4 \).

**Proof.** Let \( P \) be a nice group and let \( N = N_{G_0}(P) \), \( \partial \), \( \beta = (v_1, \ldots, v_8) \) and \( x, y, z \) be as in the proof of 3.4.2. As \( \partial \) is an \( \varepsilon_1 \)-decomposition, we can multiply each \( v_i \) by a suitable scalar to ensure that \( (v_i, v_j) = \lambda \delta_{yj} \) for some \( \lambda \) independent of \( i \) and \( j \). Now let \( P^* \leq G_0 \) be an arbitrary 2A-pure group of order 8, and let \( N^* = N_{G_0}(P^*) \), \( \partial^*, \beta^* = (v_{1*}, \ldots, v_{8*}) \), \( x^*, y^*, z^* \) be the objects associated with \( P^* \), analogous to those associated with \( P \). Thus \( \partial^* = \langle -1, x^*, y^*, z^* \rangle \), where \( x^*, y^*, z^* \) are given in (3b) with \( \beta \) replaced by \( \beta^* \), and \( \partial^* \) is the 1-decomposition \( V = \langle v_{1*}^* \rangle \oplus \cdots \oplus \langle v_{8*}^* \rangle \).

Assume in this paragraph that \( P^* \) is nice. Then \( \partial^* \) is an \( \varepsilon_1 \)-decomposition, and so by multiplying each \( v_i^* \) by a suitable scalar we can ensure that \( (v_i^*, v_j^*) = \lambda^* \delta_{yj} \) for some fixed \( \lambda^* \). Then the map \( v_i \mapsto v_i^* \) multiplies \( Q \) by \( \lambda^* \lambda^{-1} \) and its image in \( PA \) takes \( P \) to \( P^* \). Therefore

\[
P\partial \text{ is transitive on nice groups.} \tag{3c}
\]

Hence 1.3.2(iv) ensures that \( G_0 \) has \( |PA : N_{PA}(P) G_0| \) classes of nice groups. Since \( \partial \) is the unique 1-decomposition fixed by \( P \), we have \( N_{PA}(P) \leq N_{PA}(\partial) = N_{PO}(\partial) \). Therefore \( |PA : N_{PA}(P) G_0| \geq |PA : PO| = 2 \), and as \( \hat{r}_{v_i} \in C_{PO}(P) \setminus G_0 \), we have \( |PA : N_{PA}(P) G_0| \leq 4 \). Consequently

\[
G_0 \text{ has either 2 or 4 classes of nice groups.} \tag{3d}
\]

Now fix \( a \in A \) and note that \( P^a \) is 2A-pure by 3.4.1(v). Thus we may let \( P^* = P^a \). Let \( F \) be the matrix of the bilinear form \( \langle \cdot, \cdot \rangle \) with respect to \( \beta^* \). Then \( [\hat{P}^*, F] = 1 \) which means that \( F \) is a diagonal matrix. Thus \( V = \langle v_{1*}^* \rangle \perp \cdots \perp \langle v_{8*}^* \rangle \). If the spaces \( \langle v_{i*}^* \rangle \) are not all isometric, then \( |C_{N^*}(\partial^*)| \geq 2^5 \). Since \( 2^{12} |N^*| \), we have \( 2^5 |(N^*)^\varepsilon^* \). But a Sylow 2-subgroup of \( S_8 \) is transitive on 8 points, consequently the spaces \( \langle v_{i*}^* \rangle \) are isometric after all. Therefore \( \partial^* \) is an \( \varepsilon_1 \)-decomposition which means \( P^* \) is nice. Thus \( A \) acts on the nice groups and so by (3c),

\[
A \text{ acts transitively on nice groups in } G_0. \tag{3e}
\]
We return now to statement (3d). If there are just 2 classes of nice groups in \( G_0 \), then (3e) ensures that \( P \) extends to \( \frac{1}{2}A = A' = G_{A_4} \cong G_0.A_4 \). In other words, \( \pi(N_A(P)) = A_4 \). However \( \pi(e_{q_1}) \in N_A(P) \) and \( \pi(e_{q_1}) \notin A_4 \) for it is a 2-cycle. Hence there are exactly 4 classes of nice groups in \( G_0 \), which means
\[
|A : N_A(P) G_0| = 4. \tag{3f}
\]

Evidently (3e) guarantees that \( A \) acts transitively on \( N_4 \)-groups, hence there are \( |A : N_A(N)| \) classes of \( N_4 \)-groups in \( G_0 \). However 3.4.2(ii)(b) yields \( N_A(N) = N_A(P) \), and the result now follows from (3f).

This next result describes normalizers of 2A-pure groups of order 4.

**Proposition 3.4.4.** Assume that \( q \) is odd and that \( R \) is a 2A-pure 4-group in \( G_0 \). Then \( N_{G_0}(R) \) is an \( I_{22} \)-group for some \( \varepsilon \).

**Proof.** As in the proof of 3.4.2, we can assume that \( \hat{R} = \langle -1, x, y \rangle \), where \( x \) and \( y \) are given in (3b), with respect to some basis \( \beta' = (w_1, \ldots, w_8) \), possibly distinct from \( \beta \). Let \( F \) be the matrix of the bilinear form \( (, ) \) with respect to \( \beta' \). Then \( [F, \hat{R}] = 1 \) and so \( V = V_1 \perp \cdots \perp V_4 \), where \( V_i = \langle w_{2i}, w_{2i-1} \rangle \). Since \( C_{G_0}(\hat{x}) = N_{G_0}(V_i \perp V_2, V_3 \perp V_4) \), it follows that \( V_1 \perp V_2 \) and \( V_3 \perp V_4 \) are +4-spaces. Therefore \( V_3 \) is isometric to \( V_4 \) and \( V_1 \) is isometric to \( V_2 \). Similarly \( V_2 \) is isometric to \( V_3 \), and hence the spaces \( V_i \) form an \( \varepsilon_2 \)-decomposition of \( V \) for some \( \varepsilon \). Thus \( N_{G_0}(R) = N_{G_0}(V_1, V_2, V_3, V_4) \) is an \( I_{22} \)-group.

**Part 4. The Case \( G \cap \mathcal{T} \neq \emptyset \)**

Throughout Section 4 we assume that \( G_0 \leq G \leq A \) and that \( G \cap \mathcal{T} \neq \emptyset \). Thus \( \pi(G) \) contains a 3-cycle and replacing \( G \) by a suitable \( A \)-conjugate allows us to assume that \( (123) \in G \). Thus
\[
\pi(G) \in \{ \langle 123 \rangle, S_3, A_4, S_4 \}. \tag{4a}
\]
As usual, \( M \) is a maximal subgroup of \( G \) not containing \( G_0 \) and \( M_0 = M \cap G_0 \).

**4.1. The Determination of \( \mathcal{C}_2 \)**

As advertised in Section 1.5, we obtain a collection \( \mathcal{C}_2 \) of subgroups of \( G_0 \) such that \( M_0 \) is \( G_0 \)-conjugate to some member of \( \mathcal{C}_2 \). As there is no harm in putting the \( N_3 \)-groups and \( I_{22} \)-groups in \( \mathcal{C}_2 \), we can suppose for the rest of Section 4.1 that
\[
M_0 \text{ is not an } N_3 \text{ or an } I_{22} \text{-group.} \tag{4b}
\]
Assumption (4b) will serve as a convenience in several of the arguments below. It is also convenient to define

\[ H = O^2(M_0) \quad \text{and} \quad C = C_{G_0}(H), \]

and by (4a) we can fix a triality automorphism \( \tau \in M' \setminus PT \) with \( \langle \pi(\tau) \rangle = \langle (123) \rangle \). Furthermore we let \( L \) satisfy

\[ M_0 \leq L < G_0, \]

so that \( M_0 = L_M \) by 1.3.1(i). In the following Proposition we collect a number of useful facts to which we refer in subsequent arguments.

**Proposition 4.1.1.**

(i) If \( 1 \neq O_r(M_0) \in \text{Syl}_r(M_0) \) for some odd prime \( r \), then \( r \mid q(q^4 - 1) \).

(ii) We have \( H \neq 1 \). Thus \( M_0 = N_{G_0}(H) \) and \( C \leq M_0 \).

(iii) \( M_0 \) is not contained in a Borel subgroup of \( G_0 \); hence \( M_0 \) does not stabilize a flag.

(iv) Suppose that \( H \) fixes a n.d. space \( W \) and that \( 3 \leq \dim(W) \leq 5 \). Then \( \hat{H}(W) \neq \langle -1 \rangle \).

(v) \( H \) cannot be contained in a group \( N_{G_0}(U, W) \), where \( U \) is a n.d. 2-space, \( W \) is a n.d. 4-space and \( U < W \).

(vi) If \( M \cap PT \leq N_M(L) \), then \( M_0 = L \cap L^1 \cap L^{2^1} \).

(vii) If \( q \) is odd and \( M \) normalizes a 4-group \( K \) in \( G_0 \), then \( K \) is 2BCD-mixed and \( M_0 \) fixes a n.d. 2-space.

**Proof.**

(i) By 1.3.1(iv), \( O_r(M_0) \in \text{Syl}_r(G_0) \). If \( r \mid q^2 - 1 \), then by 1.3.1(iii) and the proof of 15.1.9–15.1.10 of [2], \( M_0 = N_{G_0}(J(O_r(M_0))) \) is an \( I_{2^2} \)-group, contrary to (4b). If \( r \mid q^2 + 1 \), then \( M_0 \) is an \( N_3 \)-group, also against (4b). If \( r \mid q \), then \( M_0 \) is a Borel subgroup, contrary to 1.6.1.

(ii) Otherwise, \( M_0 \in \text{Syl}_2(G_0) \) by 1.3.1(iv). If \( q \) is even, then either \( q = 2 \), in which case \( M_0 \) is a Borel subgroup, contrary to 1.6.1, or \( q > 2 \), in which case \( N_{G_0}(M_0) \geq M_0 \), against 1.3.1(iii). If \( q \) is odd, then we may assume that \( M_0 \in \text{Syl}_2(I_{+4}) \). Now \( I_{+4} \) extends to \( A \) (see 2.2.7(iii)) and by the Frattini argument, \( M_0 \) extends from \( I_{+4} \) to \( N_A(I_{+4}) \), contrary to 1.3.3(ii). The latter statements in (ii) are a consequence of 1.3.1(iii).

(iii) Otherwise \( M_0 \leq B_0 \) for some Borel subgroup \( B_0 \) of \( G_0 \). If \( O_p(M_0) = 1 \), then \( M_0 \) embeds in a group of shape \((Z_{q-1})^4\), contrary to (i) and (ii). Thus by 1.3.1(v), \( O_p(B_0) \leq O_p(M_0) \). But \( O_p(B_0) \in \text{Syl}_p(G_0) \), and so \( O_p(B_0) = O_p(M_0) \). Therefore \( M_0 = N_{G_0}(O_p(B_0)) = B_0 \), contrary to 1.6.1. The second statement now follows because the Borel subgroups are flag stabilizers (Sect. 1.6).
(v) Assume for a contradiction that $\hat{H}(W) = \langle -1_W \rangle$. Suppose first that $H' \neq 1$, so that $M_0 = N_{G_0}(H')$ by 1.3.1(iii). Hence $\Omega(W) \leq C_{G_0}(H') \leq \hat{M}_0$. Therefore $(O^2(\Omega(W)))' \leq (O^2(\hat{M}_0))' \leq H'$, which means $(O^2(\Omega(W)))'$ acts on $W$ as a subgroup of $\langle -1_W \rangle$. Thus

$$|(O^2(\Omega(W)))'| \leq 2.$$ 

(4c)

Now $3 \leq \dim(W) \leq 5$, and thus $\Omega(W)$ is isomorphic to $\Omega_5(q)$, $\Omega_6^+(q)$ or $\Omega_6^-(q)$. Thus (4c) holds only when $W$ is a +4-space and $q = 2$. However $\Omega_4^+(2) \cong S_4$ and we deduce that $H$ has a normal Sylow 3-subgroup of order at most $3^4$. Thus the same holds of $M_0$, contrary to (i). Therefore $H' = 1$ and so if $r$ is a prime divisor of $|H|$, then $M_0$ is a Sylow $r$-normalizer in $G_0$ by 1.3.1(iv). Now $|H|$ divides $|I_{+4}|$, $|I_{-4}|$ or $|R_{e3}|$, thus $r$ divides $q$, $q^2 + 1$ or $q^2 - 1$, contradicting (i) again.

(v) Otherwise $H \leq N_{G_0}(U, Y)$ where $Y = W \cap U^\perp$, hence $\hat{H}(W)$ is contained in $\langle -1_W, \Omega(U) \times \Omega(Y) \rangle$, an abelian group. Thus $\hat{H}(W) = \langle -1_W \rangle$, against (iv).

(vi) By 1.3.1(i), it suffices to show that $M$ normalizes $L \cap L^\perp \cap L^\perp$. Since $M \cap B$ (see (1g)) normalizes $L$ and is normalized by $\tau$, it follows that $M \cap B$ normalizes $L^\perp$ and $L^{\perp^2}$. Therefore $M/M \cap B$ embeds in $S_3$, acting naturally or trivially on the set $\{L, L^\perp, L^{\perp^2}\}$. The result follows.

(vii) Clearly $M = N_G(K)$ and so $M_0 = N_{G_0}(K)$. Thus $K$ is not $2A$-pure by 3.4.4 and (4b). Thus $K$ contains an involution in $2B \cup 2C \cup 2D$. However, these three involution classes are permuted cyclically by $\tau$ (see 3.4.1(vi)), and thus $K$ has one involution from each class. Thus $K$ is $2BCD$-mixed, and $M_0$ centralizes the $2B$-involution in $K$, hence $M_0$ is contained in an $R_{e2}$-group.

We now consider the possibilities for the overgroup $L$ of $M_0$.

**Proposition 4.1.2.** If $L$ is an $R_{-2}$- or $F_2$-group, then $M_0$ is an $N_1$-group.

**Proof.** By 2.2.8, $M_0$ is contained in an $R_{-2}$-group $R$ and an $F_2$-group $F$. Let $\eta(R) = \langle x \rangle$ and $\eta(F) = \langle y \rangle$ be the cyclic groups of order $s$ as in 3.2.1. Thus $M_0 \leq R \cap F = C_{G_0}^s(x) \cap C_{G_0}^s(y)$.

**Case s odd.** Obviously $x, y \in C \cap H \leq \Omega(H)$ and so $M_0 = N_{G_0}(J)$, where $J = \Omega_1(\Omega_3(\Omega(H)))$. Since $[x, y] = 1$, $R \cap F$ is an $N_1$-group hence it suffices to prove

$$J = \langle x, y \rangle.$$ 

(4d)

We have $\hat{J} = \langle -1 \rangle \times J_0$, where $J_0 \simeq J \geq \langle \hat{x}, \hat{y} \rangle$. Let $W, E = F \langle \hat{y} \rangle$ and $f$ be as in 3.2.1. Since $J_0 \leq C_{G_0}(\hat{y})$, we can diagonalize $J_0$ over $E$ with respect
to an $E$-basis $(v_1, v_2, v_3, v_4)$ of $V$. Because $W$ and $W^\perp$ are $J_0$-invariant, they are also $E$-invariant. Hence we can arrange this basis so that $v_1 \in W$ and $v_2, v_3, v_4 \in W^\perp$. Thus $\hat{x} = \text{diag}(\omega, 1, 1, 1)$ where $\omega \in E$ is a primitive $s$th root of unity, and replacing $y$ by a suitable power of itself, we can take $\hat{y}$ to be the scalar $\text{diag}(\omega, \omega, \omega, \omega)$. Define $W_i = v_i E$, so that $W_i$ is a 2-space (over $F$), $W = W_1$, and $W^\perp = W_2 \oplus W_3 \oplus W_4$. Now take $g \in J$ and write $\hat{g} = \text{diag}(\omega', \omega', \omega', \omega')$. Multiplying $\hat{g}$ by $\hat{x}^{-1} \hat{y}^{-1}$, we can assume that $i = j = 0$. If $\omega^k \neq 1 \neq \omega'$, then $\hat{H} \leq C_0(\hat{g}) \leq N_0(C_r(\hat{g})) = N_0(W_1 \oplus W_2)$. We have $f(v_3, v_2) = f(v_3^k, v_2^k) = \omega^k f(v_3, v_2)$, hence $f(v_2, v_3) = f(v_3, v_2) = 0$. Consequently $(v_2, v_3) = f(v_2, v_3) + f(v_3, v_2) = 0$. Similarly $(v_2, v_4) = 0$ and therefore $V = (W_1 \oplus W_2) \perp (W_3 \oplus W_4)$. Thus $W_1 \oplus W_2$ is an $H$-invariant 4-space and $H \leq N_{G_0}(W_1, W_1 \oplus W_2)$, contrary to 4.1.1(v). Similarly, if $\omega^k = 1 \neq \omega'$, then $V = W_1 \perp (W_2 \oplus W_3) \perp W_4$ and $H \leq N_{G_0}(W_1, W_1 \perp W_4)$, contradicting 4.1.1(v) again. The case $\omega \neq 1 = \omega'$ is entirely similar, and we conclude that $g \in \langle x, y \rangle$. Thus (4d) holds, as desired.

Case $s = 2$. Here $q + 1$ is a power of 2 and $x, y$ are involutions in $Z(M_0)$. Hence there is a minimal normal subgroup $K$ of $M$ contained in $Q_1(G_2(Z(M_0)))$, and by 1.3.1(iii) we have $M_0 - N_{G_0}(K)$. Since $q + 1$ is a power of 2, so is $n = \log_2(q)$, and thus $\text{Out}(G_0)/O_2(\text{Out}(G_0)) \cong S_3$. As $O_2(M/M_0)$ acts trivially on $K$, it follows that $\text{Out}_M(K) \cong 1$, $Z_3$ or $S_3$. If $K \leq Z(M)$, then $|K| = 2$ and by 3.4.1, $M_0$ is an $I_4^*$-group, which is absurd. Therefore $\text{Aut}_M(K) \cong Z_3$ or $S_3$ and $K$ is a 4-group. Thus by 4.1.1(vii), $K$ is $2BCD$-mixed. Since $q \equiv 3(4)$ it follows from 3.4.1(iii) and the definition in Section 3.2 that $M_0 = C_{G_0}(K)$ is an $N_1$-group.

We may assume hereafter that

$M_0$ is not contained in an $R_{-2^r}$ or an $F_2$-group. (4e)

**Proposition 4.1.3.** If $L$ is an $R_{+2^r}$- or $I_{4^s}$-group, then $q \geq 3$ and $M_0$ is an $N_{+r}$-group. If $q = 3$, then $A_4 \leq \pi(G)$.

**Proof:** By 2.2.8, $M_0$ is contained in an $R_{+2^r}$-group $R$ and an $I_{4^s}$-group $I$. Let $\eta(R) = \langle w \rangle$ and $\eta(I) = \langle z \rangle$ be the cyclic groups of order $r$ as in 3.2.1. Thus $M_0 \leq R \cap I = C_{G_0}^*(w) \cap C_{G_0}^*(z)$. Also write $R = N_{G_0}(U)$, $I = N_{G_0} \{V_1, V_2\}$ as in 3.2.1.

Suppose in this paragraph that $q \leq 3$. Then $M_0 \leq S$, where $S$ is the stabilizer in $G_0$ of the unique 1-space in $U$. By 3.1.1(vi), $M_0 < S \cap S^r \cong G_2(q)$. Now $S \leq C_{G_0}(\bar{r})$, where $r \in O$ is a reflection in a vector of norm 1, and $\pi(\bar{r}) = (12)$ (see (1i)). Thus $\pi(\langle \bar{r}, \bar{r}^3 \rangle \rangle) = \langle (123), (12) \rangle = S_3$ and by 3.1.1(iv), $N_A(S \cap S^r) = (S \cap S^r) \times \langle \bar{r}, \bar{r}^3 \rangle$. Obviously $M_0$ extends from $S \cap S^r$ to $(S \cap S^r) \times \langle \bar{r}, \bar{r}^3 \rangle$, and so by 1.3.3(ii), $S \cap S^r$ does not extend from $G_0$ to $G$. Consequently $G \not\leq G_{S_3}$, which means $A_4 \leq \pi(G)$ and $q = 3$. 
Thus it remains to show that $M_0$ is an $N_2$-group.

Case $r$ odd. As in the proof of 4.1.2 (Case $s$ odd), we have $w, z \in J = \Omega_1(Z(H))$, $M_0 = N_{G_0}(J)$, $J = \langle -1 \rangle \times J_0$, and $\langle \hat{w}, \hat{z} \rangle \leq J_0 \cong J$. Hence there is a basis $\beta = (v_1, ..., v_8)$ with respect to which $J_0$ is a group of diagonal matrices. As in the proof of 3.2.3, we may write

$$\hat{w} = \text{diag}(\omega, \omega^{-1}, 1, 1, 1, 1, 1, 1)$$

and

$$\hat{z} = \text{diag}(\omega, \omega^{-1}, \omega, \omega, \omega^{-1}, \omega^{-1}, \omega^{-1}),$$

where $U = \langle v_1, v_2 \rangle$, $V = \langle v_3, ..., v_8 \rangle$ and $\omega = \omega_r$ is a primitive $r$th root of unity in $F$. Also $\hat{H}$ fixes the t.s. planes $X = \langle v_3, v_4, v_5 \rangle$ and $Y = \langle v_6, v_7, v_8 \rangle$, and the representation of $\hat{H}$ on $X$ is dual to its representation on $Y$ (see 1.2.2). We claim

$$J = \langle w, z \rangle. \quad (4f)$$

Take $g \in J$ and write $\hat{g} = \text{diag}(\omega', \omega^{-i}, \omega', \omega, \omega^{-i}, \omega^{-k}, \omega^{-i})$. Now multiply $\hat{g}$ by $\hat{z}^{-r} \hat{w}^{j-t}$ to ensure that $i = j = 0$. Suppose that $\omega^k \neq 1 \neq \omega'$. Then $H \leq C_{G_0}(\hat{g}) \leq N_{G_0}(C_{\hat{H}}(\hat{g})) = \langle v_1, v_2, v_3, v_6 \rangle$. If $a \in \{1, 2, 3, 6\}$ then $(v_a, v_4) = (v_a^2, v_4^6) = \omega^k (v_a, v_4)$, whence $(v_a, v_4) = 0$. Similarly $(v_a, v_5) = (v_a, v_7) = (v_a, v_8) = 0$, and so $V = C_{\hat{H}}(\hat{g}) \perp \langle v_4, v_5, v_7, v_8 \rangle$, which means $C_{\hat{H}}(\hat{g})$ is a +4-space. But then $H \leq N_{G_0}(\{U, C_{\hat{H}}(\hat{g})\})$, contrary to 4.1.1(v). Now suppose that $\omega^k = 1 = \omega'$. Then $V = C_{\hat{H}}(\hat{g}) \perp \langle v_5, v_8 \rangle$, which means $H$ fixes the +4-space $V \perp \langle v_5, v_8 \rangle$, contradicting 4.1.1(v) again. The case $\omega^k \neq 1 = \omega'$ is identical, and so $\omega^k = \omega' = 1$, proving (4f). Hence $M_0$ is the $N_2$-group $C_{G_0}(w) \cap C_{G_0}(z)$.

Case $r = 2$. Here $q - 1 = 2^i$ for some $i \geq 1$, hence $n = \log_p(q) \leq 2$. Suppose that $2 \mid |Z(M_0)|$. Then there is a minimal normal subgroup $K$ of $M$ contained in $\Omega_2(O_2(Z(M_0)))$, and $M_0 = N_{G_0}(K)$. As $n \leq 2$, $\text{Out}(G_0)$ is a subgroup of $Z_2 \times S_4$, so the argument used in the proof of 4.1.2 (Case $s = 2$) shows that $M_0$ is the centralizer of a $2BCD$-mixed 4-group, and hence by (4e), $M_0$ is an $N_2$-group. Assume therefore that

$$2 \mid |Z(M_0)|. \quad (4g)$$

Then $N_{G_0}(U)$ does not centralize an involution and so 3.4.1(iii) yields $q = 3(4)$. Therefore $q = 3$ and so $w, z \in D \setminus G_0$. Now $w$ inverts $wz$, hence $w$ inverts $(wz)^2 \in G_0$. However $w, z \in C_D(M_0)$, hence $(wz)^2 \in C_{G_0}(M_0) \leq M_0 \leq C_{G_0}(\langle w, z \rangle)$, hence $w$ centralizes $(wz)^2$. Therefore $|\langle wz \rangle|^2 \leq 2$, and thus $(wz)^2 = 1$ by (4g). Therefore $[w, z] = 1$, whence $N_{D}(M_0) = M_0 \times \langle w \rangle \times \langle z \rangle$. It follows from (4g) that $\langle w, z \rangle = O_2(Z(N_{D}(M_0)))$ and so
$M$ normalizes $\langle w, z \rangle$. Therefore $M = N_G(\langle w, z \rangle)$ hence $M_0$ is the $N_2$-group $C_{g_0}(\langle w, z \rangle)$. 

Hereafter we can assume that

\[ M_0 \text{ is not contained in an } R_{+2}^- \text{ or } I_{44}^- \text{-group.} \quad (4h) \]

**Proposition 4.1.4.** If $L$ is an $R_{44}^-$ or an $R_{44}$-group, then $M_0$ is a parabolic subgroup of $G_0$ corresponding to the central node of the Dynkin diagram.

**Proof.** The arguments in this proof rely on the facts that $A$ acts on the set $\mathcal{S} \cup \mathcal{S}_1 \cup \mathcal{S}_2$ and that $A$ preserves incidence therein (see Sect. 1.6). We may write $M_0 \leq L = N_{G_0}(\langle v \rangle)$, where $v \in V$ and $\langle v \rangle \in \mathcal{P}$, and we put $V_1 = \langle v \rangle^\tau \in \mathcal{S}_1$ and $V_2 = \langle v \rangle \in \mathcal{S}_2$. Thus $M_0 \leq N_{G_0}(V_i)$ for $i = 1, 2$.

**Step 1.** $v \in V_1 \cup V_2$.

Assume otherwise. As $v \not\in V_1$, $\dim(V_1 \cap V_2) = 1$ (because $\tau$ preserves incidence) and we fix $w \in (V_1 \cap V_2) \setminus \{0\}$. Now $(v, w) = 0$ by (4h), hence $v \in w^\perp = V_1 + V_2$. Thus $w = v_1 + v_2$ where $v_i \in V_i$. It follows that $M_0$ fixes the t.s. lines $\langle v_i, w \rangle$. Observe $v \notin V_1 - V_1^\perp$ and hence $v^\perp \cap V_1$ is an $M_0$-invariant plane. Furthermore $(v_1, v) = (v_1, v_2) = Q(v) - Q(v_1) = 0$ and so $v_1 \in v^\perp$. Thus $M_0$ stabilizes the flag $(\langle w \rangle, \langle v_1, w \rangle, v^\perp \cap V_1, V_1)$, against 4.1.1(iii).

**Step 2.** $v \in V_1 \cap V_2$.

Assume for a contradiction that $v \notin V_1$, and as in Step 1 let $\langle w \rangle = V_1 \cap V_2$. Notice that Step 1 forces $v \in V_2$. So as $\tau$ preserves incidence, the t.s. point $V_1^\tau$ is contained in $V_1$. Now $V_1^\tau \not\subseteq V_2$ (since $v \notin V_1$), hence $V_1^\tau \not= \langle w \rangle$. Also $V_2^\tau \subseteq v^\perp$ by (4h) and so $M_0$ stabilizes the flag $(\langle w \rangle, \langle w \rangle \oplus V_2^\tau, V_1 \cap v^\perp, V_1)$, another contradiction. Therefore $v \in V_1$ and a similar argument (which uses $\tau^2$ instead of $\tau$) shows that $v \in V_2$.

**Step 3.** $\tau^3 \in N_{Pr}(\langle v \rangle)$.

Write $\langle u \rangle = \langle v \rangle^\tau \in \mathcal{P}$. Since $\tau$ preserves incidence, Step 2 guarantees that $u \in V_1 \cap V_2$. Thus if $\langle u \rangle \neq \langle v \rangle$, then $M_0$ fixes the flag $(\langle v \rangle, \langle u \rangle, V_1 \cap V_2, V_1)$, a contradiction.

**Step 4.** $M_0$ fixes a unique t.s. point.

Let $\langle x \rangle$ be a t.s. point fixed by $M_0$ and set $W_1 = \langle x \rangle^\tau \in \mathcal{S}_1$ and $W_2 = W_1^\tau \in \mathcal{S}_2$. We claim that

$$x \in V_1.$$  \hspace{1cm} (4i)

Otherwise, $x \not\in V_1$ and so $W = V_1 \cap x^\perp$ has dimension 3. Applying Step 2 to $x$ yields $x \in W$, hence $V_1 \cap W_2 \leq W$. And as $(v, x) = 0$ (by (4h)),

\[ x \in V_1. \]
\[ v \in W. \] However \( \langle x \rangle \) and \( V_1 \) are not incident, hence \( W_2 \) and \( V_1^2 \) are not incident. But \( V_1^2 = \langle v \rangle^2 = \langle v \rangle \) by Step 3, whence \( v \notin W_2 \). Therefore \( v \in W \setminus (V_1 \cap W_2) \) which means \( V_1 \cap W_2 \neq W \). Thus as \( \dim(V_1 \cap W_2) \) is odd, \( V_1 \cap W_2 \in \mathcal{P} \). Therefore \( M_0 \) stabilizes the flag \( (V_1 \cap W_2, (V_1 \cap W_2) \oplus \langle v \rangle, W, V_1) \), a contradiction. Therefore (4i) holds, and similarly \( x \in V_2 \). However, because \( M_0 \) stabilizes no flag, \( M_0 \) fixes a unique point in the plane \( V_1 \cap V_2 \), hence \( \langle v \rangle = \langle x \rangle \), as required.

**Step 5. Conclusion.**

By Step 4, \( M \cap P \Gamma \) fixes \( \langle v \rangle \) and hence normalizes \( L \). Therefore by 4.1.1(vi), \( M_0 = L \cap L^i \cap L^2 = \mathcal{N}_{G_0}(\langle v \rangle, V_1, V_2) \), and the result now follows from Section 1.6. □

Recall that \( R_{s3^*} \)-groups are contained in \( R_{s4^-} \)-groups (see (10)). Thus we can assume hereafter that

\[ M_0 \text{ is not contained in an } R_{s1^-}, R_{s3^-}, \text{ or } R_{s4^-} \text{-group.} \]  \hspace{1cm} (4j)

**Proposition 4.1.5.** If \( L \) is an \( R_{s1^-} \)-group, then \( M_0 \) is a \( G_2 \)-group.

**Proof.** Write \( M_0 \leq L = \mathcal{N}_{G_0}(W) \), where \( W \) is a nonsingular 1-space. By (4e), (4h), and (4j), \( M_0 \) fixes no other nonsingular 1-space. Therefore \( M \cap P \Gamma \) normalizes \( L \), whence \( M_0 = L \cap L^i \cap L^2 \) by 4.1.1(vi). Now \( L^2 \) is a \( K_1 \)-group (2.2.4), hence by 3.1.1(vi), \( G_2(q) \simeq L^i \cap L^2 = \mathcal{N}_{L}(X) \) for some nonsingular 1-space \( X \). Thus \( W = X \) and \( M_0 = L \cap L^i \simeq G_2(q) \). □

Thus we may assume that

\[ M_0 \text{ is not contained in an } R_{s1^-} \text{-group.} \]  \hspace{1cm} (4k)

**Proposition 4.1.6.** If \( L \) is an \( R_{s2^-} \)-group, then \( M_0 = L \).

**Proof.** Write \( M_0 \leq L = \mathcal{N}_{G_0}(W) \), where \( W \) is a t.s. line in \( V \). If \( W \) is the unique t.s. line fixed by \( M_0 \), then \( M \) normalizes \( L \) and the proposition follows. So assume for a contradiction that \( M_0 \) fixes another t.s. line \( U \neq W \). By (4j), \( W \cap U = W \cap U^\perp = 0 \), hence \( W \oplus U \) is a \( +4 \)-space and \( M_0 \) is irreducible on \( W \) and \( U \). Define \( Y = (W \oplus U) \perp \), a \( +4 \)-space. By (4e), (4h), and (4k), \( M_0 \) does not fix a 1-space or a n.d. 2-space in \( Y \). And if \( M_0 \) fixes a t.s. line \( Y_0 \) in \( Y \), then \( M_0 \) fixes the t.s. solid \( W \perp Y_0 \), against (4j). Therefore \( M_0 \) is irreducible on \( Y \), hence \( Y \) is the unique 4-space upon which \( M_0 \) acts irreducibly. Thus \( I = \mathcal{N}_{G_0}(U \oplus W, Y) \) is the unique \( I_{+,4} \)-group containing \( M_0 \), which means \( M \) normalizes \( I \). But then \( M = \mathcal{N}_{G}(I) \) and so \( M_0 = I \), a contradiction. □

We hereafter assume that

\[ M_0 \text{ is not contained in an } R_{s2^-} \text{-group.} \]  \hspace{1cm} (4l)
Clearly (4j) and (4l) imply that $M_0$ does not fix a nonzero t.s. space and therefore by [4],
\[ O_p(M_0) = 1. \] (4m)

**Proposition 4.1.7.** If $L$ is an $I_{64}$-group, then $\varepsilon = +$, and $M_0 = L$.

**Proof.** First observe that (4e)-(4m) imply

**Step 1.** Any nonzero irreducible $M_0$-submodule of $V$ is n.d. of dimension 3, 4, 5, or 8.

Now write $M_0 \leq L = N_{O_3}(U, W)$, where $U$ and $W = U^\perp$ are $\varepsilon 4$-spaces. Define
\[ N = N_{M_0}(U), \]
so that $|M_0 : N| \leq 2$.

**Step 2.** $N$ is irreducible on $U$ and $W$.

By Step 1 we can assume that $N \neq M_0$. Let $X \neq 0$ be an irreducible $\hat{N}$-submodule of $U$ and choose $g \in M_0 \setminus N$. Then $X \perp X^g$ is $M_0$-invariant, hence by Step 1, $X \perp X^g$ is either a n.d. 4-space or is all of $V$. The former cannot hold by 4.1.1(v), hence $X = U$, as required.

**Step 3.** $H$ is irreducible on $U$ and $W$.

Otherwise there exists an irreducible $\hat{H}$-submodule $U_1$ of $U$, with $0 < U_1 < U$. If $\dim(U_1) = 1$, then $\hat{H}$ may be diagonalized on $U$ by Clifford's Theorem. But then $\hat{H}(U) = \langle -1_U \rangle$, contrary to 4.1.1(iv). Also $U_1$ is not a n.d. 2-space by 4.1.1(v). Therefore $U_1$ is a t.s. line, which means $\varepsilon = +$. If $H$ is irreducible on $W$, then $W$ is the unique 4-space upon which $H$ acts irreducibly, hence $L$ is the unique $I_{+4}$-group containing $H$. Thus $M$ normalizes $L$, whence $M_0 = L$. So it may be assumed that $H$ is also reducible on $W$, and the previous argument shows that $\hat{H}$ acts irreducibly on a t.s. line, $W_1 \leq W$. In particular, all irreducible constituents of $\hat{H}$ on $V$ have dimension 2. But $H$ fixes the t.s. solid $U_1 \perp W_1$ and so (by applying triality) $H$ also fixes a t.s. point, whence $\hat{H}$ has an irreducible constituent of dimension 1. This contradiction completes the proof of Step 3.

**Step 4.** We can assume that $U \cong W$ as $\hat{H}$-modules and that $\hat{H}$ is absolutely irreducible on $U$.

**Case $\varepsilon = +$**. If $U \cong W$ as $\hat{H}$-modules, then $U$ and $W$ are the only nonzero irreducible $\hat{H}$-submodules of $V$, hence $L$ is the unique $I_{+4}$-group containing $H$. Therefore $M_0 = L$, as required. Thus it may be assumed that $U \cong W$ as $\hat{H}$-modules. Suppose now that $\hat{H}$ fails to be absolutely irreducible on $U$. Then by 7.6.1 of [2], $\hat{H}(U)$ embeds in a group $\Omega_2^+(q^2)$ or $GU_2(q)$. In
either case, \( \hat{C} \) contains a group \( C_1 \) satisfying \( Z_{q+1} \cong C_1 < \Omega(U) \). Since \( U \cong W \) as \( \hat{H} \)-modules, \( C_{\hat{H}}(W) = 1 \) and so \( C_1 \cap \hat{H} = 1 \). Therefore \( \hat{C} \) embeds in \( M_0/\hat{H} \), which means \( |C_1| = q + 1 \) is a power of 2. Consequently \( q \) is odd and \( n = \log_p(q) \) is also a power of 2. Thus \( M_0 \) centralizes the involution in \( Z(L) \) and reasoning as in the proof of 4.1.2 (Case \( s = 2 \)), we deduce that \( M = N_{\hat{G}}(K) \), where \( K \leqslant \Omega_1(O_2(Z(M_0))) \) and \( |K| \leqslant 4 \). Clearly \(|K| \neq 4 \) by 4.1.1(iv) and Step 1, so we have \(|K| = 2 \). Therefore \( M_0 \) is the centralizer of a 2A-involution, hence \( M_0 \) is an \( I_{+4} \)-group, as desired. Thus it may be assumed that \( \hat{H} \) is absolutely irreducible on \( U \).

**Case \( \varepsilon = - \).** If \( \hat{H} \) fails to be absolutely irreducible on \( U \), then by 7.6.1 of [2], \( \hat{H}(U) \) embeds in a group \( SO_2^1(q^2) \), which is abelian. Therefore \( \hat{H}(U) = \langle -1_U \rangle \), contrary to 4.1.1(iv). Thus we can suppose that \( \hat{H} \) is absolutely irreducible on \( U \) and similarly on \( W \). As \( M_0 \) is contained in an \( F_{-1} \)-group (see 2.2.8), \( \text{Hom}_{\hat{H}}(V) \) contains a quadratic field extension of \( F \). Therefore \( C_{GL(V)}(\hat{H}) \) contains a group \( Z_{q^2}^{-1} \). If \( U \not\cong W \) as \( \hat{H} \)-modules, then \( C_{GL(V)}(\hat{H}) \) fixes both \( U \) and \( W \) and induces scalars on these spaces. But this means \( C_{GL(V)}(\hat{H}) \cong Z_{q-1} \times Z_{q-1}^{-1} \), a contradiction. Therefore \( U \cong W \), and Step 4 now follows.

**Step 5. Conclusion.**

Fix a basis \( \beta_1 = (u_1, u_2, u_3, u_4) \) of \( U \). By Step 4, there is a basis \( \beta_2 = (w_1, w_2, w_3, w_4) \) of \( W \) such that that elements \( h \in \hat{H} \) have the form

\[
h = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \quad (b \in GL_4(q)),
\]

with respect to \( \beta = (\beta_1, \beta_2) = (u_1, ..., u_4, w_1, ..., w_4) \). We write \( h = 1 \otimes b \) and in general we write \( a \otimes b \) for the matrix

\[
\begin{pmatrix} xb & yb \\ zb & wb \end{pmatrix} \in GL(V),
\]

where \( b \in GL_4(q) \) and \( a = (x \ y) \in GL_2(q) \). With this convention, Step 4 yields \( C_{GL(V)}(\hat{H}) = GL_2(q) \otimes 1 \) and

\[
C_{\hat{H}}(\hat{H}) \leqslant \langle -1, z \rangle, \quad \text{where} \quad z = (-1)^{-1}_U 1 = -1_U. \tag{4n}
\]

Notice that 1.2.3 implies \( C_{\hat{H}}(\hat{H}) = \hat{C} = C_{\hat{M}_0}(\hat{H}) \). In particular,

\[
|\hat{C}| = 2|C_{\hat{H}}(\hat{H})|. \tag{4o}
\]

Moreover the proof of 1.7.1(i) shows that there exists \( \lambda \in F^* \) such that

\[
Q(u_i) = \lambda Q(w_i), \quad 1 \leq i \leq 4. \tag{4p}
\]
If $-\lambda^{-1} = \mu^2$ for some $\mu \in F$, then $H$ fixes the t.s. solid $\langle u_i + \mu w_i : 1 \leq i \leq 4 \rangle$. But then $H$ fixes a t.s. point, contrary to Step 3. Therefore $-\lambda$ is a non-residue, and so $q$ is odd.

**Case $\varepsilon = +$.** Here equality holds in (4n) and it follows from (4o) that $|\hat{C}|$ is 4 or 8. If $|\hat{C}| = 4$, then $\hat{C} = \langle -1, z \rangle$ and $C = \langle \hat{z} \rangle$; hence $M_0 = C_{g_0}(\hat{z}) = L$, as desired. If, however, $|\hat{C}| = 8$, then $C \cong \mathbb{Z}_4$ or $(\mathbb{Z}_2)^2$. In the former case, $M_0 = C_{g_0}(\Omega_1(C)) = C_{g_0}(\hat{z}) = L$, as required. And the latter case is ruled out by Step 1 and 4.1.1(vii).

**Case $\varepsilon = -$.** Since $-1 \not\in \Omega(U)$, we have $z \not\in \Omega$ and so (4n) yields $C_{g_0}(\hat{H}) = \langle -1 \rangle$. Therefore $|C| \leq 2$ by (4o). If $|C| = 2$, then $M_0 = C_{g_0}(C)$ is an involution centralizer which means $M_0$ is an $I_{+4}$-group. But $I_{+4} \not\leq I_{-4}$, hence $C = 1$ and so $\hat{C} = C_{g_0}(\hat{H}) = \langle -1 \rangle$. Consequently

$$|C_{SO}(\hat{H})| \leq 4. \quad (4q)$$

Let $F$ be the matrix of the bilinear form $(\ , \ )$ on $U$ with respect to $\beta_1$. By (4p), the matrix of $(\ , \ )$ on $V$ with respect to $\beta$ is $f \otimes F$ where $f = (1 \ 0) \in GL_2(q)$. Thus $C_O(\hat{H}) = \{ g \otimes 1 \in GL_2(q) \otimes 1 : g'fg = f \}$. Since $-\lambda$ is a nonresidue, $f$ is the matrix of a nondegenerate symmetric form on a 2-space which gives rise to an $O_\varepsilon^2$-geometry (see 1.2.1). Therefore $C_O(\hat{H}) \cong O_\varepsilon^{-2}(q) \cong D_{2(q+1)}$. And if $g \otimes 1 \in C_O(\hat{H})$, then $\det(g) = \pm 1$ and so $\det(g \otimes 1) = \det(g)^q = 1$. Consequently $|C_{SO}(\hat{H})| = |C_O(\hat{H})| = 2(q+1)$, against (4q). $\blacksquare$

By 2.2.8, we may assume hereafter that

$M_0$ is not contained in an $I_{+4}$, $I_{-4}$ or $F_1$-group. \hspace{1cm} (4r)

**Proposition 4.1.8.** If $L$ is an $R_{+3}$, $T$, or $S$-group, then $M_0 = L$ and $L$ is an $S_x$-group for some prime divisor $x$ of $n = \log_3(q)$.

**Proof.** We argue that

$$C_D(M_0) = 1. \quad (4s)$$

Take $x \in C_D(M_0)$ of order $r$, where either $r = 1$ or $r$ is prime. Evidently $r \neq p$ in view of (4m). Thus (4e), (4h), (4r), and 3.4.1(i), (ii), (iii), (vii) ensure that $r \neq 2$. Consequently $r$ is odd and $x \in C_{g_0}(M_0) \leq M_0$ and $\hat{M}_0 \leq C_{g_0}(\hat{x})$ by 1.2.3. Suppose for the moment that $M_0$ is reducible. By (4e), (4h), (4j), (4k), (4l), and (4r), $q$ is odd and $M_0$ acts irreducibly on $W$ and $W^\perp$, where $W$ is a n.d. 5-space. If $M_0$ fails to be absolutely irreducible on $W$, then $\hat{M}_0(W)$ embeds in $GL_1(q^5) \cong \mathbb{Z}_{q^5-1}$. But then $\hat{H}(W) = \langle -1, w \rangle$, contrary to 4.1.1(iv). Therefore $M_0$ is absolutely irreducible on $W$, and similarly on $W^\perp$. Thus by Schur's Lemma, $\hat{x}$ induces scalars on both $W$
and $W^\perp$ and so acts as $\pm 1$ on $W$ and $W^\perp$. Thus $r \leq 2$ and by a previous remark, $r = 1$. So we assume that $M_0$ is irreducible. Thus $E = \text{Hom}_{sk_0}(V)$ is a field, and if $E \neq F$ then the proof of 11.5 of [2] shows that $M_0$ is contained in a member of $C_3$; that is, $M_0$ is contained in an $F_1$- or $F_2$-group, contrary to (4r) and (4e). Therefore $E = F$, whence $x \in F \cap \Omega = \langle -1 \rangle$, and (4s) has been proved.

Now suppose that $L$ is an $R_{\pm 3^r}$, $T_r$, or $S$-group. Then by 2.2.11, $L = C_{G_0}(y)$ for some $y \in A \setminus G_0$. Observe that the groups in 1.4.1(i), (ii), (iv) are not contained in $L$, hence 1.4.2(ii) ensures that $M_0 = C_{G_0}(y) = L$. By 2.2.5 and 2.2.10, the $R_{\pm 3^r}$, $T_r$, and $S$-groups are not normalized by a triality automorphism. Consequently $M_0 = L$ is an $S_\infty$-group, as required.

Hereafter we can assume that

$$M_0 \text{ is not contained in an } R_{\pm 3^r}, T_r, \text{ or } S \text{-group.} \quad (4t)$$

Evidently (4e), (4h), (4j), (4k), (4l), (4r), and (4t) imply that

$$M_0 \text{ is irreducible on } V. \quad (4u)$$

**Proposition 4.1.9.** $L$ is not an $I_{42}$-group.

**Proof.** Otherwise, $M_0 \leq L = N_{G_0}(\partial)$, where $\partial$ is an $\varepsilon 2$-decomposition $V_1 \perp \cdots \perp V_4$ for some $\varepsilon = \pm$. By (4u) and (4r), $M_0$ acts primitively on $\{V_1, V_2, V_3, V_4\}$, hence

$$M_0^0 \cong A_4 \text{ or } S_4. \quad (4v)$$

Observe that $L$ is a Sylow $r$-normalizer in $G_0$ for all primes $r \geq 5$ dividing $|L|$. Thus $M_0$ has normal Sylow $r$-subgroups for all $r \geq 5$, hence 4.1.1(i) ensures that $M_0$ is a $\{2, 3\}$-group. Observe that $O_3(C_L(\partial))$ is abelian. Thus $O_3(L)$ is the set of 3-elements in $C_L(\partial)$. Similarly (4v) ensures that $O_3(M_0)$ is the set of 3-elements in $C_{M_0}(\partial)$, hence $O_3(M_0) \leq O_3(L)$. Now if $9 | |M_0|$, then $3 | |C_{M_0}(\partial)||$ and so $O_3(M_0) \neq 1$. Thus by 1.3.1(v), $O_3(L) \leq O_3(M_0)$. Therefore $O_3(M_0) = O_3(L)$ and so $M_0 = N_{G_0}(O_3(M_0)) = N_{G_0}(O_3(L)) = L$, against (4b). Therefore $|M_0| = 2^a 3$ for some $a$. This implies that $J = O_2(M_0) > 1$ and hence $q$ is odd by (4u). Further

$$M_0/J \cong S_3 \text{ or } Z_3, \quad (4w)$$

hence 5.5 of [2], (4u), (4h), and (4r) imply that

$J$ acts homogeneously on $V$. \quad (4x)

We now argue that

$$\Omega_1(Z(J)) \text{ is } 2A\text{-pure.} \quad (4y)$$
Otherwise, $Z(J)$ contains an involution in $2B \cup 2C \cup 2D$. Since $\tau$ normalizes $Z(J)$, $J$ centralizes involutions in each of $2B$, $2C$ and $2D$. Thus

$$|\Omega_1(Z(J))| \geq 4 \quad (4z)$$

and $\hat{J}$ fixes an $e_2$-space $W$ (see 3.4.1(iii)). By (4x), $\hat{J}$ acts faithfully on $W$ and this means $\hat{J}$ embeds in $O(W) \cong D_{2(q-\varepsilon_1)}$. Hence $\hat{J}$ is either cyclic or dihedral and thus the same holds of $J$. Therefore (4z) implies that $J$ is a 4-group, contradicting 4.1.1(vii) and (4u). Thus (4y) holds, and by 3.4.2, $|\Omega_1(Z(J))| \leq 8$. Now $|\Omega_1(Z(J))| \neq 4$ by 4.1.1(vii). Thus $|\Omega_1(Z(J))| = 2$ or 8, and so by (4w), $M_0$ centralizes an involution in $\Omega_1(Z(J))$. But then $M_0$ is contained in an $I_{4+}$-group, against (4r). This final contradiction finishes the proof.

**Proposition 4.1.10.** Assume that $q = p \geq 3$. If $L$ is an $I_{4+}$- or $E$-group, then $M_0$ is an $N_4$-group.

**Proof.** By 2.2.6, we may write $M_0 \leq L = N_{G_0}(\partial)$ where $\partial$ is the $e_1$-decomposition $V_1 \perp \cdots \perp V_8$, and we put $K = O_2(L) = C_{G_0}(\partial)$. The group $L^e$ acts faithfully on $K \cong 2^6$ as a subgroup of $O(K, Q_0, GF(2)) \cong O^{+}_{6}(2)$, stabilizing a quadratic form $Q_0$ on $K$ defined as follows: if $k = \text{diag}(\varepsilon_1, \ldots, \varepsilon_8) \in K$, then $Q_0(k)$ is 0 or 1 according as the number of eigenvalues $\varepsilon_i$ equal to 1 is congruent to 0 or 2 mod 4. One easily verifies that $Q_0$ is a nondegenerate quadratic form of defect 0. On the one hand $M_0^e$ embeds in $O(K, Q_0, GF(2))$, and on the other, $M_0^e$ embeds in $S_8$, acting faithfully on the 1-spaces $V_i$, $1 \leq i \leq 8$. By (4u), (4r) and 4.1.9, $M_0^e$ is a primitive subgroup of $S_8$ and hence by [6], one of the following occurs:

1. $2^3:7 \leq M_0^e \leq 2^3:L_3(2) \leq S_8$,
2. $L_2(7) \leq M_0^e \leq \text{PGL}_2(7) \leq S_8$,
3. $A_8 \leq M_0^e \leq S_8$.

Suppose for the moment that $O_2(M_0) = 1$. Then (b) or (c) holds. It now follows from the facts that $M_0$ is irreducible ((4u)), $M_0$ is not contained in an $F$-group ((4e) and (4r)), $M_0$ is not contained in the stabilizer of a 4-decomposition of $V$ ((4h) and (4r)), and $M_0$ is not contained in an $S$-group ((4t)), that $M_0 = M_0^e$ is a $C_0$-group. However, this contradicts 2.3.13. Therefore $O_2(M_0) \neq 1$, whence 3.1.1(v) ensures that $K \leq O_2(M_0)$. Since $N_{G_0}(O_2(M_0)) = M_0 \neq L = N_{G_0}(K)$, we have $K < O_2(M_0)$. Therefore $O_2(M_0^e) > 1$ and (a) occurs. Thus $|O_2(M_0)| = 2^9$, $M_0 = N_{G_0}(O_2(M_0)) \cong [2^9].L_3(2)$ and $M_0^e \cong 2^3:L_3(2)$. Thus $M_0^e$ fixes a t.s. 3-subspace $P$ of $K$ in the $O(K, Q_0, GF(2))$-geometry associated with $K$. It follows from the definition of $Q_0$ and 3.4.1 that $P$ is $2A$-pure. And as the 1-spaces $V_i$ are isometric, $P$ is nice (see Sect. 3.4). It now follows from 3.4.2 and 3.4.3 that $M_0 = N_{G_0}(P)$ is an $N_4$-group. 


TABLE III

<table>
<thead>
<tr>
<th>Name</th>
<th>Order</th>
<th>Non-abelian composition factors</th>
<th>Restrictions on $q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_2$</td>
<td>$\frac{1}{d^2}q^{12}(q^2-1)(q-1)^3$</td>
<td>$L_2(q)^*$</td>
<td></td>
</tr>
<tr>
<td>$R_{22}$</td>
<td>$\frac{1}{d^2}q^{12}(q^2-1)(q-1)$</td>
<td>$L_2(q)^*$</td>
<td></td>
</tr>
<tr>
<td>$N_1$</td>
<td>$\frac{2}{d^2}q^4(q^3+1)(q^2-1)(q+1)^2$</td>
<td>$U_3(q)^*$</td>
<td></td>
</tr>
<tr>
<td>$N_2$</td>
<td>$\frac{2}{d^2}q^4(q^3-1)(q^2-1)(q-1)^2$</td>
<td>$L_3(q)$</td>
<td></td>
</tr>
<tr>
<td>$G'_2$, $1 \leq i \leq d^2$</td>
<td>$q^4(q^6-1)(q^2-1)$</td>
<td>$G_2(q)'$</td>
<td></td>
</tr>
<tr>
<td>$I_{+4}$</td>
<td>$\frac{4}{d^2}q^4(q^3-1)^4$</td>
<td>$L_2(q)^*$</td>
<td></td>
</tr>
<tr>
<td>$N_3$</td>
<td>$\frac{16}{d^2}(q^2+1)^2$</td>
<td>none</td>
<td></td>
</tr>
<tr>
<td>$I_{+2}$</td>
<td>$\frac{192}{d^2}(q-1)^4$</td>
<td>none</td>
<td></td>
</tr>
<tr>
<td>$I_{-2}$</td>
<td>$\frac{192}{d^2}(q+1)^4$</td>
<td>none</td>
<td></td>
</tr>
<tr>
<td>$S_*$</td>
<td>$\frac{1}{d^2}q_0^{12}(q_0^6-1)(q_0^4-1)^2(q_0^2-1)$</td>
<td>$PO_{+}^+(q_0)$</td>
<td>$\alpha$ prime, $(\alpha, d) = 1$</td>
</tr>
<tr>
<td>$S_2$, $1 \leq i \leq 4$</td>
<td>$q^4(q-1)(q^2-1)^3(q^3-1)$</td>
<td>$PO_{+}^+(q_0)$</td>
<td>$q = q_0^2$, $q$ odd</td>
</tr>
<tr>
<td>$N_2$, $1 \leq i \leq 4$</td>
<td>$2^{12}.3^2.7$</td>
<td>$L_3(2)$</td>
<td>$q = p &gt; 2$</td>
</tr>
<tr>
<td>$K_3$, $1 \leq i \leq d^2$</td>
<td>$q^4(q^2-1)(q^2-1)$</td>
<td>$L_3(q)$</td>
<td>$2 &lt; q = 1e1(3)$</td>
</tr>
<tr>
<td>$K'_2$, $1 \leq i &lt; 2d^2$</td>
<td>$q^{12}(q_1^4+1)(q_1^2-1)(q_1^2-1)$</td>
<td>$3D_4(q_1)$</td>
<td>$q = q_1^2$</td>
</tr>
<tr>
<td>$K'_3$, $1 \leq i &lt; d^4$</td>
<td>$2^{12}.3^2.7.1$</td>
<td>$\Omega^+_8(2)$</td>
<td>$q = p &gt; 2$</td>
</tr>
<tr>
<td>$K_6$, $1 \leq i \leq 8$</td>
<td>$2^6.5.7.13$</td>
<td>$2_2(8)$</td>
<td>$q = 5$</td>
</tr>
</tbody>
</table>

* $L_2(2), L_3(3)$, and $U_3(2)$ are not simple.

4.2. Maximal amongst the Groups in $\mathcal{G}_2$

As Section 4.1 deals with the case in which $M_0$ is contained in a classical subgroup of $G_0$, and as 2.3.13 handles the case in which $M_0$ is a $C_9$-group, we may conclude

PROPOSITION 4.2.1. Suppose that $G_0 \leq G \leq A$, and $G \cap \mathcal{I} \neq \emptyset$. Then $M_0$ is $G_0$-conjugate to some member of $\mathcal{G}_2$, where $\mathcal{G}_2$ consists of the following groups in Table III.

For each $H \in \mathcal{G}_2$ we determine those $G$ for which $N_G(H)$ is maximal in $G$. In doing so, we show that the remaining rows of the results matrix (Table I) hold, thereby completing the proof of our theorem.
PROPOSITION 4.2.2. Row 4 of the results matrix holds.

Proof. Since $P_2$ is a parabolic subgroup of $G_0$ corresponding to the central node of the Dynkin diagram of $G_0$, while $R_{s2}$ corresponds to the three outer nodes, we have $P_2 \not< R_{s2}$. None of the other members of $\mathcal{C}_2$ contain a parabolic subgroup of $G_0$, hence the result follows.

PROPOSITION 4.2.3. Row 22 of the results matrix holds.

Proof. Order considerations eliminate all the groups in $\mathcal{C}_2$ as possible overgroups of $N_2$, save $G_2$ (when $q \leq 3$) and $S_x$. It follows easily from 2.4.1 that $L_3(q)$ is not involved in $S_x$, and hence $N_2 \not< S_x$. Since $G_2$ does not extend to $G = A'$ or $G_x = A$ when $q = 3$, the result follows from 3.2.3 and 4.1.3.

PROPOSITION 4.2.4. Row 26 of the results matrix holds.

Proof. Lagrange's Theorem ensures that $N_1$ is contained in no other group in $\mathcal{C}_2$, except possibly $I_{-2}$ when $q = 2$, or $K_5$ when $q = 3$.

Suppose for a contradiction that $N_1 \not< I_{-2}$ when $q = 2$, and write $N = N_1$, $I = I_{-2}$. Assume that $I = N G_0 \{ V_1, \ldots, V_4 \}$, where $V_1 \perp \cdots \perp V_4$ is a $-2$-decomposition. We have $N \leq N(I(W))$ for some $-2$-space $W$. Note that $Syl_3(N) \subseteq Syl_3(G_0)$, and that $N(I(V_1)) \cong 3^4.2^3.S_3$ contains a Sylow 3-subgroup $P$ of $I$. By Sylow's Theorem there exists $g \in I$ such that $P \leq N^g \leq N(I(W^g))$. It follows from 2.4.7 that $P$ does not fix a nonzero t.s. space or a $+1$-space or a n.d. 4-space, hence $P$ fixes a unique $-2$-space. Therefore $W^g = V_1$ whence

$$(3 \times GU_3(2)).2 \cong N^g = N(I(V_1)) \cong 3^4.2^3.S_3.$$  

But this is impossible as $O_3(N) \cong 3 \times 3^{1+2}$ while $O_3(N(I(V_1))) \cong 3^4$. If $N_1 \not< K_5$ when $q = 3$, then order considerations and 2.4.7 force $N_1 \leq Sp_6(2)$. However $Sp_6(2)$ has no subgroup $2^2 \times U_3(3).2 \cong N_1$.

PROPOSITION 4.2.5. Rows 15–18 of the results matrix hold.

Proof. The fact that $G_2(q)$ has no nontrivial $p$-modular projective representations of degree less than 6, coupled with Lagrange's Theorem, ensures that $G_2(q)$ is contained in no other member of $\mathcal{C}_2$. Thus the result follows from 3.1.1.

PROPOSITION 4.2.6. Row 61 of the results matrix holds.

Proof. Lagrange's Theorem shows that $N_4$ is contained in no other member of $\mathcal{C}_2$, except possibly $K_5 \cong \Omega_5^+(2)$ when $q \in \{3, 7\}$. In these cases
\(N_3\) is a Sylow 5-normalizer of \(G_0\). By 3.3.1, a Sylow 5-normalizer of \(K_5\) has order 400, hence \(N_3 \not\subset K_5\) when \(q = 7\). When \(q = 3\), then \(|N_3| = 400\), so we may regard \(N_3\) as a Sylow 5-normalizer of \(K_5\), hence columns VI and VII of row 61 hold by 1.3.3(ii). Because \(K_5\) does not extend to \(G_{A_4}\) or \(G_{\Sigma}\), columns XIII and XIV also hold.  

**Proposition 4.2.7.** Rows 51–54 of the results matrix hold.

*Proof.* Clearly \(N_4\) has 2-rank \(\geqslant 6\) and so \(N_4 \not\subset K_3\) (note \(q = p\) is odd). The fact that \(N_4\) is nonsolvable together with 3.4.2(ii)(c) imply that \(N_4\) is contained in none of the other groups in \(G_2\), except possibly \(K_5 \cong G_2(2)\). Suppose for a contradiction that \(N_4 \subset N_4(K_5)\), where \(G \cong G_0,4\) or \(G_0,3\). Then there exists \(\tau \in N_4 \cap G_5\), and by 2.3.8(ii), \(\tau\) induces a triality automorphism on \(K_5\). However, \(N_4\) is a parabolic subgroup of \(K_5\) which corresponds to just two nodes of the Dynkin diagram, and this means that \(N_4\) cannot be normalized by a triality automorphism of \(K_5\). This contradiction finishes the proof.  

We have now completed the proof of our theorem.

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